# INVERSION FORMULAS FOR THE CONTINUOUS WAVELET TRANSFORM ASSOCIATED WITH A DILATION MATRIX

**Somchai Suk-In**

A Thesis Submitted in Partial Fulfillment of the Requirements

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วิทยานิพนธ์นี้นำเสนอวิธีการบูรณะฟังก์ชันกำลังสองซึ่งหาปริพันธ์บนระนาบได้ จากการแปลงเวฟเลตแบบต่อเนื่อง โดยในที่นี้ นิยามตัวดำเนินการเปลี่ยนขนาดในรูปกำลังของเมทริกซ์เพิ่มขยาย และนิยามตัวดำเนินการเลื่อนขนานด้วยเวกเตอร์ในระนาบ ในการจำกัดเซตของการหาปริพันธ์อย่างเหมาะสม หรือ การดัดแปลงตัวปริพัทธ์ ค่าปริพันธ์ที่ถูกบูรณะอย่างอ่อนจะกลายเป็นค่าปริพันธ์ชนิดเลอเบกแบบจุด ซึ่งสามารถประมาณฟังก์ชันที่ต้องการด้วยความแม่นยำใดๆก็ได้ นอกจากนี้การศึกษานี้ยังได้กล่าวถึงการบูรณะโดยกรอบเวฟเลต และนำเสนอตัวอย่างของตัวก่อกำเนิดกรอบเวฟเลต

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CONTINUOUS WAVELET TRANSFORM / ADMISSIBLE FUNCTION / APPROXIMATE IDENTITY / WAVELET FRAME/

This thesis presents various methods of reconstructing a square integrable function on the plane from its continuous wavelet transform. The dilation operators are given by powers of an expanding matrix, and the translation operators by vectors in the plane. By suitably restricting the set of integration or modifying the integrand, the weak reconstruction integral becomes a pointwise Lebesgue integral approximating the given function with arbitrary accuracy. Reconstruction by wavelet frames is discussed as well, and examples of wavelet frame generators are presented.

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Somchai Suk-In

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# CHAPTER I

# INTRODUCTION

The classic Fourier transform has wide applications in the fields of science and engineering, such as signal processing for example. If $f(x) \in L^1(\mathbb{R})$, then its Fourier transform is the function $\hat{f}(\gamma)$ given by

$$\hat{f}(\gamma) = \mathcal{F}f(\gamma) = \int_{\mathbb{R}} f(x) e^{-2i\pi\gamma x}\, dx, \quad \gamma \in \mathbb{R}.$$

If $\hat{f} \in L^1(\mathbb{R})$ as well, then one can reconstruct $f$ from its Fourier transform $\hat{f}$ by

$$f(x) = \overline{\mathcal{F}}\hat{f}(x) = \int_{\mathbb{R}} \hat{f}(\gamma) e^{2i\pi\gamma x}\, d\gamma, \quad a.e.\ x \in \mathbb{R}.$$

Plancherel's theorem says that the restriction of the Fourier transform $\mathcal{F}$ to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is an isometry onto a dense subset of $L^2(\mathbb{R})$, and thus extends to a surjective isometry, also called the Fourier transform on $L^2(\mathbb{R})$.

For example, in signal processing, a function $f(x) \in L^2(\mathbb{R})$ is called a finite energy signal while $x$ is referred to as time. The value $\hat{f}(\gamma)$ is then interpreted as the contents of frequency $\gamma$ in $f$.

In the analysis of seismic data (Goupilland, Grossman and Morlet, 1984) or images (Mallat and Zhong, 1992) one deals with signals which have well localized and steep gradients. In image processing, these would occur at the edges of an object, for example. However, the Fourier transform does not reveal where such gradients occur. To see this, note that the Fourier transform of $f(x - x_0)$ is

$$\hat{f}_{x_0}(\gamma) = \int_{\mathbb{R}} f(x - x_0) e^{-2i\pi\gamma x} dx = \int_{\mathbb{R}} f(x) e^{-2i\pi\gamma(x+x_0)}\, dx = \hat{f}(\gamma) e^{-2i\pi\gamma x_0},$$

so translation of $x$ simply corresponds to a phase shift of the Fourier transform. Thus the magnitude of the Fourier transform does not show whether or where steep gradients occur.

For this reason, Grossmann and Morlet introduced the wavelet transform in 1984. Here one fixes a function $\psi \in L^2(\mathbb{R})$, and considers the 2-parameters family of dilates and translates,

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x}{a} - b\right) \qquad (a > 0, b \in \mathbb{R}).$$

Given $f \in L^2(\mathbb{R})$, its wavelet transform is the function

$$Wf(a,b) = \langle f, \psi_{a,b}\rangle = \frac{1}{\sqrt{a}}\int_{\mathbb{R}} f(x)\overline{\psi\left(\frac{x}{a} - b\right)}\, dx \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{R})$. Now if $\psi$ is well localized, say the support of $\psi$ is the interval [-1,1], then for fixed $a$ and $b$, the value of $Wf(a,b)$ depends on the values of $f$ in the interval $[(b-1)a, (b+1)a]$ only. For small values of $a$, the wavelet transform captures rapidly changing features of $f$, while for large values of $a$, it captures gradually changing features of $f$, at location determined by $b$.

Setting $a = e^t$ in (1.1), we obtain an alternative notation for the wavelet transform,

$$Wf(t,b) = e^{-t/2}\int_{\mathbb{R}} f(x)\overline{\psi\left(e^{-t}x - b\right)}\, dx, \qquad (t, b \in \mathbb{R}).$$

Thus, a natural extension of the wavelet transform to $\mathbb{R}^n$ is as follows. Given an invertible $n \times n$ matrix $A = e^B$ and a vector $b \in \mathbb{R}^n$, define dilation and translation operators by

$$(D_A f)(x) = |\det A|^{-1/2} f\left(A^{-1}x\right),$$

$$(T_b f)(x) = f\left(x - b\right),$$

for $f \in \mathrm{L}^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. The function

$$\mathrm{W}f(t,b) = \langle f, \mathrm{D}_{A^t} T_b \psi\rangle = \frac{1}{|\det A|^{t/2}} \int_{\mathbb{R}^n} f(x) \overline{\psi(A^{-t}x - b)} dx, \qquad (1.2)$$

where $t \in \mathbb{R}, b \in \mathbb{R}^n$ and $A^t = e^{tB}$, is then called the continuous wavelet transform of $f$. We say that $\psi$ is admissible if there exists a constant $c_\psi > 0$ such that $\|Wf\|_2^2 = c_\psi \|f\|_2^2$ for all $f \in L^2(\mathbb{R}^n)$. In this case, one can reconstruct a function $f$ from its wavelet transform as follows. By the polarization identity,

$$\langle Wf, Wg\rangle_{L^2(\mathbb{R}\times\mathbb{R}^n)} = c_\psi \langle f, g\rangle_{L^2(\mathbb{R}^n)},$$

for all $g \in L^2(\mathbb{R}^n)$, and hence

$$\begin{aligned} \langle f, g\rangle &= \frac{1}{c_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}^n} Wf(t,b) \overline{\langle g, D_{A^t} T_b \psi\rangle} db\, dt \\ &= \frac{1}{c_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \langle Wf(t,b) D_{A^t} T_b \psi, g\rangle \, db\, dt. \end{aligned}$$

That is,

$$f = \frac{1}{c_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}^n} Wf(t,b) D_{A^t} T_b \psi \, db\, dt \qquad (1.3)$$

as a weak integral in $L^2(\mathbb{R}^n)$.

Since (1.3) is a weak integral, it cannot be computed directly. One thus needs to know under what conditions on $f$ or $\psi$ reconstruction formula (1.3) holds as a usual integral,

$$f(x) = \frac{1}{c_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}^n} Wf(t,b) D_{A^t} T_b \psi(x) \, db\, dt \qquad (1.4)$$

for almost all $x$.

In the case of the wavelet transform in $L^2(\mathbb{R})$, it has been shown (Gasquet and Witomaski (1998)) that $f$ can be approximated arbitrarily by a usual integral. To be precise, if $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is admissible, then for each $\varepsilon > 0$,

$$f_\varepsilon(x) = \frac{1}{c_\psi} \int_\varepsilon^\infty \int_{\mathbb{R}} Wf(a,b) \psi_{a,b}(x) \frac{1}{a} \, db\, da$$

exists, $f_\varepsilon \in \mathrm{L}^2(\mathbb{R})$ and

$$\lim_{\varepsilon \to 0^+} \|f - f_\varepsilon\|_2 = 0. \tag{1.5}$$

Heil and Walnut (1989) have shown that if $\{\rho_n\}_{n=1}^\infty$ is a symmetric approximate identity for $L^2(\mathbb{R})$, then for each $n$ and almost all $x$, the function

$$f_n(x) = \int_0^\infty \int_{\mathbb{R}} Wf(a,b)(\rho_n * D_a T_b \psi)(x) \frac{1}{a}\, db\, da \tag{1.6}$$

exists, and $f_n$ converges to $f$ in $L^2(\mathbb{R})$.

It is natural to ask whether similar properties hold for the inverse wavelet transform in $L^2(\mathbb{R}^n)$. Thus, one may ask:

1. For each $\varepsilon \in \mathbb{R}$, does

$$f_\varepsilon(x) = \frac{1}{c_\psi} \int_\varepsilon^\infty \int_{\mathbb{R}^n} (Wf)(t,b) D_{A^t} T_b \psi(x)\, db\, dt \tag{1.7}$$

exist, and converge to $f$ in the square mean, as $\varepsilon \to -\infty$ ?

2. If $\{\rho_n\}$ is an approximate identity for $L^2(\mathbb{R}^n)$, and if we set

$$f_n(x) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} (Wf)(t,b)(\rho_n * D_{A^t} T_b \psi)(x)\, db\, dt, \tag{1.8}$$

is then

$$\lim_{n \to \infty} \|f - f_n\|_2 = 0\,? \tag{1.9}$$

Can one estimate how well $f_n$ approximates $f$? That is, given $\{\rho_n\}$ and $\varepsilon > 0$, can one find $N$ such that

$$\|f - f_n\|_2 < \varepsilon, \quad \forall n \geq N\,? \tag{1.10}$$

Another way to reconstruct $f$ from its wavelet transform is by means of frames. Here, one tries to reconstruct $f$ by an infinite series, which is computationally much simpler than reconstruction by an integral. Given $b > 0$, we say that the collection $\{D_{A^n}T_{mb}\psi\}_{n\in\mathbb{Z},m\in\mathbb{Z}^n}$ is a wavelet frame, if there exist constants $\alpha,\beta>0$ such that

$$\alpha\|f\|_2^2 \leq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n}|Wf(n,mb)|^2 \leq \beta\|f\|_2^2, \tag{1.11}$$

for all $f\in L^2(\mathbb{R}^n)$. It is known (Grossmann and Morlet (1984), Hernandez and Weiss (1996)) that in this case, $f$ can be obtained from its sequence of frame coefficients $\{Wf(n,mb)\}_{n\in\mathbb{Z},m\in\mathbb{Z}^n}$ as a series in $L^2(\mathbb{R}^n)$ by means of the dual frame.

Heil and Walnut have shown that in case of the one-dimensional wavelet transform (1.1), the collection $\{\psi_{a^n,mb}\}_{n,m\in\mathbb{Z}}$ is a frame in $L^2(\mathbb{R})$, provided that

1. supp$(\hat{\psi})\subset(-L,-l)\cup(l,L)$, where $0<l<L<\dfrac{1}{2b}$,
2. there exist $\alpha,\beta$ such that $\quad 0<\alpha\leq\sum_{n\in\mathbb{Z}}|\hat{\psi}(a^n\gamma)|^2\leq\beta \qquad$ for $a.e.$ $\gamma\in\mathbb{R}$.

The question is now whether this result generalizes to the multidimensional transform (1.2). This is called the discretrization problem.

In this thesis, we extend the well known reconstruction formulas for the classic continuous wavelet transform to $L^2(\mathbb{R}^2)$, and in the special case where the matrix $A$ is diagonalizable, to $L^2(\mathbb{R}^n)$ as well. We give conditions on $f$ and $\psi$ which guarantee that the weak reconstruction integral exists as a usual integral. Given a symmetric approximate identity $\{\rho_n\}$, we show that the approximate reconstruction (1.9) holds. Finally, we give sufficient conditions on $\psi\in L^2(\mathbb{R}^2)$ for wavelet frames to exist, and present examples of such frames.

This thesis is organized as follows. Chapter II introduces the necessary background knowledge from real analysis and linear algebra. In Chapter III, we give the definition of the classic continuous wavelet transform, and its generalization to $L^2(\mathbb{R}^n)$. In Chapter IV, we present approximate and exact reconstruction formulas for the continuous wavelet transform in $L^2(\mathbb{R}^2)$. As examples, we present the construction of a symmetric approximate identity and of several wavelet frame generators.

# CHAPTER II

# BACKGROUND

In this chapter, we review the theoretical background from real analysis and linear algebra which will be used throughout this thesis. Results are mostly stated without proof, which can be found in standard literature. Additional details and proofs can be founded in Apostol (1997), Cohn (1980), Folland (1999), Gasquet and Witomski (1998), and Wade (1999).

## 2.1 Basic Concepts from Real Analysis

Throughout, when considered as column vectors, elements of $\mathbb{R}^n$ will be denoted by $x, y$ or $z$, while when considered as row vectors, they will be denoted by the symbols $\gamma$ and $\eta$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ will be denoted by $\|x\|$ or $\|x\|_2$, while its maximum norm will be denoted by $\|x\|_\infty$. Hence,

$$\|x\| = \|x\|_2 = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

and

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|,$$

where $x = (x_1, x_2, ..., x_n)^T$.

**Definition 2.1.** Let $U \subset \mathbb{R}^n$ be open and let $F : U \to \mathbb{R}^m$ (respectively, $F : U \to \mathbb{C}^m$). We say that $F$ is *differentiable* at $x_o \in U$ if there exists an $m \times m$ real (respectively, complex) matrix $T$, depending on $x_0$ such that

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - T(x - x_0)\|}{\|x - x_0\|} = 0.$$

It is easily seen that if the components of $F$ are given by $f_1, f_2, ..., f_m$ and if $F$ is differentiable at $x_0 \in U$, then the partial derivatives $\frac{\partial f_i}{\partial x_j}(x_0)$ exist for any $i \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}$ and are given by the entries of the matrix $T$:

$$F'(x_0) \equiv T = \left( \frac{\partial f_i}{\partial x_j}(x_0) \right)_{ij}.$$

We call the matrix $F'(x_0)$ the *Jacobian matrix*. Its determinant

$$J_F(x_0) \equiv \det\left(F'(x_0)\right)$$

is called the *Jacobian of $F$ at $x_0$*.

**Definition 2.2.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. The *support* of $f$, denoted by $\text{supp}(f)$, is the set

$$\text{supp}(f) = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

Here, $\overline{A}$ denotes the closure of a set $A$. We say that $f$ has compact support if $\text{supp}(f)$ is a compact set.

**Definition 2.3.** Let $p \in \{1, 2, \ldots\}$. We set

1. $C^p(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \textit{ is } p \textit{ times continuously differentiable}\}$.

2. $C_c^p(\mathbb{R}^n) = \{f \in C^p(\mathbb{R}^n) : f \textit{ has compact support}\}$.

3. $C^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \textit{ is infinitely differentiable}\}$.

4. $C_c^\infty(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : f \textit{ has compact support}\}$.

5. $C_c(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \textit{ is continuous and has compact support}\}$.

**Theorem 2.1.** If $f \in C_c(\mathbb{R}^n)$, then $f$ is uniformly continuous.

*Proof.* Given $\epsilon > 0$, for each $x \in \text{supp}(f)$ there exists $\delta_x > 0$ such that $|f(x - y) - f(x)| < \frac{1}{2}\epsilon$ whenever $|y| < \delta_x$, by continuity of $f$. Since $\text{supp}(f)$ is compact, there exist $x_1, ..., x_N$ such that the balls of radius $\frac{1}{2}\delta_{x_j}$ about $x_j$ cover supp$(f)$. If $\delta = \frac{1}{2}\min\{\delta_{x_j}\}$, then, one easily sees that $|f(x-y) - f(x)| < \epsilon$ whenever $|y| < \delta, \quad \forall x \in \mathbb{R}^n$. $\square$

**Definition 2.4 (Equivalent Norms).** Let $X$ be a vector space. Two norms $\|\cdot\|$ and $\|\cdot\|_o$ on $X$ are said to be equivalent, if there exist constants $a, b > 0$ such that

$$a\|x\| \leq \|x\|_o \leq b\|x\|,$$

for all $x \in X$.

**Theorem 2.2.** On a finite dimensional vector space $X$, any two norms $\|\cdot\|$ and $\|\cdot\|_o$ are equivalent. In particular, any two norms on $\mathbb{R}^n$ are equivalent.

For example, $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$ for all $x \in \mathbb{R}^n$.

**Theorem 2.3 (Change of Variables).** Let $U \subset \mathbb{R}^n$ be open, and $F : U \to \mathbb{R}^n$ be in $C^1(\mathbb{R}^n)$, injective, with $J_F(x) \neq 0$ for all $x \in U$. Set $V = F(U)$. If $f : V \to \mathbb{C}$ is Lebesgue measurable, then $f \circ F : U \to \mathbb{C}$ is Lebesgue measurable. Furthermore

$$\int_V f(x)d\lambda(x) = \int_U (f \circ F)(x)|J_F(x)|d\lambda(x)$$

in the sense that if one of these Lebesgue integral exists, then both exist and are equal.

Note that if $F$ itself is a linear map, and if $A$ is the matrix associated with $F$, then $J_F = A$ so that

$$\int_{\mathbb{R}^n} f(x)d\lambda(x) = \int_{\mathbb{R}^n} f(Ax)|\text{det}A|d\lambda(x)$$

**Theorem 2.4 ($C^\infty$ Version of Urysohn's Lemma).** Let V be open in $\mathbb{R}^n$, and $H \subset V$ be compact and nonempty. Then there exists $h \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq h(x) \leq 1$ for all $x \in \mathbb{R}^n$, $h(x) = 1$ for all $x \in H$ and supp$(h) \subset V$.

**Theorem 2.5 (Mean Value Theorem).** Let $V$ be nonempty and open in $\mathbb{R}^n$ and $f : V \to \mathbb{R}^m$ be differentiable on $V$. If $a, x \in V$ and the line segment from $a$ to $x$, $L(x, a)$ is contained in $V$, then given any $u \in \mathbb{R}^m$ there exists $c \in L(x, a)$ such that

$$u \cdot (f(x) - f(a)) = u \cdot (J_f(c)(x - a)).$$

If $m = 1$, then $J_f = \nabla f$ (gradient of $f$), and choosing $u = 1$ we have

$$f(x) - f(a) = J_f(c)(x - a).$$

## 2.2 Spaces of Integrable Functions

In this section we review some theorems from integration theory which will be used in this thesis. We assume that the reader is familiar with basic concepts from measure theory, as discussed in Cohn (1980), or Folland (1999), for example.

**Definition 2.5.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p < \infty$. Then $L^p(X, \mathcal{M}, \mu)$ is the set of equivalence classes of $\mathcal{M}$-measurable functions $f : X \to \mathbb{C}$ (resp. $f : X \to \mathbb{R}$) such that $|f|^p$ is integrable. Here, two functions $f$ and $g$ are called equivalent, written $f \sim g$, if $f(x) = g(x)$ a.e. For ease of notation, we usually confuse a function $f$ with its equivalence class in $L^p(X, \mathcal{M}, \mu)$, and simply write

$$L^p(X) = L^p(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{C} | f \text{ is } \mathcal{M}\text{-measurable}, \int_X |f|^p d\mu < \infty\}.$$

Then the number

$$\|f\|_p = \left( \int_X |f|^p d\mu \right)^{1/p}.$$

is a norm on $L^p(X, \mathcal{M}, \mu)$, and $L^p(X, \mathcal{M}, \mu)$ is Banach space.

**Definition 2.6.** Let $(X, \mathcal{M}, \mu)$ be a measure space. An $\mathcal{M}$-measurable function $f : X \to \mathbb{C}$ is said to be essentially bounded, if there exists $M \geq 0$ such that

$|f(x)| \leq M$ a.e. Such a number $M$ is called an essential bound for $f$. Set

$$L^\infty(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{C} | f \text{ is } \mathcal{M}\text{-measurable and essential bounded}\}$$

where again, we have identified functions which are equal a.e. Then

$$\|f\|_\infty = \inf\{M : M \text{ is an essential bounded of } f\}$$

is a norm on $L^\infty(X, \mathcal{M}, \mu)$, and $L^\infty(X, \mathcal{M}, \mu)$ is a Banach space.

Note that $\mathbb{R}^n$ and $\mathbb{C}^n$ can be viewed as $L^p$-spaces. To see this, let $X = \{1, 2, 3, ..., n\}$, $\mathcal{M}$ be its power set,i.e. $\mathcal{M} = \mathcal{P}(X)$, and $\mu$ be the counting measure. There is a 1-1 correspondence between functions $f : X \to \mathbb{R}\,(\text{or}\,\mathbb{C})$ and vectors $(x_1, x_2, x_3, ..., x_n)$ in $\mathbb{R}^n$ $(\text{or}\,\mathbb{C}^n)$ given by $x_i = f(i)$. With this identification and since the integral with respect to the counting measure is simply a sum,

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|\right)^{1/p}, \quad 1 \leq p < \infty$$

and

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

$\forall x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ (resp. $\mathbb{C}^n$).

In this thesis, we deal with the measure space $(\mathbb{R}^n, \mathcal{M}_\lambda, \lambda)$, where $\mathcal{M}_\lambda$ is the $\sigma-$algebra of Lebesgue measurable subsets of $\mathbb{R}^n$, and $\lambda$ the Lebesgue measure. For simplicity, we set $L^p(\mathbb{R}^n) \equiv L^p(\mathbb{R}^n, \mathcal{M}_\lambda, \lambda)$.

