NON-LINEAR TRANSFORMATIONS OF THE TERRELL EFFECT: A COMPARATIVE STUDY

Mr. Seckson Sukkhasena

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for the Degree of Master of Science in Physics

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วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

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วิทยานิพนธ์ฉบับนี้เป็นการศึกษา 1). การสังเกตหลักพื้นฐานในการถ่ายภาพของวัตถุของเทอร์เรลล์ ซึ่งแสงที่แผ่ออกมาจากจุดต่างๆ บนวัตถุจะต้องแผ่ออกมาที่เวลาต่างกันเป็นลำดับ เพื่อมาถึงยังจุดสังเกตพร้อมกัน 2). การแปลงโลเร็นตซ์ของทฤษฎีสัมพัทธภาพ และ 3). การค้นพบ ความสัมพันธ์ระหว่างระนาบ 2 มิติ ในกรอบสังเกต กับแสงที่แผ่ออกมาจากวัตถุ โดยได้มาจากความสอดคล้องกันของการส่งผ่านแสงที่แผ่ออกมาจากวัตถุไปยังระนาบ 2 มิติ ความสัมพันธ์นี้ สามารถประยุกต์ใช้กับวัตถุใดๆก็ได้ ไม่ว่าวัตถุนั้นจะมีรูปร่างซับซ้อนมากแค่ไหนก็ตามในลักษณะการเคลื่อนที่แบบสัมพัทธ์กันเทียบกับกรอบสังเกตที่ความเร็วไม่เจาะจงจนถึงระดับความเร็วใกล้แสง ความสัมพันธ์ที่ได้มานั้นไม่เหมือนกับการแปลงโลเร็นตซ์ โดยมีลักษณะเป็นการแปลงแบบไม่เชิงเส้น ซึ่งได้มีการเปรียบเทียบกันระหว่างการแปลงโลเร็นตซ์กับการแปลงแบบกาลิเลียนโดยพิจารณารวมถึงการแผ่ของแสงมีความเร็วจำกัดด้วย จากการเพ่งเล็งที่ตำแหน่งเจาะจงหนึ่งบนไม้บรรทัดที่กำลังเคลื่อนที่ แสดงให้เห็นถึงการหดตัวโลเร็นตซ์อาจมองเห็นได้ที่ตำแหน่งดังกล่าว ภาพของวัตถุที่ปรากฏในลักษณะที่มีความซับซ้อนและแปลกประหลาดเป็นผลมาจากการเกิดปรากฏการณ์คล้ายปรากฏการณ์โดปป์เลอร์สำหรับสเกลซึ่งทำให้เกิดการปิดบังการหดตัวโลเร็นตซ์ และสุดท้ายได้มีการแก้ปัญหาของความขัดแย้งกันของปัญหารถไฟให้หมดไปได้เป็นครั้งแรก ซึ่งปัญหานี้มีรากฐานมาจากงานวิจัยของเทอร์เรลล์กับไวซ์สคอปฟ์และได้ถูกย้ำถึงปัญหานี้อีกครั้งจากงานวิจัยของแมททิวส์กับลัคส์มานัน โดยส่วนใหญ่ได้มีการกล่าวถึงปัญหานี้เมื่อประมาณ 30 ปีมาแล้ว

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Taking into account of (i) Terrell's basic observation that in photographing an object different points on the object must "emit" light at different times in order to reach the observation point simultaneously, (ii) the Lorentz transformations of relativity and (iii) the piercing of these light rays an appropriate 2D plane in the observation frame, a derivation of the corresponding mapping onto such a 2D plane is derived. The latter may be applied to any object no matter how complicated in relative motion, to the observation frame, at arbitrary speeds including extreme relativistic ones. Unlike the Lorentz transformations, which are linear in character, the present ones are necessarily non-linear. For completeness these fully relativistic transformations are compared with the corresponding Galilean ones which, however, take into account the finiteness of the propagation speed of light. By concentrating over a specific point of a moving ruler, it is shown how the Lorentz contraction may be visible about such a point. The complexity and highly non-trivial aspect of the latter arises because a Doppler-like effect for scale is observed, which, in general, masks the Lorentz contraction. Finally a resolution of the long standing so-called "train" paradox, having its roots in the early work of Terrell and Weisskopf and emphasized by Mathews and Lakshmanan almost thirty years ago, is provided for the first time.

สาขาวิชาฟิสิกส์ ลายมือชื่อนักศึกษา...

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Seckson Sukkhasena

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Chapter I
Introduction

There are many experiments which verify the time dilation predicted by relativity involving the decay of elementary particles, e.g., Bailey et al. (1977) and in a beautiful experiment by Hafele and Keating in 1972 using (caesmic) clocks, which clearly show the "slowing down of time" of moving objects. No physicist in his right mind would doubt another important consequence of relativity - that is of the so-called Lorentz contraction. As the legendary physicist R. P. Feynman puts it in his famous lectures on physics (1965): "All the physicists who could not accept relativity are now dead". Unlike the time dilation effect, the actual visibility of the Lorentz contraction itself in experiments remains undisputably a great challenge. Even the theoretical description of the simplest experiment of photographing the Lorentz contraction turned out to be far from obvious. As early as 1922, Lorentz (1931) stated that this contraction could be photographed. There are indications, as pointed out by Terrell (1959), that even Einstein left us with this impression. In 1960, Weisskopf (1960, cf. 1961), in his early review in Physics Today, on this states: "We all believed that, according to special relativity, an object in motion appears to be contracted in the direction of motion". The word "appears" has caused a lot of confusion in the physics literature over the years.

In a truly remarkable paper of Terrell in 1959, the latter has investigated the visibility of the Lorentz contraction. The main contribution of Terrell to this fundamental problem was the following: in order to see an object, unlike measuring its length, all the light rays coming from the object to the observation point have to reach this point simultaneously. That is, in order that the light rays reach the observation point simultaneously, they have to leave the different parts of the object at different times due to the finite speed *c* of the propagation of light itself. The latter fact is referred to as the time-delay mechanism. This looks so simple today that it is surprising that it took over a half of a century since Einstein's work, before any statements were made about the visual appearance of the Lorentz contraction. [It is worth recalling that in measuring the length of an object one determines the positions of its extremities simultaneously, unlike the seeing of an object]. Ever since, the visual appearance of relativistically moving objects, with its associated time delay mechanism, has been justifiably referred to as the Terrell Effect. Since the appearance of Terrell's paper (1959) and Weisskopf's review paper in 1960, many papers (e.g., Yngström, 1962; Scott and Viner, 1965; Scott and Driel, 1970; McGill, 1968; Mathews and Lakshmanan, 1972; Hollenbach, 1976; Hickey, 1979) have been published on the subject, and more papers will undoubtedly continue to appear (e.g., Burke and Strod, 1991; Howard et al.,1995), on this challenging, and certainly very intriguing and fascinating problem. The earlier studies (e.g., Terrell, 1959; Weisskopf 1960; Penrose 1959) did not pay detailed attention to the method of observation which was particularly illuminated later in (e.g., Scott and Viner, 1965; McGill and Driel, 1968; Mathews and Lakshmanan, 1972; Hollenbach, 1976; Hickey, 1979), and generalized with a more precise definition of the relativistically moving object as projected on a two–dimensional surface (the latter being particularly emphasized in Hickey's paper in 1979) as on a photograph.

For orientation, we recall the strategy in the study of the Terrell effect is to consider all the rays "emerging" from the object to reach the observer simultaneously. In this work, we consider the latter to be infinitesimal as a point. When all the light rays from the object reach the observation point, this aperture is instantaneously closed. All the light rays are then collected on a sensitive detecting plate (plane) perpendicular to the optic axis of observation. We consider the object under study fixed in a frame F' in relative motion to the observation frame F along the *x*-axis.

As stated about the word "appears" or the statement "appears as on a photograph" have caused some confusion over the years. To be precise, the latter are meant in the following manner in the present investigation and are based on taking into account these three points:

(1) Terrell's observation that different points on the object, in relative motion, must "emit" light at different times in order to reach an observation point simultaneously,

(2) the Lorentz transformations,

(3) the piercing of these light rays an appropriate 2D-plane in the observation frame.

The necessarily non-linear transformations resulting from the application of the above three points will be referred to as the non-linear Terrell transformations.

One of the most puzzling aspects about the earlier investigations, concerning this problem, is the so-called "train" paradox. This paradox has its roots in the early work of Terrell (1959) and Weisskopf (1960,1961), and was emphasized by Mathews and Lakshmanan (1972) almost thirty years ago. In its simplest terms, the paradox arises in the following manner. One often reads in the above quoted papers that an object appears to be rotated when in motion relative to an observation frame due to the fact that different points on the object must "emit" light at different times in order to reach an observation point simultaneously. Such an inference seems to indicate that a rectangular block, for example, sliding on (smooth) rails and the edges of its bottom in contact with them, appear off of them due to the relative motion with the rails stationary relative to the observer and hence the paradox – [as a train, for example, off of the rails]. The same reasoning may be applied, as shown in an illuminating application given in this thesis (see, e.g., Fig. 4.1(b) also (a)), Fig. 4.4.(b)), to an object with a horizontal flat top touching a "smooth" flat horizontal stationary plane. Again this rotational effect would seem to imply, in particular cases, as if one end of the object has miraculously broken and gone through the flat plane due to the relative motion. Although the demonstration of the absence of a paradox seems non-trivial, a rigorous and complete resolution of the long standing “train” paradox will be given.

The purpose of this thesis is to give a complete derivation of the explicit (non-linear) transformations arising from the application of the three points mentioned earlier which may be applied to any object, in relative motion, no matter how complicated the object is. As already mentioned above, these transformations will be appropriately referred to as the non-linear Terrell transformations. The closest investigation to these transformations was given by Hickey (1979) which, however, applies methods of mapping out the tangents to points on the object. The latter also provides no room for resolving the “train” paradox. For completeness and for a comparative study with the fully relativistic case we also specialize these transformations to the Galilean case by formally incorporating into them the finiteness of the propagation speed c of light (the so-called time delay mechanism). Three major applications of the derived transformations are given corresponding to a set of houses, to a pyramid and to a train. These applications clearly and quite generally explain the roles of the transformations and the physical consequence of the Lorentz contraction when the Terrell effect is taken into consideration. These figures constitute an integral part of this investigation. A very important contribution of this work is to provide a resolution of the long standings “train” paradox. It is also explicitly shown how the Lorentz contraction may be visible by concentrating over a specific point of a rapidly moving ruler.

The plan of the thesis is as follows. Chapter II deals with the intricacies of the Galilean and Lorentz transformations starting from the very basics of relativity. This chapter will be essential in all of our subsequent analysis. In Chapter III, we provide the complete non-linear Terrell transformations for both the Galilean and relativistic cases spelling out all of the fine details. Chapter IV deals with the very important applications of these transformations to the set of three objects mentioned above. This chapter is appropriately entitled: “Applications and Comparative Study – Seeing is Believing”, and is an integral part of the thesis. The resolution of the long standing “train” paradox is provided in Chapter V. This chapter also contains some pertinent analytical properties of the non-linear transformations which help us understand more clearly their applications to the corresponding figures given in Chapter IV. The final chapter, Chapter VI, deals with our conclusion and summarizes some of our results.

Chapter II
The Galilean and Lorentz Transformations

2.1 Introduction

The purpose of this chapter is to give a brief introduction to those aspects of special relativistic physics and closely related aspects culminating into the famous Lorentz transformations and their classical counterparts, the Galilean transformations. This is essential in our subsequent analysis and for our very basic understanding of the subject.

For the description of processes taking place in nature, one must have a system of references. By a system of reference we understand a system of coordinates serving to indicate the position of a particle in space, as well as clocks fixed in this system serving to indicate the time.

There exist systems of references in which a freely moving body, i.e., a moving body, which is not acted upon by external forces, proceeds with constant velocity. Such references system, are said to be inertial.

If two reference systems move uniformly relative to each other and if one of them is an inertial system, then clearly the other is also inertial (in this system too every free motion will be linear and uniform). In this way we can obtain arbitrarily many inertial systems of reference, moving uniformly relative to one another.

Experiment shows that the so-called principle of relativity is valid. According to this principle all the laws of nature are identical in all inertial systems of reference. In other words, the equations expressing the laws of nature are invariant with respect to transformations of coordinates and time from one inertial system to another. This means that the equation describing any law of nature, when written in terms of coordinates and time in different inertial reference systems, has one and the same form.

The interaction of material particles is described in ordinary mechanics by means of a potential energy of interaction, which appears as a function of the coordinates of the interacting particles. It is easy to see that this manner of describing interactions contains the assumption of instantaneous propagation of interactions. For the forces exerted on each of the particles by the other particles at a particular instant of time depend, according to this description, only on the positions of the particles at this one instant. A change in the position of any of the interacting particles influences the other particles immediately.

However, experiment shows that instantaneous interactions do not exist in nature. Thus a mechanics based on the assumption of instantaneous propagation of interactions contains within itself a certain inaccuracy. In actuality, if any change takes place in one of the interacting bodies, it will influence the other bodies only after the lapse of a certain interval of time. It is only after this time interval that processes caused by the initial change begin to take place in the second body. Dividing the distance between the two bodies by this time interval, we obtain the velocity of propagation of the interaction.

We note that this velocity should, strictly speaking, be called the maximum velocity of propagation of interaction. It determines only that interval of time after which a change occurring in one body begins to manifest itself in another. It is clear that the existence of a maximum velocity of propagation of interactions implies, at the same time, that motions of bodies with greater velocity than this are in general impossible in nature. For, if such a motion could occur, then by means of it, one could realize an interaction with a velocity exceeding the maximum possible velocity of propagation of interactions.

Interactions propagating from one particle to another are frequently called “signals”, sent out from the first particle, and “informing” the second particle of changes, which the first has experienced. The velocity of propagation of interaction is then referred to as the signal velocity.

The special theory of relativity asserts that the speed of light (in vacuum) is the same in all inertial frames. Thus the velocity of propagation of interaction is a universal constant. This constant velocity of light is usually designated by the letter c, and its exact numerical value is

$$c = 2.99792458 \times 10^{8} \quad m/s.$$

The large value of this velocity explains the fact that in practice, classical mechanics appears to be sufficiently accurate in most cases. The velocities with, which we have occasion to deal with in every day life are usually so small compared with the velocity of light. The assumption that the latter is infinite does not materially affect the accuracy of the results.

The combination of the principle of relativity with the finiteness of the velocity of propagation of interactions is called the principle of relativity of Einstein (it was formulated by Einstein in 1905). In contrast to the principle of relativity of Galileo, which was based on an infinite velocity of propagation of interactions.

The mechanics based on the Einsteinian principle of relativity (we shall usually refer to it simply as the principle of relativity) is called relativistic. In the limiting case when the velocities of the moving bodies are small compared with the velocity of light, we can neglect the effect on the motion of the finiteness of the velocity of propagation. Then relativistic mechanics goes over into the usual non-relativistic mechanics, based on the assumption of instantaneous propagation of interaction; this mechanics is called Newtonian or classical. The limiting transition from relativistic to classical mechanics can be produced formally by taking the limit $c \to \infty$ in the formulae of relativistic mechanics.

In classical mechanics distance is already relative, i.e., the spatial relations between different events depend on the system of reference in which they are described. The statement that two non-simultaneous events occur at one and the same point in space or, in general, at a definite distance from each other, acquires a meaning only when we indicate the system of reference which is being used.

On the other hand, time is absolute in classical mechanics; in other words, the properties of time are assumed to be independent of the system of reference; there is one time for all reference frames. This means that if any two phenomena occur simultaneously for any one observer, then they occur simultaneously also for all others. In general, the interval of time between two given events is assumed to be identical for all systems of reference in classical mechanics.

It is easy to show, however, that the idea of an absolute time is in complete contradiction to the Einstein principle of relativity. For this it is sufficient to recall that in classical mechanics, based on the concept of an absolute time, a general law of combination of velocities is valid, according to which the velocity of a composite motion is simply equal to the (vector) sum of the velocities which constitute this motion. This law, being universal, should also be applicable to the propagation of interactions. From this it would follow that the velocity of propagation of light must be different in different inertial systems of reference, in contradiction to the principle of relativity. In this matter experiment completely confirms the principle of relativity. Measurements first performed by Michelson (1887) showed, in particulars complete lack of dependence of the velocity of light on its direction of propagation; whereas according to classical mechanics the velocity of light should be smaller in the direction of the earth’s motion than in the opposite direction.

Thus the principle of relativity leads to the result that time is not absolute. Time elapses differently in different systems of references. Consequently the statement that a definite time interval has elapsed between two given events acquires meaning only when the reference frame to which this statement applies is indicated. In particular, events, which are simultaneous in one reference frame, will not in general be simultaneous in other frames.

2.1.1 Intervals

In what follows we shall frequently use the concept of an event. An event is described by the place where it occurred, and the time when it occurred. Thus an event occurring in a certain material particle is defined by the three coordinates of that particle and the time when the event occurs.

It is frequently useful for reasons of presentation to use a fictitious four-dimensional space, on the axes of which marked three space coordinates and time. In this space events are represented by points, are called world points. In this fictitious four-dimensional space there corresponds to each particle a certain curve, called a world-line. The points of this line determine the coordinates of the particle at all moments of time. It is easy to show that to a particle in uniform rectilinear motion there corresponds a straight world-line.

We now express the principle of the invariance of the velocity of light in mathematical form. For this purpose we consider two reference systems K and K' moving relative to each other with constant velocity. We choose the coordinate axes so that the axes X and X' coincide, while the Y and Z axes are parallel to Y' and Z'; we designate the time in the systems K and K' by t and t'.

Let the first event consist of sending out a signal, propagating with light velocity, from a point having coordinates x_1 y_1 z_1 in the K system, at time t_1 in this system. We observe the propagation of this signal in the K system. Let the second event consist of the arrival of the signal at point $x_2\ y_2\ z_2$ at the moment of time t_2. The signal propagates with velocity c; the distance covered by it is therefore $c(t_1 - t_2)$. On the other hand, this same distance equals $\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{\frac{1}{2}}$. Thus we can write the following relation between the coordinates of the two events in the K system:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2 = 0. \qquad (2.1)$$

The same two events, i.e., the propagation of the signal, can be observed from the K' system:

Let the coordinates of the first event in the K' system be $x'_1\, y'_1\, z'_1\, t'_1$, and of the second: $x'_2\, y'_2\, z'_2\, t'_2$. Since the velocity of light is the same in the K and K' systems, we have, similarly to Eq.(2.1):

$$(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2 - c^2(t'_2 - t'_1)^2 = 0. \qquad (2.2)$$

If $x_1\ y_1\ z_1\, t_1$ and $x_2\ y_2\ z_2\, t_2$ are the coordinates of any two events, then the quantity

$$s_{12} = \left[c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2\right]^{\frac{1}{2}}, \qquad (2.3)$$

is called the <u>interval</u> between these two events.

Thus it follows from the principle of invariance of the velocity of light that if the interval between two events is zero in one coordinate system, then it is equal to zero in all other systems.