**Theorem 2.6 (Hölder's Inequality).** Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p \leq \infty$, and $q$ be the conjugate of $p$, that is, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $f \in L^p(X, \mathcal{M}, \mu)$ and $g \in L^q(X, \mathcal{M}, \mu)$ then $fg \in L^1(X, \mathcal{M}, \mu)$ and

$$\int |fg| d\mu \leq \|f\|_p \|g\|_q.$$

If $p = q = 2$ then this is called the *Cauchy-Schwartz Inequality.*

**Theorem 2.7 (Monotone Convergence Theorem).** Let $\{f_n\} \geq 0$ be a sequence of $\mathcal{M}$-measurable functions such that $0 \leq f_n \leq f_{n+1}$ for all $n$. Then $\displaystyle\int (\lim_{n\to\infty} f_n) d\mu = \lim_{n\to\infty} \int f_n d\mu.$

**Theorem 2.8.** Let $1 \leq p < \infty$. Then $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

**Theorem 2.9 (Fubini's Theorem).** Let $f : \mathbb{R}^{m+n} \to \mathbb{C}$ be measurable. Then

1. $f_y(x) = f(x,y)$ from $\mathbb{R}^m \to \mathbb{C}$ is measurable for each fixed $y \in \mathbb{R}^n$ (and hence $x \longmapsto |f(x,y)|$ is measurable $\forall\, y \in \mathbb{R}^n$) and $g_x(y) = f(x,y)$ from $\mathbb{R}^n \to \mathbb{C}$ is measurable for each fixed $x \in \mathbb{R}^m$ (and hence $y \longmapsto |f(x,y)|$ is measurable $\forall\, x \in \mathbb{R}^m$).

2. The functions $h(y) = \displaystyle\int_{\mathbb{R}^m} |f_y(x)|\, d\lambda(x) = \int_{\mathbb{R}^m} |f(x,y)|\, d\lambda(x)$ from $\mathbb{R}^n$ to $\mathbb{C}$ and
$k(x) = \displaystyle\int_{\mathbb{R}^n} |g_x(y)|\, d\lambda(y) = \int_{\mathbb{R}^n} |f(x,y)|\, d\lambda(y)$ from $\mathbb{R}^n$ to $\mathbb{C}$ are measurable.

It follows that

**i)** $\displaystyle\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |f(x,y)|\, d\lambda(x) d\lambda(y)$

**ii)** $\displaystyle\int_{\mathbb{R}^m} \int_{\mathbb{R}^n} |f(x,y)|\, d\lambda(y) d\lambda(x)$

**iii)** $\displaystyle\int_{\mathbb{R}^{m+n}} |f(x,y)|\, d\lambda(x,y)$

all exist (possibly $\infty$).

If at least one of **i)**,**ii)** or **iii)** is finite, then

a) $f_y(x) = f(x,y) \in L^1(\mathbb{R}^m)$ for almost all $y$,
$g_x(y) = f(x,y) \in L^1(\mathbb{R}^n)$ for almost all $x$

b) $\tilde{h}(y) = \displaystyle\int_{\mathbb{R}^m} f(x,y)\, d\lambda(x) \in L^1(\mathbb{R}^n)$
$\tilde{k}(x) = \displaystyle\int_{\mathbb{R}^n} f(x,y)\, d\lambda(y) \in L^1(\mathbb{R}^m)$,
$f(x,y) \in L^1(\mathbb{R}^{m+n})$

c) Double and iterated integrals are equal

$$\int_{\mathbb{R}^{m+n}} f(x,y)\, d\lambda(x,y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} f(x,y)\, d\lambda(x)\lambda(y) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} f(x,y)\, d\lambda(y)d\lambda(x).$$

When integrating over $\mathbb{R}^n$, we often denote the Lebesgue integral simply by $\int_{\mathbb{R}^n} f(x)\, dx$ instead of $\int_{\mathbb{R}^n} f(x)\, d\lambda(x)$.

**Definition 2.7 (Convolution).** Let $f$ and $g$ be measurable functions on $\mathbb{R}^n$. The *convolution* of $f$ and $g$ is the function $f * g$ defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} f(y)g(x-y)dy$$

for all $x$ for which the integral exists. Various conditions can be imposed on $f$ and $g$ to guarantee that $f * g$ is defined at least almost everywhere. For example, if $f$ is bounded and compactly supported, $g$ can be any locally integrable function.

The elementary properties of convolutions are summarized in the following theorem. Let's us first introduce some notation : If $K, L \subset \mathbb{R}^n$, we set $K + L = \{x + y : x \in K,\, y \in L\}$.

**Theorem 2.10.** Assuming that all integrals in question exist, we have

a. $f * g = g * f$.

b. $(f * g) * h = f * (g * h)$.

c. $\text{supp}(f * g) \subset \overline{\text{supp}(f) + \text{supp}(g)}$

**Theorem 2.11 (Young's inequality).** Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n), 1 \leq p, q, r \leq \infty$, and $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r} + 1$. Then $f * g \in L^r(\mathbb{R}^n)$, and $\|f * g\|_r \leq \|f\|_p \|g\|_q$.

The following two properties are direct consequences of Young' inequality.

1. If $f \in L^p(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n), 1 \leq p < \infty$, then $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_p \leq \|f\|_p \|g\|_1$.

2. if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, $1 \le p, q \le \infty$, then $f*g \in L^\infty(\mathbb{R}^n)$, and $\|f*g\|_\infty \le \|f\|_p \|g\|_q$.

**Definition 2.8 (Approximate Identity).** We say that $\{\rho_n\}_{n=1}^\infty \subset L^1(\mathbb{R}^n)$ is an *approximate identity* for $L^p(\mathbb{R}^n)$ , if $\{\rho_n\}$ satisfies

a. $\rho_n > 0, \quad \forall n$

b. $\displaystyle\int_{\mathbb{R}^n} \rho_n(x)dx = 1, \quad \forall n$

c. $\displaystyle\lim_{n\to\infty} \|(\rho_n * f) - f\|_p = 0, \quad \forall f \in L^p(\mathbb{R}^n).$

## 2.3 Frames and Weak Integrals

The concept of frames is a generalization of that of a basis of a Hilbert space. For details, see Heil and Walnut (1989), or Hernandez and Weiss (1996).

**Definition 2.9 (Frames).** A collection of elements $\{\varphi_j : j \in \mathbb{N}\}$ in a Hilbert space $\mathcal{H}$ is called a *frame* if there exist constants $\alpha$ and $\beta$, $0 < \alpha \le \beta < \infty$, such that

$$\alpha\|f\|^2 \le \sum_{j\in\mathbb{N}} |\langle f, \varphi_j\rangle|^2 \le \beta\|f\|^2 \qquad \forall f \in \mathcal{H}.$$

The constants $\alpha, \beta$ are called frame bounds. If $\alpha = \beta$ , we say that the frame is tight. We can reconstruct $f$ from its frame coefficients $\{\langle f, \varphi_j\rangle\}_{j\in\mathbb{N}}$ as follows :

**Theorem 2.12 (Dual Frame).** Let $\{\varphi_j : j \in \mathbb{N}\}$ be a frame on a Hilbert space $\mathcal{H}$ with frame bounds $\alpha$ and $\beta$. Then $\exists S \in B(\mathcal{H})$ such that the collection $\{\widetilde{\varphi_j} := S^{-1}(\varphi_j) : j \in \mathbb{N}\}$ is also a frame with bounds $\dfrac{1}{\beta}$ and $\dfrac{1}{\alpha}$ and $f = \displaystyle\sum_{j\in\mathbb{N}} \langle f, \varphi_j\rangle \widetilde{\varphi_j} = \sum_{j\in\mathbb{N}} \langle f, \widetilde{\varphi_j}\rangle \varphi_j \quad \forall f \in \mathcal{H}$. The family $\{\widetilde{\varphi_j} : j \in \mathbb{N}\}$ is called the *dual frame* to $\{\varphi_j : j \in \mathbb{N}\}$.

**Definition 2.10 (Weakly Measurable Function and Weak Integral).** Let $\mathcal{H}$ be a Banach space, $(X, \mathcal{M}, \mu)$ a measure space, and $\varphi : X \to \mathcal{H}$. We say $\varphi$ is *weakly measurable*, if for each $g \in \mathcal{H}^*$, where $\mathcal{H}^*$ is the dual space of $\mathcal{H}$, the function $F_g : X \to \mathbb{C}$ given by $F_g(x) := g(\varphi(x))$ is $\mathcal{M}-$measurable. If $\varphi(x)$ is weakly measurable, and if there exists $f \in \mathcal{H}$ such that

$$g(f) = \int_X g(\varphi(x))\, d\mu(x) \qquad \forall g \in \mathcal{H}^*$$

then we say that $f(x) = \displaystyle\int_X \varphi(x)\, d\mu$ as a *weak integral.*

Note that if $\mathcal{H}$ is a Hilbert space, then by the Riesz representation theorem, all bounded linear functionals on $\mathcal{H}$ are of the form $f \mapsto \langle f, g \rangle \qquad g \in \mathcal{H}$. Thus, $\varphi : X \mapsto \mathcal{H}$ is weakly measurable if and only if $x \in X \mapsto \langle \varphi(x), g \rangle$ is measurable for all $g \in \mathcal{H}$ and $f(x) = \int_X \varphi(x)\, d\mu$ as a *weak integral* if and only if

$$\langle f, g \rangle = \int_X \langle \varphi(x), g \rangle\, d\mu \tag{2.1}$$

for all $g \in \mathcal{H}$.

## 2.4 The Fourier Transform

**Definition 2.11.** The *Fourier Transform* of $f \in L^1(\mathbb{R}^n)$ is defined by

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) := \int_{\mathbb{R}^n} f(x) e^{-2i\pi\gamma\cdot x} dx \quad (\gamma \in \mathbb{R}^n).$$

If we write $\gamma$ as a row vector, and $x$ as a column vector, then the dot product $\gamma \cdot x$ is simply multiplication of a row vector with a column vector and we can write $\hat{f}(\gamma) = \displaystyle\int_{\mathbb{R}^n} f(x) e^{-2i\pi\gamma x}\, dx$. Similarly, the *inverse Fourier transform* of $f \in L^1(\mathbb{R}^n)$ is given by

$$\overline{\mathcal{F}}f(x) = \check{f}(x) := \int_{\mathbb{R}^n} f(\gamma) e^{2i\pi\gamma\cdot x}\, d\gamma.$$

**Theorem 2.13 (Plancherel's Theorem).** If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ , then $\hat{f} \in L^2(\mathbb{R}^n)$ and $\|\hat{f}\|_2 = \|f\|_2$. In fact, the restriction of the Fourier transform,

$$\mathcal{F} : L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

extends uniquely to an isometry,

$$\widetilde{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

between Hilbert spaces. Furthermore $\widetilde{\mathcal{F}}(\widetilde{\mathcal{F}}(f))(x) = f(-x),\ a.e \quad \forall f \in L^2(\mathbb{R}^n)$.

In the following, this Fourier transform on $L^2(\mathbb{R}^n)$ will also be denoted by $\mathcal{F}$, and we will set $\hat{f} = \mathcal{F}(f), \quad \forall f \in L^2(\mathbb{R}^n)$.

**Definition 2.12 (Bandlimited Function).** A function $f \in L^2(\mathbb{R}^n)$ is called *bandlimited*, if there exists $M \subset \mathbb{R}^n$, $M$ is compact, such that $\hat{f}(\gamma) = 0$ a.e on $M^c$.

## 2.5 Exponential Matrices

Recall that if $a > 0$, then $a^t = e^{t \ln a}$ for all $t \in \mathbb{R}$. One uses this idea to define real powers $A^t$ of a matrix $A$.

**Theorem 2.14.** Let $A$ be an $n \times n$ real or complex matrix. Then the series

$$\sum_{k=0}^{\infty} \frac{A^k}{k!}$$

converges. (Here we make the convention that $A^0 = I$, the identity matrix)

*Proof.* Let $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices with complex entries. Since $M_n(\mathbb{C})$ is a finite dimensional vector space, any two norms are equivalent, and $M_n(\mathbb{C})$ is complete. Choose the operator norm on $M_n(\mathbb{C})$. Then

$$\|AB\| \leq \|A\|\|B\|$$

for all $A, B \in M_n(\mathbb{C})$. Next let $A \in M_n(\mathbb{C})$. It is enough to show that the sequence of partial sums $S_N = \sum_{k=0}^{N} \frac{A^k}{k!}$ $(N = 1, 2, 3, ...)$ is Cauchy. Now, for any $N$, $l \in \mathbb{N}$,

$$\begin{aligned} \|S_{N+l} - S_N\| &= \left\| \sum_{k=N+1}^{N+l} \frac{1}{k!} A^k \right\| \\ &\leq \sum_{k=N+1}^{N+l} \frac{1}{k!} \|A^k\| \\ &\leq \sum_{k=N+1}^{N+l} \frac{1}{k!} \|A\|^k. \end{aligned} \tag{2.2}$$

Since the series $\sum_{k=0}^{\infty} \frac{\|A\|^k}{k!}$ converges to $e^{\|A\|}$ in $\mathbb{R}$, its sequence of partial sums is Cauchy. That is, given $\varepsilon > 0$, $\exists N_0$ such that

$$\sum_{k=N+1}^{N+l} \frac{1}{k!} \|A\|^k < \varepsilon \qquad \forall N \geq N_0, l \in \mathbb{N}.$$

Then by (2.2), $\|S_{N+l} - S_N\| < \varepsilon \qquad \forall N \geq N_0, l \in \mathbb{N}$ which shows that $\{S_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $M_n(\mathbb{C})$ . Since $M_n(\mathbb{C})$ is complete, it follows that $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ converges. $\square$

**Definition 2.13.** Given an $n \times n$ matrix $A$, we define its *exponential* $e^A$ to be the $n \times n$ matrix given by the convergent matrix series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Now if $A = e^B$, then we define $A^t = e^{tB}$.

**Proposition 2.1.** Let $A = e^B$ Then $A^{s+t} = A^s A^t$ for any $s, t \in \mathbb{R}$. In particular, $A \in \mathrm{GL}_n(\mathbb{R})$, the set of invertible $n \times n$ matrices.

*Proof.* Let $F = \sum_{k=0}^{\infty} \frac{(sB)^k}{k!}$, $G = \sum_{k=0}^{\infty} \frac{(tB)^k}{k!}$ and $H = \sum_{k=0}^{\infty} C_k$ where $C_k = \frac{(s+t)^k B^k}{k!}$. We need to show that $H = FG$.

By the Binomial Theorem, $C_k = \displaystyle\sum_{j=0}^{k} \frac{(sB)^j (tB)^{k-j}}{j!(k-j)!}$ for all $k$. For each $n \in \mathbb{N}$, we set

$$F_n = \sum_{j=0}^{n} \frac{(sB)^j}{j!},\ G_n = \sum_{k=0}^{n} \frac{(tB)^k}{k!} \text{ and } H_n = \sum_{k=0}^{n} C_k = \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(sB)^j (tB)^{k-j}}{j!(k-j)!}.$$

By induction on $n$, it is easy to see that

$$H_n = \sum_{j=0}^{n}\sum_{k=j}^{n} \frac{(sB)^j}{j!}\frac{(tB)^{k-j}}{(k-j)!}.$$

Thus,

$$\begin{aligned} H_n &= \sum_{j=0}^{n} \frac{(sB)^j}{j!} \sum_{k=j}^{n} \frac{(tB)^{k-j}}{(k-j)!} \\ &= \sum_{j=0}^{n} \frac{(sB)^j}{j!} G_{n-j} \\ &= \sum_{j=0}^{n} \frac{(sB)^j}{j!} G_{n-j} + F_n G - F_n G \\ &= F_n G + \sum_{j=0}^{n} \frac{(sB)^j}{j!}(G_{n-j} - G). \end{aligned}$$

Since $H_n \to H$ and $F_n \to F$ as $n \to \infty$, it suffices to show that

$$\lim_{n\to\infty} \sum_{j=0}^{n} \frac{(sB)^j}{j!}(G_{n-j} - G) = 0.$$

Let $\varepsilon > 0$ be given. Since the sequence $\{G_n\}$ converges, it is bounded, hence there exists $M > 0$ such that

$$\|G_{n-j} - G\| \leq M$$

for all integers $n > j > 0$. Since $L = \displaystyle\sum_{j=0}^{\infty} \frac{\|sB\|^j}{j!}$ is finite, there exist $N \in \mathbb{N}$ such that $l \geq N$ implies $\|G_l - G\| < \dfrac{\varepsilon}{2L}$ and $\displaystyle\sum_{j=N+1}^{\infty} \frac{\|sB\|^j}{j!} < \frac{\varepsilon}{2M}$.

Then for all $n > 2N$,

$$\begin{aligned}
\left\|\sum_{j=0}^{n} \frac{(sB)^j}{j!}(G_{n-j} - G)\right\| &= \left\|\sum_{j=0}^{N} \frac{(sB)^j}{j!}(G_{n-j} - G) + \sum_{j=N+1}^{n} \frac{(sB)^j}{j!}(G_{n-j} - G)\right\| \\
&< \frac{\varepsilon}{2L}\sum_{j=0}^{N} \frac{\|sB\|^j}{j!} + M \sum_{j=N+1}^{n} \frac{\|sB\|^j}{j!} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, $H_n$ converge to $FG$. That is, $A^{s+t} = e^{(s+t)B} = e^{sB}e^{tB} = A^s A^t$. □

**Theorem 2.15.** Let $A = e^B$ be an exponential matrix. If $P$ is an invertible matrix, and if we set $\tilde{A} = PAP^{-1}$, then $\tilde{A}$ is also an exponential matrix and in fact, $\tilde{A} = e^{\tilde{B}}$ where$\tilde{B} = PBP^{-1}$. Furthermore, $\tilde{A}^t = PA^tP^{-1}$ for all $t \in \mathbb{R}$.

*Proof.* Observe that

$$\begin{aligned}
PA^tP^{-1} &= P\left(\sum_{k=0}^{\infty} \frac{t^k B^k}{k!}\right)P^{-1} \\
&= \sum_{k=0}^{\infty} \frac{t^k PB^kP^{-1}}{k!} \\
&= \sum_{k=0}^{\infty} \frac{t^k (PBP^{-1})^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(t\tilde{B})^k}{k!} \\
&= e^{t\tilde{B}} \\
&= \tilde{A}^t
\end{aligned}$$

where we have used the fact that matrix multiplication is continuous. Choosing $t = 1$, we obtain $\tilde{A} = e^{\tilde{B}}$. Then by definition 2.13, $\tilde{A}^t = e^{t\tilde{B}} = PA^tP^{-1}$. □

The interested reader may notice that the discussion of this section also applies to elements of $\mathcal{B}(X)$, the set of bounded linear operators on a normed space $X$, with identical proofs.

## 2.6 The Real Jordan Normal Form

Any real or complex square matrix is similar to an upper triangular matrix, but not necessarily similar to a diagonal matrix. In an advanced linear algebra course one usually proves that every complex square matrix is similar to a matrix which is nearly diagonal, namely has nonzero entries only in the diagonal and directly above the diagonal, called the Jordan normal form of the matrix. A similar characterization exists for real matrices as outlined below.

**Definition 2.14.** A real Jordan block is a real upper triangular square matrix $[b_{ij}]$ of one of the two following forms,

$$B = \begin{bmatrix} \lambda & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \lambda \end{bmatrix} \quad \text{with} \quad \lambda \in \mathbb{R},$$

or

$$B = \begin{bmatrix} D & I_2 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ (0) & & & D \end{bmatrix} \quad D = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \quad \alpha, \beta \in \mathbb{R} \quad \text{with} \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By a suitable change of basis, every real matrix can be brought into block diagonal form, where all blocks of this form:

**Theorem 2.16.** Let $A$ be an $n \times n$ real matrix. Then $A$ is similar to a block diagonal matrix of the form $J = diag(J_1, J_2, ..., J_m)$ with each $J_k$ being a real Jordan block, $k = 1, 2, ..., m$. The Jordan blocks are determined by the eigenvalues $\lambda$ of $A$. A real eigenvalue gives rise to a real Jordan block of the first type while a complex pair $\alpha \pm i\beta$ of eigenvalues gives rise to a real Jordan block of the second type. The matrix $J$ is called the real Jordan normal form of $A$.

*Proof.* Since we will mainly deal with $2 \times 2$ matrices, let us give a proof of the theorem for this particularly simple case only. That is, we will show that every real $2 \times 2$ matrix is similar to a matrix of one of the following forms,

$$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{or} \quad B = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

depending on whether $A$ has two, one, or no (linearly independent) eigenvectors.

**Case 1.** $A$ has two eigenvectors, say $v_1, v_2$. Let $\lambda_1$ and $\lambda_2$ denote the corresponding real eigenvalues. The matrix $P = [v_1\ v_2]$ is invertible because $v_1, v_2$ are linearly independent, and obviously $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

**Case 2.** $A$ has only one eigenvector, call it $v_1$. Then $A$ has only one eigenvalue, call it $\lambda$. Let $w$ be any other vector which is not a scalar multiple of $v_1$, so that $\{v_1, w\}$ is a basis of $\mathbb{R}^2$, and $w$ is not an eigenvector. We claim that $(A - \lambda I)w = kv_1$ for some scalar $k$.

To proof the claim, suppose to contrary that $(A - \lambda I)w = kv_1 + lw$ where $l \neq 0$. Set $z = \frac{k}{l}v_1 + w$. Then

$$\begin{aligned} (A - \lambda I)z = (A - \lambda I)(\frac{k}{l}v_1 + w) &= \frac{k}{l}(A - \lambda I)v_1 + (A - \lambda I)w \\ &= (A - \lambda I)w \\ &= kv_1 + lw \\ &= l(\frac{k}{l}v_1 + w) = lz \end{aligned}$$

which shows that $z$ is an eigenvector for $A$ with eigenvalue $\lambda + l \neq \lambda$, contradicting the uniqueness of the eigenvalue $\lambda$. This proves the claim.