If two events are infinitesimally close to each other, then the interval ds between them is

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 . \qquad (2.4)$$

The form of expressions Eqs.(2.3) and (2.4) permit us to regard the interval, from the formal point of view, as the distance between two points in a fictitious four-dimensional space (whose axes are labeled by $x, y, z,$ and the product ct). But there is a basic difference between the rule for forming this quantity and the rule in ordinary geometry: in forming the square of the interval, the squares of the coordinate differences along the different axes are summed, not with the same sign, but rather with varying signs. (The four-dimensional geometry described by the quadratic form Eq.(2.4) was introduced by H. Minkowski, in connection with the theory of relativity. This geometry is called pseudo-Euclidean, in contrast to ordinary Euclidean geometry)

As already shown, if $ds = 0$ in one inertial system, then $ds' = 0$ in any other system. On the other hand, ds and ds' are infinitesimals of the same order. From these two conditions it follows that ds^2 and ds'^2 must be proportional to each other:

$$ds^2 = a ds'^2 , \qquad (2.5)$$

where the coefficient a can depend only on the absolute value of the relative velocity of the two inertial systems. It cannot depend on the coordinates or the time, since then different points in space and different moments in time would not be equivalent, which would be in contradiction to the homogeneity of space and time. Similarly, it cannot depend on the direction of the relative velocity, since that would contradict the isotropy of space.

Let us consider three reference systems K, K_1, K_2, and let v_1 and v_2 be the velocities of systems K_1 and K_2 relative to K.

We then have:

$$ds^2 = a(v_1)ds_1^2, \qquad ds^2 = a(v_2)ds_2^2. \tag{2.6}$$

Similarly we can write

$$ds_1^2 = a(v_{12})ds_2^2, \tag{2.7}$$

where v_{12} is the absolute value of the velocity of K_2 relative to K_1. Comparing these relations with one another, we find that we must have

$$\frac{a(v_2)}{a(v_1)} = a(v_{12}). \tag{2.8}$$

But v_{12} depends not only on the absolute values of the vectors v_1 and v_2, but also on the angle between them. However, this angle does not appear on the left side of formula (2.8). It is therefore clear that this formula can be correct only if the function $a(v)$ reduces to a constant, which is equal to unity according to this same formula.

Thus,

$$ds^2 = ds'^2, \tag{2.9}$$

and from the equality of the infinitesimal intervals there follows the equality of finite intervals: $s = s'$.

Thus we arrive at a very important result: the interval between two events is the same in all inertial systems of reference, i.e., it is invariant under transformation from one inertial system to any other. This invariance is the mathematical expression of the constancy of the velocity of light.

Again let $x_1\, y_1\, z_1\, t_1$ and $x_2\, y_2\, z_2\, t_2$ be the coordinates of two events in a certain reference system K. Does there exist a coordinate system K', in which these two events occur at one and the same point in space? We introduce the notation

$$t_2 - t_1 = t_{12}\,, \qquad (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = l_{12}^2. \tag{2.10}$$

Then the interval between events in the K system is:

$$s_{12}^2 = c^2 t_{12}^2 - l_{12}^2,$$

and in the K' system

$$s'^2_{12} = c^2 t'^2_{12} - l'^2_{12},$$

whereupon, because of the invariance of intervals,

$$c^2 t_{12}^2 - l_{12}^2 = c^2 t'^2_{12} - l'^2_{12},$$

If two events occur at the same point in the K' system, that is, we require $l'_{12} = 0$, then

$$s_{12}^2 = c^2 t_{12}^2 - l_{12}^2 = c^2 t'^2_{12} > 0\,.$$

Consequently a system of reference with the required property exists if $s_{12}^2 > 0$, that is, if the interval between the two events is a real number. Real intervals are said to be timelike.

Thus, if the interval between two events is timelike, then there exists a system of reference in which the two events occur at one and the same place. The time which elapses between the two events in this system is

$$t'_{12} = \frac{1}{c}\sqrt{c^2 t_{12}^2 - l_{12}^2} = \frac{s_{12}}{c}. \qquad (2.11)$$

If two events occur in one and the same body, then the interval between them is always timelike, for the distance which the body moves between the two events cannot be greater than ct_{12}, since the velocity of the body cannot exceed c. So we always have

$$l_{12} < ct_{12}.$$

Let us now ask whether or not we can find a system of reference in which the two events occur at one and the same time. As before, we have for the K and K' systems $c^2 t_{12}^2 - l_{12}^2 = c^2 t'^2_{12} - l'^2_{12}$. We want to have $t'_{12} = 0$, so that

$$s_{12}^2 = -l'^2_{12} < 0.$$

Consequently the required system can be found only for the case when the interval s_{12} between the two events is an imaginary number. Imaginary intervals are said to be spacelike.

Thus if the interval between two events is spacelike, there exists a reference system in which the two events occur simultaneously. The distance between the points where the events occur in this system is

$$l'_{12} = \sqrt{l_{12}^2 - c^2 t_{12}^2} = i s_{12}. \qquad (2.12)$$

The division of intervals into spacelike and timelike intervals is, because of their invariance, an absolute concept. This means that the timelike or spacelike character of an interval is independent of the reference system.

Let us take some event O as our origin of time and space coordinates. In other words, in the four-dimensional system of coordinates, the axes of which are marked x, y, z, t, the world point of the event O is the origin of coordinates. Let us now consider what relation other events bear to the given event O. For visualization, we shall consider only one space dimension and the time, marking them on two axes (Fig. 2.1). Uniformly rectilinear motion of a particle, passing through $x = 0$ at $t = 0$, is represented by a straight line going through O, and inclined to the t axis at an angle whose tangent is the velocity of the particle. Since the maximum possible velocity is c, there is a maximum angle, which this line can subtend with the t axis. In Fig.2.1, the two lines representing the propagation of two signals are shown

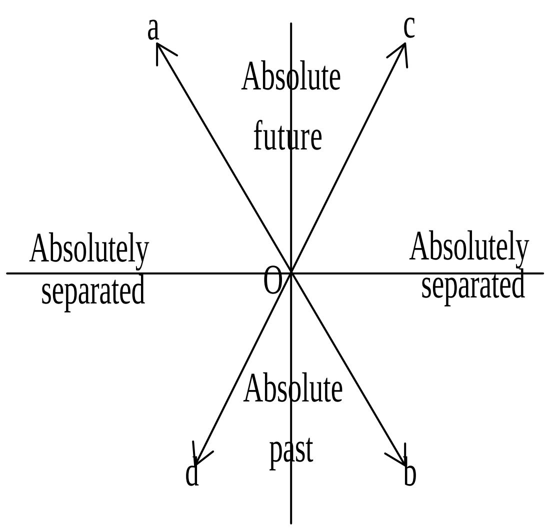


Fig. 2.1. The light cone

(with the velocity of light) in opposite directions passing through the event O (i.e., going through $x = 0$ at $t = 0$). All lines representing the motion of particles can lie only in the regions aOc and dOb. On the lines ab and cd, $x = \pm ct$. First consider events whose world points lie within the region aOc. It is easy to show that for all the points of this region $c^2t^2 - x^2 > 0$. In other words, the interval between any event in this region, and the event O is timelike. In this region $t > 0$, i.e., all the events in this region occur "after" the event O. But two events which are separated by a timelike interval cannot occur simultaneously in any reference system. Consequently it is impossible to find a reference system in which any of the events in region aOc occurred "before" the event O, i.e., at time $t < 0$. Thus all the events in region aOc are future events relative to O in all reference systems. Therefore this region can be called the absolute future relative to O.

In exactly the same way, all events in the region bOd are in the absolute past relative to O; i.e., events in this region occur before the event O in all systems of reference.

Next consider regions dOa and cOb. The interval between any event in this region, and the event O is spacelike. These events occur at different points in space in every reference system. Therefore these regions can be said to be absolutely remote relative to O. However, the concepts "simultaneous", "earlier", and "later" are relative for these regions. For any event in these regions there exist systems of reference in which it occurs after the event O, systems which occurs earlier than O, and finally one reference system in which it occurs simultaneously with O.

Note that if we consider all three space coordinates instead of just one, then instead of the two intersecting lines of Fig. 2.1, we would have a "cone" represented by the equation $x^2 + y^2 + z^2 - c^2t^2 = 0$ in the four-dimensional coordinate system $x, y, z, t,$ with the axis of the cone coinciding with the t axis. (This cone is called the light cone.) The region of absolute future and absolute past are then represented by the two interior portions of this cone.

Two events can be related causally to each other only if the interval between them is timelike; this follows immediately from the fact that no interaction can propagate with a velocity greater than the velocity of light. As we have just seen, it is precisely for these events that the concepts “earlier” and “later” have an absolute significance, which is a necessary condition for the concepts of cause and effect to have meaning. A remarkable property of the Minkowski space-time is that the triangular inequality, known to hold in Euclidean space, is reversed (Manoukian, 1993) for three causally related events and lies in the heart of the twin “paradox” problem.

2.1.2 Proper Time

Suppose that in a certain inertial reference system we observe clocks which are moving relative to us in an arbitrary manner. At each different moment of time this motion can be considered as uniform. Thus at each moment of time we can introduce a coordinate system rigidly linked to the moving clocks, which with the clocks constitutes an inertial reference system.

In the course of an infinitesimal time interval dt (as read by a clock in our rest frame) the moving clocks go a distance $\sqrt{dx^2 + dy^2 + dz^2}$. Let us ask what time interval dt' is indicated for this period by the moving clocks. In a system of coordinates linked to the moving clocks, the coordinates are at rest, i.e., $dx' = dy' = dz' = 0$. Because of the invariance of intervals

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt'^2,$$

from which

$$dt' = dt\sqrt{1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2}}.$$

But

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} = v^2,$$

where v is the velocity of the moving clock; therefore

$$dt' = \frac{ds}{c} = dt\sqrt{1 - v^2/c^2}\ . \qquad (2.13)$$

Integrating this expression, we can obtain the time interval indicated by the moving clocks when the elapsed time according to a clock at rest is $t_2 - t_1$:

$$t'_2 - t'_1 = \int_{t_1}^{t_2} dt\sqrt{1 - v^2/c^2}\ . \qquad (2.14)$$

The time read by a clock moving with a given object is called the proper time for this object. Formulae (2.13) and (2.14) express the proper time in terms of the time for a system of reference from which the motion is observed.

As we see from Eq.(2.13) or (2.14), the proper time of a moving object is always less than the corresponding interval in the rest system. In other words, moving clocks go more slowly than those at rest.

Suppose some clocks are moving in uniform rectilinear motion relative to an inertial system K. A reference frame K' linked to K is also inertial. Then from the point of view of an observer in the K system the clocks in the K' system fall behind. And conversely, from the point of view of the K' system, the clocks in K lag. To convince ourselves that there is no contradiction, let us note the following. In order to establish that the clocks in the K' system lag behind those in the K system, we must proceed in the following fashion. Suppose that at a certain moment the clock in K' passes by the clock in K, and at that moment the readings of the two clocks coincide. To compare the rates of the two clocks in K and K' we must once more compare the readings of the same moving clock in K' with the clocks in K. But now we compare this clock with different clocks in K. Then we find that the clock in K' lags behind the clocks in K with which it is being compared. We see that to compare the rates of clocks in two reference frames we require several clocks in one frame, and one in the other, and that, therefore, this process is not symmetric with respect to the two systems. The clock that appears to lag is always the one, which is being compared with different clocks in the other system.

If we have two clocks, one of which describes a closed path returning to the starting point (the position of the clock, which remained at rest), then clearly the moving clock appears to lag relative to the one at rest. The reverse reasoning, in which the moving clock would be considered to be at rest (and vice versa) is now impossible, since the clock describing a closed trajectory does not carry out a uniform rectilinear motion, so that a coordinate system linked to it will not be inertial.

Since according to special relativity, the laws of nature are the same only for inertial reference frames, the frames linked to the clock at rest (inertial frame), and to the moving clock (non-inertial) have different properties and the argument, which leads to the result that the clock at rest must lag is not valid.

The time interval read by a clock is equal to the integral

$$\frac{1}{c}\int_a^b ds\,,$$

taken along the world line of the clock. If the clock is at rest then its world line is clearly a line parallel to the t axis; if the clock carries out a nonuniform motion in a closed path, and returns to its starting point, then its world line will be a curve passing through the two points, on the straight world line of a clock at rest, corresponding to the beginning and end of the motion. On the other hand, we saw that the clock at rest always indicates a greater time interval than the moving one. Thus we arrive at the result that the integral

$$\int_a^b ds\,,$$

taken between a given pair of world points, has its maximum value if it is taken along the straight world line joining these two points. (It is assumed, of course, that the points a and b, and the curves joining them are such that all elements ds along the curves are timelike. This property of the integral is connected with the pseudo-Euclidean character of the four-dimensional geometry. In Euclidean space the integral would, of course, be a minimum along the straight line.

2.2 The Galilean Transformations

Let us begin with motions having constant velocities, as they are described by the Galilean relativity. Since we deal with constant relative velocity between the observer and the observed system, the corrections to the description according to Einstein will be those emerging from the special theory of relativity

Consider a bus parked at a station. Let us designate the point where the rear edge of the bus is as point O. Two observers are supposed to report on the motions which take place in the system: observer A sits in the bus and observer B stands paralleled to A outside the bus. It is clear that as long as the bus is parked, the reports of both observers will be identical. Suppose now that the bus (moves with a constant velocity u, passing B at time $t = 0$). If the bus moves along a straight line, we can perform all our measurements along the line of the motion of the bus. Let us designate this line as the x-axis with coordinate points labeled by x. Until the time $t = 0$, the point marked by O was the same point for the two observers: it was the point where the rear edge of the bus was. On the other hand if the bus is moving, observer A will assign it to the rear edge of the bus (which moves together with him) while observer B will assign it to the point on the ground where the rear edge of the bus was while the bus parked. To avoid confusion, let us mark the rear edge of the bus by O' and O will designate the point marked by observer B. The point O' will be the origin for the measurements of observer A, and all his measurements will be related to this point. (The same will be true for all the observers who stay with him in the moving system, the bus). The point O will be the origin for the measurements of observer B, and for the measurements of all the observers who stay with him in the rest system, the earth. From now on, we shall treat the earth and all the objects attached to it as the rest system and the bus, and all the objects staying in it as the moving system. All the entities determined by the observers staying in the moving system will be designated by a prime $(')$.

At the moment $t = 0$ both points O and O' coincide $(O = O')$. If we ask observer A to designate his position, he will report that, according to his measurements, he is located at some distance from point O'. Let us designate this distance by x'. On the other hand, when observer B marks the position of observer A, he will report the distance of observer A from the point O. Let us call this distance x. How do the distances x and x' related to each other? The distance x includes the distance x', and in addition it includes the distance of the rear edge of the bus from the starting point. This additional distance is the distance between O and O', and it is equal to the speed of the bus times the duration of the motion (the velocity is constant, and the motion began at $t = 0$), which is $u \cdot t$:
Therefore

$$x' = x - ut. \qquad (2.15)$$

The interrelation between x and x' is symmetric, and hence:

$$x = x' + ut, \qquad (2.16)$$

and

$$y = y', \qquad (2.17)$$

$$z = z', \qquad (2.18)$$

$$t = t'. \qquad (2.19)$$

How will the two observers report on velocities? Suppose a ball is rolling in the bus with linear velocity v' (relative to the bus), and in the same direction of the bus motion. It is clear that observer A will report that the ball moves (relative to him) with velocity v'. The velocity of the ball as measured by observer B, however, consists of the sum of the velocity of the bus, and the velocity of the ball relative to the bus:

$$v = v' + u\ , \qquad (2.20)$$

and of course:

$$v' = v - u\ . \qquad (2.21)$$

Eqs.(2.15) to (2.21) are called the Galilean transformations for the position and velocity or the transformations which connect one inertial frame with another. Eqs. (2.20) and (2.21) give "the law of addition of velocities." They can be obtained from equations (2.15) and (2.16) by differentiating them with respect to time which means, by calculating the rate of change of the position on the condition that the time in the moving bus and on the earth, are the same. Stating that the time is an absolute entity (the time is the same in both systems and is independent of the measuring system) is actually a hidden assumption, which lies at the basis of Newtonian mechanics. During the hundreds of years since Newtonian mechanics was formulated, and until the beginning of the twentieth century, this assumption was considered a self-evident one, and even today it is commonly accepted intuitively. Actually, one of the biggest difficulties in studying the STR (special theory of relativity) is to accept the conclusion that time is not an absolute entity, and that the results of time measurements depend upon the motion of the observer. Research on the evolution of the concept of time, and its measuring was conducted by Professor G. Szamosi from the University of Windsor, Ontario, Canada, and published in his book "The Twin Dimensions" (1986).

Until the end of the nineteenth century, there seemed to be no difficulty with Galilean transformation equations and they suited the observations well. As was later discovered, the reason for this fact was that all the phenomena investigated were concerned with low velocities, except for the light motion. As for measurements concerned with light velocity, the degree of accuracy was so low that the contradictions between these equations, and the observations were not observed. The problems arose when equations (2.20) and (2.21) were used in accurate experiments concerning the motion of light.

When one wants to relate these equations to light motion, one has first to determine what light is: is it a wave phenomenon or a corpuscular one? If the light is a corpuscular phenomenon, then its velocity (like the velocity of all other particles) depends upon the velocity of the light source. In such a case, by using the additional law for velocities one finds that the velocity of light relative to the observer equals the velocity of the light relative to the source, plus the velocity of the source relative to the observer.

If light is a wave phenomenon, then its additional law for velocities should be that of waves. When a wave moves in a medium, its velocity is defined relative to the medium, and is determined by the properties of the medium. The wave velocity as measured by an observer is equal to the sum of the wave velocity relative to the medium and the velocity of the observer relative to the medium. At the beginning of the nineteenth century, it was established experimentally that light is a wave phenomenon, and hence people expected that the Galilean additional law of velocities for waves would be the correct law to use for light motion. The acceptance of the assumption that light is a wave phenomenon implied also the assumption that there is a medium in which the light moves as a wave. This medium was termed “the Ether,” and it was assumed that it fills the whole space, and that it can be considered as an absolute rest system to which the motions of all objects can be related. Towards the end of the nineteenth century, scientists believed that light is a wave moving in the ether, and it was concluded that its motion could be treated according to the addition law of velocities for waves.

In 1887 the famous experiment of Michelson and Moreley was performed. In this experiment the scientists tried to measure the velocity of the earth relative to the ether, where the technique of the experiment was based on the addition law of velocities for light. The degree of the precision of the experiment was very high, and significant results were expected. Yet the results of the experiment were null: no velocity of the earth relative to the ether was observed. Since then, the same experiment was repeated again and again with higher and higher precision, but always the same null results were obtained: the ether, to which the motion of the earth was supposed to be related, was not found. The results of this experiment were considered a mystery; the one that bothered Einstein greatly as he took his first steps in science.

The answer to the mystery was given by Einstein in 1905 in the form of the STR. This theory was based on two assumptions:

1. The validity of the principle of relativity that all inertial frames are equivalent.
2. The speed of light in vacuum is constant, and is the same in all systems moving with constant velocities.

The acceptance of the second assumption implies that the addition law for velocities should be corrected in such a way that the velocity of light will remain the same on transforming from one system to another. For this purpose, the transformation Eqs. (2.20) and (2.21) were also corrected. From this modification it followed that time could not be an absolute entity, and that the time duration, measured for some given event, depends upon the situation of motion of the observer. (Actually, Einstein arrived first at the conclusion that the solution of the contradiction might be obtained only after abolishing the hidden assumption that time is an absolute entity. The correction of the equations was already done by him on the basis of the relativistic character of time.)

2.3 The Lorentz Transformations

Our purpose is now to obtain the formulae of transformations from one inertial reference system to another, that is, a formula by means of which, knowing the coordinates x, y, z, t, of a certain event in the K system, we can find the coordinates x', y', z', t' of the same event in another inertial system K'.

In classical mechanics this question is resolved very simply as shown in the previous section. It is easy to verify that this transformations, as was to be expected, does not satisfy the requirements of the theory of relativity; violates the constancy of the speed of light, and it does not leave the interval between events invariant.

We shall obtain the relativistic transformations precisely as a consequence of the requirement that they leave the interval between events invariant.

The interval between events can be looked upon as the distance between the corresponding pair of world points in a four-dimensional system of coordinates. Consequently we may say that the required transformation must leave unchanged all distances in the four-dimensional $x, y, z, ct,$ space. But such transformations consist only of parallel displacements, and rotations of the coordinate system. Of these, the displacement of the coordinate system parallel to itself is of no interest, since it leads only to a shift in the origin of the space coordinates, and a change in the time reference point. Thus the required transformation must be expressible mathematically as a rotation of the four-dimensional $x, y, z, ct,$ coordinate system.

Every rotation is the four-dimensional space can be resolved into six rotations, in the planes xy, zy, xz, tx, ty, tz (just as every rotation in ordinary space can be resolved into three rotations in the planes $(xy, zy$ and $xz)$. The first three of these rotations transform only the space coordinates; they correspond to the usual space rotations.

Let us consider a rotation in the tx plane; under this, the y and z coordinates do not change. In particular, this transformation must leave unchanged the difference $(ct)^2 - x^2$, the square of the "distance" of the point (ct, x) from the origin. The relation between the old, and the new coordinates is given in most general form by the formulae:

$$x = x'\cosh y + ct'\sinh y\ , \qquad ct = x'\sinh y + ct'\cosh y\ , \tag{2.22}$$

where y is the "angle of rotation"; a simple check shows that in fact $c^2t^2 - x^2 = c^2t'^2 - x'^2$. Formula (2.22) differs from the usual formulae for transformation under rotation of the coordinate axes in having hyperbolic functions in place of trigonometric ones. This is the difference between pseudo-Euclidean and Euclidean geometry.

We try to find the formula of transformations from an inertial reference frame K to a system K' moving relative to K with velocity *v* along the x axis. In this case clearly only the coordinate x and the time t are subject to change. Therefore this transformation must have the form of Eq.(2.22). Now it remains only to determine the angle y, which can depend only on the relative velocity *v*.

Let us consider the motion in the K system of the origin of the K' system. For $x' = 0$, formulae (2.22) take the form:

$$x = ct'\sinh y\ , \qquad ct = ct'\cosh y\ , \tag{2.23}$$

or dividing one by the other,

$$\frac{x}{ct} = \tanh y \, , \tag{2.24}$$

But x/t is clearly the velocity v of the K' system relative to K.
So

$$\tanh y = \frac{v}{c} . \tag{2.25}$$

From this

$$\sinh y = \frac{v/c}{\sqrt{1-v^2/c^2}} , \qquad \cosh y = \frac{1}{\sqrt{1-v^2/c^2}} . \tag{2.26}$$

Substituting the latter in Eq.(2.22), we find:

$$x = g(x' + vt'), \quad y = y', \quad z = z', \quad t = g\left(t' + \frac{v}{c^2} x'\right), \quad g = \frac{1}{\sqrt{1-v^2/c^2}} . \tag{2.27}$$

These are the required transformation formulae. They are called the Lorentz transformations, and are of fundamental importance for what follows.

The inverse formulae, expressing x', y', z', t' in term of $x, y, z, t,$ are most easily obtained by changing v to -v (since the K system moves with velocity -v relative to the K' system). The same formulae can be obtained directly by solving equation (2.27) for x', y', z', t'.

It is easy to see from Eq.(2.27) that on making the transition to the limit $c \to \infty$, the formulae for the Lorentz transformations actually go over to the Galilean transformations.

For $v > c$ the coordinates x, t in Eq.(2.27) are imaginary; this corresponds to the fact that motion with a velocity greater than the velocity of light is impossible. Moreover, one cannot use a reference system that is moving with the velocity of light as in that case the denominators in Eq.(2.27) would go to zero.

For velocities v small compared with the velocity of light, we can use in place of Eq.(2.27) the approximate formulae:

$$x = x' + vt', \quad y = y', \quad z = z', \quad t = t' + \frac{v}{c^2}x'. \qquad (2.28)$$

Suppose there is a rod at rest in the K system, parallel to the x-axis. Let its length, measured in this system, be $\Delta x = x_2 - x_1$ (x_2 and x_1 are the coordinates of the two ends of the rod in the K system). We now determine the length of this rod as measured in the K' system. To do this we must find the coordinates of the two ends of the rod (x'_2 and x'_1) in this system at one, and the <u>same</u> time t', i.e., simultaneously. From Eq. (2.27) we find:

$$x_1 = g(x'_1 + vt'), \quad x_2 = g(x'_2 + vt'). \qquad (2.29)$$

The length of the rod in the K' system is $\Delta x' = x_2' - x_1'$; subtracting x_1 from x_2 we find

$$\Delta x = g(\Delta x').$$

The proper length of a rod is its length in a reference system in which it is at rest. Let us denote it by $l_0 = \Delta x$, and the length of the rod in any other reference frame K' by l. Then

$$l = \frac{l_0}{g}. \qquad (2.30)$$

Thus a rod has its greatest length in the reference system in which it is at rest. Its length in a system in which it moves with velocity v is decreased by the factor $\frac{1}{g}$. This result of the theory of relativity is called the Lorentz contraction.

Since the transverse dimensions do not change because of its motion, the volume $\boldsymbol{n}$ of a body decreases according to the similar formula

$$\boldsymbol{n} = \frac{\boldsymbol{n}_0}{g}, \qquad (2.31)$$

where $\boldsymbol{n}_0$ is the proper volume of the body.

From the Lorentz transformation we can obtain anew the results already known to us concerning the proper time. Very briefly, suppose a clock to be at rest in the K' system. We take two events occurring at one, and the same point x', y', z' in the K' system. The time between these events in the K' system is $\Delta t' = t_2' - t_1'$. Now we find the time Δt which elapse between these two events in the K system. From Eq.(2.27), we have

$$t_1 = g\left(t'_1 + \frac{v}{c^2}x'\right), \quad t_2 = g\left(t'_2 + \frac{v}{c^2}x'\right), \qquad (2.32)$$

or, subtracting one from the other,

$$t_2 - t_1 = \Delta t = g\Delta t', \qquad (2.33)$$

in complete agreement with Eq.(2.13).

Finally we mention another general property of the Lorentz transformations which distinguishes them from the Galilean transformations. The latter have the general property of commutativity, i.e., the combined result of two successive Galilean transformations (with different velocities v_1 and v_2) does not depend on the order in which the transformations are performed. On the other hand, the result of two successive Lorentz transformations do depend, in general, on their order. This is already apparent purely mathematically from our formal description of these transformations as rotations of the four-dimensional coordinate system: we know that the result of two rotations (about different axes) depends on the order in which they are carried out. The sole exception is the case of transformations with parallel vectors v_1 and v_2 (which are equivalent to two rotations of the four-dimensional coordinate system about the same axis).

Chapter III
Non-Linear Terrell Transformations

3.1 Introduction

The purpose of this chapter is to provide a complete derivation of the transformations resulting from the applications of the following three points:

(1) Terrell's observation that different points on the object, in relative motion to the observation frame, must "emit" light at different points in order to reach an observation point simultaneously. That is, distant points, to the observation point, must "emit" light prior to those closer points.

(2) The Lorentz transformations (and then of the corresponding Galilean transformations).

(3) The piercing of these light rays an appropriate 2D-plane in the observation frame. This is illustrated in Fig. 3.1 below. The optic axis is perpendicular to this plane and is taken parallel to the *xy*-plane. $\overline{O}$ denotes the observation point.

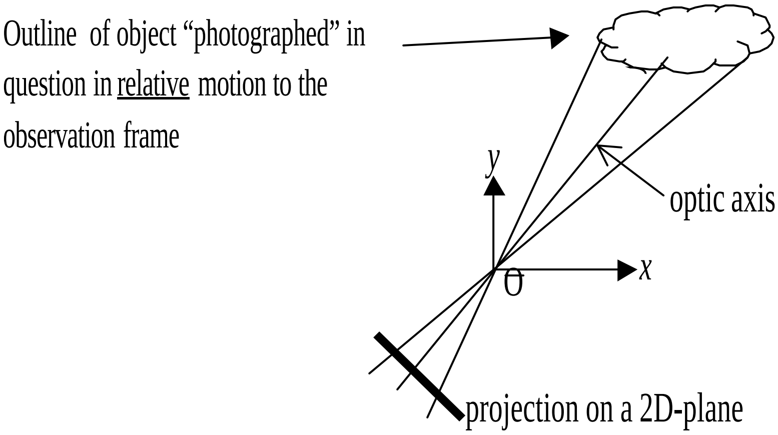


Fig. 3.1. Top view: $\overline{O}$ denotes the observation point at a vertical distance h above the origin of the observation frame. The plane is fixed in the observation frame and is perpendicular to the optic axis.

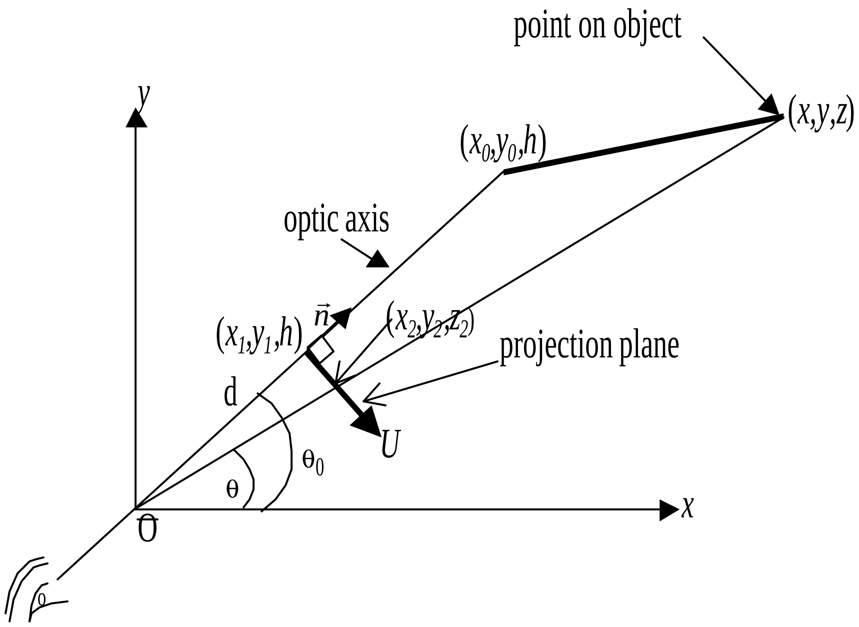


Fig. 3.2. The *UV*-plane shown in 2 dimensions (the top view).

As mentioned above, the optic axis is taken to be parallel to the *xy*-plane. We denote by (x,y,z): the coordinates point <u>on the object</u> as given in the observation frame. The axes of the projection plane are denoted by *U* and *V*. The *U*-axis is parallel to the *xy*-plane, and the *V*-axis, is parallel to the *z*-axis, is perpendicular to it. We denote by (x_0, y_0, h): the coordinate point specifying the tip of the optic axis in the observation frame. From the figure we get

$$\tan\theta_0 = \frac{y_0}{x_0} = \frac{y_1}{x_1}, \tag{3.1}$$

or

$$y_1^2 = \frac{y_0^2}{x_0^2} x_1^2, \tag{3.2}$$

and

$$x_1^2 + y_1^2 = d^2. \tag{3.3}$$

Where *d* is the distance from the observation point to the origin of the *UV*-plane along the optic axis.

Solving Eqs.(3.1) to (3.3) for y_1 and x_1, we get

$$y_1 = \frac{y_0}{\sqrt{x_0^2 + y_0^2}} d \,, \tag{3.4}$$

and

$$x_1 = \frac{x_0}{\sqrt{x_0^2 + y_0^2}} d \,. \tag{3.5}$$

According to Fig.3.2, we may use the Pythagoras theorem to obtain the following equation in the *xy*–plane:

$$d^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 = x_2^2 + y_2^2 \,. \tag{3.6}$$

Upon expanding Eq.(3.6), we get

$$d^2 + x_2^2 - 2x_2 x_1 + x_1^2 + y_2^2 - 2y_2 y_1 + y_1^2 - x_2^2 - y_2^2 = 0 \,, \tag{3.7}$$

$$d^2 - 2x_2 x_1 - 2y_2 y_1 + x_1^2 + y_1^2 = 0 \;. \tag{3.8}$$

Upon substituting Eqs.(3.3), (3.4) and (3.5) into Eq.(3.8) we obtain

$$d^2 - \frac{2x_2 x_0 d}{\sqrt{x_0^2 + y_0^2}} - \frac{2y_2 y_0 d}{\sqrt{x_0^2 + y_0^2}} + d^2 = 0 \,,$$

$$2d^2 - \frac{2x_2 x_0 d}{\sqrt{x_0^2 + y_0^2}} - \frac{2y_2 y_0 d}{\sqrt{x_0^2 + y_0^2}} = 0 \,, \tag{3.9}$$

which leads to

$$d^2 - \frac{x_2 x_0 d}{\sqrt{x_0^2 + y_0^2}} - \frac{y_2 y_0 d}{\sqrt{x_0^2 + y_0^2}} = 0\,. \tag{3.10}$$

Again according to Fig.3.2,

$$\tan \boldsymbol{q} = \frac{y}{x} = \frac{y_2}{x_2}, \tag{3.11}$$

or

$$x_2 = y_2 \frac{x}{y}\,. \tag{3.12}$$

Simplifying Eq.(3.10) gives

$$d^2 - \frac{d(x_2 x_0 + y_2 y_0)}{\sqrt{x_0^2 + y_0^2}} = 0\,,$$

$$d^2 = \frac{d(x_2 x_0 + y_2 y_0)}{\sqrt{x_0^2 + y_0^2}}, \tag{3.13}$$

or

$$y_2 = \frac{d\sqrt{x_0^2 + y_0^2} - x_2 x_0}{y_0}\,. \tag{3.14}$$

Insert x_2, as given in Eq.(3.12), to obtain

$$y_2 = \frac{d\sqrt{x_0^2 + y_0^2}}{y_0} - \frac{x_0}{y_0}\frac{x}{y} y_2\,, \tag{3.15}$$

with the solution

$$y_2 = \frac{yd\sqrt{x_0^2 + y_0^2}}{xx_0 + yy_0}. \tag{3.16}$$

Next solve for x_2 by inserting $y_2 = x_2 \frac{y}{x}$ into Eq.(3.14). This gives

$$x_2 = \frac{xd\sqrt{x_0^2 + y_0^2}}{xx_0 + yy_0}. \tag{3.17}$$

From Fig.3.2 we can write the following expression for U:

$$U^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2. \tag{3.18}$$

Expanding Eq.(3.18), gives

$$U^2 = x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2. \tag{3.19}$$

Insert Eqs.(3.4), (3.5), (3.16) and (3.17) into Eq.(3.19) to obtain the following chain of equalities :

$$U^2 = \frac{x^2(x_0^2 + y_0^2)d^2}{(xx_0 + yy_0)^2} - \frac{2x\sqrt{x_0^2 + y_0^2}\,d}{xx_0 + yy_0} \cdot \frac{x_0 d}{\sqrt{x_0^2 + y_0^2}} + \frac{x_0^2}{x_0^2 + y_0^2} d^2$$

$$+ \frac{y^2(x_0^2 + y_0^2)d^2}{(xx_0 + yy_0)^2} - \frac{2y\sqrt{x_0^2 + y_0^2}\,d}{xx_0 + yy_0} \cdot \frac{y_0 d}{\sqrt{x_0^2 + y_0^2}} + \frac{y_0^2}{x_0^2 + y_0^2} d^2, \tag{3.20}$$

$$= \frac{(x_0^2 x^2 + x_0^2 y^2 + y_0^2 x^2 + y_0^2 y^2)d^2}{(xx_0 + yy_0)^2} - d^2$$

$$= \frac{(x_0^2 x^2 + x_0^2 y^2 + y_0^2 x^2 + y_0^2 y^2)d^2 - (x_0^2 x^2 + 2xx_0 yy_0 + y^2 y_0^2)d^2}{(xx_0 + yy_0)^2}$$

$$= \frac{(x_0^2 y^2 + y_0^2 x^2 - 2xx_0 yy_0)d^2}{(xx_0 + yy_0)^2},$$

or

$$U = \frac{(xy_0 - yx_0)d}{xx_0 + yy_0}. \tag{3.21}$$

Given the U-coordinate value corresponding to a point on the object. To find the V-coordinate value corresponding to a point on the object we refer to the figure (Fig.3.3) below.

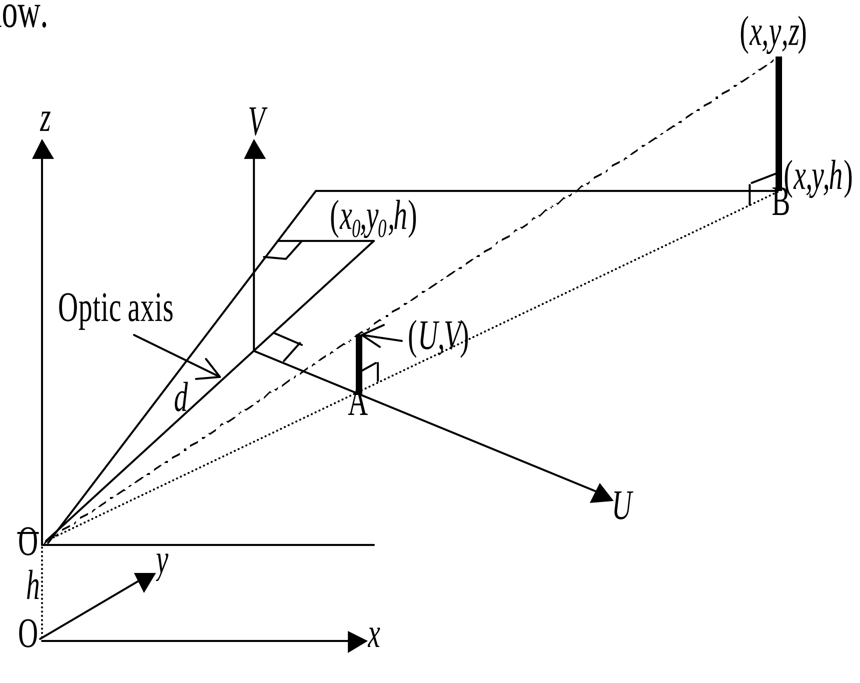


Fig. 3.3. Projection onto the UV-plane as shown in the actual 3D configuration .