If $k = 0$, then $w$ is an eigenvector of $A$ belonging to $\lambda$, contradicting uniqueness of the eigenvector. Thus $k \neq 0$, and we can set $v_2 = \frac{1}{k}w + v_1$. Then by the claim, $Av_2 = A(\frac{1}{k}w+v_1) = \frac{1}{k}Aw+Av_1 = \frac{1}{k}(\lambda w+kv_1)+\lambda v_1 = \lambda v_2+v_1$. Now, as $v_1, v_2$ are linearly independent, the matrix $P = [v_1\ v_2]$ is invertible. Then $AP = A[v_1\ v_2] = [Av_1\ Av_2] = [\lambda v_1\ v_1 + \lambda v_2]$ while $PB = [v_1\ v_2]\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = [\lambda v_1\ v_1 + \lambda v_2]$. Hence, $AP = PB$, that is, $P^{-1}AP = B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

**Case 3.** $A$ has no eigenvector. Then the eigenvalues of $A$ are complex, say $\lambda, \overline{\lambda} = \alpha \pm i\beta$. Applying the argument of case 1 to this complex case, there exists an invertible complex matrix $P$ such that $A = P\begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}P^{-1}$, and the columns of $P$ are the complex eigenvectors $v_1$ and $v_2$ of $A$, say $P = [v_1\ v_2]$. Observe that if $x$ is an eigenvector belonging to an eigenvalue $\lambda$, then as $A$ has real entries, $A\bar{x} = \overline{Ax} = \overline{\lambda x} = \bar{\lambda}\bar{x}$, that is, $\bar{x}$ is an eigenvector belonging to the eigenvalue $\bar{\lambda}$. It follows that if $v_1 = \begin{bmatrix} r_1 + is_1 \\ r_2 + is_2 \end{bmatrix}$ then we can choose $v_2 = \bar{v_1} = \begin{bmatrix} r_1 - is_1 \\ r_2 - is_2 \end{bmatrix}$.

Now if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $Av_1 = \lambda v_1$ gives

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} r_1 + is_1 \\ r_2 + is_2 \end{bmatrix} = (\alpha + i\beta)\begin{bmatrix} r_1 + is_1 \\ r_2 + is_2 \end{bmatrix}$$

that is

$$(ar_1 + br_2) + i(as_1 + bs_2) = (\alpha r_1 - \beta s_1) + i(\alpha s_1 + \beta r_1)$$

$$(cr_1 + dr_2) + i(cs_1 + ds_2) = (\alpha r_2 - \beta s_2) + i(\alpha s_2 + \beta r_2).$$

Comparing real and imaginary parts,

$$ar_1 + br_2 = \alpha r_1 - \beta s_1$$

$$as_1 + bs_2 = \alpha s_1 + \beta r_1$$

$$cr_1 + dr_2 = \alpha r_2 - \beta s_2$$

$$cs_1 + ds_2 = \alpha s_2 + \beta r_2$$

or in matrix notation,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix} = \begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}. \tag{2.3}$$

Set $P = \begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix}$. Note that $\det P \neq 0$ since $v_1$ and $v_2$ are linearly independent. Then (2.3) gives

$$A = P \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} P^{-1}.$$

□

We note that each block of the form

$$D = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

can be rewritten as

$$D = |\lambda| \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

where $|\lambda| = \sqrt{\alpha^2 + \beta^2}$ and $\cos\theta = \dfrac{\alpha}{|\lambda|}, \sin\theta = \dfrac{\beta}{|\lambda|}$.

# CHAPTER III

# THE CONTINUOUS WAVELET TRANSFORM

In this chapter, we review the usual continuous wavelet transform on $L^2(\mathbb{R})$ as first introduced by Grossmann and Morlet (1984) and also described in Gasquet-Witomaski (1998) and Heil-Walnut (1989). We then describe the extension of the wavelet transform to $L^2(\mathbb{R}^n)$.

## 3.1 The Continuous Wavelet Transform on $L^2(\mathbb{R})$

Let us first introduce the operators on $L^2(\mathbb{R})$ which are essential in the discussion of the wavelet transform.

**Definition 3.1 (Dilation, Translation, Modulation).** Given $a > 0$ and $b \in \mathbb{R}$, we define operators $D_a, T_b, E_b$ on $L^2(\mathbb{R})$ by

1. $(D_a f)(x) = \frac{1}{\sqrt{a}} f\left(\frac{x}{a}\right) \qquad \forall x \in \mathbb{R}$ (Dilation by $a$)

2. $(T_b f)(x) = f(x-b), \qquad \forall x \in \mathbb{R}$ (Translation by $b$)

3. $(E_b f)(x) = e^{2i\pi bx} f(x), \qquad \forall x \in \mathbb{R}$ (Modulation by $b$)

for $f \in L^2(\mathbb{R})$.

All three operators map $L^2(\mathbb{R})$ isometrically onto itself. Furthermore, they intertwine with the Fourier transform as follows:

a) $\widehat{D_a f} = D_{a^{-1}} \hat{f}$

b) $\widehat{T_b f} = E_{-b} \hat{f}$

c) $\widehat{E_b f} = T_b \hat{f}$

for all $f \in L^2(\mathbb{R})$. The proof will given in the section 3.3.

**Definition 3.2 (Wavelet Transform on $L^2(\mathbb{R})$).** Fix $\psi \in L^2(\mathbb{R})$, and call it the mother wavelet. Consider the 2-parameter family of dilates and translates of $\psi$,

$$\psi_{a,b}(x) = (D_a T_b \psi)(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x}{a} - b\right) \qquad (a > 0, b \in \mathbb{R}).$$

The wavelet transform of a function $f \in L^2(\mathbb{R})$ is the function $Wf : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$ given by

$$Wf(a,b) =< f, \psi_{a,b} >= \frac{1}{\sqrt{a}} \int_{\mathbb{R}} f(x) \overline{\psi\left(\frac{x}{a} - b\right)}\, dx.$$

The mapping $W$ is called the *continuous wavelet transform* associated with $\psi$. Obviously, it is linear.

We would like to reconstruct the function $f$ from its wavelet transform $Wf(a,b)$. Consider the measurable space $(\mathbb{R}^+ \times \mathbb{R}, \mathcal{M}_\lambda)$, with measure $d\mu = \frac{1}{a} da\, db$. By this we mean that for $E \in \mathcal{M}_\lambda$,

$$\mu(E) = \int_{\mathbb{R}^+} \int_{\mathbb{R}} \chi_E(a,b) \frac{1}{a}\, db\, da$$

Following the definition of the Lebesgue integral, one obtains that

$$\int_{\mathbb{R}^+ \times \mathbb{R}} f(a,b)\, d\mu(a,b) = \int_{\mathbb{R}^+} \int_{\mathbb{R}} f(a,b) \frac{1}{a}\, db\, da$$

for all $f \in L^2(\mathbb{R}^+ \times \mathbb{R}, \mathcal{M}_\lambda, \mu)$.

Suppose we have shown that the wavelet transform $W$ associated with $\psi$ satisfies

$$\|Wf\|_{L^2(\mathbb{R}^+ \times \mathbb{R}, \mu)} = \sqrt{c_\psi} \|f\|_{L^2(\mathbb{R})} \tag{3.1}$$

for all $L^2(\mathbb{R})$, that is, $W$ is a multiple of an isometry of $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^+ \times \mathbb{R})$. By the polarization identity

$$\langle f, g \rangle = \frac{1}{4}\left[\|f+g\|^2 - \|f-g\|^2\right] + \frac{i}{4}\left[\|f+ig\|^2 - \|f-ig\|^2\right]$$

we then obtain that

$$\langle Wf, Wg\rangle_{L^2(\mathbb{R}^+\times\mathbb{R},\mu)} = c_\psi \langle f, g\rangle_{L^2(\mathbb{R})}, \qquad \text{for all } f, g \in L^2(\mathbb{R}). \tag{3.2}$$

To see this, suppose that (3.1) holds. Then

$$\begin{aligned}
&\langle Wf, Wg\rangle_{L^2(\mathbb{R}^+\times\mathbb{R},\mu)} \\
&= \frac{1}{4}\left[\|Wf + Wg\|^2_{L^2(\mathbb{R}^+\times\mathbb{R},\mu)} - \|Wf - Wg\|^2_{L^2(\mathbb{R}^+\times\mathbb{R},\mu)}\right] \\
&\quad + \frac{i}{4}\left[\|Wf + iWg\|^2_{L^2(\mathbb{R}^+\times\mathbb{R},\mu)} - \|Wf - iWg\|^2_{L^2(\mathbb{R}^+\times\mathbb{R},\mu)}\right] \\
&= \frac{1}{4}\left[\|W(f+g)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R},\mu)} - \|W(f-g)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R},\mu)}\right] \\
&\quad + \frac{i}{4}\left[\|W(f+ig)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R},\mu)} - \|W(f-ig)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R},\mu)}\right] \\
&= \frac{1}{4}\left[c_\psi\|(f+g)\|^2 - c_\psi\|(f-g)\|^2\right] + \frac{i}{4}\left[c_\psi\|(f+ig)\|^2 - c_\psi\|(f-ig)\|^2\right] \\
&= c_\psi\left[\frac{1}{4}\left(\|(f+g)\|^2 - \|(f-g)\|^2\right) + \frac{i}{4}\left(\|(f+ig)\|^2 - \|(f-ig)\|^2\right)\right] \\
&= c_\psi \langle f, g\rangle_{L^2(\mathbb{R})} \qquad \forall f, g \in L^2(\mathbb{R}).
\end{aligned} \tag{3.3}$$

Conversely, if (3.2) holds, then choosing $g = f$, we immediately obtain (3.1).

Now suppose that (3.1) holds. Then by (3.2),

$$\begin{aligned}
c_\psi\langle f, g\rangle &= \int_0^\infty \int_{\mathbb{R}} (Wf)(a,b)\overline{(Wg)(a,b)}\frac{1}{a}\, db\, da \\
&= \int_0^\infty \int_{\mathbb{R}} (Wf)(a,b)\langle \psi_{a,b}, g\rangle \frac{1}{a}\, db\, da \\
&= \int_0^\infty \int_{\mathbb{R}} \langle (Wf)(a,b)\psi_{a,b}, g\rangle \frac{1}{a}\, db\, da,
\end{aligned}$$

that is

$$\langle f, g\rangle = \frac{1}{c_\psi}\int_0^\infty \int_{\mathbb{R}} \langle (Wf)(a,b)\psi_{a,b}, g\rangle \frac{1}{a}\, db\, da$$

for all $f, g \in L^2(\mathbb{R})$. Thus

$$f = \frac{1}{c_\psi}\int_0^\infty \int_{\mathbb{R}} (Wf)(a,b)\psi_{a,b}(x)\frac{1}{a}\, db\, da \tag{3.4}$$

as *a weak integral.*

The next theorem classifies those $\psi \in L^2(\mathbb{R})$ which satisfy identity (3.1).

**Theorem 3.1 (Admissibility Condition ).** Let $\psi \in L^2(\mathbb{R})$. Then

$$\|Wf\|_{L^2(\mathbb{R}^+\times\mathbb{R},\mu)} = \sqrt{c_\psi}\|f\|_{L^2(\mathbb{R})}$$

for all $f \in L^2(\mathbb{R})$ if and only if

$$\int_0^\infty \frac{|\hat{\psi}(a)|^2}{|a|}da = \int_{-\infty}^0 \frac{|\hat{\psi}(a)|^2}{|a|}da = c_\psi. \tag{3.5}$$

*Proof.* See Heil-Walnut (1989). □

Because of this theorem, we call $\psi \in L^2(\mathbb{R})$ admissible if it satisfies condition (3.5), for some $c_\psi > 0$. Condition (3.5) is called the admissibility condition. From the previous discussion, it follows that if $\psi$ is admissible, then every $f \in L^2(\mathbb{R})$ can be reconstructed from its wavelet transform by means of the weak integral (3.4). The next theorem shows that $f$ can even be approximated arbitrarily by a usual integral, provided that $\psi$ is integrable.

**Theorem 3.2 (Approximate reconstruction).** Let $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be admissible, that is

$$\int_0^\infty \frac{|\hat{\psi}(a)|^2}{|a|}da = \int_{-\infty}^0 \frac{|\hat{\psi}(a)|^2}{|a|}da =: c_\psi < \infty.$$

Then given $\varepsilon > 0$, the integral

$$f_\varepsilon(x) = \int_\varepsilon^\infty \int_{\mathbb{R}} Wf(a,b)\psi_{a,b}(x)\frac{1}{a}db\,da$$

exists for almost all $x \in \mathbb{R}$, $f_\varepsilon \in L^2(\mathbb{R})$, and $\lim\limits_{\varepsilon\to 0^+} \|f - f_\varepsilon\|_2 = 0$.

*Proof.* See Gasquet-Witomaski (1998). □

## 3.2 The Continuous Wavelet Transform on $L^2(\mathbb{R}^n)$

We now extend the definition of the wavelet transform to functions in $L^2(\mathbb{R}^n)$. First, we must extend the operators $D_a, T_b$ and $E_b$ of section 3.1 to $L^2(\mathbb{R}^n)$.

**Definition 3.3 (Dilation, Translation, Modulation and Inversion).** Given fixed $A \in \mathrm{GL}_n(\mathbb{R})$, $y, z, \gamma \in \mathbb{R}^n$, we define operators $D_A, T_y, E_\gamma, M$ on $L^2(\mathbb{R}^n)$ by

1. $(D_A f)(x) = \frac{1}{|\det A|^{1/2}} f(A^{-1}x), \qquad \forall x \in \mathbb{R}^n$ (Dilation by $A$)
   $(D_A f)(\gamma) = \frac{1}{|\det A|^{1/2}} f(\gamma A^{-1}), \qquad \forall \gamma \in \mathbb{R}^n$
2. $(T_y f)(x) = f(x-y), \qquad \forall x \in \mathbb{R}^n$ (Translation by $y$)
   $(T_\xi f)(\gamma) = f(\gamma - \xi), \qquad \forall \gamma \in \mathbb{R}^n$
3. $(E_\gamma f)(x) = e^{2i\pi\gamma x} f(x), \qquad \forall x \in \mathbb{R}^n$ (Modulation by $\gamma$)
   $(E_z f)(\gamma) = e^{2i\pi\gamma z} f(\gamma), \qquad \forall \gamma \in \mathbb{R}^n$
4. $(Mf)(x) = f(-x), \qquad \forall x \in \mathbb{R}^n$ (Inversion)

for $f \in L^2(\mathbb{R}^n)$.

Observe that we have defined two types of dilation operators, depending on whether elements of $\mathbb{R}^n$ are considered column vectors, or row vectors. Note that in the first case, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
(D_A D_B f)(x) &= \frac{1}{|\det A|^{1/2}} (D_B f)(A^{-1}x) \\
&= \frac{1}{|\det A|^{1/2}} \frac{1}{|\det B|^{1/2}} f(B^{-1}A^{-1}x) \\
&= \frac{1}{|\det AB|^{1/2}} f((AB)^{-1}x) \\
&= (D_{AB} f)(x)
\end{aligned}$$

that is, $D_A D_B = D_{AB}$. In the second case, for $\gamma \in \mathbb{R}^n$,

$$\begin{aligned}
(D_A D_B f)(\gamma) &= \frac{1}{|\det A|^{1/2}}(D_B f)(\gamma A^{-1}) \\
&= \frac{1}{|\det A|^{1/2}}\frac{1}{|\det B|^{1/2}} f(\gamma A^{-1}B^{-1}) \\
&= \frac{1}{|\det BA|^{1/2}} f(\gamma (BA)^{-1}) \\
&= (D_{BA} f)(\gamma)
\end{aligned}$$

that is, $D_A D_B = D_{BA}$. However, if $A$ and $B$ commute, then obviously $D_A D_B = D_{AB}$. In order to obtain $D_A D_B = D_{AB}$ in the second case, some authors define dilation by $(D_A f)(\gamma) = |\det A|^{1/2} f(\gamma A)$. However, this would complicate the notation in what follows.

In a similar way, one shows that, $T_y T_z = T_{y+z}$ and $E_\gamma E_\xi = E_{\gamma+\xi}$ for all $y, z, \gamma, \xi \in \mathbb{R}^n$.

**Proposition 3.1** The operators $D_A, T_y, E_\gamma$ and $M$ are isometries mapping $L^2(\mathbb{R}^n)$ onto itself. Furthermore for all $f \in L^2(\mathbb{R}^n)$, we have

a) $\langle D_A f, g\rangle = \langle f, D_{A^{-1}} g\rangle$ and $\widehat{D_A f} = D_{A^{-1}} \hat{f}$

b) $\langle T_y f, g\rangle = \langle f, T_{-y} g\rangle$ and $\widehat{T_y f} = E_{-y} \hat{f}$

c) $\langle E_\gamma f, g\rangle = \langle f, E_{-\gamma} g\rangle$ and $\widehat{E_\gamma f} = T_\gamma \hat{f}$

d) $\langle M f, g\rangle = \langle f, M g\rangle$ and $\widehat{M f} = M \hat{f}$.

*Proof.* It is straight forward to check that these operators are linear.

a) For all $f \in L^2(\mathbb{R}^n)$, we shall have by theorem 2.3,

$$\|D_A f\|_2^2 = \int_{\mathbb{R}^n} |D_A f(x)|^2\, dx = |\det A|^{-1} \int_{\mathbb{R}^n} |f(A^{-1}x)|^2\, dx = \int_{\mathbb{R}^n} |f(x)|^2\, dx = \|f\|_2^2$$

so that $D_A$ is an isometry, and

$$D_A(D_{A^{-1}} f) = D_{AA^{-1}} f = D_I f = f$$

which shows that $D_A$ is surjective. Furthermore,

$$\langle D_A f, g\rangle = \int_{\mathbb{R}^n} \frac{1}{|\det A|^{1/2}} f(A^{-1}x)\overline{g(x)}\,dx = \int_{\mathbb{R}^n} |\det A|^{1/2} f(x)\overline{g(Ax)}\,dx = \langle f, D_{A^{-1}g}\rangle$$

for all $f, g \in L^2(\mathbb{R}^n)$.

Assume now that $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $D_A f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ as well, so that by definition of the Fourier transform on $L^1(\mathbb{R}^n)$,

$$\begin{aligned}
(\widehat{D_A f})(\gamma) &= \int_{\mathbb{R}^n} (D_A f)(x) e^{-2i\pi\gamma x}\,dx \\
&= \int_{\mathbb{R}^n} |\det A|^{-1/2} f(A^{-1}x) e^{-2i\pi\gamma x}\,dx \qquad (x \to Ax) \\
&= \int_{\mathbb{R}^n} |\det A|^{1/2} f(x) e^{-2i\pi\gamma Ax}\,dx \\
&= |\det A|^{1/2} \hat{f}(\gamma A) \\
&= D_{A^{-1}} \hat{f}(\gamma).
\end{aligned}$$

Next let $f \in L^2(\mathbb{R}^n)$ be arbitrary. Then there exists a sequence $\{f_n\}$ in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ converging to $f$ in $L^2(\mathbb{R}^n)$, and by the continuity of all operators involved,

$\widehat{D_A f} = \mathcal{F}(D_A f) = \mathcal{F}(D_A(\lim_{n\to\infty} f_n)) = \lim_{n\to\infty} \mathcal{F}(D_A f_n) = \lim_{n\to\infty} \widehat{D_A f_n} = \lim_{n\to\infty} D_{A^{-1}} \hat{f}_n = D_{A^{-1}}\mathcal{F} f = D_{A^{-1}} \hat{f}$.

b) First we show $T_y$ is isometry on $L^2(\mathbb{R}^n)$. For all $f \in L^2(\mathbb{R}^n)$ we have

$$\|T_y f\|_2^2 = \int_{\mathbb{R}^n} |T_y f(x)|^2\,dx = \int_{\mathbb{R}^n} |f(x-y)|^2\,dx = \int_{\mathbb{R}^n} |f(x)|^2\,dx = \|f\|_2^2$$

by translation invariance of the integral, so that $T_y$ is on isometry, and

$$T_y(T_{-y} f) = T_{y-y} f = T_0 f = f,$$

which show that $T_y$ is surjective. Also,

$$\langle T_y f, g\rangle = \int_{\mathbb{R}^n} f(x-y)\overline{g(x)}dx = \int_{\mathbb{R}^n} f(x)\overline{g(x+y)}dx = \langle f, T_{-y} g\rangle.$$

Next let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $T_y f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, as well, and by definition of the Fourier transform on $L^1(\mathbb{R}^n)$

$$\begin{aligned}
(\widehat{T_y f})(\gamma) &= \int_{\mathbb{R}^n} (T_y f)(x) e^{-2i\pi\gamma x}\, dx \\
&= \int_{\mathbb{R}^n} f(x-y) e^{-2i\pi\gamma x}\, dx \qquad (x \to x+y) \\
&= \int_{\mathbb{R}^n} f(x) e^{-2i\pi\gamma x} e^{-2i\pi\gamma y}\, dx \\
&= e^{-2i\pi\gamma y} \int_{\mathbb{R}^n} f(x) e^{-2i\pi\gamma x}\, dx \\
&= (E_{-y}\hat{f})(\gamma).
\end{aligned}$$

Using a continuity argument in case a), it now follows that $\widehat{T_y f} = E_{-y}\hat{f}$ for all $f \in L^2(\mathbb{R}^n)$.

c) Since $|e^{it}| = 1$ for all $t \in \mathbb{R}$, we have

$$\|E_\gamma f\|_2^2 = \int_{\mathbb{R}^n} |E_\gamma f(x)|^2\, dx = \int_{\mathbb{R}^n} |f(x)e^{2i\pi\gamma x}|^2\, dx = \int_{\mathbb{R}^n} |f(x)|^2\, dx = \|f\|_2^2$$

so that $E_\gamma$ is an isometry, and

$$E_\gamma(E_{-\gamma} f)(x) = E_\gamma(f(x)e^{-2i\pi\gamma x}) = f(x)e^{-2i\pi\gamma x}e^{2i\pi\gamma x} = f(x)$$

which shows that $E_\gamma(E_{-\gamma} f) = f$, hence $E_\gamma$ is surjective. Also,

$$\langle E_\gamma f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}e^{2i\pi\gamma x} dx = \int_{\mathbb{R}^n} f(x)\overline{g(x)e^{-2i\pi\gamma x}} dx = \langle f, E_{-\gamma} g\rangle.$$

Next for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$(\widehat{E_\gamma f})(\xi) = \int_{\mathbb{R}^n} (E_\gamma f)(x)e^{-2i\pi\xi x}\, dx = \int_{\mathbb{R}^n} f(x)e^{-2i\pi(\xi-\gamma)x}\, dx = \hat{f}(\xi-\gamma) = (T_\gamma \hat{f})(\xi)$$

and arguing as in case a), it follows that $\widehat{E_\gamma f} = T_\gamma \hat{f}$ for all $f \in L^2(\mathbb{R}^n)$.

d) First we show $M$ is isometry on $L^2(\mathbb{R}^n)$. For all $f \in L^2(\mathbb{R}^n)$ we have

$$\|Mf\|_2^2 = \int_{\mathbb{R}^n} |(Mf)(x)|^2\, dx = \int_{\mathbb{R}^n} |f(-x)|^2\, dx = \int_{\mathbb{R}^n} |f(x)|^2\, dx = \|f\|_2^2$$

and

$$M(Mf(x)) = M(f(-x)) = f(x)$$

which show that $M$ is surjective. Also,

$$\langle Mf, g\rangle = \int_{\mathbb{R}^n} f(-x)\overline{g(x)}\,dx = \int_{\mathbb{R}^n} f(x)\overline{g(-x)}\,dx = \langle f, Mg\rangle.$$

Next for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\begin{aligned}(\widehat{Mf})(\xi) &= \int_{\mathbb{R}^n} Mf(x)e^{-2i\pi\xi x}\,dx = \int_{\mathbb{R}^n} f(-x)e^{-2i\pi\xi x}\,dx = \int_{\mathbb{R}^n} f(x)e^{-2i\pi(-\xi)x}\,dx \\ &= (M\hat{f})(\xi)\end{aligned}$$

and arguing as in case a), it follows that $\widehat{Mf} = M\hat{f}$ for all $f \in L^2(\mathbb{R}^n)$. □

**Proposition 3.2** Given fixed $A \in \mathrm{GL}_n(\mathbb{R})$, $y, z \in \mathbb{R}^n$, we have

$D_{A^t}M = MD_{A^t}$, $T_yM = MT_{-y}$, $E_zM = ME_{-z}$ and $\hat{\bar{f}} = \overline{M\hat{f}}$.