From the above figure we can write

$$(\overline{O}B)^2 = x^2 + y^2, \tag{3.22}$$

$$(\overline{O}A)^2 = d^2 + U^2, \tag{3.23}$$

and

$$\frac{V}{z-h} = \frac{\overline{O}A}{\overline{O}B}, \tag{3.24}$$

Inserting Eqs.(3.22) and (3.23) into (3.24), gives

$$V = \frac{\sqrt{d^2 + U^2}}{\sqrt{x^2 + y^2}}(z-h). \tag{3.25}$$

Insert Eq.(3.21) into (3.25) to obtain

$$V = \frac{\sqrt{d^2 + \left(\frac{xy_0 - yx_0}{xx_0 + yy_0}\right)^2 d^2}}{\sqrt{x^2 + y^2}}(z-h), \tag{3.26}$$

$$= \frac{\sqrt{d^2 (xx_0 + yy_0)^2 + (xy_0 - yx_0)^2 d^2}}{\sqrt{(xx_0 + yy_0)^2}\sqrt{x^2 + y^2}}(z-h)$$

$$= \frac{\sqrt{d^2 (x^2 x_0^2 + 2xx_0 yy_0 + y^2 y_0^2) + \left(x^2 y_0^2 - 2xy_0 yx_0 + y^2 x_0^2\right) d^2}}{\sqrt{(xx_0 + yy_0)^2}\sqrt{x^2 + y^2}}(z-h)$$

$$= \frac{\sqrt{d^2 (x^2 x_0^2 + y^2 y_0^2 + x^2 y_0^2 + y^2 x_0^2)}}{\sqrt{(xx_0 + yy_0)^2}\sqrt{x^2 + y^2}}(z-h)$$

$$= \frac{\sqrt{d^2(x^2+y^2)(x_0^2+y_0^2)}}{\sqrt{(xx_0+yy_0)^2}\sqrt{x^2+y^2}}(z-h).$$

Finally we get the projection on the *V*-axis to be

$$V = \frac{d\sqrt{(x_0^2+y_0^2)}}{xx_0+yy_0}(z-h). \tag{3.27}$$

Note : *h* is the height where the “observation point” is located along the *z*-axis.

We now consider the unit vector n, perpendicular to the *UV*-plane, specifying the direction of the optic axis:

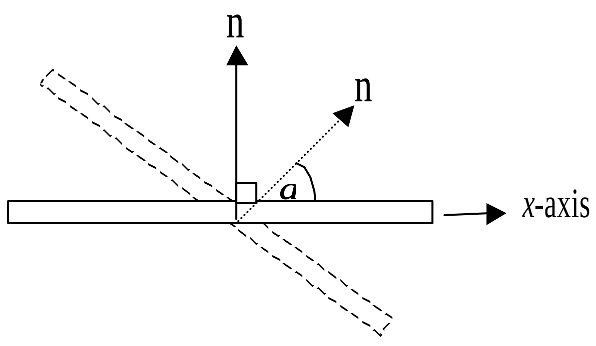


Fig. 3.4. Normal vector to the *UV*-plane.

We can express the formula of *U* in terms of the unit vector n by dividing Eq.(3.21) by $\sqrt{x_0^2+y_0^2}$, to get

$$U = \left(\frac{x\frac{y_0}{\sqrt{x_0^2+y_0^2}} - y\frac{x_0}{\sqrt{x_0^2+y_0^2}}}{x\frac{x_0}{\sqrt{x_0^2+y_0^2}} + y\frac{y_0}{\sqrt{x_0^2+y_0^2}}} \right) d, \tag{3.28}$$

According to Fig.3.2, may express the unit vector n to the projection plane as

$$\mathrm{n} = \left(\frac{x_0}{\sqrt{x_0^2 + y_0^2}}, \frac{y_0}{\sqrt{x_0^2 + y_0^2}}, 0 \right) = (n_1, n_2, 0) = (\cos \alpha, \sin \alpha, 0) \;, \qquad (3.29)$$

or

$$n_1 = \frac{x_0}{\sqrt{x_0^2 + y_0^2}} = \cos \alpha \quad \text{and} \quad n_2 = \frac{y_0}{\sqrt{x_0^2 + y_0^2}} = \sin \alpha \,. \qquad (3.30)$$

Where α denotes the angle between the unit vector n and the x-axis. Substituting Eq.(3.30) into Eq.(3.28) we obtain

$$U = \left(\frac{xn_2 - yn_1}{xn_1 + yn_2} \right) d \,. \qquad (3.31)$$

Similarly for the V formula we get

$$V = \frac{d \dfrac{\sqrt{x_0^2 + y_0^2}}{\sqrt{x_0^2 + y_0^2}}}{x \dfrac{x_0}{\sqrt{x_0^2 + y_0^2}} + y \dfrac{y_0}{\sqrt{x_0^2 + y_0^2}}} (z - h) \,, \qquad (3.32)$$

$$V = \frac{d(z - h)}{xn_1 + yn_2} \,, \qquad (3.33)$$

Different values for n_1 and n_2, such that $(n_1)^2 + (n_2)^2 = 1$, specify different orientations of the projection (observation) UV-plane.

Now we have to express the (x, y, z) values in terms of the (x', y', z') values with the latter corresponding to the point on the object in its proper, i.e., rest frame. The relative motion is taken equivalently as follows. Either the object is moving, relative to the observation frame, along the x-axis to the right with speed v or the frame is moving, relative to the object, to the left with the same speed.

3.2 Relativistic Transformations on the Projection Plane

Consider the proper inertial frame F' of an object and an observational inertial frame F with relative speed v, the proper frame F' of the object is moving to the right of the x-axis. Let (x', y', z'), (x, y, z) denote, respectively, the corresponding labelings of an arbitrary point on the object. The observation point $\overline{O}$ is at a height h, along the z-axis (see Fig.3.5), above the origin O of the F frame. When the origins O, O', at $t = 0, t' = 0$, of the F, F' frames coincide, the observer at $\overline{O}$ sees these origins coincide he takes a snap shot of the object. Since the observer at $\overline{O}$ sees the origins coincide only at a later time equal to h/c, the time that light was emitted from (x, y, z) to reach $\overline{O}$ is given by

$$t = -\frac{\sqrt{x^2 + y^2 + (z-h)^2}}{c} + \frac{h}{c}, \qquad (3.34)$$

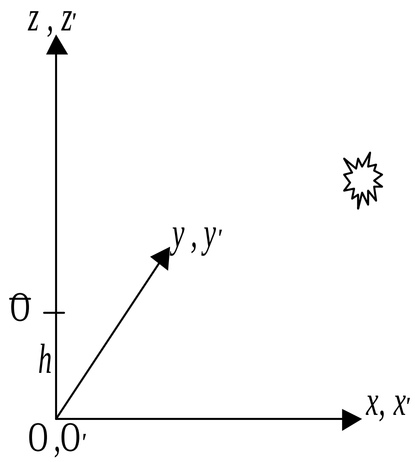


Fig. 3.5. The origins of the proper frame of the object and observation frame are shown to coincide. Object at rest in the F' frame. When the observer sees the origins O and O' coincide (at time $t = t' = 0$), he takes a "snap shot" of the object.

we now recall the inverse of the Lorentz transformations (see Eq.(2.27))

$$x' = g(x - vt)\,, \tag{3.35}$$

$$y' = y\,, \tag{3.36}$$

$$z' = z\,, \tag{3.37}$$

To solve for x in terms of x', y', z' we insert Eqs.(3.34), (3.36) and (3.37) into (3.35), to get

$$x' = g\left\{x - v\left(\frac{h}{c} - \frac{\sqrt{x^2 + y'^2 + (z' - h)^2}}{c}\right)\right\}, \tag{3.38}$$

expanding, and simplifying Eq.(3.38), we get

$$gx - (x' + gbh) = -gb\sqrt{x^2 + y'^2 + (z' - h)^2}\,. \tag{3.39}$$

Squaring both side of Eq.(3.39) , we get

$$g^2 x^2 - 2gx(x' + gbh) + (x' + gbh)^2 - g^2 b^2 (x^2 + y'^2 + (z' - h)^2) = 0\,. \tag{3.40}$$

From the definition $\quad g = \dfrac{1}{\sqrt{1 - v^2/c^2}} = \dfrac{1}{\sqrt{1 - b^2}}\,,$

and $\quad g^2 = \dfrac{1}{1 - b^2}\,,$

we obtain

$$1-b^2=\frac{1}{g^2}\,. \qquad (3.41)$$

Upon using Eqs.(3.40) and (3.41) we may write

$$x^2-2g(x'+gbh)x+(x'+gbh)^2-g^2b^2\left[y'^2+(z'-h)^2\right]=0\,, \qquad (3.42)$$

to obtain

$$x=g(x'+gbh)\pm\sqrt{g^2(x'+gbh)^2-(x'+gbh)^2+g^2b^2\left[y'^2+(z'-h)^2\right]}\,, \qquad (3.43)$$

$$x=g(x'+gbh)\pm\sqrt{(g^2-1)(x'+gbh)^2+g^2b^2\left[y'^2+(z'-h)^2\right]}\,, \qquad (3.44)$$

From the identity $g^2-1=b^2g^2$, Eq.(3.44) can be rewritten as

$$x=g(x'+gbh)\pm gb\sqrt{(x'+gbh)^2+y'^2+(z'-h)^2}\;, \qquad (3.45)$$

$$x=g\left[(x'+gbh)\pm b\sqrt{(x'+gbh)^2+y'^2+(z'-h)^2}\right]. \qquad (3.46)$$

Since the motion of the object, relative to the observation frame is to the right we have

$$x=g\left[(x'+gbh)-b\sqrt{(x'+gbh)^2+y'^2+(z'-h)^2}\right]. \qquad (3.47)$$

In conclusion, the Lorentz transformations, reduce to

$$\left.\begin{aligned} x &= g\left[(x'+gbh) - b\sqrt{(x'+gbh)^2 + y'^2 + (z'-h)^2}\right] \\ y &= y' \\ y_0 &= y'_0 \\ z &= z' \\ z_0 &= z'_0 = h \end{aligned}\right\} \qquad (3.48)$$

Upon substituting Eq.(3.48) into Eqs.(3.31) and (3.33) we get

$$U = \left(\frac{g\left[(x'+gbh) - b\sqrt{(x'+gbh)^2 + y'^2 + (z'-h)^2}\right] n_2 - y' n_1}{g\left[(x'+gbh) - b\sqrt{(x'+gbh)^2 + y'^2 + (z'-h)^2}\right] n_1 + y' n_2} \right) d\,, \qquad (3.49)$$

$$V = \frac{d\,(z'-h)}{g\left[(x'+gbh) - b\sqrt{(x'+gbh)^2 + y'^2 + (z'-h)^2}\right] n_1 + y' n_2}\,. \qquad (3.50)$$

The expressions for U and V in Eqs.(3.49), (3.50) provide the mapping of a point (x', y', z') on the object onto the *UV*-plane in the observation frame. We specialize below the above general formulae to particular cases of interest.

3.3 Galilean Transformations, with <u>no</u> Time Delay

In this case we have simply to take the limit $c \to \infty$. That is we have the formulae

$$U = \left(\frac{x'n_2 - y'n_1}{x'n_1 + y'n_2} \right) d \,, \tag{3.51}$$

and

$$V = \frac{(z'-h)d}{x'n_1 + y'n_2} \,. \tag{3.52}$$

Where the primed variables denote points on the object in its proper frame where the object in question is fixed. The above corresponds simply and formally to no relative motion of the observation frame relative to the object in question.

3.4 Galilean Transformations, with Time Delay

For the Galilean transformations, which involve the time-delay mechanism (c is finite but large), we have to set the Lorentz factor equal to one (g =1).

Thus in the *UV*-plane, according to Eqs. (3.49) and (3.50), we may write

$$U = \left(\frac{\left[(x'+bh) - b\sqrt{(x'+bh)^2 + y'^2 + (z'-h)^2}\right] n_2 - y'n_1}{\left[(x'+bh) - b\sqrt{(x'+bh)^2 + y'^2 + (z'-h)^2}\right] n_1 + y'n_2} \right) d \,, \tag{3.53}$$

and

$$V = \frac{d(z'-h)}{\left[(x'+bh) - b\sqrt{(x'+bh)^2 + y'^2 + (z'-h)^2}\right] n_1 + y'n_2} \,. \tag{3.54}$$

3.5 The Relativistic Case

For the fully relativistic theory involving the Lorentz transformations and time delay, the *U* and *V* variables are as already derived,

$$U = \left(\frac{g\left[(x'+gbh) - b\sqrt{(x'+gbh)^2 + y'^2 + (z'-h)^2}\right] n_2 - y' n_1}{g\left[(x'+gbh) - b\sqrt{(x'+gbh)^2 + y'^2 + (z'-h)^2}\right] n_1 + y' n_2} \right) d\,, \qquad (3.55)$$

and

$$V = \frac{d\,(z'-h)}{g\left[(x'+gbh) - b\sqrt{(x'+gbh)^2 + y'^2 + (z'-h)^2}\right] n_1 + y' n_2}\,. \qquad (3.56)$$

Unlike the Lorentz transformations $(t', x', y', z') \rightarrow (t, x, y, z)$, the transformations $(x', y', z') \rightarrow (U, V)$, given in Eqs.(3.55) and (3.56), are obviously non – linear.

In the next chapter we will use all of the above formulae to carry out a detailed comparative study.

Chapter IV
Applications and Comparative Study - Seeing is Believing

In this chapter, we make a systematic use of the projection of the 3D objects onto the *UV*-plane, in relative motion, as described in the previous chapter. The *UV*-plane is fixed in the observation frame. The object is assumed to move to the right with speed βc or equivalently that the observation frame is moving to the left with the same speed as the physical situation may dictate. We study objects, which are rich enough in structure for a detailed conclusive analysis. We consider speeds given through $\beta = 0$, 0.3, 0.5, 0.8, 0.9, 0.99, 0.999, respectively. The unit vector specifying the direction of the optic axis is denoted by $\mathbf{n}$ where $\mathbf{n} = (n_1, n_2, 0)$, n_1 and n_2 (see Eq.(3.30)) and α denotes the angle of the optic axis to the *x*-axis. Throughout this chapter, the distance d from the projection plane to the observer is chosen equal to 0.7 units.

We provide three applications. For greater generality, we consider the optic axis, specified by the unit vector $\mathbf{n}$, parallel to the *xy*-plane, to take three different directions corresponding to the angles 90°, 86° and 94°. Pertinent remarks concerning these figures will be made in Chapter V when some important analytical properties of the non-linear Terrell transformations will be established as well as establishing of the long standing resolution of the “train” paradox.

In the figures we note that the crossed small circle denotes the origin of the *UV*-plane. On the other hand the small crossed lines denote some point, such as a mid point on the object. When these two location-points objects are at different positions for $\beta = 0$ versus $\beta \neq 0$, this is simply a confirmation of the famous “aberration of light”.

4.1 Application I

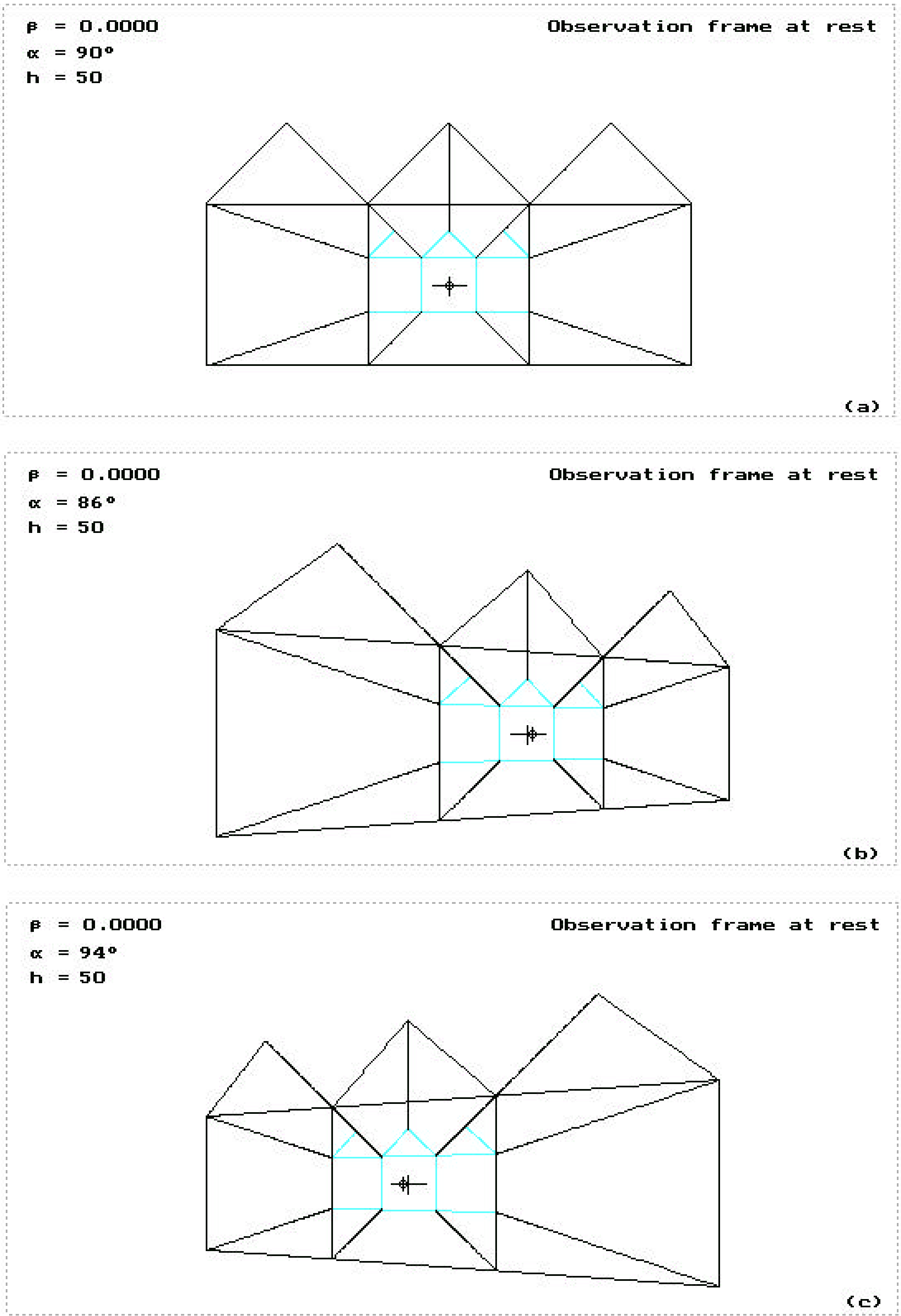


Fig. 4.1. Set of houses for : (a) $\alpha = 90^o$. (b) $\alpha = 86^o$. (c) $\alpha = 94^o$, where $\mathrm{n} = (\cos\alpha, \sin\alpha, 0)$.

4.1.1 Galilean Treatment with Time Delay

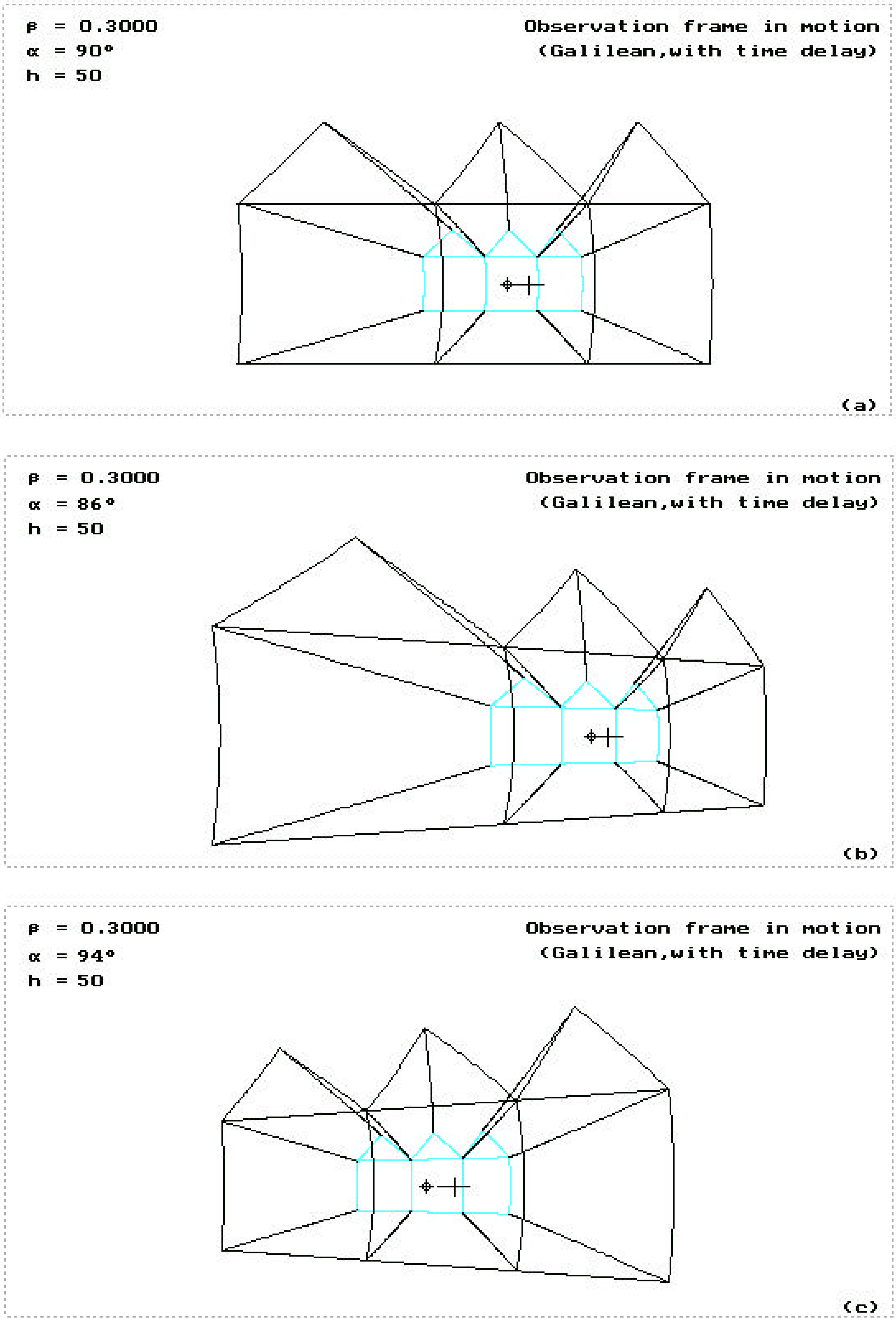


Fig. 4.2. Galilean, with time delay: $\beta = 0.3$. Set of houses for: (a) $\alpha = 90^{o}$. (b) $\alpha = 86^{o}$. (c) $\alpha = 94^{o}$, where $n = (\cos\alpha, \sin\alpha, 0)$.

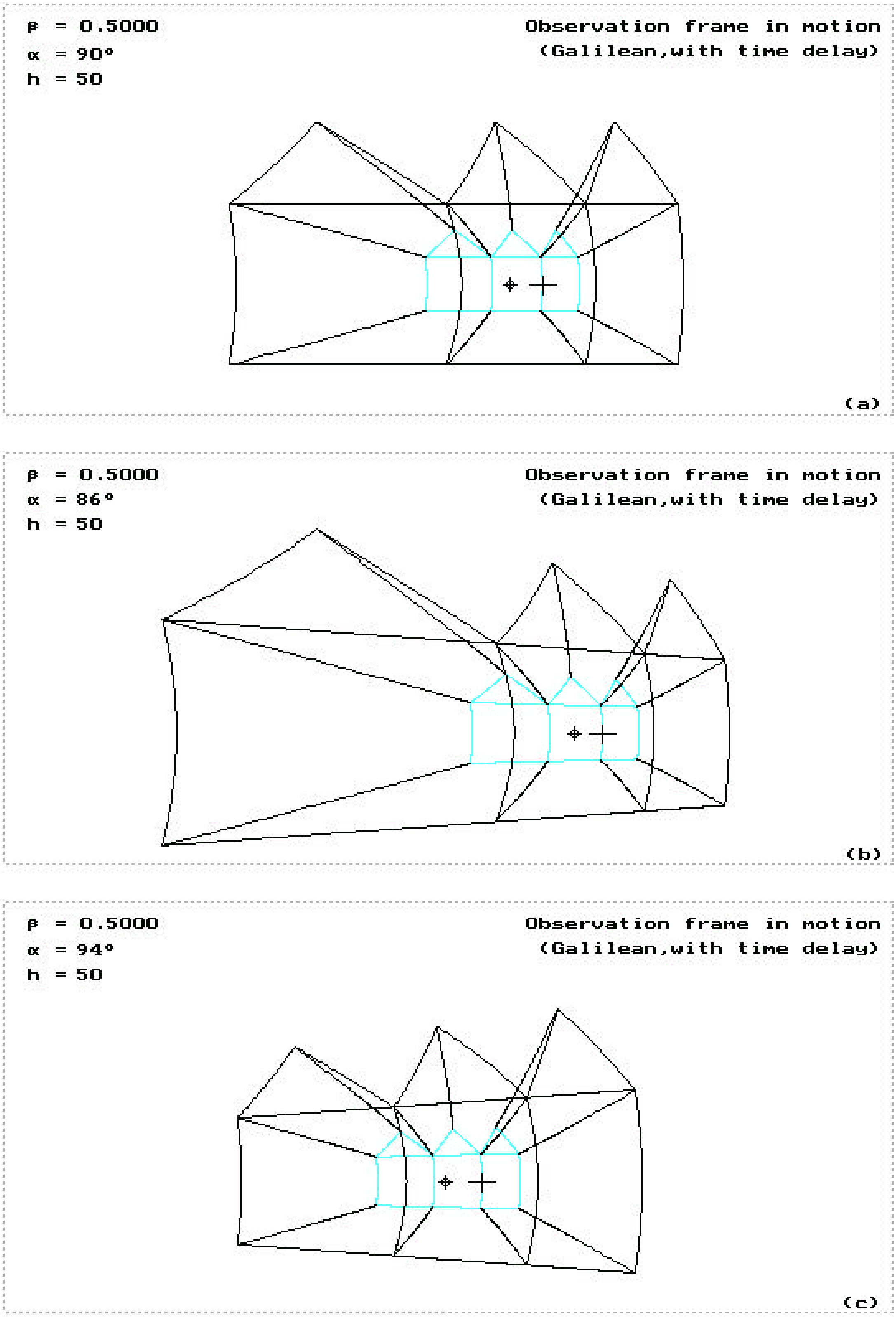


Fig. 4.3. Galilean, with time delay: $\boldsymbol{b} = 0.5$. Set of houses for: (a) $\boldsymbol{a} = 90^o$. (b) $\boldsymbol{a} = 86^o$. (c) $\boldsymbol{a} = 94^o$, where $n = (\cos \boldsymbol{a}, \sin \boldsymbol{a}, 0)$.

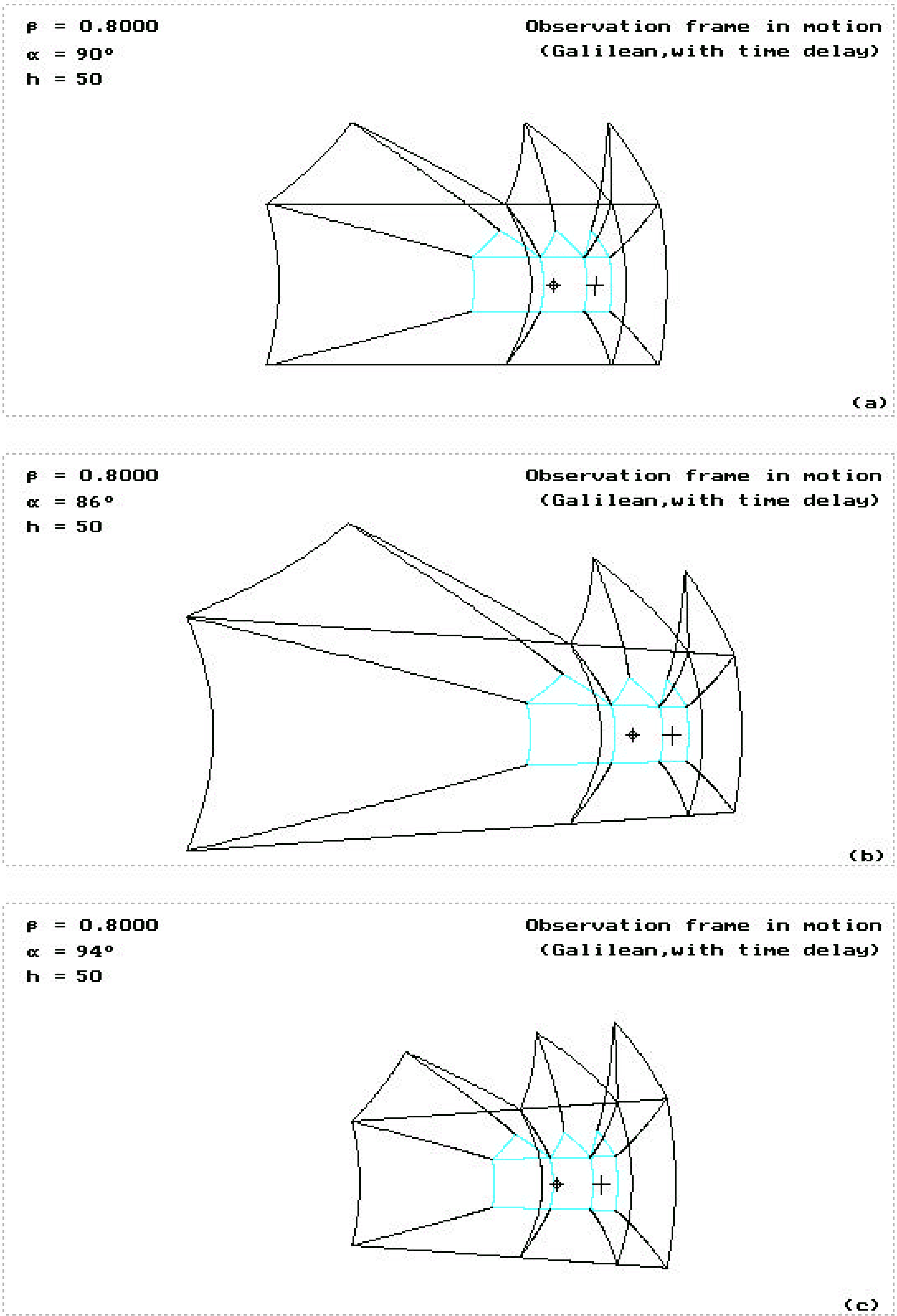


Fig. 4.4. Galilean, with time delay: $\beta = 0.8$. Set of houses for: (a) $\alpha = 90^\circ$. (b) $\alpha = 86^\circ$. (c) $\alpha = 94^\circ$, where $n = (\cos\alpha, \sin\alpha, 0)$.

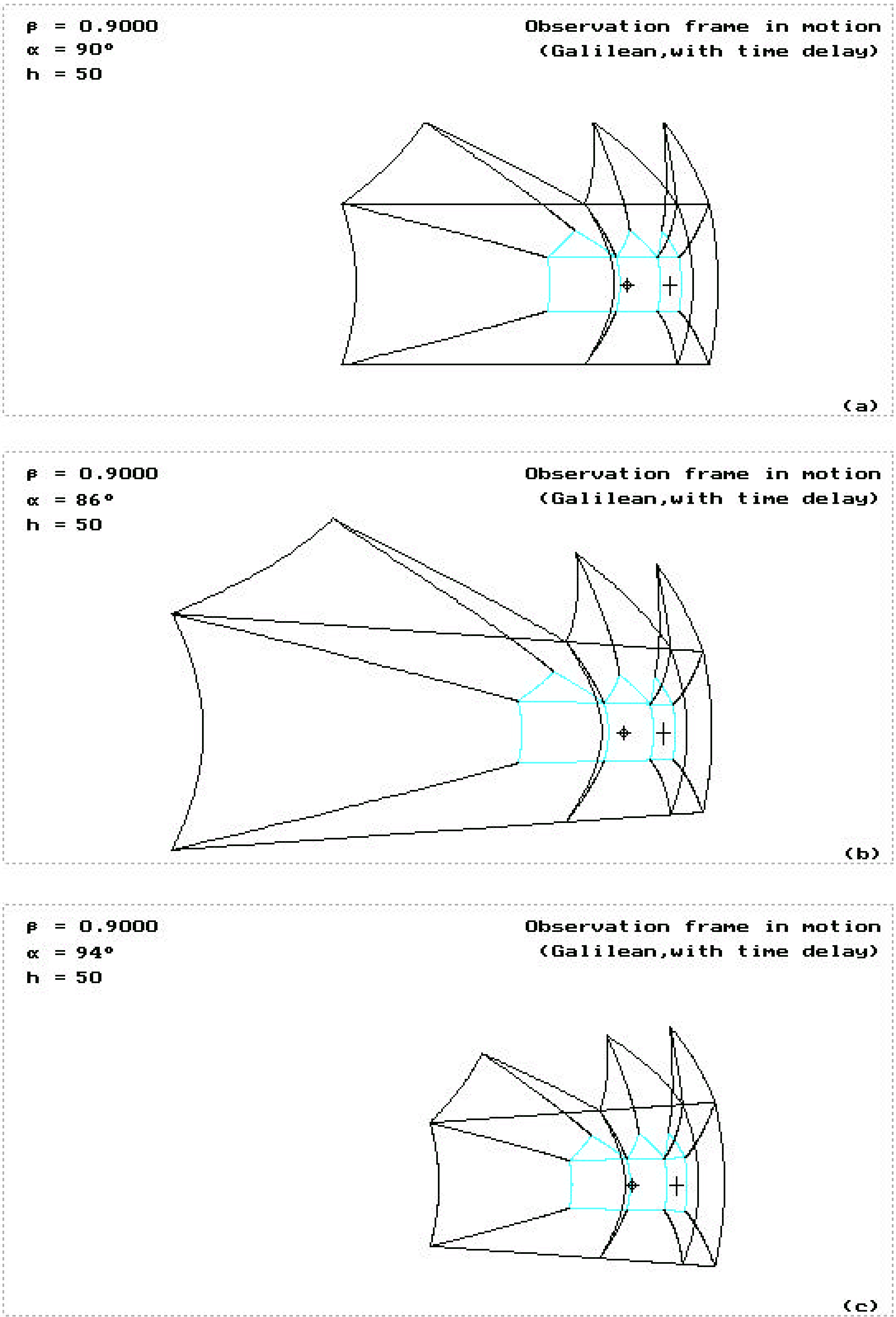


Fig. 4.5. Galilean, with time delay: $\beta = 0.9$. Set of houses for: (a) $\alpha = 90^o$. (b) $\alpha = 86^o$. (c) $\alpha = 94^o$, where $n = (\cos\alpha, \sin\alpha, 0)$.

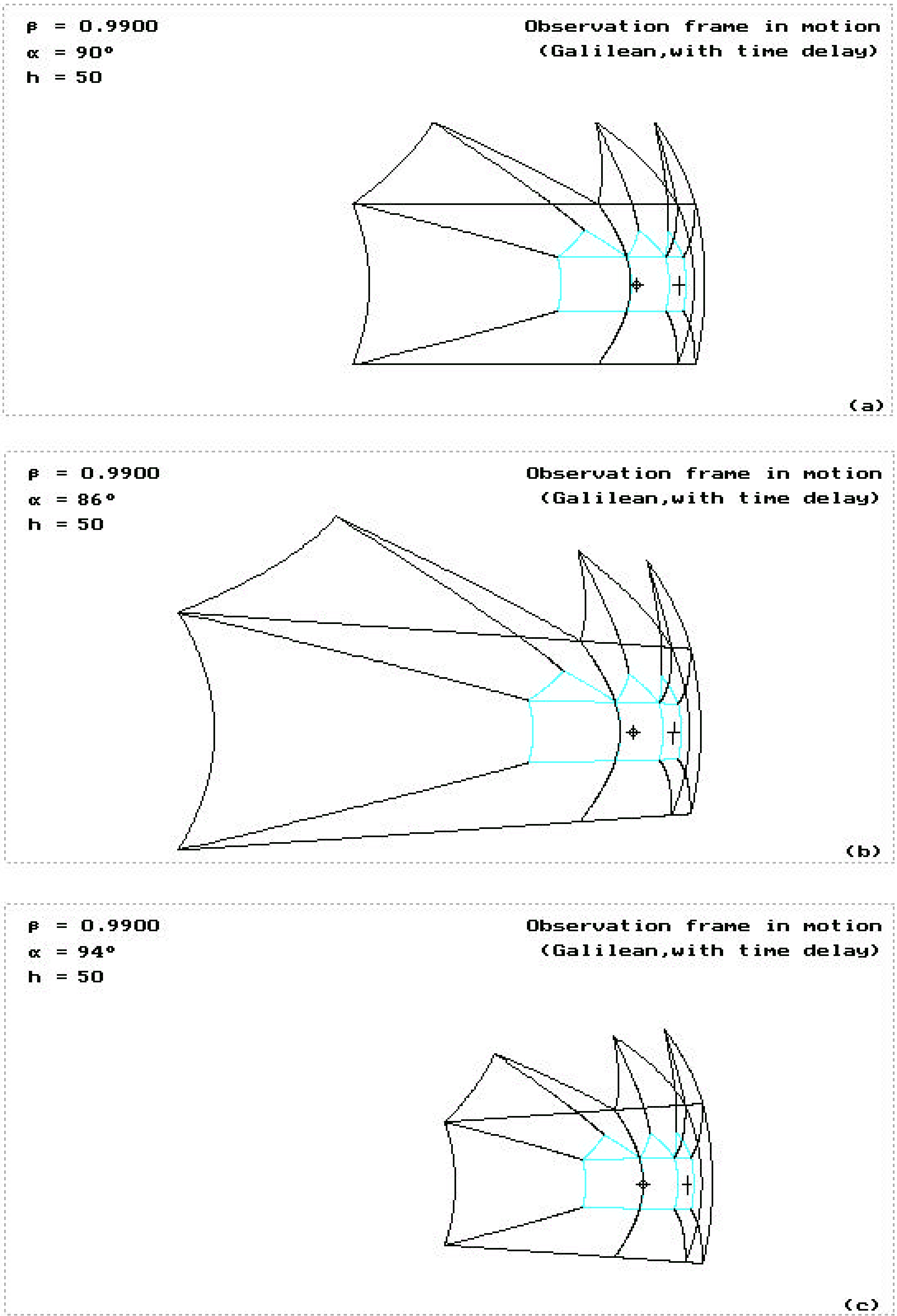


Fig. 4.6. Galilean, with time delay: $\beta = 0.99$. Set of houses for: (a) $\alpha = 90^{o}$. (b) $\alpha = 86^{o}$. (c) $\alpha = 94^{o}$, where $n = (\cos\alpha, \sin\alpha, 0)$.