*Proof.* All these assertions are easily checked. For example,

$$\hat{\bar{f}}(\gamma) = \int_{\mathbb{R}^n} \bar{f}(x)e^{-2i\pi\gamma\cdot x}\,dx = \overline{\int_{\mathbb{R}^n} f(x)e^{-2i\pi(-\gamma)\cdot x}\,dx} = \overline{\hat{f}(-\gamma)} = \overline{M\hat{f}}(\gamma)$$

for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and by continuity, for all $f \in L^2(\mathbb{R}^2)$. □

We are now ready to introduce the wavelet transform on $L^2(\mathbb{R}^n)$. Throughout, we fix an $n \times n$ exponential matrix $A$, say $A = e^B$ for some matrix $B$, which will be used to define dilations.

**Definition 3.4 (Wavelet Transform on $L^2(\mathbb{R}^n)$).** Fix $\psi \in L^2(\mathbb{R}^n)$, $\psi \neq 0$ and call it the mother wavelet. Set $\psi_{t,b}(x) = (D_{A^t}T_b\psi)(x)$ so that $\psi_{t,b}(x) = \dfrac{1}{|\det A|^{t/2}}\psi\left(A^{-t}x - b\right)$. The wavelet transform of $f \in L^2(\mathbb{R}^n)$ is the function $Wf$ defined on $\mathbb{R} \times \mathbb{R}^n$ by

$$Wf(t,b) = \langle f, \mathrm{D}_{A^t}T_b\psi\rangle = \frac{1}{|\det A|^{t/2}} \int_{\mathbb{R}^n} f(x)\overline{\psi(A^{-t}x - b)}dx.$$

The operator $W : f \mapsto Wf$ is called the *continuous wavelet transform associated with* $\psi$, and is obviously linear.

The reconstruction of $f$ from its wavelet transform as a weak integral can be obtained as for the wavelet transform on $L^2(\mathbb{R})$, and we give a detailed explanation and proofs below. It turns out that because of our different choice of dilation, the measure required on $\mathbb{R} \times \mathbb{R}^n$ is simply the Lebesgue measure.

Suppose, we have shown that the wavelet transform $W$ associated with $\psi$ is a multiple of an isometry of $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R} \times \mathbb{R}^n)$,

$$\|W_\psi f\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} = \sqrt{c_\psi}\|f\|_{L^2(\mathbb{R}^n)}$$

for some $c_\psi > 0$ and all $f \in L^2(\mathbb{R}^n)$. As in section 3.2, by the polarization identity this condition is equivalent to

$$\langle Wf, Wg\rangle_{L^2(\mathbb{R}\times\mathbb{R}^n)} = c_\psi\langle f, g\rangle_{L^2(\mathbb{R}^n)}, \qquad \text{for all } f, g \in L^2(\mathbb{R}^n). \tag{3.6}$$

Computing the inner product on the left hand side of (3.6), we obtain

$$\begin{aligned} c_\psi\langle f, g\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} (Wf)(t,b)\overline{(Wg(t,b))}\, db\, dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} (Wf)(t,b)\langle \psi_{t,b}, g\rangle\, db\, dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \langle (Wf)(t,b) D_{A^t} T_b \psi, g\rangle\, db\, dt \end{aligned}$$

that is

$$\langle f, g\rangle = \frac{1}{c_\psi}\int_{\mathbb{R}}\int_{\mathbb{R}^n} \langle (Wf)(t,b) D_{A^t} T_b \psi, g\rangle\, db\, dt$$

for all $f, g \in L^2(\mathbb{R}^n)$. Thus

$$f = \frac{1}{c_\psi}\int_{\mathbb{R}}\int_{\mathbb{R}^n} Wf(t,b) D_{A^t} T_b \psi\, db\, dt$$

as *a weak integral* in $L^2(\mathbb{R}^n)$, so we have reconstructed $f$ from its wavelet transform. In order to characterize those functions $\psi$ for which $W$ is a multiple of an isometry, we prove the following generalization of theorem 3.1.

**Theorem 3.3 (Admissibility Condition).** Let $\psi \in L^2(\mathbb{R}^n)$. Then

$$\|Wf\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} = \sqrt{c_\psi}\|f\|_{L^2(\mathbb{R}^n)} \qquad \forall f \in L^2(\mathbb{R}^n)$$

if and only if

$$\int_{\mathbb{R}} |\hat{\psi}(\gamma A^t)|^2\, dt =: c_\psi \qquad a.e.\ \gamma \tag{3.7}$$

*Proof.* Let $V = \{f \in L^2(\mathbb{R}^n) : \hat{f} \in L^\infty(\mathbb{R}^n)\}$. Then $V$ is dense in $L^2(\mathbb{R}^n)$. If $f \in V$, then

$$\begin{aligned}
\|Wf\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} &= \int_{\mathbb{R}\times\mathbb{R}^n} |Wf(t,y)|^2\, d(t,y) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} |Wf(t,y)|^2\, dy\, dt \qquad \text{(By Fubini's theorem)} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} |\langle f, D_{A^t}T_y\psi\rangle|^2\, dy\, dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} |\langle \hat{f}, \widehat{D_{A^t}T_y\psi}\rangle|^2\, dy\, dt \qquad \text{(By Plancherel's theorem)} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} |\langle \hat{f}, D_{A^{-t}}E_{-y}\hat{\psi}\rangle|^2\, dy\, dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} |\langle D_{A^t}\hat{f}, E_{-y}\hat{\psi}\rangle|^2\, dy\, dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \left|\int_{\hat{\mathbb{R}}^n} |\det A|^{-t/2}\hat{f}(\gamma A^{-t})\overline{\hat{\psi}}(\gamma)e^{2i\pi\gamma y}\, d\gamma\right|^2 dy\, dt.
\end{aligned}$$

Set $F_t(\gamma) = |\det A|^{-t/2}\hat{f}(\gamma A^{-t})\overline{\hat{\psi}}(\gamma)$. Note that by the Cauchy-Schwarz inequality, $F_t(\gamma) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Thus, the inside integral is the inverse Fourier transform of $F_t$,

$$\begin{aligned}
\|Wf\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \left|\int_{\hat{\mathbb{R}}^n} F_t(\gamma)e^{2i\pi\gamma y}\, d\gamma\right|^2 dy\, dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \left|\check{F}_t(y)\right|^2\, dy\, dt \\
&= \int_{\mathbb{R}} \|\check{F}_t\|_2^2\, dt \qquad \text{(By Plancherel's theorem)} \\
&= \int_{\mathbb{R}} \|F_t\|_2^2\, dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} |F_t(\gamma)|^2\, d\gamma\, dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} |\det A|^{-t}|\hat{f}(\gamma A^{-t})|^2|\hat{\psi}(\gamma)|^2\,d\gamma\,dt \qquad (\gamma \to \gamma A^t)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} |\hat{f}(\gamma)|^2|\hat{\psi}(\gamma A^t)|^2\,d\gamma\,dt\\
&= \int_{\mathbb{R}^n} |\hat{f}(\gamma)|^2\left[\int_{\mathbb{R}} |\hat{\psi}(\gamma A^t)|^2\,dt\right]d\gamma. \qquad \text{(By Fubini's theorem)} \qquad (3.8)
\end{aligned}
$$

Now suppose, (3.7) holds. Then (3.8) becomes

$$
\|Wf\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\hat{f}(\gamma)|^2 c_\psi\,d\gamma = c_\psi\|\hat{f}\|_2^2 = c_\psi\|f\|_2^2 \qquad (3.9)
$$

for all $f \in V$, Now if $f \in L^2(\mathbb{R}^n)$, pick a sequence $\{f_n\}$ in $V$ such that $\lim_{n\to\infty}\|f_n - f\|_2 = 0$. By (3.9), $\{Wf_n\}$ is Cauchy in $L^2(\mathbb{R}\times\mathbb{R}^n)$, and hence there exists $g \in L^2(\mathbb{R}\times\mathbb{R}^n)$ such that $\lim_{n\to\infty}\|Wf_n - g\|_2 = 0$. On the other hand, by continuity of the inner product,

$$
Wf_n(t,b) = \langle f_n, D_{A^t}T_b\psi\rangle \to \langle f, D_{A^t}T_b\psi\rangle = Wf(t,b)
$$

for all $(t,b)\in\mathbb{R}\times\mathbb{R}^n$. Then by uniqueness of limits, $g = Wf \quad a.e.$ Then

$$
\|Wf\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} = \|\lim_{n\to\infty} Wf_n\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} = \lim_{n\to\infty}\|Wf_n\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} = c_\psi\|f\|_2^2.
$$

Hence the wavelet transform is a multiple of an isometry.

Conversely, suppose that the wavelet transform is a multiple of an isometry, that is, there exists $k > 0$ such that

$$
\|Wf\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} = k\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^n).
$$

Then (3.8) shows that

$$
k\|\hat{f}\|^2 = k\|f\|^2 = \int_{\mathbb{R}^n} |\hat{f}(\gamma)|^2\left[\int_{\mathbb{R}} |\hat{\psi}(\gamma A^t)|^2\,dt\right]d\gamma
$$

for all $f \in V$, that is,

$$
\int_{\mathbb{R}^n} |\hat{f}(\gamma)|^2\left[\int_{\mathbb{R}} |\hat{\psi}(\gamma A^t)|^2\,dt - k\right]d\gamma = 0
$$

for all $f \in V$. Choosing $f$ such that $\hat{f}$ is the characteristic function of a bounded measurable set $E$, we have

$$\int_E \left[ \int_{\mathbb{R}} |\hat{\psi}(\gamma A^t)|^2 \, dt - k \right] d\gamma = 0$$

for every bounded measurable set $E$. Hence,

$$\int_{\mathbb{R}} |\hat{\psi}(\gamma A^t)|^2 \, dt - k = 0 \qquad a.e.\, \gamma$$

that is,

$$\int_{\mathbb{R}} |\hat{\psi}(\gamma A^t)|^2 \, dt = k \qquad a.e.\, \gamma$$

so that (3.7) holds, with $c_\psi = k$. $\square$

Note that theorem 3.1 follows from the above theorem, if we replace $a$ by $e^t$ in (3.5). Because of this theorem, we give the following definition.

**Definition 3.5 (Admissibility Condition).** Let $A \in \mathrm{GL}_n(\mathbb{R}^n)$ be an exponential matrix. A function $\psi \in L^2(\mathbb{R}^n)$ is called admissible for $A$ if there exists a constant $c_\psi > 0$ such that

$$\int_{\mathbb{R}} |\hat{\psi}(\gamma A^t)|^2 \, dt =: c_\psi \qquad a.e.\, \gamma$$

**Theorem 3.4 (Existence of Admissible Function on $\mathbb{R}^n$).** Let $A \in \mathrm{GL}_n(\mathbb{R})$ be an exponential. Then there exists an admissible function $\psi$ if and only if $|\mathrm{det} A| \neq 1$.

*Proof.* See Laugesen, Weaver, Weiss and Wilson (2002). $\square$

# CHAPTER IV

# WAVELET RECONSTRUCTION

In this chapter, we will study direct and approximate reconstruction formulas for the continuous wavelet transform. We will formulate conditions on an admissible function $\psi$ and the function $f$ which allow for reconstruction of $f$ from its wavelet transform as a usual integral, or as a limit of usual integrals. For simplicity, we will focus on the wavelet transform in $L^2(\mathbb{R}^2)$.

## 4.1 Approximate Reconstruction

**Definition 4.1 (Expanding Matrix).** We call an $n \times n$ matrix an expanding matrix, if all its (real or complex) eigenvalues have modulus greater than 1.

Recall from the previous chapter that if $A = e^B$ is a $n \times n$ real matrix, and $\psi \in L^2(\mathbb{R}^n)$, we set

$$\psi_{t,b}(x) = (D_{A^t} T_b \psi)(x) = |\det A|^{-t/2} \psi(A^{-t}x - b).$$

The wavelet transform of $f \in L^2(\mathbb{R}^n)$ is

$$Wf(t,b) = \langle f, \psi_{t,b} \rangle = \frac{1}{|\det A|^{t/2}} \int_{\mathbb{R}^n} f(x) \overline{\psi(A^{-t}x - b)} \, dx,$$

and we have the reconstruction formula

$$f(x) = \frac{1}{c_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}^n} Wf(t,b) D_{A^t} T_b \psi(x) \, db \, dt \tag{4.1}$$

as *a weak integral.*

In order to reconstruct $f$ from its wavelet transform, we would like the integral in (4.1) to be a usual integral. This is not possible in general. However, we will see that if $A$ is expanding and $\hat{\psi}$ satisfies a weak decay condition at infinity, then we can approximate $f$ arbitrarily by a usual integral in case $n = 2$. First we prove a few of lemmas which will be needed in the proof of theorem 4.1.

**Lemma 4.1.** Let $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$, $a > 1$, and $\varepsilon < 0$ be given. Then for each $\tilde{a}$, $1 < \tilde{a} < a$, there exists a constant $\tilde{k}$ such that

$$\|\gamma A^t\|_2 \geq \tilde{k}\tilde{a}^t \|\gamma\|_2 \qquad \forall t \geq \varepsilon.$$

*Proof.* We want to estimate $\|\gamma A^t\|_2$ where $\gamma = (\gamma_1, \gamma_2)$. For simplicity, we first switch to the maximum norm $\|\cdot\|_\infty$. Pick $n_o > 1$ such that $\tilde{a} = a^{\frac{n_o - 1}{n_o}}$, and set

$$M_0 = \frac{-2\varepsilon}{a} > 0, \qquad m_0 = -1 - \frac{n_o}{ea \ln a}$$

where $n_o \in (1, \infty)$ is arbitrary, but fixed. Set

$$S_1 = \{\gamma :\, 0 < \gamma_1 \leq 1,\, \gamma_2 = m_0\}$$

$$S_2 = \{\gamma :\, \gamma_1 = 1,\, m_0 \leq \gamma_2 < M_0,\}$$

$$S_3 = \{\gamma :\, 0 < \gamma_1 \leq 1,\, \gamma_2 = M_0\}$$

$$S_4 = S_1 \cup S_2 \cup S_3$$

$$S_5 = -S_4 = \{-\gamma :\, \gamma \in S_4\}$$

$$S = S_4 \cup S_5$$

*Claim*: There exists $k > 0$ such that $\|\gamma A^t\|_\infty \geq k, \quad \forall t \geq \varepsilon, \quad \forall \gamma \in S.$

Note that since $\|(-\gamma)A^t\| = \|\gamma A^t\| \quad \forall t$, it is enough to consider $\gamma \in S_4$. Furthermore, observe that

$$\gamma A^t = (\gamma_1\, \gamma_2) \begin{bmatrix} a^t & ta^{t-1} \\ 0 & a^t \end{bmatrix} = (\gamma_1 a^t, \gamma_1 t a^{t-1} + \gamma_2 a^t)$$

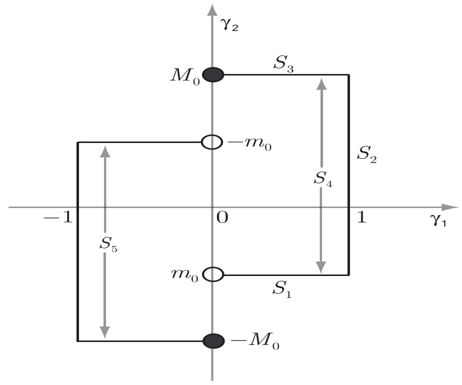


**Figure 4.1** The set $S$

so that

$$\|\gamma A^t\|_\infty = \max\{\gamma_1 a^t, \big|\gamma_1 \frac{t}{a} + \gamma_2\big| a^t\}. \qquad (\gamma_1 \geq 0)$$

Case a) $\gamma \in S_2$. Then $\gamma_1 = 1$ so that

$$\|\gamma A^t\|_\infty \geq \gamma_1 a^t = a^t \geq a^\varepsilon$$

for all $t \geq \varepsilon$.

Case b) $\gamma \in S_3$. Then $0 < \gamma_1 \leq 1$ while $\gamma_2 = M_0 = \dfrac{-2\varepsilon}{a}$, so that

$$\frac{t}{a}\gamma_1 + \gamma_2 = \frac{t}{a}\gamma_1 - \frac{2\varepsilon}{a} \geq \frac{\varepsilon}{a}\gamma_1 - \frac{2\varepsilon}{a} \geq \frac{\varepsilon}{a} - \frac{2\varepsilon}{a} = \frac{-\varepsilon}{a} > 0,$$

for all $t \geq \varepsilon$, as $\varepsilon < 0$. Hence for all $t \geq \varepsilon$,

$$\|\gamma A^t\|_\infty \geq \big|\gamma_1 \frac{t}{a} + \gamma_2\big| a^t \geq \left|\frac{-\varepsilon}{a}\right| a^t \geq \frac{|\varepsilon|}{a} a^\varepsilon.$$

Case c) $\gamma \in S_1$. Distinguish two cases, depending on the value of $t$.

i) Suppose $t \geq -n_o \log_a \gamma_1 \geq 0$ (as $0 < \gamma_1 \leq 1$). In this case, we estimate the value of $\gamma_1 a^t$:

$$a^t = a^{t\frac{n_o-1}{n_o}} a^{\frac{t}{n_o}} \geq \left(a^{\frac{n_o-1}{n_o}}\right)^t a^{-\log_a \gamma_1} = \frac{\tilde{a}^t}{\gamma_1}$$

so that $1 < \tilde{a} < a$. Hence,

$$\|\gamma A^t\|_\infty \geq \gamma_1 a^t \geq \tilde{a}^t \geq \tilde{a}^\varepsilon$$

for all $t \geq \varepsilon$.

ii). Suppose $\varepsilon \leq t < -n_o \log_a \gamma_1$. In this case, we estimate the value of $a^t \left|\frac{t}{a}\gamma_1 + \gamma_2\right|$. We have

$$\begin{aligned}\frac{t}{a}\gamma_1 + \gamma_2 &< \frac{-n_o \log_a \gamma_1}{a}\gamma_1 + \gamma_2, \qquad (\gamma_1 > 0)\\ &= -\frac{n_o \ln \gamma_1}{a \ln a}\gamma_1 + m_0\\ &= -\frac{n_o \ln \gamma_1}{a \ln a}\gamma_1 - 1 - \frac{n_o}{ea \ln a}.\end{aligned}$$

Now the function $f(x) = x \ln x$ , $0 < x \leq 1$, has one critical number at $x = \dfrac{1}{e}$, where it has an absolute minimum, $f(1/e) = -1/e$. Thus

$$\frac{n_o \ln \gamma_1}{a \ln a}\gamma_1 \geq \frac{-n_o}{ea \ln a},$$

that is,

$$-\frac{n_o}{ea \ln a} - \frac{n_o \ln \gamma_1}{a \ln a}\gamma_1 \leq 0,$$

which shows that $\frac{t}{a}\gamma_1 + \gamma_2 \leq -1$ in this case. Hence

$$\|\gamma A^t\|_\infty \geq a^t \left|\frac{t}{a}\gamma_1 + \gamma_2\right| \geq a^t \geq a^\varepsilon.$$

Note that the proofs of all three cases show that for all $t \geq \varepsilon$ and $\gamma \in S$,

$$\|\gamma A^t\|_\infty \geq k_0 \tilde{a}^t, \qquad \forall t \geq \varepsilon$$

where $k_0 = \max\left\{1, \frac{|\varepsilon|}{a}\right\}$, provided that $\varepsilon < 0$. Setting $k = k_0 \tilde{a}^\varepsilon$, the claim follows.

Next we claim that there exists $k_1 > 0$ such that $\|\gamma A^t\|_\infty \geq k_1 \tilde{a}^t \|\gamma\|_\infty, \ \forall t \geq \varepsilon, \ \forall \gamma \in \mathbb{R}^2$.

The claim is obvious if $\gamma = 0$. Thus, we may assume that $\gamma \neq 0$.

We first assume that $\gamma \in S$. Set $s_0 = \max_{\gamma \in S} \|\gamma\|_\infty = \max\{M_0, m_0, 1\} > 1$. By the previous claim, we have for all $\gamma \in S$ and $t \geq \varepsilon$,

$$\|\gamma A^t\|_\infty \geq k_0 \tilde{a}^t \geq \frac{k_0}{s_0}\tilde{a}^t \|\gamma\|_\infty = \tilde{a}^t k_1 \|\gamma\|_\infty \tag{4.2}$$

where we have set $k_1 = \dfrac{k_0}{s_0}$. In general, let $\gamma \in \mathbb{R}^2\backslash\{0\}$ be arbitrary. Since $\mathbb{R}^2\backslash\{0\} = \bigcup_{\alpha>0} \alpha S$ where $\alpha S = \{\alpha\eta : \eta \in S\}$, there exist $\alpha > 0$ and $\eta \in S$ such that $\gamma = \alpha\eta$. Then by (4.2),

$$\|\gamma A^t\|_\infty = \|(\alpha\eta)A^t\|_\infty = \alpha\|\eta A^t\|_\infty \geq \alpha\tilde{a}^t k_1\|\eta\|_\infty = \tilde{a}^t k_1\|\alpha\eta\|_\infty = \tilde{a}^t k_1\|\gamma\|_\infty$$

for all $t \geq \varepsilon$. This proves the claim.