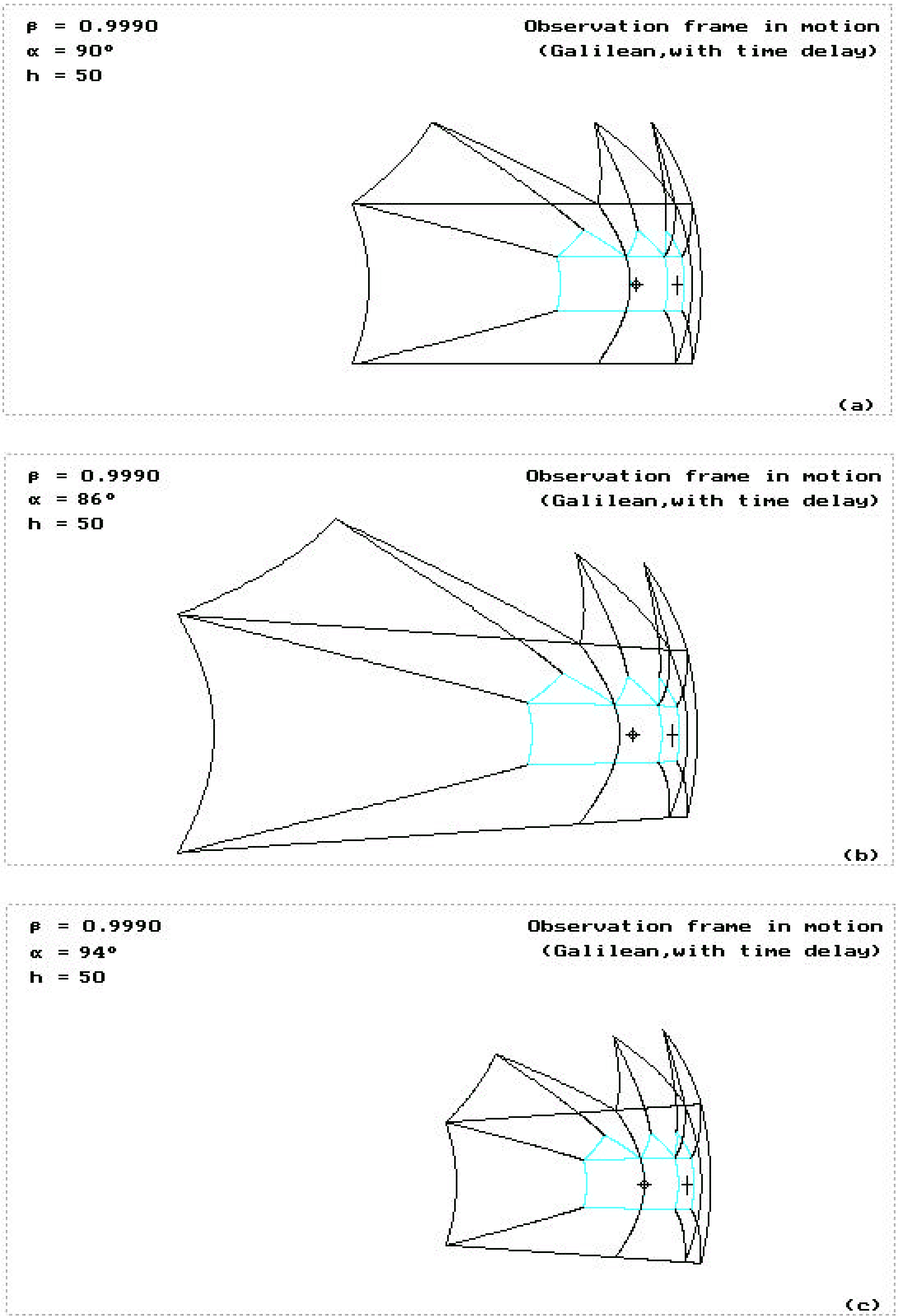


Fig. 4.7. Galilean, with time delay: $\beta = 0.999$. Set of houses for: (a) $\alpha = 90^{o}$. (b) $\alpha = 86^{o}$. (c) $\alpha = 94^{o}$, where $n = (\cos\alpha, \sin\alpha, 0)$.

4.1.2 Relativistic Treatment

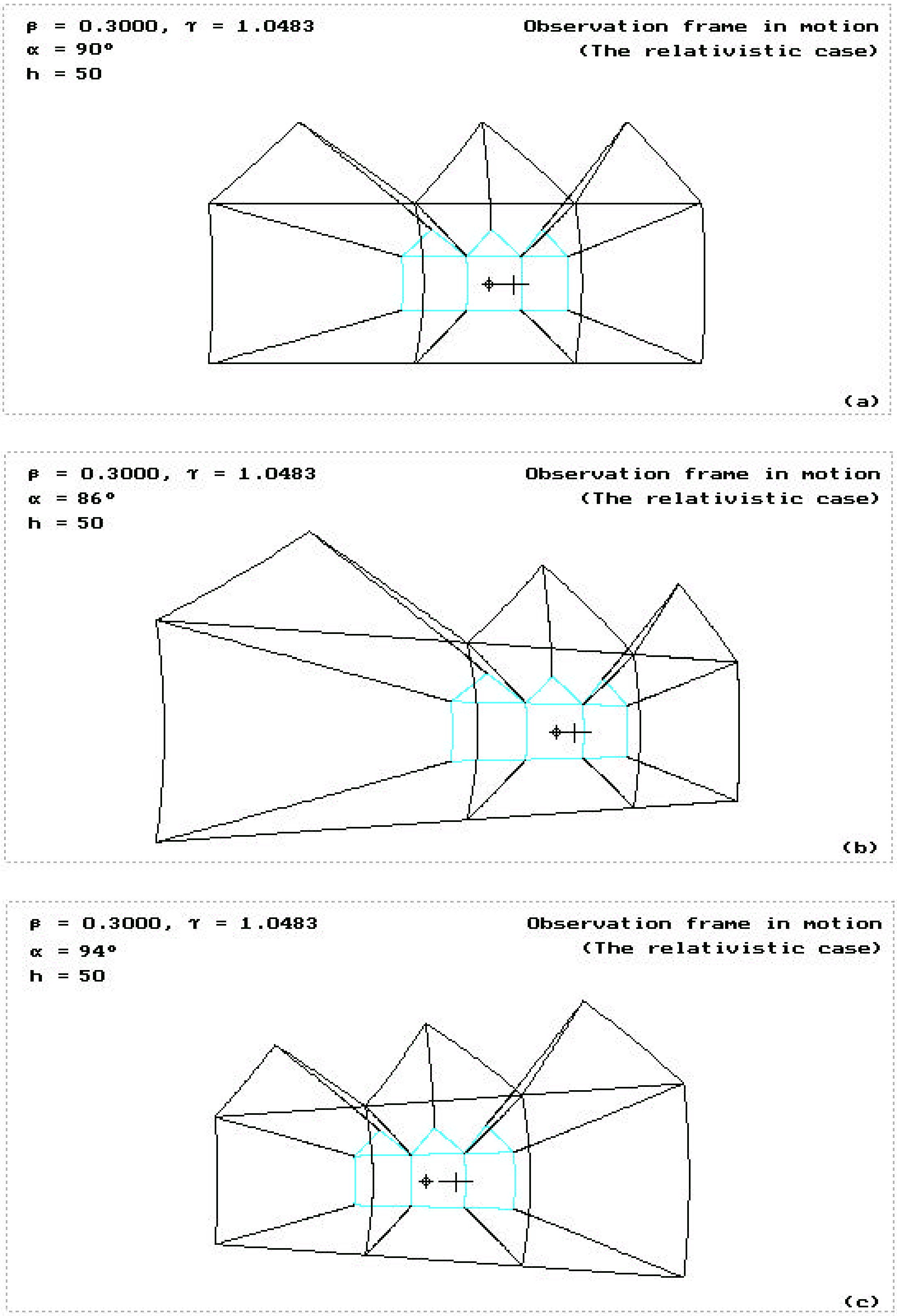


Fig. 4.8. The relativistic case: $\beta = 0.3$. Set of houses for: (a) $\alpha = 90^{o}$. (b) $\alpha = 86^{o}$. (c) $\alpha = 94^{o}$, where $n = (\cos\alpha, \sin\alpha, 0)$.

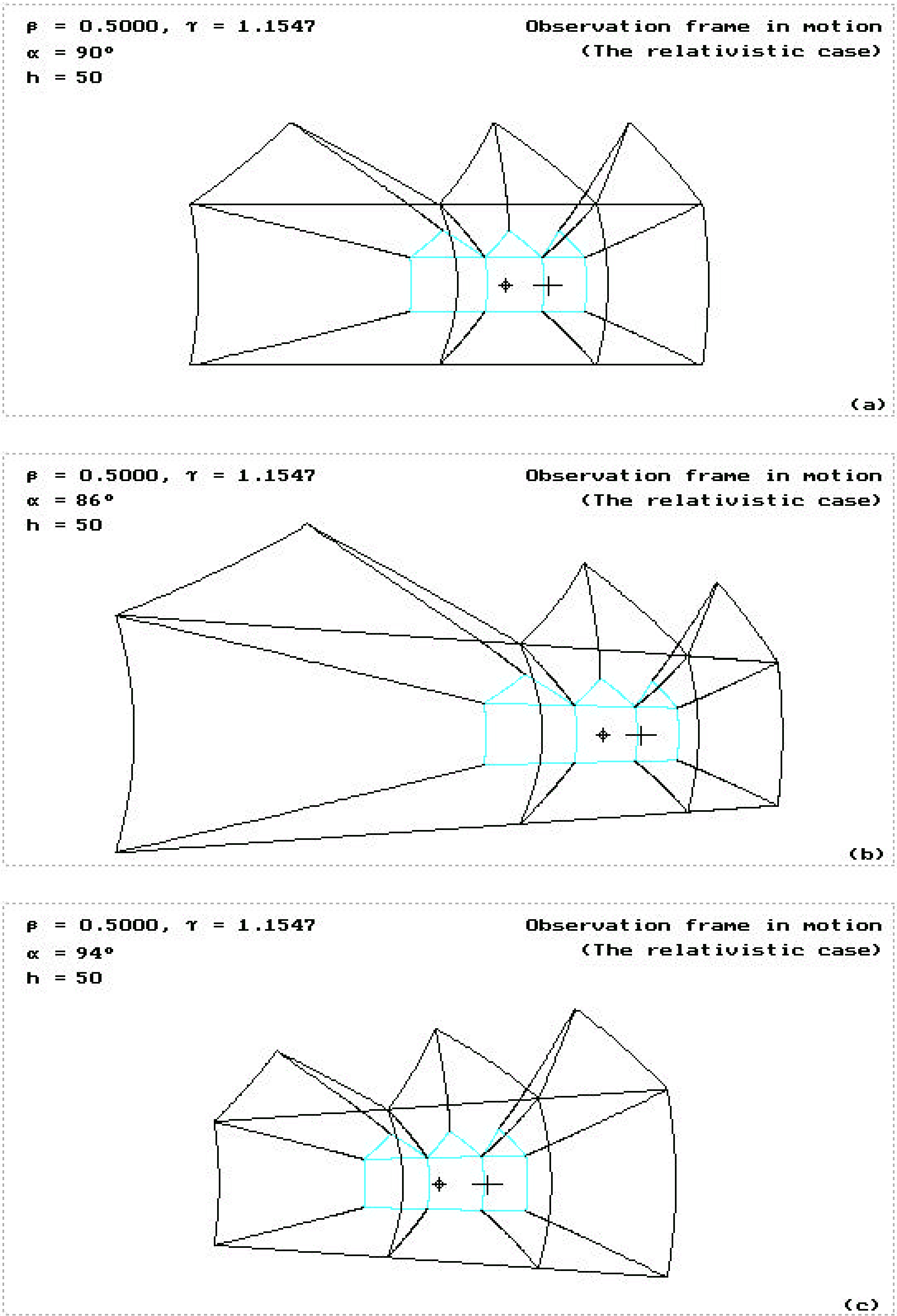


Fig. 4.9. The relativistic case: $b = 0.5$. Set of houses for: (a) $a = 90^{o}$. (b) $a = 86^{o}$. (c) $a = 94^{o}$, where $n = (\cos a, \sin a, 0)$.

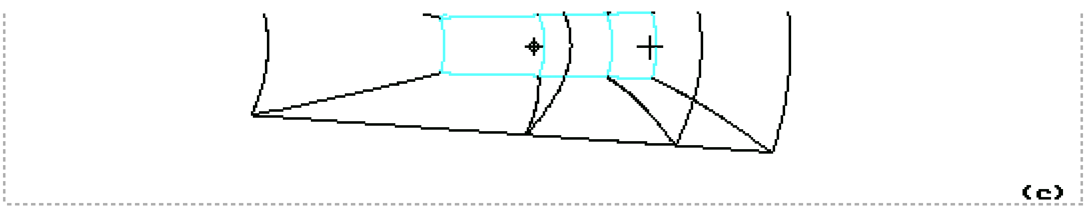


Fig. 4.10. The relativistic case: $b = 0.8$. Set of houses for: (a) $a = 90^o$. (b) $a = 86^o$.
(c) $a = 94^o$, where $n = (\cos a, \sin a, 0)$.

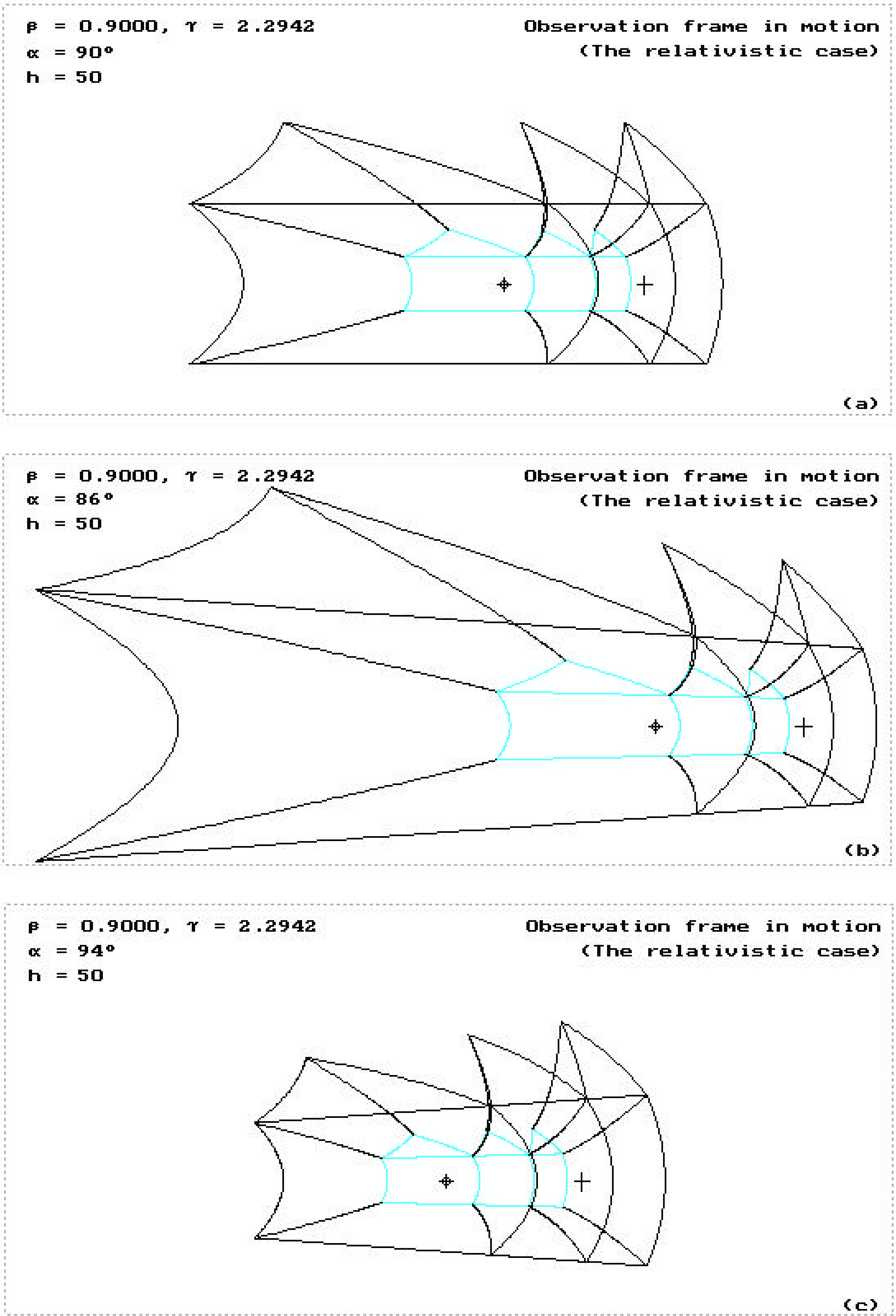
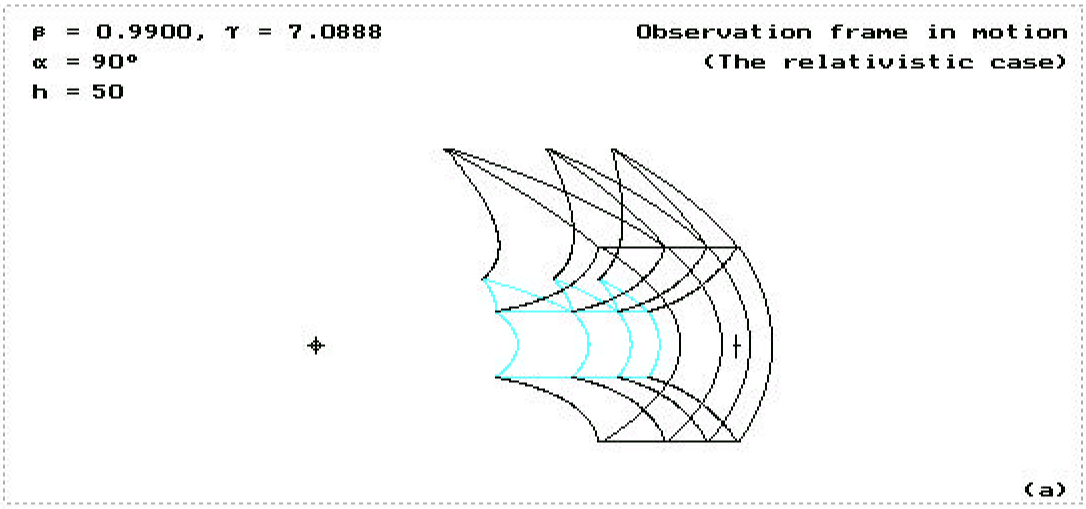
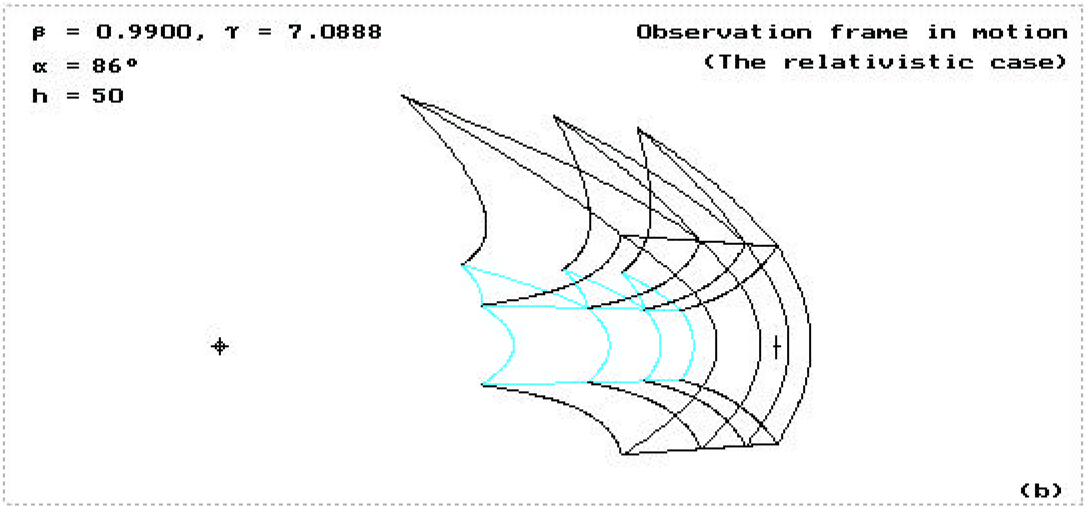


Fig. 4.11. The relativistic case: $\boldsymbol{b} = 0.9$. Set of houses for: (a) $\boldsymbol{a} = 90^{o}$. (b) $\boldsymbol{a} = 86^{o}$. (c) $\boldsymbol{a} = 94^{o}$, where $\mathrm{n} = (\cos \boldsymbol{a}, \sin \boldsymbol{a}, 0)$.