Now as the Euclidean and maximum norms on $\mathbb{R}^2$ are equivalent, $\frac{1}{\sqrt{2}}\|\gamma\|_2 \leq \|\gamma\|_\infty \leq \|\gamma\|_2$, we have by the claim,

$$\|\gamma A^t\|_2 \geq \|\gamma A^t\|_\infty \geq \tilde{a}^t k_1\|\gamma\|_\infty \geq \frac{1}{\sqrt{2}}\tilde{a}^t k_1\|\gamma\|_2.$$

That is,

$$\|\gamma A^t\|_2 \geq \tilde{a}^t\tilde{k}\|\gamma\|_2$$

for all $\gamma \in \mathbb{R}^2$ and all $t \geq \varepsilon$, where we have set $\tilde{k} = \dfrac{k_1}{\sqrt{2}}$. $\square$

**Lemma 4.2.** Let $P \in \mathrm{GL}_n(\mathbb{R})$. For any norm $\|\cdot\|$ on $\mathbb{R}^n$, there exist constants $\alpha, \beta > 0$ such that

$$\beta\|\gamma\| \leq \|\gamma P\| \leq \alpha\|\gamma\| \qquad \forall\gamma \in \mathbb{R}^n.$$

*Proof.* One easily checks that $\gamma \mapsto \|\gamma P\|$ defines a norm on $\mathbb{R}^n$, by invertibility of $P$. Since all norms on $\mathbb{R}^n$ are equivalent, the assertion follows. $\square$

**Lemma 4.3.** Consider $f(\gamma) = \dfrac{1}{1+\|\gamma\|_2^k}$ $(\gamma \in \mathbb{R}^2, k \in \mathbb{R})$. Then $f \in L^2(\mathbb{R}^2)$ if and only if $k > 1$.

*Proof.* For simplicity, denote the Euclidean norm by $\|\cdot\|$, and use polar coordinates, $\gamma = (r\cos\theta, r\sin\theta)$. Then

$$\int_{\mathbb{R}^2} \frac{1}{(1+\|\gamma\|^k)^2}\,d\gamma = \int_0^{2\pi}\int_0^\infty \frac{r}{(1+r^k)^2}\,dr\,d\theta$$

Case 1. If $k > 1$, then $2k - 1 > 1$ and hence

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\infty} \frac{r}{(1+r^k)^2}\,dr\,d\theta &\leq \int_0^{2\pi}\int_0^{\infty} \frac{r}{1+r^{2k}}\,dr\,d\theta \\
&= \int_0^{2\pi}\left(\int_0^1 \frac{r}{1+r^{2k}}\,dr + \int_1^{\infty} \frac{r}{1+r^{2k}}\,dr\right)d\theta \\
&\leq \int_0^{2\pi}\left(\int_0^1 r\,dr + \int_1^{\infty} \frac{1}{r^{2k-1}}\,dr\right)d\theta \\
&= \int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{2(k-1)}\right)d\theta \\
&< \infty.
\end{aligned}$$

Case 2. If $k \leq 1$, then

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\infty} \frac{r}{(1+r^k)^2}\,dr\,d\theta &\geq \int_0^{2\pi}\int_0^{\infty} \frac{r}{(1+r)^2}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^{\infty} \frac{u-1}{u^2}\,du\,d\theta \qquad (u = 1+r) \\
&= \int_0^{2\pi} \lim_{b\to\infty}\left(\ln|b| + \frac{1}{b} - 1\right)d\theta \\
&= \infty.
\end{aligned}$$

□

**Theorem 4.1.** Let $A$ be an expanding real $2 \times 2$ exponential matrix. Let $\psi \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ be admissible, that is,

$$\int_{\mathbb{R}} |\hat{\psi}(\gamma A^t)|^2\,dt = c_\psi < \infty \qquad a.e.\ \gamma,$$

and suppose that

$$|\hat{\psi}(\gamma)| \leq \frac{k}{1 + \|\gamma\|^{\frac{1}{2}+s}} \tag{4.3}$$

for some constants $k$ and $s > 0$. Let $f \in L^2(\mathbb{R}^2)$ be given. Then for each $\varepsilon < 0$ and $x \in \mathbb{R}^2$, the integral

$$f_\varepsilon(x) = \frac{1}{c_\psi}\int_\varepsilon^{\infty}\int_{\mathbb{R}^2} (Wf)(t,b) D_{A^t} T_b \psi(x)\,db\,dt \tag{4.4}$$

exists, $f_\varepsilon$ is square integrable and $\lim_{\varepsilon\to-\infty} \|f - f_\varepsilon\|_2 = 0$.

*Proof.* We must first show that the integral (4.4) defining $f_\varepsilon$ exists. By Plancherel's theorem and the formulas following definition (3.3),

$$\begin{aligned}
Wf(t,b) &= \langle f, D_{A^t}T_b\psi\rangle \\
&= \langle \hat{f}, D_{A^{-t}}E_{-b}\hat{\psi}\rangle \\
&= |\det A|^{t/2}\int_{\mathbb{R}^2}\hat{f}(\gamma)\overline{\hat{\psi}}(\gamma A^t)e^{2i\pi\gamma A^t b}\,d\gamma \\
&= |\det A|^{t/2}\check{\phi}_t(A^t b) \\
&= (D_{A^{-t}}\check{\phi}_t)(b)
\end{aligned}$$

where $\phi_t(\gamma) = \hat{f}(\gamma)\overline{\hat{\psi}}(\gamma A^t) \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. For each $x \in \mathbb{R}^2$, set

$$J_x(t) = \int_{\mathbb{R}^2} Wf(t,b)\psi_{t,b}(x)\,db. \tag{4.5}$$

Note that this integral is defined for almost all $t$. In fact, since $Wf(t,b) \in L^2(\mathbb{R}\times\mathbb{R}^2)$, then by Fubini's theorem, $b \mapsto Wf(t,b) \in L^2(\mathbb{R}^2)$ a.e. On the other hand, $b \mapsto \psi_{t,b}(x) = |\det A|^{-t/2}\psi(A^{-t}x - b) \in L^2(\mathbb{R}^2)$. It follows from the Cauchy Schwarz inequality that $J_x(t)$ is defined for almost all $t$. Then

$$\begin{aligned}
J_x(t) &= \int_{\mathbb{R}^2}(D_{A^{-t}}\check{\phi}_t)(b)\psi_{t,b}(x)\,db \\
&= \int_{\mathbb{R}^2}(D_{A^{-t}}\check{\phi}_t)(b)|\det A|^{-t/2}\psi(A^{-t}x - b)\,db \\
&= |\det A|^{-t/2}\int_{\mathbb{R}^2}(D_{A^{-t}}\check{\phi}_t)(b)(M\psi)(b - A^{-t}x)\,db \\
&= |\det A|^{-t/2}\int_{\mathbb{R}^2}(D_{A^{-t}}\check{\phi}_t)(b)(T_{A^{-t}x}M\psi)(b)\,db \\
&= |\det A|^{-t/2}\langle(D_{A^{-t}}\check{\phi}_t), \overline{T_{A^{-t}x}M\psi}\rangle \\
&= |\det A|^{-t/2}\langle\mathcal{F}(D_{A^{-t}}\check{\phi}_t), \mathcal{F}(\overline{T_{A^{-t}x}M\psi})\rangle \qquad \text{(By Plancherel's Theorem)} \\
&= |\det A|^{-t/2}\langle D_{A^t}\mathcal{F}(\check{\phi}_t), \overline{M\mathcal{F}(T_{A^{-t}x}M\psi)}\rangle \\
&= |\det A|^{-t/2}\langle(D_{A^t}\phi_t), \overline{ME_{-A^{-t}x}M\hat{\psi}}\rangle \\
&= |\det A|^{-t/2}\langle(D_{A^t}\phi_t), \overline{M^2E_{A^{-t}x}\hat{\psi}}\rangle
\end{aligned}$$

$$
\begin{aligned}
&= |\det A|^{-t} \int_{\mathbb{R}^2} \phi_t(\gamma A^{-t})(E_{A^{-t}x}\hat{\psi})(\gamma)\, d\gamma \qquad (\gamma \mapsto \gamma A^t) \\
&= \int_{\mathbb{R}^2} \phi_t(\gamma)(E_{A^{-t}x}\hat{\psi})(\gamma A^t)\, d\gamma \\
&= \int_{\mathbb{R}^2} \hat{f}(\gamma)\overline{\hat{\psi}(\gamma A^t)}\hat{\psi}(\gamma A^t) e^{2i\pi\gamma x}\, d\gamma \\
&= \int_{\mathbb{R}^2} \hat{f}(\gamma)|\hat{\psi}(\gamma A^t)|^2 e^{2i\pi\gamma x}\, d\gamma.
\end{aligned}
$$

By (4.4),

$$
f_\varepsilon(x) = \frac{1}{c_\psi}\int_\varepsilon^\infty J_x(t)\, dt = \frac{1}{c_\psi}\int_\varepsilon^\infty \int_{\mathbb{R}^2} \hat{f}(\gamma)|\hat{\psi}(\gamma A^t)|^2 e^{2i\pi\gamma x}\, d\gamma\, dt. \tag{4.6}
$$

We want to apply Fubini's theorem to show that the integral (4.6) exists. For this, we need to check that

$$
I = \int_{\mathbb{R}^2} |\hat{f}(\gamma)| \int_\varepsilon^\infty |\hat{\psi}(\gamma A^t)|^2\, dt\, d\gamma < \infty. \tag{4.7}
$$

The value of $I$ is estimated in two parts:

1. Integrate over a bounded set, namely the disc $B_m(0) = \{\gamma \in \mathbb{R}^2 : \|\gamma\| \leq m\}$ for some $m > 0$. We obtain

$$
\begin{aligned}
I_1 &= \int_{\|\gamma\|\leq m} |\hat{f}(\gamma)| \left( \int_\varepsilon^\infty |\hat{\psi}(\gamma A^t)|^2\, dt \right) d\gamma \\
&\leq c_\psi \int_{\|\gamma\|\leq m} |\hat{f}(\gamma)|\, d\gamma \\
&\leq c_\psi \left( \int_{\|\gamma\|\leq m} 1\, d\gamma \right)^{1/2} \left( \int_{\|\gamma\|\leq m} |\hat{f}(\gamma)|^2\, d\gamma \right)^{1/2} \\
&= c_\psi(\sqrt{\pi}m)\|f\|_2 < \infty.
\end{aligned}
$$

where we have used the Cauchy Schwarz inequality.

2. Integrate over the complement $U$ of the disc $B_m(0)$, $U = \{\gamma \in \mathbb{R}^2 : \|\gamma\| > m\}$. We distinguish 3 cases, depending on the Jordan normal form of $A$.

Case I. The Jordan normal form of $A$ is diagonal. Then $A = PBP^{-1} = P\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}P^{-1}$ where $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Then by theorem 2.15, $A^t = PB^tP^{-1} =$

$P\begin{bmatrix} a^t & 0 \\ 0 & b^t \end{bmatrix}P^{-1}$, and thus

$$\|\gamma A^t\| = \|\gamma P B^t P^{-1}\| \geq \beta_1\beta_2\alpha^t\|\gamma\|$$

where $\beta_1$ and $\beta_2$ are the constants for $P^{-1}$ and $P$ as in Lemma 4.2, respectively, and $\alpha = \min(a, b) > 1$. We obtain

$$\begin{aligned} I_2 &= \int_{\|\gamma\|>m} |\hat{f}(\gamma)| \left( \int_{\varepsilon}^{\infty} |\hat{\psi}(\gamma A^t)|^2 \, dt \right) d\gamma \\ &\leq \int_{\|\gamma\|>m} |\hat{f}(\gamma)| \left( \int_{\varepsilon}^{\infty} \left| \frac{k}{1 + \|\gamma A^t\|^{\frac{1}{2}+s}} \right|^2 dt \right) d\gamma \qquad \text{(by 4.3)} \\ &\leq k^2 \int_{\|\gamma\|>m} |\hat{f}(\gamma)| \left( \int_{\varepsilon}^{\infty} \left| \frac{1}{(\beta_1\beta_2\alpha^t\|\gamma\|)^{\frac{1}{2}+s}} \right|^2 dt \right) d\gamma \\ &= \frac{k^2}{(\beta_1\beta_2)^{1+2s}} \int_{\|\gamma\|>m} |\hat{f}(\gamma)| \frac{1}{\|\gamma\|^{1+2s}} \left( \int_{\varepsilon}^{\infty} \frac{1}{\alpha^{t(1+2s)}} \, dt \right) d\gamma \\ &= \frac{k^2}{(\beta_1\beta_2)^{1+2s}(1+2s)\ln\alpha} \alpha^{-(1+2s)\varepsilon} \int_U |\hat{f}(\gamma)| \frac{1}{\|\gamma\|^{1+2s}} \, d\gamma. \end{aligned}$$

Setting $a_0 = \dfrac{k^2}{(\beta_1\beta_2)^{1+2s}(1+2s)\ln\alpha} \alpha^{-(1+2s)\varepsilon}$ and applying the Cauchy Schwarz inequality, we obtain, by lemma 4.3,

$$I_2 \leq a_0\|f\|_2 \left\| \frac{1}{\|\gamma\|^{1+2s}} \right\|_{L^2(U)} < \infty.$$

Case II. The Jordan normal form of $A$ is an upper diagonal matrix, $A = PBP^{-1}$ with $B = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$, $a > 1$ Then by theorem 2.15, $A^t = PB^tP^{-1} = P\begin{bmatrix} a^t & ta^{t-1} \\ 0 & a^t \end{bmatrix}P^{-1}$ and thus by lemma 4.1,

$$\|\gamma A^t\| = \|\gamma P B^t P^{-1}\| \geq \beta_1\beta_2\tilde{k}\tilde{a}^t\|\gamma\|$$

where $\beta_1$ and $\beta_2$ are the constants for $P^{-1}$ and $P$ in lemma 4.2, respectively, and

$1 < \tilde{a} < a$. Computing as in the case I, we obtain

$$
\begin{aligned}
I_2 &= \int_{\|\gamma\|>m} |\hat{f}(\gamma)| \left( \int_{\varepsilon}^{\infty} |\hat{\psi}(\gamma A^t)|^2 \, dt \right) d\gamma \\
&= \int_{\|\gamma\|>m} |\hat{f}(\gamma)| \left( \int_{\varepsilon}^{\infty} \left| \frac{k}{1 + \|\gamma A^t\|^{\frac{1}{2}+s}} \right|^2 dt \right) d\gamma, \qquad \text{(by 4.3)} \\
&\leq \int_{\|\gamma\|>m} |\hat{f}(\gamma)| k^2 \left( \int_{\varepsilon}^{\infty} \left| \frac{1}{(\beta_1\beta_2\tilde{k}\tilde{a}^t\|\gamma\|)^{\frac{1}{2}+s}_2} \right|^2 dt \right) d\gamma \\
&= \int_{\|\gamma\|>m} |\hat{f}(\gamma)| \frac{k^2}{(\beta_1\beta_2\tilde{k})^{1+2s}\|\gamma\|^{1+2s}} \left( \int_{\varepsilon}^{\infty} \frac{1}{\tilde{a}^{t(1+2s)}} \, dt \right) d\gamma, \qquad \text{(Cauchy-Schwarz)} \\
&\leq a_1 \|f\|_2 \left\| \frac{1}{\|\gamma\|^{1+2s}} \right\|_{L^2(U)} < \infty,
\end{aligned}
$$

provided $\varepsilon < 0$, where $a_1 = \dfrac{1}{(1+2s)\ln \tilde{a}} \tilde{a}^{-(1+2s)\varepsilon} \dfrac{k^2}{(\beta_1\beta_2\tilde{k})^{1+2s}}$.

Case III. The Jordan normal form of $A$ has a rotation part, that is, $A = PBP^{-1}$ where $B = \alpha \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, $\alpha > 1$. Since the rotation matrix is an isometry, proceeding as in case I) we have $\|\gamma A^t\| \geq \alpha^t \beta_1\beta_2\|\gamma\|$ and we continue with the estimate of $I_2$ exactly as in case I).

Combining all three cases, it follows that $I = I_1 + I_2 < \infty$. Thus by Fubini's Theorem, the integral (4.5) defining $f_\varepsilon(x)$ exists, and we can interchange the order of integration in $f_\varepsilon(x)$, so that

$$
\begin{aligned}
f_\varepsilon(x) &= \frac{1}{c_\psi} \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^2} \hat{f}(\gamma) |\hat{\psi}(\gamma A^t)|^2 e^{2i\pi\gamma x} \, d\gamma \, dt \\
&= \frac{1}{c_\psi} \int_{\mathbb{R}^2} \hat{f}(\gamma) e^{2i\pi\gamma x} \left( \int_{\varepsilon}^{\infty} |\hat{\psi}(\gamma A^t)|^2 \, dt \right) d\gamma \\
&= \frac{1}{c_\psi} \left( \overline{\mathcal{F}}\big(\hat{f}\theta_\varepsilon\big) \right)(x) \qquad \forall x \in \mathbb{R}^2
\end{aligned}
$$

where $\theta_\varepsilon(\gamma) = \int_\varepsilon^\infty |\hat{\psi}(\gamma A^t)|^2 \, dt$. Here we have used the fact that, by (4.7), $\hat{f}\theta_\varepsilon \in L^1(\mathbb{R}^2)$. Observe that since $\theta_\varepsilon(\gamma) < c_\psi$ a.e., then $\hat{f}\theta_\varepsilon \in L^2(\mathbb{R}^2)$ as well, hence $f_\varepsilon \in L^2(\mathbb{R}^2)$.

It is left to show that $f_\varepsilon$ converges to $f$ in $L^2(\mathbb{R}^2)$ as $\varepsilon \to -\infty$. By Plancherel's theorem,

$$
\begin{aligned}
\|f-f_\varepsilon\|_2^2 &= \left\|\overline{\mathcal{F}}(\hat{f})-\frac{1}{c_\psi}\overline{\mathcal{F}}(\hat{f}\theta_\varepsilon)\right\|_2^2\\
&= \left\|\overline{\mathcal{F}}(\hat{f}-\frac{1}{c_\psi}\hat{f}\theta_\varepsilon)\right\|_2^2\\
&= \left\|\hat{f}-\frac{1}{c_\psi}\hat{f}\theta_\varepsilon\right\|_2^2\\
&= \left\|\hat{f}\left(\frac{c_\psi-\theta_\varepsilon}{c_\psi}\right)\right\|_2^2\\
&= \int_{\mathbb{R}^2}|\hat{f}(\gamma)|^2\left|\frac{c_\psi-\theta_\varepsilon(\gamma)}{c_\psi}\right|^2 d\gamma.
\end{aligned}
$$

Note that for almost all $\gamma$, $\displaystyle\lim_{\varepsilon\to-\infty}\int_\varepsilon^\infty|\hat{\psi}(\gamma A^t)|^2\,dt = c_\psi$, so that $\{c_\psi-\theta_\varepsilon(\gamma)\}$ decreases to zero a.e. as $\varepsilon\to-\infty$. Thus by the Monotone Convergence Theorem,

$$
\begin{aligned}
\lim_{\varepsilon\to-\infty}\|f-f_\varepsilon\|_2^2 &= \lim_{\varepsilon\to-\infty}\int_{\mathbb{R}^2}|\hat{f}(\gamma)|^2\left|\frac{c_\psi-\theta_\varepsilon(\gamma)}{c_\psi}\right|^2 d\gamma\\
&= \int_{\mathbb{R}^2}\lim_{\varepsilon\to-\infty}|\hat{f}(\gamma)|^2\left|\frac{c_\psi-\theta_\varepsilon(\gamma)}{c_\psi}\right|^2 d\gamma\\
&= 0.
\end{aligned}
$$

We have shown that $f_\varepsilon$ converges to $f$ in $L^2(\mathbb{R}^2)$ as $\varepsilon\to-\infty$. □

If $A$ is contracting, i.e. if all eigenvalues of $A$ have modulus less than one, then a similar statement holds:

**Theorem 4.2.** Let $A$ be a $2\times 2$ exponential matrix whose eigenvalues all have modulus less than 1. Let $\psi\in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ be admissible, that is

$$\int_{\mathbb{R}}|\hat{\psi}(\gamma A^t)|^2\,dt = c_\psi < \infty \qquad a.e.\,\gamma,$$

and suppose that

$$|\hat{\psi}(\gamma)| \le \frac{k}{1+\|\gamma\|^{\frac{1}{2}+s}}$$

for some constants $k$ and $s>0$. Let $f\in L^2(\mathbb{R}^2)$ be given. Then for each $\varepsilon>0$ and $x\in\mathbb{R}^2$, the integral

$$f_\varepsilon(x) = \frac{1}{c_\psi}\int_{-\infty}^{\varepsilon}\int_{\mathbb{R}^2}(Wf)(t,b)D_{A^t}T_b\psi(x)\,db\,dt \tag{4.8}$$

exists, $f_\varepsilon$ is square integrable and $\lim_{\varepsilon\to\infty} \|f - f_\varepsilon\|_2 = 0$.

*Proof.* Set $\tilde{A} = A^{-1}$. Then $\tilde{A}$ is also exponential, and the eigenvalues of $\tilde{A}$ all have modulus greater one, that is, $\tilde{A}$ is expanding. As the Lebesgue measure is inversion invariant,

$$\int_{\mathbb{R}} |\hat{\psi}(\gamma\tilde{A}^t)|^2\, dt = \int_{\mathbb{R}} |\hat{\psi}(\gamma A^{-t})|^2\, dt = \int_{\mathbb{R}} |\hat{\psi}(\gamma A^t)|^2\, dt = c_\psi < \infty \qquad a.e.\, \gamma,$$

that is, $\hat{\psi}$ is also admissible for $\tilde{A}$. By the previous theorem,

$$\tilde{f}_{-\varepsilon}(x) = \frac{1}{c_\psi}\int_{-\varepsilon}^{\infty}\int_{\mathbb{R}^2} \langle f, D_{\tilde{A}^t}T_b\psi\rangle D_{\tilde{A}^t}T_b\psi(x)\, db\, dt$$

exists for every $\varepsilon > 0$ and $x \in \mathbb{R}^n$, and

$$\lim_{\varepsilon\to\infty} \|f - \tilde{f}_{-\varepsilon}\|_2 = 0.$$

Replacing $t$ by $-t$ we have by inversion invariance of the Lebesgue integral that

$$\tilde{f}_{-\varepsilon}(x) = \frac{1}{c_\psi}\int_{-\infty}^{\varepsilon}\int_{\mathbb{R}^2} \langle f, D_{A^t}T_b\psi\rangle D_{A^t}T_b\psi(x)\, db\, dt = f_\varepsilon(x),$$

and hence $f_\varepsilon$ exists, and $\lim_{\varepsilon\to\infty} \|f - f_\varepsilon\|_2 = 0$. $\square$

The next theorem says that if $f$ is bandlimited, and $\hat{\psi}$ is supported away from zero, then the reconstruction formula (4.1) is always a usual integral.

**Theorem 4.3.** Keep the assumption of theorem 4.1, and assume in addition that $\hat{\psi}(\gamma) = 0$ a.e. in some neighborhood of zero. If $f \in L^2(\mathbb{R}^2)$ is bandlimited, then $f_\varepsilon = f$ for sufficiently large negative $\varepsilon$, that is, for almost every $x \in \mathbb{R}^2$,

$$f(x) = \frac{1}{c_\psi}\int_{\mathbb{R}}\int_{\mathbb{R}^2} Wf(t,b) D_{A^t}T_b\psi(x)\, db\, dt \qquad a.e.$$

as a usual integral.

*Proof.* Let $f \in L^2(\mathbb{R}^2)$ be bandlimited. That is, there exists $M > 0$ such that $\hat{f}(\gamma) = 0$ for almost all $\gamma$ with $\|\gamma\| \geq M$. By assumption, there exist $\delta > 0$

such that $\hat{\psi}(\gamma) = 0$ for almost all $\|\gamma\| < \delta$. Let $f_\varepsilon$ be as in theorem 4.1, so that $f_\varepsilon(x) = \dfrac{1}{c_\psi}\displaystyle\int_\varepsilon^\infty \int_{\mathbb{R}^2} (Wf)(t,b) D_{A^t} T_b \psi(x)\, db\, dt$. Note that as $f$ is bandlimited, $\hat{f} \in L^1(\mathbb{R}^2)$ as well, so that $f(x) = \displaystyle\int_{\mathbb{R}^2} \hat{f}(\gamma) e^{2i\pi\gamma x}\, d\gamma$ a.e. We now show that there exists $\varepsilon_o$ such that the integral

$$\frac{1}{c_\psi}\int_{-\infty}^{\varepsilon}\int_{\mathbb{R}^2} (Wf)(t,b) D_{A^t} T_b \psi(x)\, db\, dt \tag{4.9}$$

exists and equals zero for all $\varepsilon \le \varepsilon_o$ and $x \in \mathbb{R}^2$.

The integral (4.9) is similar to formula $f_\varepsilon$ in equation (4.6), so computing as in the derivation of equation (4.6), we obtain

$$\begin{aligned} (4.9) \quad &= \frac{1}{c_\psi}\int_{-\infty}^{\varepsilon}\int_{\mathbb{R}^2} \hat{f}(\gamma)|\hat{\psi}(\gamma A^t)|^2 e^{2i\pi\gamma x}\, d\gamma\, dt \\ &= \frac{1}{c_\psi}\int_{-\infty}^{\varepsilon}\int_{\|\gamma\|_2 \le M} \hat{f}(\gamma)|\hat{\psi}(\gamma A^t)|^2 e^{2i\pi\gamma x}\, d\gamma\, dt. \end{aligned}$$

where we have used the fact that $f$ is bandlimited. We first show that for $\varepsilon$ sufficiently negative,

$$I = \int_{\|\gamma\| \le M} |\hat{f}(\gamma)| \int_{-\infty}^{\varepsilon} |\hat{\psi}(\gamma A^t)|^2\, dt\, d\gamma = 0. \tag{4.10}$$

We again distinguish 3 cases, depending on the Jordan normal form of $A$.

Case I. $A = PBP^{-1} = P\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}P^{-1}$. Then by theorem 2.15, $A^t = PB^tP^{-1}$, and thus

$$\|\gamma A^t\| = \|\gamma P B^t P^{-1}\| \le \alpha_1 \alpha_2 \alpha^t \|\gamma\|$$

where $\alpha_1$ and $\alpha_2$ are the constants for $P^{-1}$ and $P$ as in lemma 4.2, respectively, and $\alpha = \max(a,b) > 1$. Now if $t < \dfrac{\ln\delta - \ln(\alpha_1\alpha_2 M)}{\ln\alpha}$, then $\alpha_1\alpha_2\alpha^t\|\gamma\| < \delta$ provided that $\|\gamma\| < M$ and hence $\hat{\psi}(\gamma A^t) = 0$. Thus, if we choose $\varepsilon_o < \dfrac{\ln\delta - \ln(\alpha_1\alpha_2 M)}{\ln\alpha}$, then for $\varepsilon \le \varepsilon_o$

$$I = \int_{\|\gamma\| \le M} |\hat{f}(\gamma)| \int_{-\infty}^{\varepsilon} |\hat{\psi}(\gamma A^t)|^2\, dt\, d\gamma = 0.$$

Case II. $A = PBP^{-1}$ with $B = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$, $a > 1$ Then by theorem 2.15, $A^t = PB^tP^{-1} = P\begin{bmatrix} a^t & ta^{t-1} \\ 0 & a^t \end{bmatrix}P^{-1}$, and thus

$$\begin{aligned}
\|\gamma B^t\|^2 &= (\gamma_1 a^t)^2 + (\gamma_1 a^{t-1}t + \gamma_2 a^t)^2 \\
&= a^{2t}({\gamma_1}^2 + {\gamma_2}^2) + 2t\gamma_1\gamma_2 a^{2t-1} + {\gamma_1}^2 t^2 a^{2t-2} \\
&\leq a^{2t}\|\gamma\|^2 + 4t\|\gamma\|^2 a^{2t-1} + t^2\|\gamma\|^2 a^{2t-2} \\
&= \left(a^{2t} + 4ta^{2t-1} + t^2 a^{2t-2}\right)\|\gamma\|^2 \\
&\leq a^{2t}\left(\frac{t}{a} + 2\right)^2 \|\gamma\|^2.
\end{aligned}$$

Hence,

$$\|\gamma A^t\| = \|\gamma PB^tP^{-1}\| \leq \alpha_1\alpha_2 a^t \left(\frac{t}{a} + 2\right)\|\gamma\|$$

where $\alpha_1$ and $\alpha_2$ are the constants for $P^{-1}$ and $P$ as in lemma 4.2, respectively. Since $\lim_{t\to-\infty} a^t \left(\frac{t}{a} + 2\right) = 0$, there exists $\varepsilon_o$ such that $\|\gamma A^t\| < \delta$ for all $t \leq \varepsilon_o$, and $\|\gamma\| \leq M$. Then for all $\varepsilon \leq \varepsilon_o$,

$$I = \int_{\|\gamma\|\leq M} |\hat{f}(\gamma)| \int_{-\infty}^{\varepsilon} |\hat{\psi}(\gamma A^t)|^2 \, dt \, d\gamma = 0.$$

Case III. The Jordan normal form of $A$ has a rotation part, that is, $A = PBP^{-1}$ where $B = \alpha\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, $\alpha > 1$. Since the rotation matrix is an isometry, proceeding as in case I) we have $\|\gamma A^t\| \leq \alpha_1\alpha_2\alpha^t\|\gamma\|$. We continue with the computation of $I$ exactly as in case I), that is, we choose $\varepsilon_o < \dfrac{\ln\delta - \ln(\alpha_1\alpha_2 M)}{\ln\alpha}$ and obtain for all $\varepsilon \leq \varepsilon_o$ that

$$I = \int_{\|\gamma\|\leq M} |\hat{f}(\gamma)| \int_{-\infty}^{\varepsilon} |\hat{\psi}(\gamma A^t)|^2 \, dt \, d\gamma = 0.$$

Combining all 3 cases, we have shown that there exists $\varepsilon_o$ such that the integral in (4.10) is zero for all $\varepsilon < \varepsilon_o$. Using Fubini's theorem, it follows that the integral

(4.9) exists, and we can interchange the order of integration in (4.9) so that, for all $\varepsilon \leq \varepsilon_o$.

$$\begin{aligned}
\left|\int_{-\infty}^{\varepsilon}\int_{\mathbb{R}^2}\hat{f}(\gamma)|\hat{\psi}(\gamma A^t)|^2 e^{2i\pi\gamma x}\,d\gamma\,dt\right| &= \left|\int_{\|\gamma\|\leq M}\int_{-\infty}^{\varepsilon}\hat{f}(\gamma)e^{2i\pi\gamma x}|\hat{\psi}(\gamma A^t)|^2\,dt\,d\gamma\right| \\
&\leq \int_{\|\gamma\|\leq M}\int_{-\infty}^{\varepsilon}\left|\hat{f}(\gamma)\right|\left|\hat{\psi}(\gamma A^t)\right|^2\,dt\,d\gamma \\
&= 0.
\end{aligned}$$

It follows that $f_\varepsilon = f_{\varepsilon_o}$ for all $\varepsilon \leq \varepsilon_o$, hence $0 = \lim_{\varepsilon\to-\infty}\|f - f_\varepsilon\|_2 = \lim_{\varepsilon\to-\infty}\|f - f_{\varepsilon_o}\|_2 = \|f - f_{\varepsilon_o}\|$, that is, $f(x) = f_{\varepsilon_0}(x) = f_\varepsilon(x)$ a.e. for $\varepsilon \leq \varepsilon_o$. □

In the case of an arbitrary expanding $n \times n$ matrix, it is difficult to track how quickly the points $\gamma A^t$ tends to infinity as $t$ grows. However, if $A$ is similar to a diagonal matrix, then this is not difficult, and we have the extension of theorem 4.1 given in theorem 4.4 below. However, first we must determine the correct decay condition on $\hat{\psi}$.

**Lemma 4.4.** Fix $m > 0$ and set $U = \{\gamma \in \mathbb{R}^n : \|\gamma\|_\infty \geq m\}$. Then $g(\gamma) = \dfrac{1}{\|\gamma\|_\infty^k} \in L^2(U)$ for all $k > \dfrac{n}{2}$.

*Proof.* Observe that if $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$, then $|\gamma_1\gamma_2...\gamma_n| \leq (\max\{|\gamma_1|, |\gamma_2|, ..., |\gamma_n|\})^n = \|\gamma\|_\infty^n$. Now suppose, $k > \dfrac{n}{2}$. Then $r = \dfrac{2k}{n} > 1$ and integrating over $U$,

$$\begin{aligned}
\int_U\left(\frac{1}{\|\gamma\|_\infty^k}\right)^2 d\gamma &= \int_U \frac{1}{\|\gamma\|_\infty^{2k}}\,d\gamma \\
&\leq \int_{|\gamma_n|\geq m}\cdots\int_{|\gamma_1|\geq m}\frac{1}{|\gamma_1\gamma_2\ldots\gamma_n|^r}\,d\gamma_1\ldots d\gamma_n \\
&= \int_{|\gamma_n|\geq m}\cdots\int_{|\gamma_2|\geq m}\frac{2}{|\gamma_2\gamma_3\ldots\gamma_n|^r(r-1)m^{r-1}}\,d\gamma_2\ldots d\gamma_n \\
&= \int_{|\gamma_n|\geq m}\cdots\int_{|\gamma_3|\geq m}\frac{4}{|\gamma_3\gamma_4\ldots\gamma_n|^r(r-1)^2 m^{2(r-1)}}\,d\gamma_3\ldots d\gamma_n
\end{aligned}$$

$$= \int_{|\gamma_n|\geq m} \cdots \int_{|\gamma_4|\geq m} \frac{8}{|\gamma_4\gamma_5\ldots\gamma_n|^r(r-1)^3m^{3(r-1)}}\, d\gamma_4\ldots d\gamma_n$$

$$\vdots$$

$$= \frac{2^n}{(r-1)^n m^{n(r-1)}} < \infty,$$

where we have used the fact that $\displaystyle\int_{|\gamma_i|\geq m} \frac{1}{|\gamma_i|^r}\, d\gamma_i = \frac{2}{(r-1)m^{r-1}}$ for $r > 1$. This shows that $g(\gamma) \in L^2(U)$.

Because the two norms $\|\gamma\|_\infty$ and $\|\gamma\|_2$ are equivalent, it follows also that $g(\gamma) = \dfrac{1}{\|\gamma\|_2^k} \in L^2(U)$ if $k > \dfrac{n}{2}$. □

**Theorem 4.4.** Let $A$ be an $n \times n$ expanding exponential diagonalizable matrix. Suppose that the function $\psi \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ is admissible, say $\displaystyle\int_{\mathbb{R}} |\hat{\psi}(\gamma A^t)|^2\, dt = c_\psi < \infty$, and

$$|\hat{\psi}(\gamma)| \leq \frac{k}{1+\|\gamma\|^{\frac{n}{4}+s}} \qquad a.e.$$

where $k, s > 0$ are constant. Let $f \in L^2(\mathbb{R}^n)$. Then for each $\varepsilon \in \mathbb{R}$,

$$f_\varepsilon(x) = \frac{1}{c_\psi}\int_\varepsilon^\infty \int_{\mathbb{R}^n} (Wf)(t,b) D_{A^t} T_b \psi(x)\, db\, dt$$

exists, $f_\varepsilon \in L^2(\mathbb{R}^n)$ and $\displaystyle\lim_{\varepsilon\to-\infty} \|f - f_\varepsilon\|_2 = 0$.

*Proof.* The proof is essentially the same as that of theorem 4.1. Since $A$ is diagonalizable, there exist a diagonal matrix $B =$diag$(a_1, ..., a_n)$ and $P \in$GL$_n(\mathbb{R})$ such that $A = PBP^{-1}$. We rewrite $f_\varepsilon$ as

$$f_\varepsilon(x) = \frac{1}{c_\psi}\int_\varepsilon^\infty \int_{\mathbb{R}^2} \hat{f}(\gamma)|\hat{\psi}(\gamma A^t)|^2 e^{2i\pi\gamma x}\, d\gamma\, dt,$$

and we need to check whether Fubini's theorem applies. That is, we need to verify that

$$I = \int_{\mathbb{R}^n} |\hat{f}(\gamma)| \int_\varepsilon^\infty |\hat{\psi}(\gamma A^t)|^2\, dt\, d\gamma$$

is finite. Again, $I$ is estimated in two parts. Fix $m > 0$. We integrate over $\{\gamma \in \mathbb{R}^n : \|\gamma\|_\infty \leq m\}$ and apply the Cauchy-Schwarz inequality,

$$\begin{aligned}
I_1 &= \int_{\|\gamma\|_\infty \leq m} |\hat{f}(\gamma)| \left( \int_\varepsilon^\infty |\hat{\psi}(\gamma A^t)|^2 \, dt \right) d\gamma \\
&\leq c_\psi \int_{\|\gamma\|_\infty \leq m} |\hat{f}(\gamma)| \, d\gamma \\
&\leq c_\psi \left( \int_{\|\gamma\|_\infty \leq m} 1 \, d\gamma \right)^{1/2} \left( \int_{\|\gamma\|_\infty \leq m} |\hat{f}(\gamma)|^2 \, d\gamma \right)^{1/2} \\
&\leq c_\psi (2m)^{n/2} \left( \int_{\|\gamma\|_\infty \leq m} |\hat{f}(\gamma)|^2 \, d\gamma \right)^{1/2} \\
&= c_\psi (2m)^{n/2} \|f\|_2 < \infty.
\end{aligned}$$

Next we integrate over $U = \{\gamma \in \mathbb{R}^n : \|\gamma\|_\infty > m\}$. By theorem 2.15 and Lemma 4.2,

$$\|\gamma A^t\|_\infty = \|\gamma P B^t P^{-1}\|_\infty \geq \beta_1 \beta_2 \alpha^t \|\gamma\|_\infty$$

for some constants $\beta_1, \beta_2 > 0$, and where $\alpha = \min\{a_1, ..., a_n\} > 1$. We obtain

$$\begin{aligned}
I_2 &= \int_{\|\gamma\|_\infty > m} |\hat{f}(\gamma)| \left( \int_\varepsilon^\infty |\hat{\psi}(\gamma A^t)|^2 \, dt \right) d\gamma \\
&\leq \int_{\|\gamma\|_\infty > m} |\hat{f}(\gamma)| \left( \int_\varepsilon^\infty \left| \frac{k}{1 + \|\gamma A^t\|^{\frac{n}{4}+s}} \right|^2 dt \right) d\gamma, \qquad \left( |\hat{\psi}(\gamma)| \leq \frac{k}{1 + \|\gamma\|^{\frac{n}{4}+s}} \right) \\
&\leq \int_{\|\gamma\|_\infty > m} |\hat{f}(\gamma)| \left( \int_\varepsilon^\infty \left| \frac{k}{(\beta_1 \beta_2 \alpha^t \|\gamma\|)^{\frac{n}{4}+s}} \right|^2 dt \right) d\gamma \\
&= \frac{k^2}{(\beta_1 \beta_2)^{\frac{n}{2}+2s}} \int_U \underbrace{|\hat{f}(\gamma)|}_{\in L^2(\mathbb{R}^n)} \frac{1}{\|\gamma\|_\infty^{\frac{n}{2}+2s}} \left( \int_\varepsilon^\infty \frac{1}{\alpha^{t(\frac{n}{2}+2s)}} \, dt \right) d\gamma
\end{aligned}$$

Since $\alpha > 1$, then $\displaystyle\int_\varepsilon^\infty \frac{1}{\alpha^{t(\frac{n}{2}+2s)}} \, dt = \frac{1}{(\frac{n}{2} + 2s) \ln \alpha} \alpha^{-(\frac{n}{2}+2s)\varepsilon}$. Furthermore, by lemma 4.4, $g(\gamma) = \dfrac{1}{\|\gamma\|^{\frac{n}{2}+s}} \in L^2(U)$, and hence by Cauchy-Schwarz inequality,

$$I_2 \leq \frac{k^2 \alpha^{-(\frac{n}{2}+2s)\varepsilon}}{(\beta_1 \beta_2)^{\frac{n}{2}+2s} (\frac{n}{2} + 2s) \ln \alpha} \|f\|_2 \|g\|_{L^2(U)}.$$

It follows that $I = I_1 + I_2 < \infty$. By Fubini's Theorem, the integral defining $f_\varepsilon(x)$ exists, and we can interchange the order of integration in $f_\varepsilon(x)$, so that

$$\begin{aligned} f_\varepsilon(x) &= \frac{1}{c_\psi}\int_\varepsilon^\infty \int_{\mathbb{R}^2} \hat{f}(\gamma)|\hat{\psi}(\gamma A^t)|^2 e^{2i\pi\gamma x}\,d\gamma\,dt \\ &= \frac{1}{c_\psi}\int_{\mathbb{R}^2} \hat{f}(\gamma)e^{2i\pi\gamma x}\left(\int_\varepsilon^\infty |\hat{\psi}(\gamma A^t)|^2\,dt\right)d\gamma \\ &= \frac{1}{c_\psi}\left(\overline{\mathcal{F}}(\hat{f}\theta_\varepsilon)\right)(x)\ \forall x \in \mathbb{R}^n \end{aligned}$$

where $\theta_\varepsilon(\gamma) = \displaystyle\int_\varepsilon^\infty |\hat{\psi}(\gamma A^t)|^2\,dt$. Proceeding as in theorem 4.1, one shows that $f_\varepsilon \in L^2(\mathbb{R}^n)$, and converges to $f$ in $L^2(\mathbb{R}^n)$ as $\varepsilon \to -\infty$.

$\square$

## 4.2 Reconstruction Using Approximate Identities

There are nearly as many examples of approximate identities as there are integrable functions. This makes it easy, in most cases, to find approximate identities that satisfy any additional conditions we might require.

**Lemma 4.5.** Let $C$ be an $n \times n$ expanding diagonal matrix, and $M \subset \mathbb{R}^n$ be compact. Given $\delta > 0$, there exists $N \in \mathbb{N}$ such that $\|C^{-k}x\| < \delta$ for all $x \in M$.

*Proof.* We must pick $N \in \mathbb{N}$ such that $C^{-k}M \subset B_\delta(0)$ for all $k \geq N$. For this, let $C = \text{diag}(a_1, ...a_n)$. Set $a = \min\{a_1, ..., a_n\}$, and set $\widetilde{M} = \max_{x\in M}\|x\|$. Note that $C^{-k}x = (a_1^{-k}x_1, ..., a_n^{-k}x_n)$, and hence

$$\|C^{-k}x\|_2^2 = a_1^{-2k}x_1^2 + ... + a_n^{-2k}x_n^2 \leq a^{-2k}\|x\|^2 \leq a^{-2k}\widetilde{M}.$$

Note that,

$$a^{-2k}\widetilde{M} < \delta \Leftrightarrow k > -\frac{1}{2}\log_a\Big(\frac{\delta}{\widetilde{M}}\Big).$$

Hence, if we choose

$$N > \log_a\Big(\frac{\widetilde{M}}{\delta}\Big)^{1/2}$$

then $\|C^{-k}x\| < \delta$ for all $k \geq N$ and $x \in M$. $\square$

**Example 4.1** (Even Approximate Identity) Fix a compact set $M_0 \subset \mathbb{R}^n$ and a function $\rho_o \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $\rho_o \geq 0$ , $\mathrm{supp}(\rho_o) \subset M_0$. There exists a great variety of such function $\rho$, for example, by Urysohn's lemma, there exists $\rho_o \in C_c^\infty(\mathbb{R}^n)$ satisfying these properties. Next set $\rho(x) = \rho_o(x) + \rho_o(-x)$. Then $\mathrm{supp}(\rho) \subset M$ where $M = M_0 \cup (-M_0)$, and $\rho(-x) = \rho(x)$ for all $x$. Fix an $n \times n$ expanding diagonal matrix $C$, and set

$$\rho_k(x) = \frac{|\det C|^k \rho(C^k x)}{\|\rho\|_1}. \tag{4.11}$$

Then $\mathrm{supp}(\rho_k) \subset C^{-k}M$. Let us show that $\{\rho_k\}$ is an approximate identity for $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. First we show that $\int_{\mathbb{R}^n} \rho_k(x)\, dx = 1$. In fact, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \rho_k(x)\, dx &= \frac{1}{\|\rho\|_1} \int_{\mathbb{R}^n} \Big| |\det C|^k \rho(C^k x) \Big|\, dx \qquad (x \to C^{-k}x) \\
&= \frac{1}{\|\rho\|_1} \int_{\mathbb{R}^n} |\rho(x)|\, dx \\
&= 1
\end{aligned}$$

Next we show $\lim_{k\to\infty} \|f - f * \rho_k\|_p = 0$.