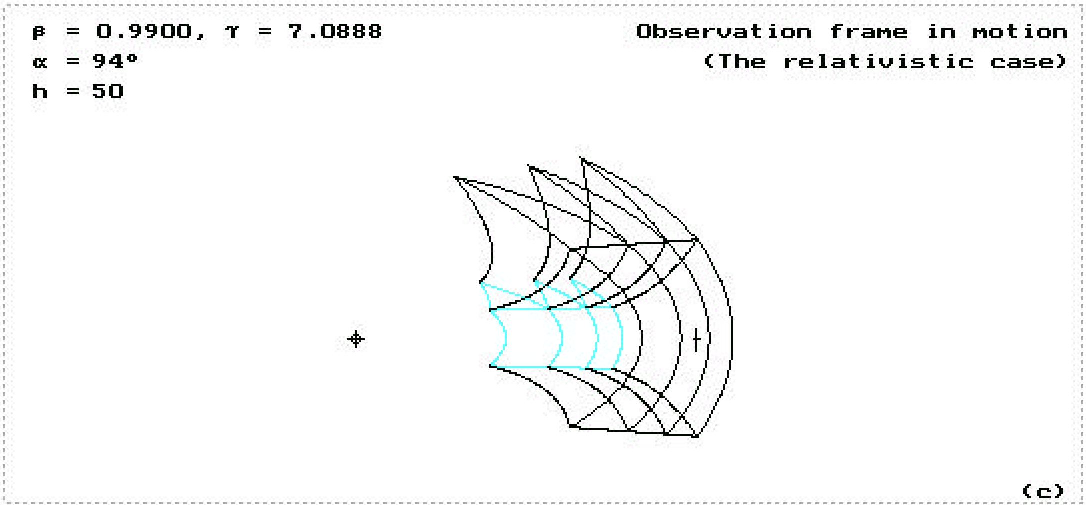


Fig. 4.12. The relativistic case: $b = 0.99$. Set of houses for: (a) $a = 90^{o}$. (b) $a = 86^{o}$. (c) $a = 94^{o}$, where $n = (\cos a, \sin a, 0)$.

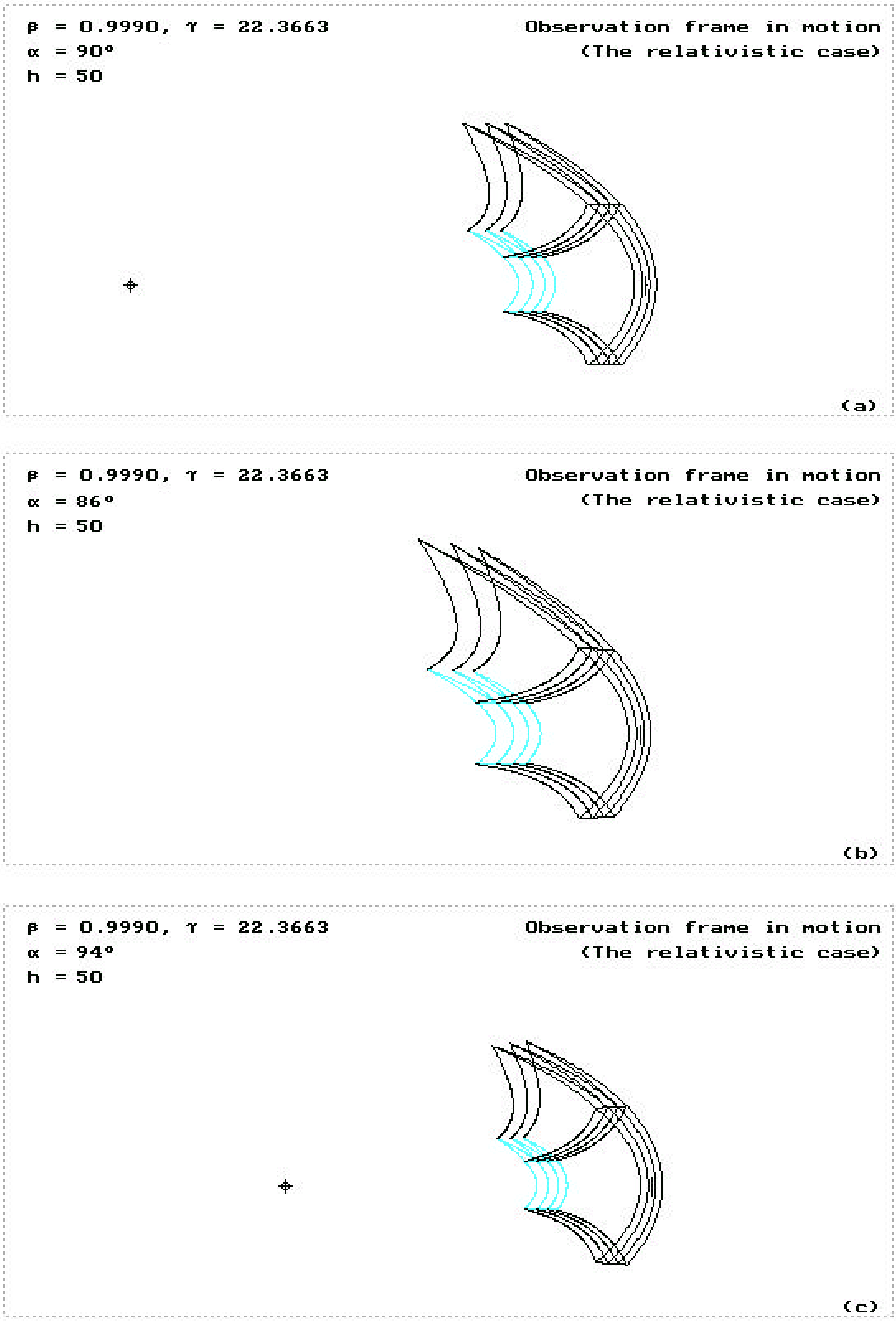


Fig. 4.13. The relativistic case: $b = 0.999$. Set of houses for: (a) $a = 90^{\circ}$. (b) $a = 86^{\circ}$. (c) $a = 94^{\circ}$, where $\mathrm{n} = (\cos a, \sin a, 0)$.

4.2 Application II

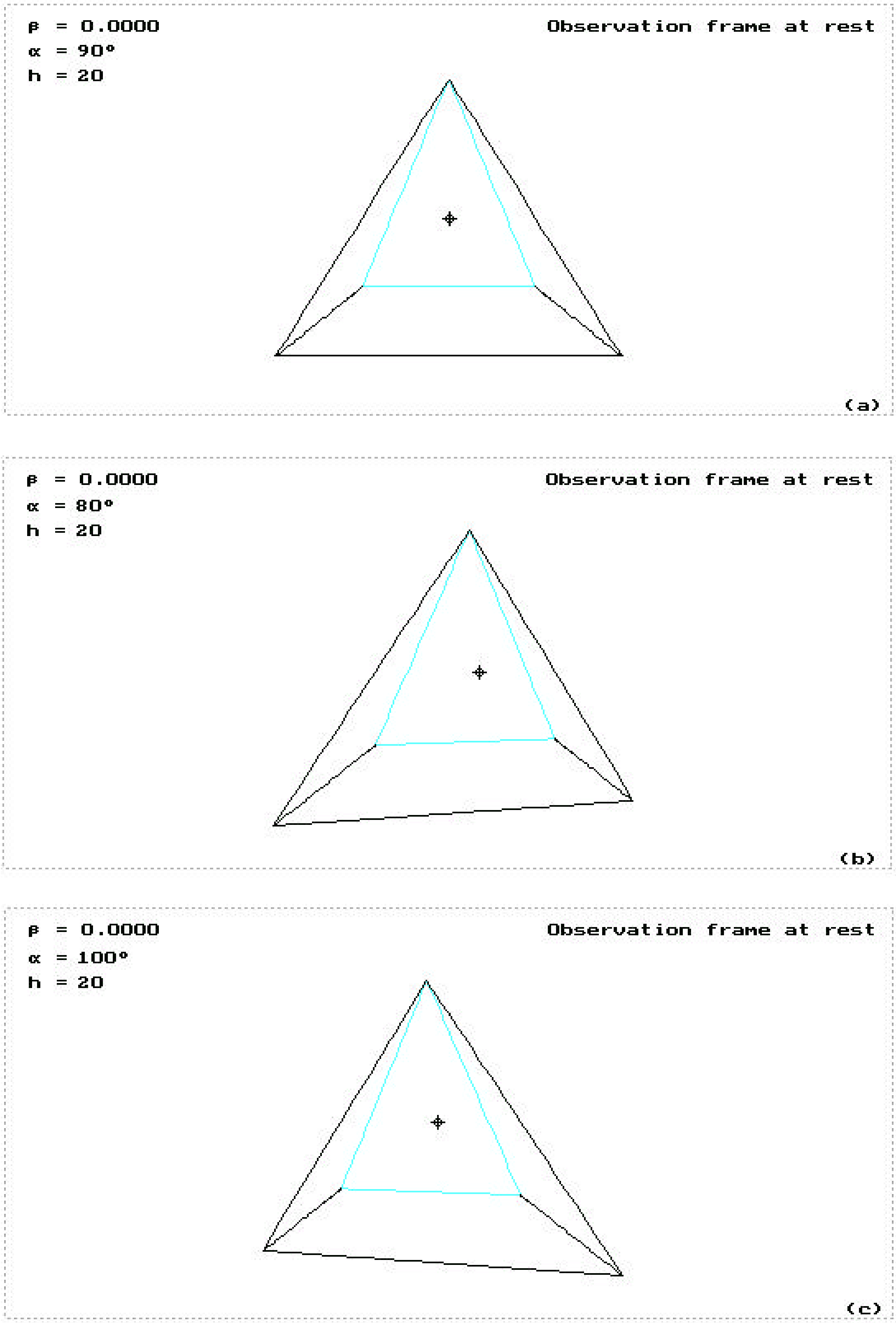


Fig. 4.14. Pyramid for: (a) $\alpha = 90^{o}$. (b) $\alpha = 80^{o}$. (c) $\alpha = 100^{o}$, where n $= (\cos\alpha, \sin\alpha, 0)$.

4.2.1 Galilean Treatment with Time Delay

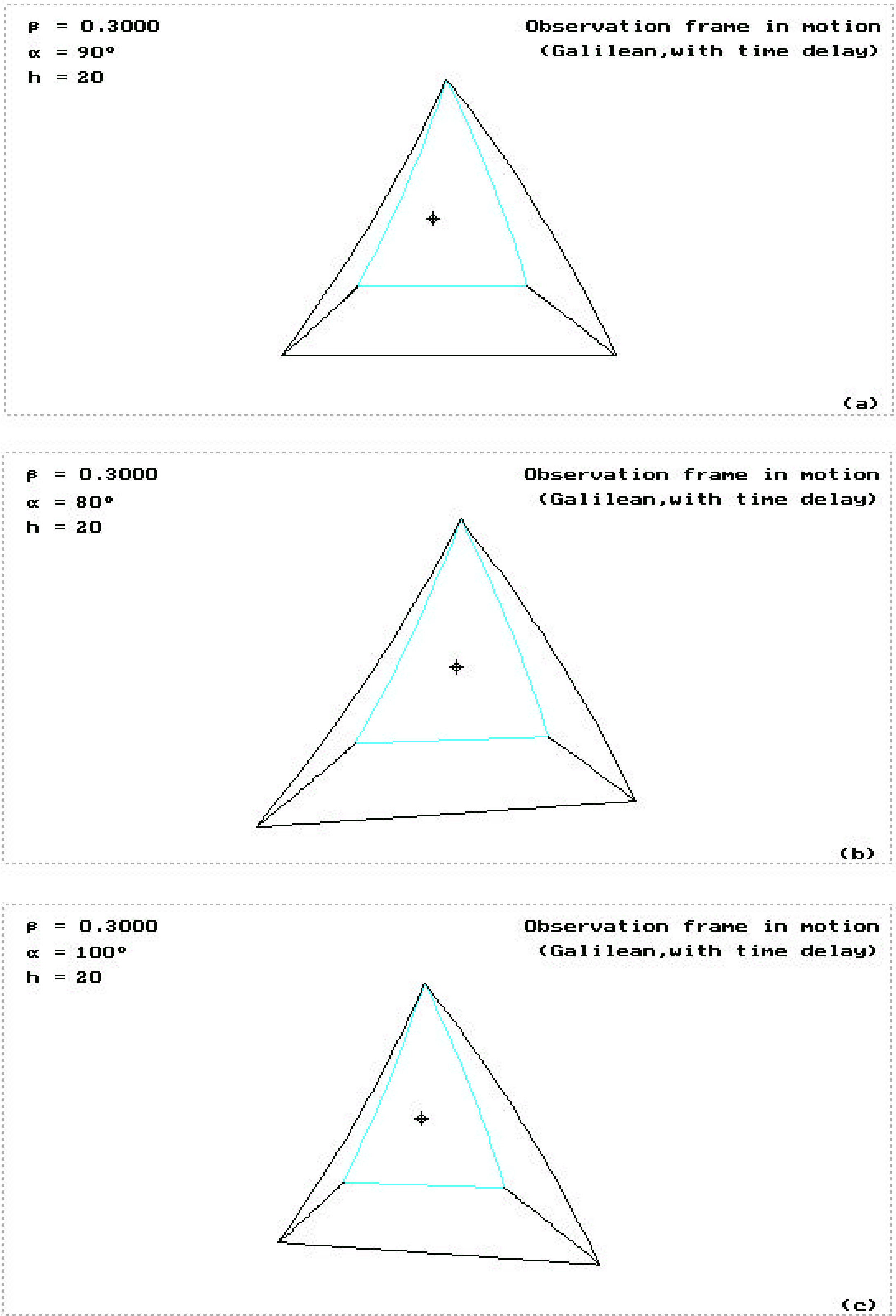


Fig. 4.15. Galilean, with time delay: $\boldsymbol{b} = 0.3$. Pyramid for: (a) $\boldsymbol{a} = 90^o$. (b) $\boldsymbol{a} = 80^o$.
(c) $\boldsymbol{a} = 100^o$, where n = $(\cos \boldsymbol{a}, \sin \boldsymbol{a}, 0)$.

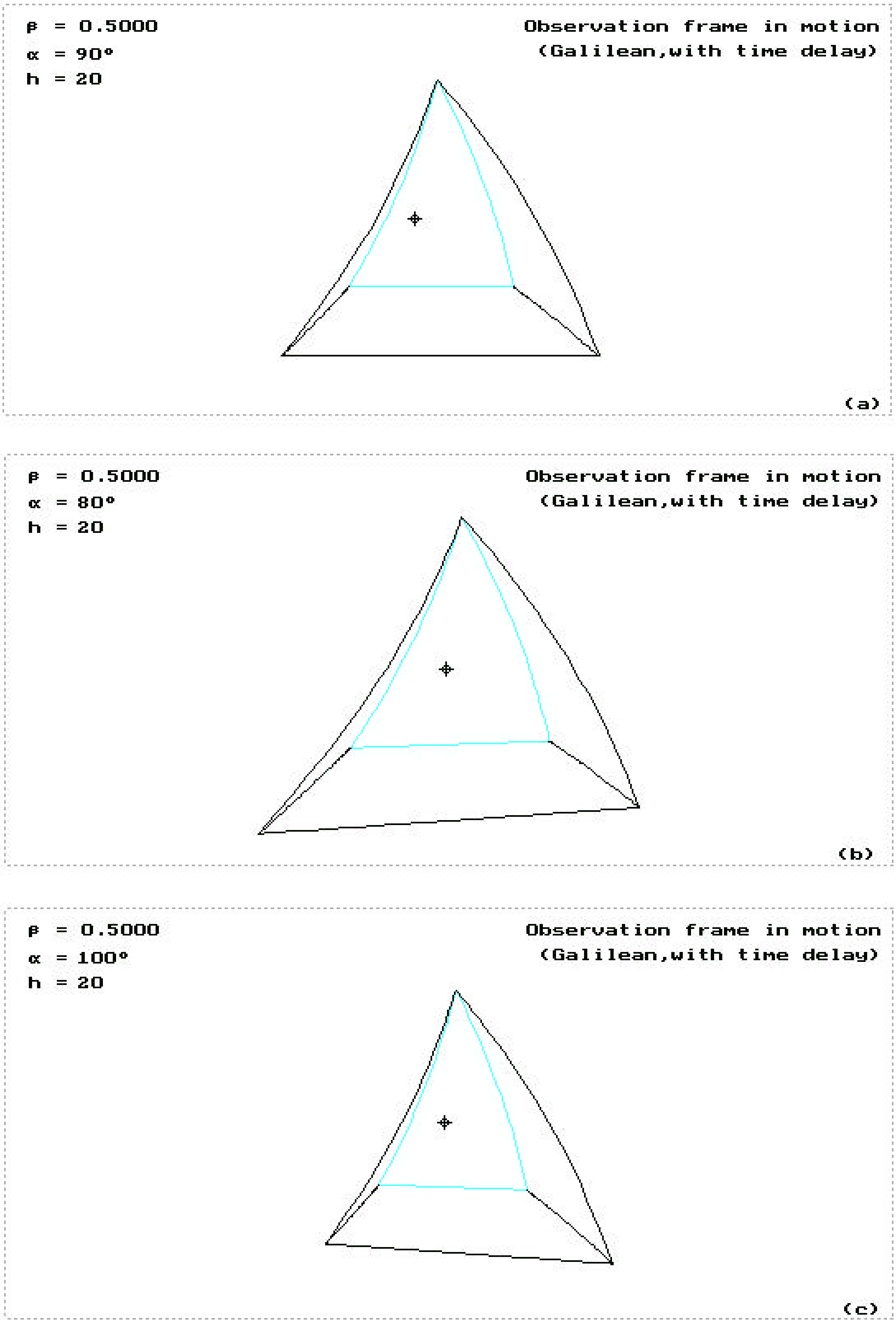


Fig. 4.16. Galilean, with time delay: $\boldsymbol{b} = 0.5$. Pyramid for: (a) $\boldsymbol{a} = 90^o$. (b) $\boldsymbol{a} = 80^o$. (c) $\boldsymbol{a} = 100^o$, where n$= (\cos \boldsymbol{a}, \sin \boldsymbol{a}, 0)$.