Let $f \in L^p(\mathbb{R}^n)$ be arbitrary. By density of $C_c(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$, there exists $f_\varepsilon \in C_c(\mathbb{R}^n)$ such that $\|f - f_\varepsilon\|_p < \frac{\varepsilon}{4}$. By Young's inequality,

$$\|f * \rho_k - f_\varepsilon * \rho_k\|_p = \|(f - f_\varepsilon) * \rho_k\|_p \leq \|f - f_\varepsilon\|_p \|\rho_k\|_1 = \|f - f_\varepsilon\|_p. \tag{4.12}$$

Now set $S$= $\mathrm{supp}(f_\varepsilon)$ and consider $g_k = f_\varepsilon * \rho_k$. Then by theorem 2.10 and as $C$ is expanding, $\mathrm{supp}(g_k) \subset S + C^{-k}M \subset S + M$. Set $K = (S + M) \cup S$. Then $\mathrm{supp}(f_\varepsilon) \subset K$ and $\mathrm{supp}(g_k) \subset K$ for all $k$, so that

$$\int_{\mathbb{R}^n} |f_\varepsilon(x) - g_k(x)|^p\, dx \leq \lambda(K) \sup_{x \in K} |f_\varepsilon(x) - g_k(x)|^p. \tag{4.13}$$

Now since $\int_{\mathbb{R}^n} \rho_k(t)\,dt = 1$, we can write

$$\begin{aligned} f_\varepsilon(x) - g_k(x) &= f_\varepsilon(x) - (f_\varepsilon * \rho_k)(x) \\ &= \int_{\mathbb{R}^n} f_\varepsilon(x)\rho_k(t)\,dt - \int_{\mathbb{R}^n} f_\varepsilon(x-t)\rho_k(t)\,dt \\ &= \int_{\mathbb{R}^n} (f_\varepsilon(x) - f_\varepsilon(x-t))\rho_k(t)\,dt \\ &= \int_{C^{-k}M} (f_\varepsilon(x) - f_\varepsilon(x-t))\rho_k(t)\,dt \end{aligned}$$

as $\operatorname{supp}(\rho_k) \subset C^{-k}M$. Thus,

$$\begin{aligned} |f_\varepsilon(x) - g_k(x)| &\le \int_{C^{-k}M} |(f_\varepsilon(x) - f_\varepsilon(x-t))\rho_k(t)|\,dt \\ &\le \sup_{t\in C^{-k}M} |f_\varepsilon(x) - f_\varepsilon(x-t)| \int_{C^{-k}M} \rho_k(t)\,dt \\ &= \sup_{t\in C^{-k}M} |f_\varepsilon(x) - f_\varepsilon(x-t)|. \end{aligned}$$

for all $k$, and hence

$$\sup_{x\in K} |f_\varepsilon(x) - g_k(x)| \le \sup_{\substack{x\in K \\ t\in C^{-k}M}} |f_\varepsilon(x) - f_\varepsilon(x-t)|. \tag{4.14}$$

Now since $f_\varepsilon$ is uniformly continuous, $\exists\delta > 0$ such that $|f_\varepsilon(x) - f_\varepsilon(x-t)| < \dfrac{\varepsilon}{2(\lambda(K))^{1/p}}$ $\forall x$ provided $\|t\| < \delta$. Note that since $C$ is expanding, by the lemma 4.5 there exists $N \in \mathbb{N}$ such that $C^{-k}M \subset B_\delta(0)$ for all $k \ge N$, so that

$$|f_\varepsilon(x) - f_\varepsilon(x-t)| < \frac{\varepsilon}{2(\lambda(K))^{1/p}}, \quad \forall t \in C^{-k}M, \forall x \in \mathbb{R}^n$$

and hence

$$\sup_{t\in C^{-k}M} |f_\varepsilon(x) - f_\varepsilon(x-t)| \le \frac{\varepsilon}{2(\lambda(K))^{1/p}}. \tag{4.15}$$

By (4.13), (4.14) and (4.15) then

$$\|f_\varepsilon - g_k\|_p^p \le \lambda(K) \sup_{x\in K} \frac{\varepsilon^p}{2^p(\lambda(K))} = \frac{\varepsilon^p}{2^p}$$

and hence

$$\|f_\varepsilon - g_k\|_p \le \frac{\varepsilon}{2} \tag{4.16}$$

for all $k \geq N$. Applying the triangle inequality and (4.12),

$$\begin{aligned}
\|f - f * \rho_k\|_p &= \|f - f_\varepsilon + f_\varepsilon - f_\varepsilon * \rho_k + f_\varepsilon * \rho_k - f * \rho_k\|_p \\
&\leq \|f - f_\varepsilon\|_p + \|f_\varepsilon - f_\varepsilon * \rho_k\|_p + \|f_\varepsilon * \rho_k - f * \rho_k\|_p \\
&\leq 2\|f - f_\varepsilon\|_p + \|f_\varepsilon - f_\varepsilon * \rho_k\|_p \\
&\leq 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

for $k \geq N$. As $\varepsilon$ is arbitrary, it follows that $\lim\limits_{k\to\infty} \|f - f * \rho_k\|_p = 0$. This shows the existence of an even approximate identity.

**Theorem 4.5.** Let $A \in \mathrm{GL}_n(\mathbb{R})$, $|\det A| \neq 1$ and suppose that $\psi \in L^2(\mathbb{R}^n)$ is admissible, that is, $\int_{\mathbb{R}} |\hat{\psi}(\gamma A^t)|\,dt = c_\psi < \infty$ a.e $\gamma$. Let $\{\rho_k\}_{k=1}^{\infty} \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ be an approximate identity for $L^2(\mathbb{R}^n)$ such that $\rho_k(-x) = \rho_k(x)$ for all $k$. Then

$$f_k(x) = \frac{1}{c_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}^n} Wf(t,b)(\rho_k * D_{A^t} T_b \psi)(x)\,db\,dt$$

exists for all $k$ and all $x \in \mathbb{R}^n$, and $f_k \to f$ in $L^2(\mathbb{R}^n)$.

*Proof.* An approximate identity with the required properties exists by example 1. Now as $\rho_k$ is even and the wavelet transform is a multiple of an isometry, we have

$$\begin{aligned}
(f * \rho_k)(x) &= \int_{\mathbb{R}^n} f(y)\rho_k(x - y)\,dy \\
&= \int_{\mathbb{R}^n} f(y)\rho_k(y - x)\,dy \\
&= \langle f, \overline{T_x \rho_k} \rangle \\
&= \frac{1}{c_\psi} \langle Wf, W(\overline{T_x \rho_k}) \rangle_{L^2(\mathbb{R}\times\mathbb{R}^n)} \\
&= \frac{1}{c_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}^n} Wf(t,b) \langle D_{A^t} T_b \psi, \overline{T_x \rho_k} \rangle_{L^2(\mathbb{R}^n)}\,db\,dt \\
&= \frac{1}{c_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}^n} Wf(t,b) \left( \int_{\mathbb{R}^n} D_{A^t} T_b \psi(y) \rho_k(y - x)\,dy \right) db\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{c_\psi}\int_{\mathbb{R}}\int_{\mathbb{R}^n} Wf(t,b)\Bigg(\int_{\mathbb{R}^n} D_{A^t}T_b\psi(y)\rho_k(x-y)\,dy\Bigg)\,db\,dt \\
&= \frac{1}{c_\psi}\int_{\mathbb{R}}\int_{\mathbb{R}^n} Wf(t,b)(\rho_k * D_{A^t}T_b\psi)(x)\,db\,dt \\
&= f_k(x).
\end{aligned}
$$

Since $\{\rho_k\}$ is an approximate identity, it follows that

$$\lim_{k\to\infty}\|f-f_k\|_2 = \lim_{k\to\infty}\|f-f*\rho_k\|_2 = 0.$$

$\square$

Let us analyze how well $f_k$ estimates $f$, at least for some particular $f$, where $\{\rho_k\}$ is the approximate identity of example 4.1 That is, given $\varepsilon > 0$, we want to find $k$ such that $\|f-f_k\|_2 < \varepsilon$.

I. Keep the notation of example 4.1. Let $f \in C_c(\mathbb{R}^n)$, then $f_\varepsilon = f$ and $g_k = f_k$. Let $\varepsilon > 0$ be given. Since $f$ is uniformly continuous, there exists $\delta > 0$ such that $\|t\| < \delta$ implies $|f(x)-f(x-t)| < \dfrac{\varepsilon}{(\lambda(K))^{1/2}}$ for all $x \in \mathbb{R}^n$. Pick $N$ such that $C^{-k}M \subset B_\delta(0)$ for all $k \geq N$. Then by (4.13) and (4.14),

$$\|f-f_k\|_2 \leq \sup_{t\in C^{-k}M}|f(x)-f(x-t)|(\lambda(K))^{1/2} < \varepsilon$$

for all $k \geq N$.

II. Fix $m > 0$ and a compact subset $S$ of $\mathbb{R}^n$, and set $S_m = \{f \in C_c^1(\mathbb{R}^n),\ \|\nabla f\| \leq m,\ \mathrm{supp}(f) \subset S\}$. Now given $\varepsilon > 0$, choose $\delta = \dfrac{\varepsilon}{m(\lambda(K))^{1/2}}$ where $K = (S+M)\cup S$. We must pick $N \in \mathbb{N}$ such that $C^{-k}M \subset B_\delta(0)$ for all $k \geq N$. By the proof of lemma 4.5, we must choose

$$N > \log_a\Big(\frac{\widetilde{M}m(\lambda(K))^{1/2}}{\varepsilon}\Big)^{1/2}$$

where $\widetilde{M} = \max_{x\in M}\|x\|$. Now let $f \in S_m$ be arbitrary. By the Mean Value Theorem, for each $x, t$ there exists $c$ on the line segment connecting $x$ and $x-t$ such that

$f(x) - f(x-t) = (D_r f)(c)t$ where $D_r$ is the directional derivative in direction of the line segment connecting $x$ and $x - t$. Thus,

$$\begin{aligned} |f(x) - f(x-t)| &\leq \|D_r f(c)\| \|t\| \\ &\leq \|\nabla f\| \|t\| \\ &< m\delta \\ &\leq \frac{\varepsilon}{(\lambda(K))^{1/2}} \end{aligned}$$

provided that $\|t\| < \delta$. Then by (4.14)

$$\|f - f_k\|_2 \leq \sup_{t \in C^{-k}M} |f(x) - f(x-t)| (\lambda(K))^{1/2} < \varepsilon$$

for all $k \geq N$.

## 4.3 Wavelet Frames

Another way to reconstruct a function $f$ from its wavelet transform is by means of frames. Here, one tries to reconstruct $f$ by an infinite series, which is computationally much simpler than reconstruction by an integral. We will now identify a class of $2 \times 2$ matrices and admissible functions in $L^2(\mathbb{R}^2)$ which provide for wavelet frames. Let us say that $g \in L^2(\mathbb{R}^2)$ *generates a wavelet frame*, if $\{D_{A^n} T_{bm} g\}_{n \in \mathbb{Z}, m \in \mathbb{Z}^2}$ is a frame for $L^2(\mathbb{R}^2)$, where $b > 0$ is constant.

**Theorem 4.6.** Let $I = \left[\frac{-1}{2b}, \frac{1}{2b}\right] \times \left[\frac{-1}{2b}, \frac{1}{2b}\right]$ for some $b > 0$, $g \in L^2(\mathbb{R}^2)$ be such that $\operatorname{supp}(\hat{g}) \subset I$ and suppose there exist $\alpha, \beta$ such that

$$0 < \alpha \leq \sum_{n \in \mathbb{Z}} |\hat{g}(\gamma A^n)|^2 \leq \beta < \infty \qquad \text{for a.e. } \gamma. \tag{4.17}$$

Then $\forall f \in L^2(\mathbb{R}^2)$,

$$\frac{\alpha}{b^2} \|f\|_2^2 \leq \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^2} |\langle f, D_{A^n} T_{bm} g \rangle|^2 \leq \frac{\beta}{b^2} \|f\|_2^2.$$

That is, $\{D_{A^n}T_{bm}g\}_{n\in\mathbb{Z},m\in\mathbb{Z}^2}$ is a wavelet frame in $L^2(\mathbb{R}^2)$ with frame bounds $\dfrac{\alpha}{b^2}$ and $\dfrac{\beta}{b^2}$.

*Proof.* Since $\operatorname{supp}(D_{A^n}\hat{f}\cdot\hat{g}) \subset I$ and $\hat{g}$ is essentially bounded by (4.17), then $D_{A^n}\hat{f}\cdot\hat{g} \in L^2(I)$. It is well known that the collection of functions $\{e_m(\gamma)\}_{m\in\mathbb{Z}^2}$ with $e_m(\gamma) = be^{2i\pi\gamma bm}$ is an orthonormal basis for $L^2(I)$. Then by Parserval's identity, for each $h \in L^2(I)$,

$$\sum_{m\in\mathbb{Z}^2} |\langle h, e_m\rangle_{L^2(I)}|^2 = \|h\|^2_{L^2(I)}$$

and thus

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2} |\langle f, D_{A^n}T_{bm}g\rangle|^2 &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2} |\langle \hat{f}, D_{A^{-n}}E_{-bm}\hat{g}\rangle|^2 \qquad \text{(By Plancherel's Theorem)}\\
&= \sum_{n\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}^2} |\langle D_{A^n}\hat{f}, E_{-bm}\hat{g}\rangle|^2\right)\\
&= \sum_{n\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}^2} |\langle D_{A^n}\hat{f}, E_{bm}\hat{g}\rangle|^2\right)\\
&= \frac{1}{b^2}\sum_{n\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}^2} |\langle (D_{A^n}\hat{f})\overline{\hat{g}}, be^{2i\pi\gamma\cdot bm}\rangle|^2\right)\\
&= \frac{1}{b^2}\sum_{n\in\mathbb{Z}} \|(D_{A^n}\hat{f})\overline{\hat{g}}\|^2_{L^2(I)}\\
&= \frac{1}{b^2}\sum_{n\in\mathbb{Z}}\left(\int_I \left|(D_{A^n}\hat{f})(\gamma)\overline{\hat{g}}(\gamma)\right|^2 d\gamma\right) \qquad (\operatorname{supp}(\hat{g}) \subset I)\\
&= \frac{1}{b^2}\sum_{n\in\mathbb{Z}}\left(\int_{\mathbb{R}^2} \left|(D_{A^n}\hat{f})(\gamma)\overline{\hat{g}}(\gamma)\right|^2 d\gamma\right)\\
&= \frac{1}{b^2}\sum_{n\in\mathbb{Z}}\left(\int_{\mathbb{R}^2} \left[|\det A|^{-n/2}|\hat{f}(\gamma A^{-n})\hat{g}(\gamma)|\right]^2 d\gamma\right)\\
&= \frac{1}{b^2}\sum_{n\in\mathbb{Z}}\left(\int_{\mathbb{R}^2} |\hat{f}(\gamma)\hat{g}(\gamma A^n)|^2\, d\gamma\right) \qquad (\gamma \to A^n\gamma)\\
&= \frac{1}{b^2}\int_{\mathbb{R}^2}\left(|\hat{f}(\gamma)|^2 \sum_{n\in\mathbb{Z}} |\hat{g}(\gamma A^n)|^2\right) d\gamma
\end{aligned}$$

so that, by assumption,

$$\frac{\alpha}{b^2}\int_{\mathbb{R}^2}|\hat{f}(\gamma)|^2\,d\gamma \le \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2}|\langle f, D_{A^n}T_{bm}g\rangle|^2 \le \frac{\beta}{b^2}\int_{\mathbb{R}^2}|\hat{f}(\gamma)|^2\,d\gamma$$

or

$$\frac{\alpha}{b^2}\|f\|_2^2 \le \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2}|\langle f, D_{A^n}T_{bm}g\rangle|^2 \le \frac{\beta}{b^2}\|f\|_2^2$$

for all $f \in L^2(\mathbb{R}^2)$. $\square$

In the following, we will give examples of such frame generators.

**Example 4.2** (A tight frame generator). Let $I = \left[\frac{-1}{2},\frac{1}{2}\right]\times\left[\frac{-1}{2},\frac{1}{2}\right]$ and $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a, b > 1$. Set $I' = \left[\frac{-1}{2a},\frac{1}{2a}\right]\times\left[\frac{-1}{2b},\frac{1}{2b}\right]$ , and $\widetilde{I} = I\backslash I'$. We will show that the function $g \in L^2(\mathbb{R}^2)$ defined by

$$\hat{g}(\gamma) = \chi_{\tilde{I}}(\gamma) = \begin{cases} 1, & \gamma \in \tilde{I} \\ 0, & \gamma \notin \tilde{I} \end{cases}$$

generates a tight wavelet frame for $L^2(\mathbb{R}^2)$.

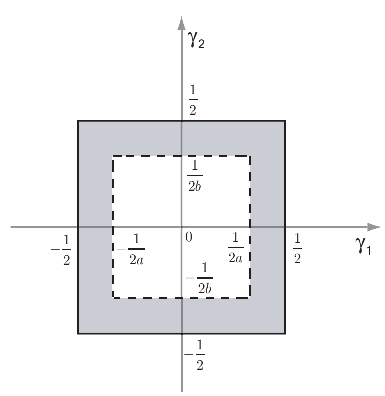


**Figure 4.2** The set $\widetilde{I}$

For this, let $\gamma \in \mathbb{R}^2, \gamma = (\gamma_1,\gamma_2) \ne 0$ be arbitrary. We will show that $\sum_{n\in\mathbb{Z}}|\hat{g}(\gamma A^n)|^2 = 1$.

*Claim : There exists a unique $k_1 \in \mathbb{Z}$ such that $\frac{1}{2a} < |\gamma_1 a^{k_1}| \leq \frac{1}{2}$.*

To compute $k_1$, note that

$$|\gamma_1 a^{k_1}| > \frac{1}{2a} \Leftrightarrow k_1 > -\log_a(2|\gamma_1|) - 1.$$

On the other hand,

$$|\gamma_1 a^{k_1}| \leq \frac{1}{2} \Leftrightarrow k_1 \leq \log_a(2|\gamma_1|).$$

Set $M = -\log_a(2|\gamma_1|)$. It follows that $\dfrac{1}{2a} < |\gamma_1 a^{k_1}| \leq \dfrac{1}{2}$ if and only if $k_1 \in (M-1, M]$. Since this interval contains exactly one integer $k_1$, the claim follows. Similarly, there exists a unique $k_2 \in \mathbb{Z}$ such that $\dfrac{1}{2b} < |\gamma_2 b^{k_2}| \leq \dfrac{1}{2}$. Set $k_0 = \max\{k_1, k_2\}$. Next we show : $\gamma A^k \in \tilde{I}$ if and only if $k = k_0$. Observe that $\gamma A^k = (\gamma_1 a^k, \gamma_2 b^k)$.

Case 1. if $k_0 = k_1$ then $\dfrac{1}{2a} < |\gamma_1 a^{k_1}| \leq \dfrac{1}{2}$ while $0 < |\gamma_2 b^{k_1}| \leq |\gamma_2 b^{k_2}| \leq \dfrac{1}{2}$. Hence $\gamma A^{k_0} \in \tilde{I}$. Note that if $k > k_1$, then $|\gamma_1 a^k| \geq |\gamma_1 a^{k_1+1}| > \dfrac{1}{2a}a = \dfrac{1}{2}$, hence $\gamma A^k \notin \tilde{I}$ while if $k < k_1$, then $|\gamma_1 a^k| \leq |\gamma_1 a^{k_1-1}| = \dfrac{1}{a}|\gamma_1 a^{k_1}| \leq \dfrac{1}{2a}$ and $|\gamma_2 b^k| < |\gamma_2 b^{k_1-1}| \leq |\gamma_2 b^{k_2-1}| \leq \dfrac{1}{2b}$, hence $\gamma A^k \notin \tilde{I}$.

Case 2. If $k_0 = k_2$ then $0 < |\gamma_1 a^{k_2}| \leq |\gamma_1 a^{k_1}| \leq \dfrac{1}{2}$ while $\dfrac{1}{2b} < |\gamma_2 b^{k_2}| \leq \dfrac{1}{2}$. Hence $\gamma A^{k_0} \in \tilde{I}$. Note that if $k > k_2$, then $|\gamma_2 b^k| \geq |\gamma_2 b^{k_2+1}| > \dfrac{1}{2}$, hence $\gamma A^k \notin \tilde{I}$ while if $k < k_2$, then $|\gamma_2 b^k| \leq \dfrac{1}{2b}$ and $|\gamma_1 a^k| \leq \dfrac{1}{2a}$, hence $\gamma A^k \notin \tilde{I}$.

This proves that there exists a unique $k_0$ such that $\gamma A^{k_0} \in \tilde{I}$. Since $\gamma A^k \in \tilde{I} \Leftrightarrow k = k_0$, then $\chi_{\tilde{I}}(\gamma A^n) = \delta_{n,k_0}$. Thus,

$$\sum_{n=-\infty}^{\infty} |\hat{g}(\gamma A^n)|^2 = \sum_{n=-\infty}^{\infty} |\chi_{\tilde{I}}(\gamma A^n)|^2 = \sum_{n=-\infty}^{\infty} \delta_{n,k_0} = 1.$$

By the theorem, $\{D_{A^n} T_m g\}$ is a tight frame with frame bounds $\alpha = \beta = 1$.

Note that since $\hat{g}$ is a characteristic function, $g$ vanishes only slowly at infinity. We thus want to present another example, namely of a smooth frame generator.

**Remark 4.1** : Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with $a, b > 1$ and let $S^1 = \{\gamma = (\gamma_1, \gamma_2) \mid {\gamma_1}^2 + {\gamma_2}^2 = 1\}$ denote the unit circle, and $D = \{\gamma A^t : \gamma \in S^1, 0 \leq t \leq 1\}$.

1) For each $\gamma \neq 0$, there exists a unique $t_0 \in \mathbb{R}$ such that $\gamma_0 := \gamma A^{t_0} \in S^1$.

Furthermore,

$$\int_{-\infty}^{\infty} f(\gamma A^t)\, dt = \int_{-\infty}^{\infty} f(\gamma_0 A^t)\, dt.$$

for every $f \geq 0$ measurable.

2) For each $\gamma \neq 0$, there exists a unique $n_0 \in \mathbb{Z}$ such that $\widetilde{\gamma} := \gamma A^{n_0} \in D$.

Furthermore,

$$\sum_{n=-\infty}^{\infty} f(\gamma A^n) = \sum_{n=-\infty}^{\infty} f(\widetilde{\gamma} A^n)$$

for every $f \geq 0$.

*Proof.* 1) Since $\gamma \mapsto \|\gamma A^t\|$ is continuous, $\lim_{t\to-\infty} \|\gamma A^t\| = 0$ and $\lim_{t\to\infty} \|\gamma A^t\| = \infty$, by the intermediate value theorem, there exists $t_0$ such that $\|\gamma A^{t_0}\| = 1$. On the other hand, $t \mapsto \|\gamma A^t\|$ is strictly increasing, hence $t_0$ is unique. Then if $\gamma_0 := \gamma A^{t_0}$ we have for every measurable $f \geq 0$,

$$\begin{aligned} \int_{-\infty}^{\infty} f(\gamma A^t)\, dt &= \int_{-\infty}^{\infty} f(\gamma_0 A^{-t_0} A^t)\, dt \\ &= \int_{-\infty}^{\infty} f(\gamma_0 A^{t-t_0})\, dt \\ &= \int_{-\infty}^{\infty} f(\gamma_0 A^t)\, dt \qquad (t \to t + t_0). \end{aligned}$$

2) Let $\gamma_0, t_0$ be as in part 1). Then $\gamma A^n \in D \Leftrightarrow \gamma A^{t_0} A^{n-t_0} \in D \Leftrightarrow 0 \leq n - t_0 < 1 \Leftrightarrow t_0 \leq n < t_0 + 1$. Now the interval $[t_0, t_0 + 1)$ contains exactly one integer $n_0$, that is, $n_0$ is the unique integer such that $\gamma A^{n_0} \in D$. Set $\widetilde{\gamma} := \gamma A^{n_0}$. Then for all $f \geq 0$,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} f(\gamma A^n) &= \sum_{n=-\infty}^{\infty} f(\widetilde{\gamma} A^{-n_0} A^n) \\ &= \sum_{n=-\infty}^{\infty} f(\widetilde{\gamma} A^{n-n_0}) \qquad (n \to n + n_0) \\ &= \sum_{n=-\infty}^{\infty} f(\widetilde{\gamma} A^n). \end{aligned}$$

$\square$

**Example 4.3** (A frame generator $g$ with $\hat{g} \in C_c^\infty(\mathbb{R}^2)$). Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $b \geq a > 1$. Pick $M$ such that $\frac{1}{b^2} < M < \frac{1}{b}$ and $N$ such that $1 < N < a$. Pick $\varphi \in C_c^\infty(\mathbb{R})$ with $\text{supp}(\varphi) \subset [M, N]$, $0 \leq \varphi \leq 1$, $\varphi = 1$ on $[1/b, 1]$, $\varphi = 0$ on $\mathbb{R} \backslash (M, N)$ and $\varphi$ is increasing on $[M, 1/b]$ and decreasing $[1, N]$.

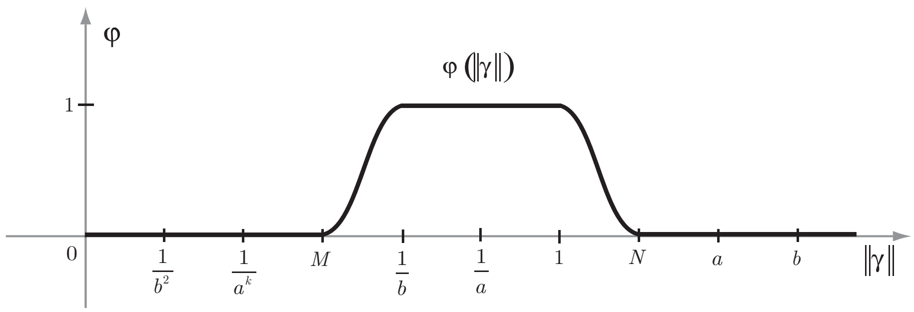


**Figure 4.3** The function $\varphi$

This is certainly possible as shown in the proof of Urysohn's lemma (Wade, 1999). Set $\hat{g}(\gamma) = \varphi(\|\gamma\|)$ where $\gamma = (\gamma_1, \gamma_2)$. We show that there exists $k \in \mathbb{N}$ such that

$$1 \leq \int_{-\infty}^{\infty} \varphi(\|\gamma A^t\|)^2 \, dt \leq k + 1 \tag{4.18}$$

and

$$1 \leq \sum_{n=-\infty}^{\infty} \varphi(\|\gamma A^n\|)^2 \leq k + 1 \tag{4.19}$$

for all $\gamma \neq 0$.

First let $\gamma \in S^1$. Then $\gamma$ can be written uniquely as $\gamma = (\cos\theta, \sin\theta)$ where $0 \leq \theta < 2\pi$. Then $\|\gamma A^t\|^2 = a^{2t}\cos^2\theta + b^{2t}\sin^2\theta$ so that $\forall t > 0$,

$$\|\gamma A^t\|^2 \leq b^{2t}\cos^2\theta + b^{2t}\sin^2\theta = b^{2t}$$

$$\|\gamma A^t\|^2 \geq a^{2t}\cos^2\theta + a^{2t}\sin^2\theta = a^{2t}$$

and hence

$$a^t \leq \|\gamma A^t\| \leq b^t \qquad \forall t > 0 \tag{4.20}$$

Similarly,

$$b^t \leq \|\gamma A^t\| \leq a^t \qquad \forall t < 0. \tag{4.21}$$

Thus,

$$\hat{g}(\gamma A^t) = \begin{cases} 1, & -1 \leq t < 0 \\ 0, & t < \log_a M \ \text{ or } \ \log_a N < t \\ \in [0,1], & \text{else.} \end{cases}$$

Pick $k$ such that $\dfrac{1}{a^k} < M < \dfrac{1}{b}$. By (4.20) and as $\varphi$ is decreasing (to be precise, nonincreasing) on $(1, \infty)$ then

$$\varphi(b^t) \leq \varphi(\|\gamma A^t\|) \leq \varphi(a^t) \qquad t \in (0, \infty). \tag{4.22}$$

By (4.21) and as $\varphi$ is increasing on $(0,1)$ then

$$\varphi(b^t) \leq \varphi(\|\gamma A^t\|) \leq \varphi(a^t) \qquad t \in (-\infty, 0). \tag{4.23}$$

If $t < -k$, then by (4.21), $\|\gamma A^t\| \leq a^{-k} < M$, hence $\varphi(\|\gamma A^t\|) = 0$. Similarly, if $t > 1$, then by (4.20), $\|\gamma A^t\| \geq a > N$, hence $\varphi(\|\gamma A^t\|) = 0$. Thus

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi(\|\gamma A^t\|)^2 \, dt &= \int_{-k}^{1} \varphi(\|\gamma A^t\|)^2 \, dt \\ &= \underbrace{\int_{-k}^{0} \varphi(\|\gamma A^t\|)^2 \, dt}_{I_1} + \underbrace{\int_{0}^{1} \varphi(\|\gamma A^t\|)^2 \, dt}_{I_2}. \end{aligned} \tag{4.24}$$

We estimate $I_1$. By (4.23)

$$\int_{-k}^{0} \varphi(b^t)^2 \, dt \leq \int_{-k}^{0} \varphi(\|\gamma A^t\|)^2 \, dt \leq \int_{-k}^{0} \varphi(a^t)^2 \, dt. \tag{4.25}$$

We estimate the term on the left of the inequality (4.25). As $\varphi = 1$ on $[\frac{1}{b}, 1]$ then

$$\int_{-k}^{0} \varphi(b^t)^2 \, dt \geq \int_{-1}^{0} \varphi(b^t)^2 \, dt \underset{\substack{t \in (-1,0) \\ \text{then } \varphi(b^t) = 1}}{=} \int_{-1}^{0} 1 \, dt = 1.$$

Similarly, we estimate the term on the right of the inequality by

$$\int_{-k}^{0} \varphi(a^t)^2 \, dt \leq \int_{-k}^{0} 1 \, dt = k.$$

Hence,

$$1 \leq \int_{-k}^{0} \varphi(\|\gamma A^t\|)^2 \, dt \leq \ k. \tag{4.26}$$

Next, we estimate $I_2$. Since $0 \leq \varphi \leq 1$, the

$$0 \leq \int_{0}^{1} \varphi(\|\gamma A^t\|)^2 \, dt \leq \int_{0}^{1} 1 \, dt = 1. \tag{4.27}$$

By (4.26) and (4.27),

$$1 \leq \int_{-k}^{1} \varphi(\|\gamma A^t\|)^2 \, dt \leq k + 1.$$

Hence by (4.24),

$$1 \leq \int_{-\infty}^{\infty} \varphi(\|\gamma A^t\|)^2 \, dt \leq k + 1 \tag{4.28}$$

for all $\gamma \in S^1$, that is, by (4.28)

$$1 \leq \int_{-\infty}^{\infty} |\hat{g}(\gamma A^t)|^2 \, dt \leq k + 1. \tag{4.29}$$

It now follows from remark 4.1 that (4.29) holds for every $\gamma \neq 0$.

Next we show that the left inequality (4.19) holds. Let $\gamma \neq 0$ and pick $t_0$ such that $\gamma_0 = \gamma A^{t_0} \in S^1$. Let $n_0 = \lfloor t_0 \rfloor$ where $\lfloor \cdot \rfloor$ denotes the least integer function. Then $t_0 - 1 < n_0 \leq t_0$. Note that $\|\gamma A^{t_0 - 1}\| = \|\gamma A^{t_0} A^{-1}\| = \|\gamma_0 A^{-1}\|$, while by (4.21),

$$\frac{1}{b} \leq \|\gamma_0 A^{-1}\| \leq \frac{1}{a}.$$

Since, the function $t \to \|\gamma A^t\|$ is increasing, we have

$$\frac{1}{b} \leq \|\gamma_0 A^{-1}\| = \|\gamma A^{t_0 - 1}\| \leq \|\gamma A^{n_0}\| \leq \|\gamma A^{t_0}\| = 1.$$

Since $\varphi = 1$ on $[1/b, 1]$, then $\varphi(\|\gamma A^{n_0}\|) = 1$. Hence,

$$\sum_{n=-\infty}^{\infty} \varphi(\|\gamma A^n\|)^2 \geq 1. \tag{4.30}$$

Next we show that the right inequality holds in (4.19). First, let $\gamma \in D$. Then $\gamma A^n = (a^{t+n} \cos \theta, b^{t+n} \sin \theta)$ for some $t$, $0 \leq t < 1$. If $n + t \notin [-k, 1)$ then

$\varphi(\|\gamma A^{n+t}\|) = 0$. Now $n + t \in [-k, 1)$ for exactly $k + 1$ values of $n$, namely for $n \in \{-k, -k+1, ..., 0\}$, so that

$$\sum_{n=-\infty}^{\infty} \varphi(\|\gamma A^n\|)^2 = \sum_{n=-k}^{1} \varphi(\|\gamma A^n\|)^2 \leq k + 1 \qquad \forall \gamma \in D, \tag{4.31}$$

as $|\varphi| \leq 1$. The inequalities (4.30) and (4.31) show that (4.19) holds for every $\gamma \in D$. It then follows from remark 4.1 that (4.19) holds for every $\gamma \neq 0$. Since $\text{supp}(\hat{g}) \subset \{\gamma \in \mathbb{R}^2 : \|\gamma\| < N\} \subset \left[\frac{-2N}{2}, \frac{2N}{2}\right] \times \left[\frac{-2N}{2}, \frac{2N}{2}\right]$, it follows by (4.19) and theorem 4.6 that $\{D_{A^n} T_{\frac{m}{2N}} g\}_{n\in\mathbb{Z}, m\in\mathbb{Z}^2}$ is a wavelet frame with frame bounds $4N^2$ and $4(k+1)N^2$.

**Example 4.4** (A frame generator $g$ with $\hat{g} \in C_c^{\infty}(\mathbb{R}^2)$). Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ be an expanding matrix. Without loss of generality, suppose that $b \geq a > 1$. Let $D = \{(a^s \cos\theta, b^s \sin\theta) : \quad 0 \leq s \leq 1\}$ as in remark 4.1, so its closure is

$$\bar{D} = \{(a^s \cos\theta, b^s \sin\theta) : \quad 0 \leq s \leq 1\}. \tag{4.32}$$

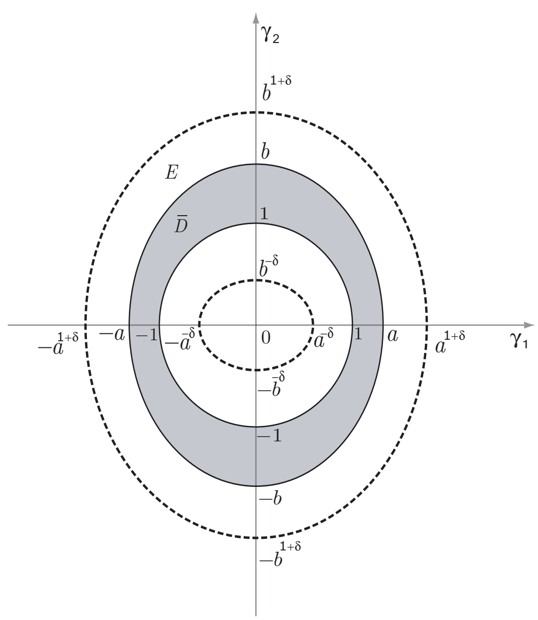


**Figure 4.4** The sets $\overline{D}$ and $E$

Then $\bar{D}$ is a compact elliptic annulus. Next we enlarge this annulus to a slightly bigger open set, by fixing $\delta$, $0 < \delta < \frac{1}{2}$ and setting

$$E = \{(a^s \cos\theta, b^s \sin\theta), \quad -\delta < s < 1+\delta\}. \tag{4.33}$$

By Urysohn's lemma, there exist $\hat{g} \in C_c^\infty(\mathbb{R}^2)$ such that $0 \leq \hat{g} \leq 1 \;\; \forall \gamma \in \mathbb{R}^2$ and $\hat{g}(\gamma) = 1 \;\; \forall \gamma \in \bar{D}$ , and $\operatorname{supp}(\hat{g}) \subset E$.

We will show that the function $g$ generates a wavelet frame for $L^2(\mathbb{R}^2)$. Let $\gamma \in \bar{D}$. Then $\gamma = (a^s \cos\theta, b^s \sin\theta)$ for some $0 \leq s \leq 1$, and $\gamma A^t = (a^{s+t} \cos\theta, b^{s+t} \sin\theta)$. Note that

$$\gamma A^t \in \bar{D} \Leftrightarrow 0 \leq s+t \leq 1 \Leftrightarrow -s \leq t \leq 1-s$$

Also,

$$\gamma A^t \in E \Leftrightarrow -\delta < s+t < 1+\delta \Leftrightarrow -s-\delta < t < 1+\delta-s.$$

Hence,

$$\hat{g}(\gamma A^t) = 1 \qquad \forall t \in [-s, 1-s] \tag{4.34}$$

and

$$\hat{g}(\gamma A^t) = 0 \qquad \forall t \notin (-s-\delta, 1+\delta-s). \tag{4.35}$$

First let us find $\alpha, \beta$ such that $\alpha \leq \displaystyle\int_{-\infty}^{\infty} |\hat{g}(\gamma A^t)|^2 \, dt \leq \beta$. By (4.34) we have

$$\int_{-\infty}^{\infty} |\hat{g}(\gamma A^t)|^2 \, dt \geq \int_{-s}^{1-s} |\hat{g}(\gamma A^t)|^2 \, dt = \int_{-s}^{1-s} 1 \, dt = 1$$

and by (4.35), and since $\hat{g} \leq 1$

$$\int_{-\infty}^{\infty} |\hat{g}(\gamma A^t)|^2 \, dt \leq \int_{-\infty}^{\infty} |\chi_E(\gamma A^t)|^2 \, dt = \int_{-\delta-s}^{1+\delta-s} 1 \, dt = 1+2\delta.$$

Hence,

$$1 \leq \int_{-\infty}^{\infty} |\hat{g}(\gamma A^t)|^2 \, dt \leq 1+2\delta \qquad \forall \gamma \in \bar{D}.$$

It now follows from remark 4.1 that

$$1 \leq \int_{-\infty}^{\infty} |\hat{g}(\gamma A^t)|^2 \, dt \leq 1 + 2\delta \qquad \forall \gamma \neq 0.$$

Next we will show that

$$1 \leq \sum_{n=-\infty}^{\infty} |\hat{g}(\gamma A^n)|^2 \leq 2 \qquad \forall \gamma \neq 0.$$

Let $\gamma \neq 0$. As shown in remark 4.1, there exists a unique $n_0 \in \mathbb{Z}$ such that $\gamma_0 = \gamma A^{n_0} \in D$. Then

$$\sum_{n=-\infty}^{\infty} |\hat{g}(\gamma A^n)|^2 \geq |\hat{g}(\gamma A^{n_0})| = |\hat{g}(\gamma_0)| = 1. \tag{4.36}$$

Note that $\gamma_0 = (a^s \cos\theta, b^s \sin\theta)$ for some unique $0 \leq s < 1$. and $0 \leq \theta < 2\pi$.

If $s > \frac{1}{2}$ then $s + n \in (-\delta, 1+\delta)$, implies $n \in \{-1, 0\}$.

If $s \leq \frac{1}{2}$ then $s + n \in (-\delta, 1+\delta)$, implies $n \in \{0, 1\}$.

Hence $\gamma_0 A^n \in E$ for at most two values of $n$, namely, $n_1 = 0$ and $n_2 = -1$ (resp. $n_2 = 1$), so that $\hat{g}(\gamma_0 A^n) = 0$ all other values of $n$. Thus by remark 4.1,

$$\sum_{n=-\infty}^{\infty} |\hat{g}(\gamma A^n)|^2 = \sum_{n=-\infty}^{\infty} |\hat{g}(\gamma_0 A^n)|^2 = |\hat{g}(\gamma_0)|^2 + |\hat{g}(\gamma_0 A^{n_2})|^2 \leq 1 + 1 = 2.$$

Together with (4.36), we have shown that

$$1 \leq \sum_{n=-\infty}^{\infty} |\hat{g}(\gamma A^n)|^2 \leq 2 \qquad \forall \gamma \neq 0.$$

Now as $E \subset \left[\frac{-1}{2\tilde{b}}, \frac{1}{2\tilde{b}}\right] \times \left[\frac{-1}{2\tilde{b}}, \frac{1}{2\tilde{b}}\right]$ where $\tilde{b} = \dfrac{1}{2b^{1+\delta}}$, it follows from theorem 4.6 that $\{D_{A^n} T_{\tilde{b}m} g\}_{n \in \mathbb{Z}, m \in \mathbb{Z}^2}$ is a wavelet frame with frame bounds $4b^{2+2\delta}$ and $8b^{2+2\delta}$.

# CHAPTER V

# CONCLUSION

The objective of this thesis was to discuss methods of reconstructing a function $f \in L^2(\mathbb{R}^2)$ from its wavelet transform $Wf(t,b)$ not by a weak integral, but through approximation by usual integrals or infinite series.

For this, we considered three methods: Modification of the set of integration in the weak integral, introduction of an approximate identity into the integrand, and construction of wavelet frames. We have obtained the following results:

1. In theorem 4.1 we showed that $f_\varepsilon(x) \to f(x)$ in the mean square norm as $\varepsilon \to -\infty$, where

$$f_\varepsilon(x) = \frac{1}{c_\psi} \int_\varepsilon^\infty \int_{\mathbb{R}^2} (Wf)(t,b) D_{A^t} T_b \psi(x)\, db\, dt$$

   provided that the dilation matrix $A$ is expanding, that the wavelet $\psi$ is admissible and its Fourier transform satisfies a weak decay condition at infinity.

2. In theorem 4.2, we proved a similar approximation result in case $A^{-1}$ is expanding.

3. In theorem 4.3, we showed that under the additional assumption that $\hat{\psi}$ vanish in a neighborhood of zero and $f$ be bandlimited, then $f_\varepsilon = f$ for sufficiently negative $\varepsilon$, and $f$ can be reconstructed by the usual integral

$$f(x) = \frac{1}{c_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}^2} Wf(t,b) D_{A^t} T_b \psi(x)\, db\, dt \qquad a.e.$$

4. In theorem 4.4, we extended theorem 4.1 to $L^2(\mathbb{R}^n)$ under the assumption that the matrix $A$ is diagonalizable and expanding.

5. In theorem 4.5, we showed that if $\{\rho_k\}$ is a symmetric approximate identity, and the matrix $A$ is diagonalizable, then $f_k \to f$ in the mean square norm, where
$$f_k(x) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} Wf(t,b)(\rho_k * D_{A^t} T_b \psi)(x)\, db\, dt$$
as a usual integral.

6. In theorem 4.6, assuming the matrix $A$ is diagonal, we identified admissible functions $\psi$ which provide for wavelet frames. As examples, we constructed several wavelet frame generators $\psi$.

The results in this thesis are extensions to $L^2(\mathbb{R}^2)$ of theorems presented in Gasquet, C. and Witomaski, P. (1998) and Heil, C.E. and Walnut, D.F. (1989) for $L^2(\mathbb{R})$. It is conceivable that these results can be extended to $L^2(\mathbb{R}^n)$ for an arbitrary expanding dilation matrix $A$. The difficultly here is to extend lemma 4.1 to higher dimensions, that is to estimate how quickly the points $\gamma A^t$ tend to infinity, as $t \to \infty$.

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