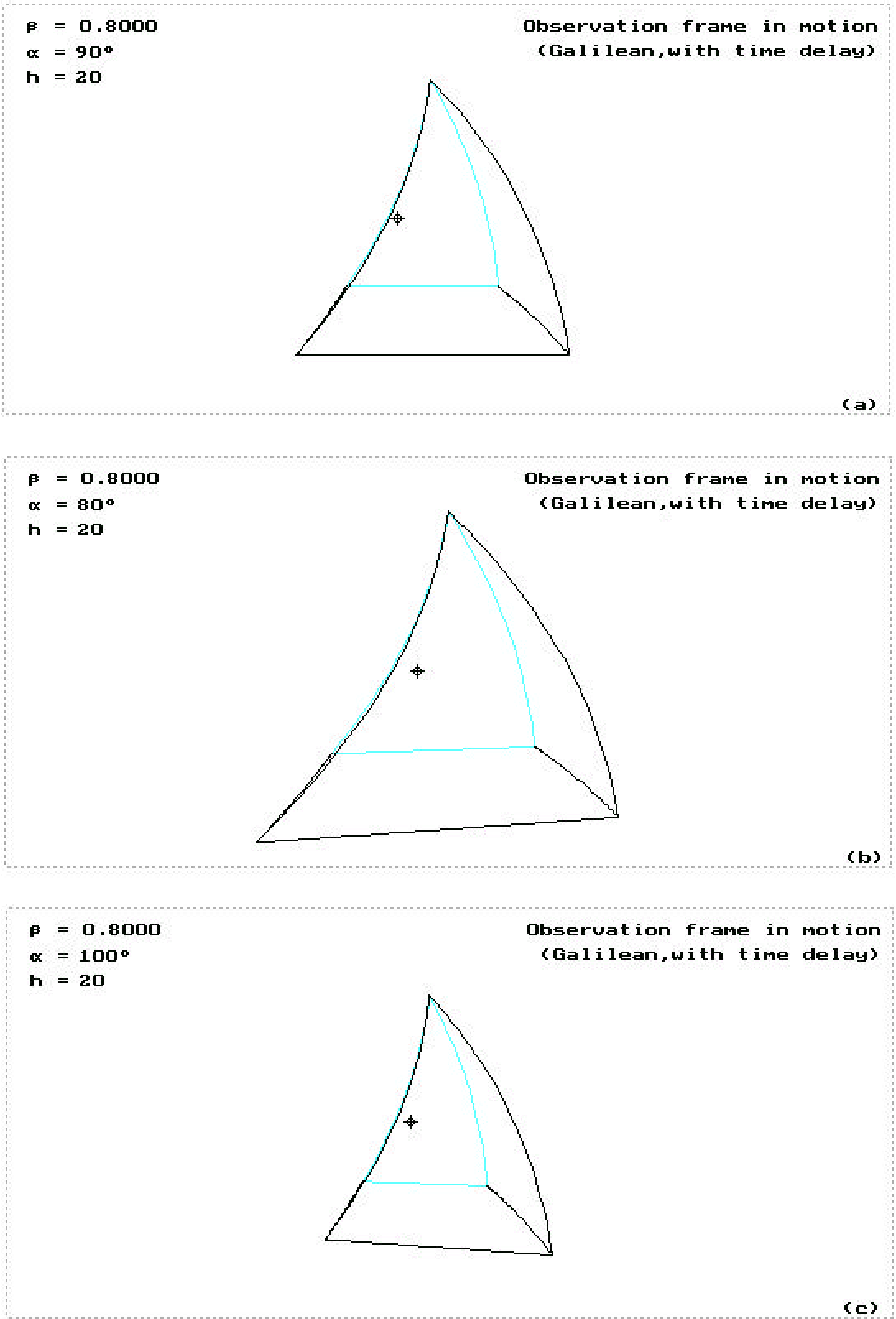


Fig. 4.17. Galilean, with time delay: $\beta = 0.8$. Pyramid for: (a) $\alpha = 90^o$. (b) $\alpha = 80^o$. (c) $\alpha = 100^o$, where $\mathrm{n} = (\cos\alpha, \sin\alpha, 0)$.

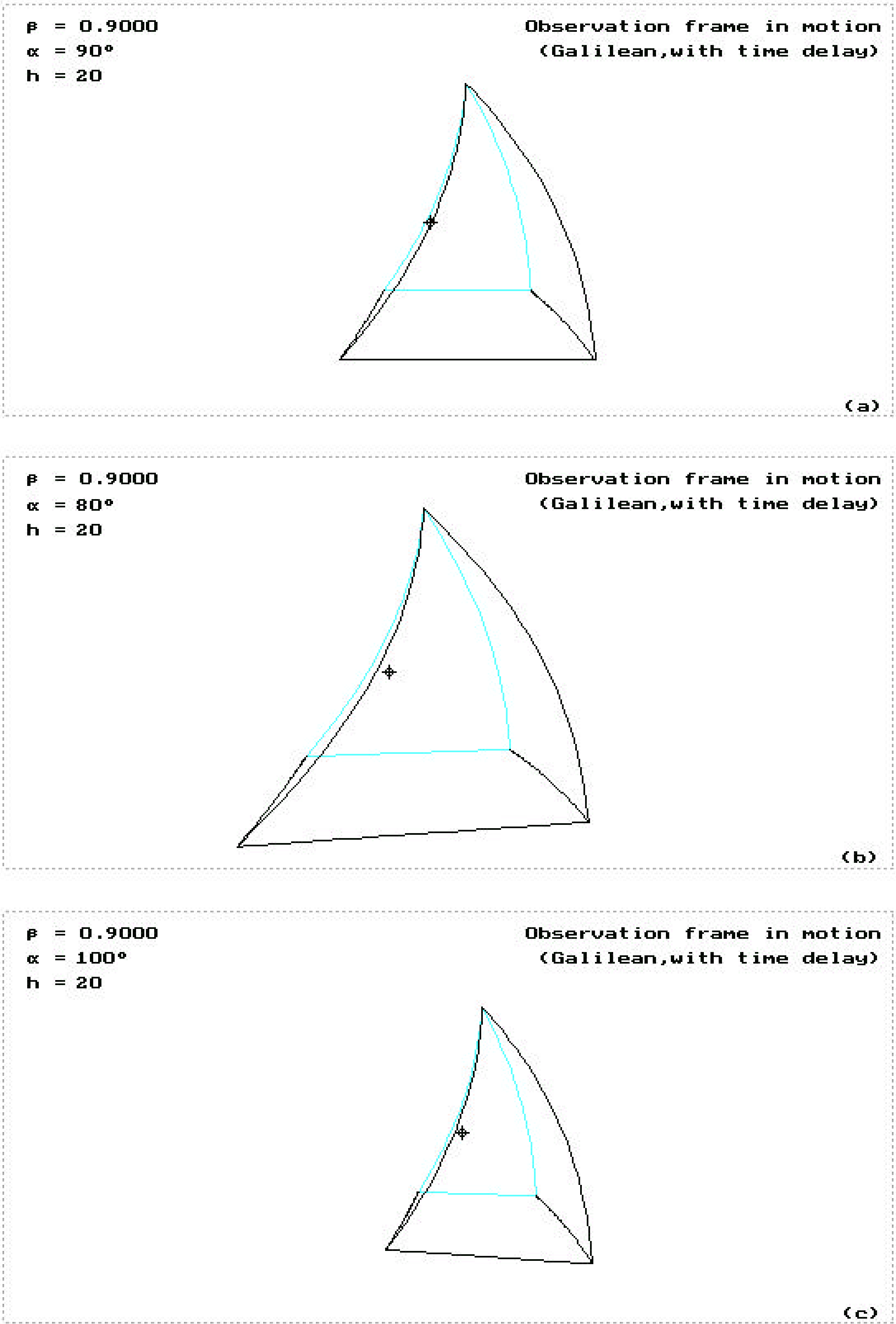


Fig.4.18. Galilean, with time delay: $\beta = 0.9$. Pyramid for: (a) $\alpha = 90^o$. (b) $\alpha = 80^o$. (c) $\alpha = 100^o$, where $\mathrm{n} = (\cos\alpha, \sin\alpha, 0)$.

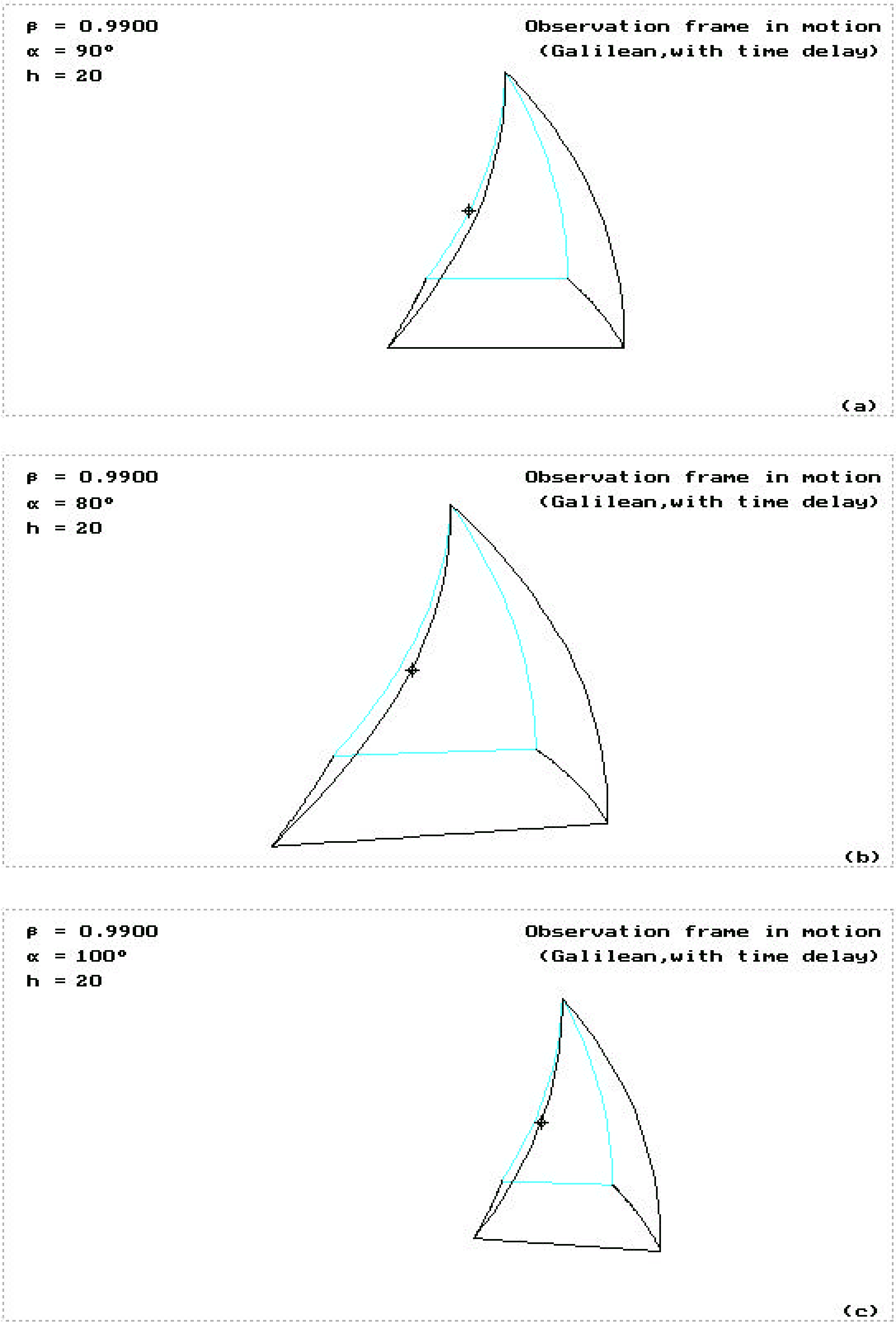


Fig. 4.19 Galilean, with time delay: $\boldsymbol{b} = 0.99$. Pyramid for: (a) $\boldsymbol{a} = 90^o$. (b) $\boldsymbol{a} = 80^o$. (c) $\boldsymbol{a} = 100^o$, where $n = (\cos \boldsymbol{a}, \sin \boldsymbol{a}, 0)$.

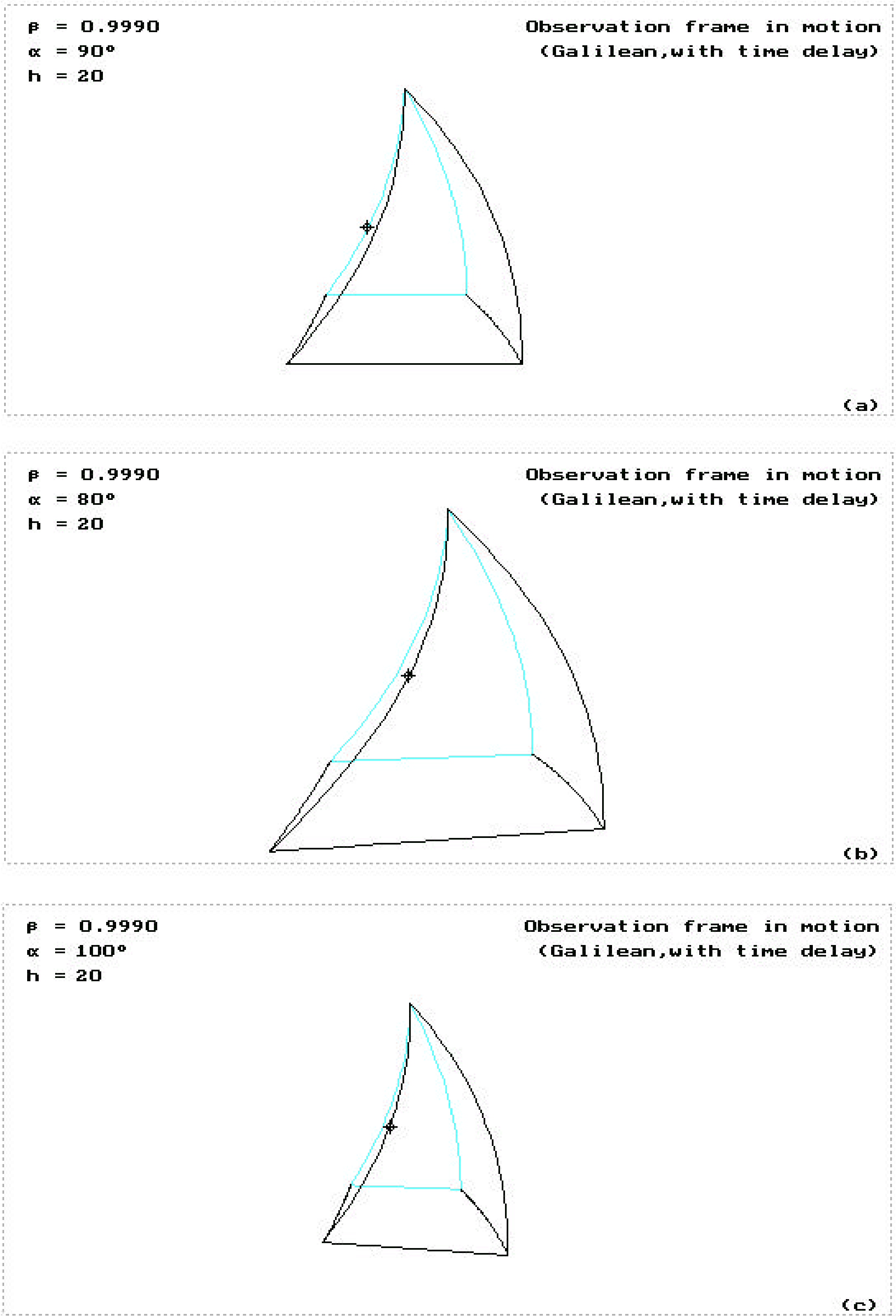


Fig. 4.20 Galilean, with time delay: $\beta = 0.999$. Pyramid for: (a) $\alpha = 90^{o}$. (b) $\alpha = 80^{o}$. (c) $\alpha = 100^{o}$, where $\mathrm{n} = (\cos\alpha, \sin\alpha, 0)$.

4.2.2 Relativistic Treatment

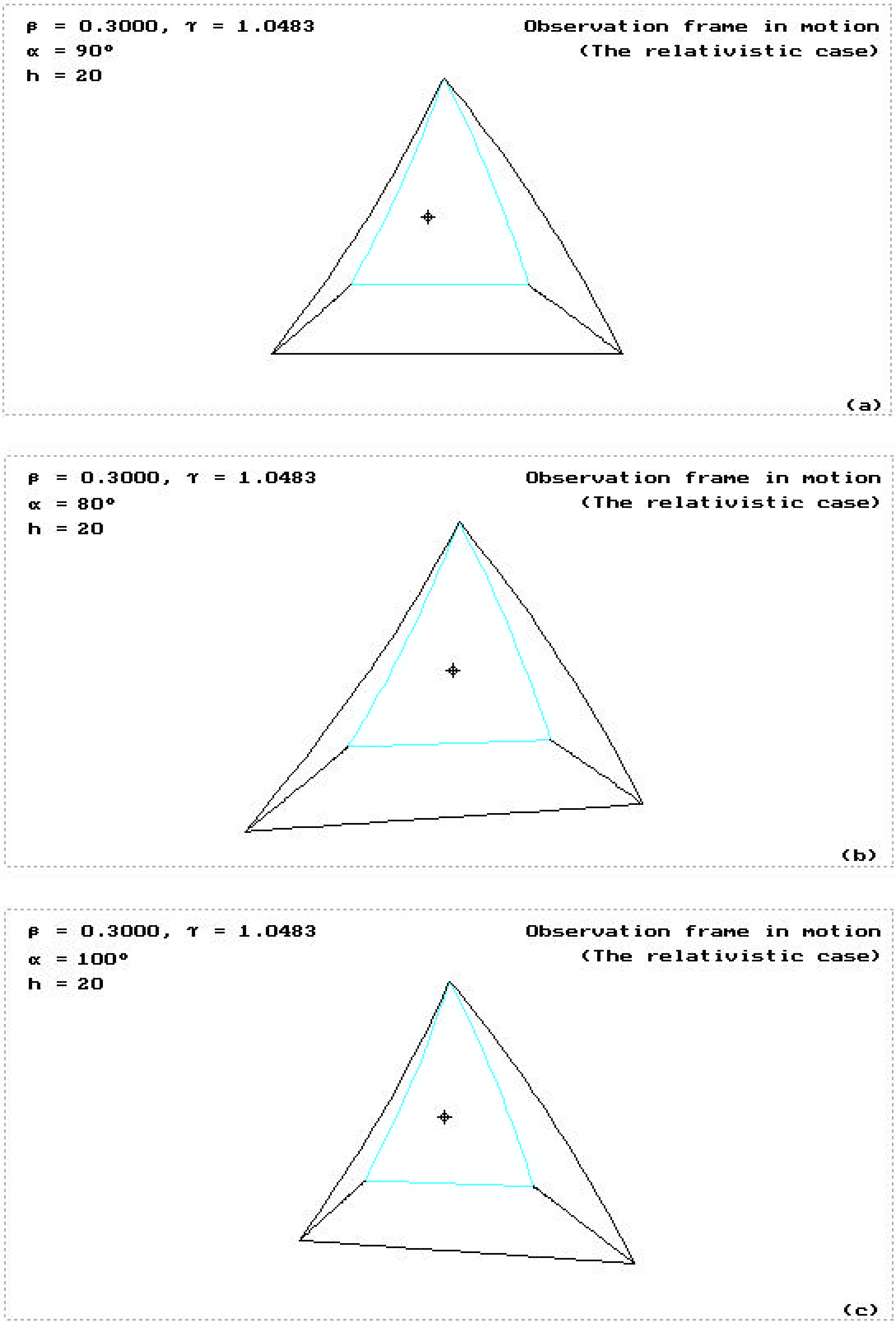


Fig. 4.21. The relativistic case: $\boldsymbol{b} = 0.3$. Pyramid for: (a) $\boldsymbol{a} = 90^o$. (b) $\boldsymbol{a} = 80^o$. (c) $\boldsymbol{a} = 100^o$, where n $= (\cos \boldsymbol{a}, \sin \boldsymbol{a}, 0)$.

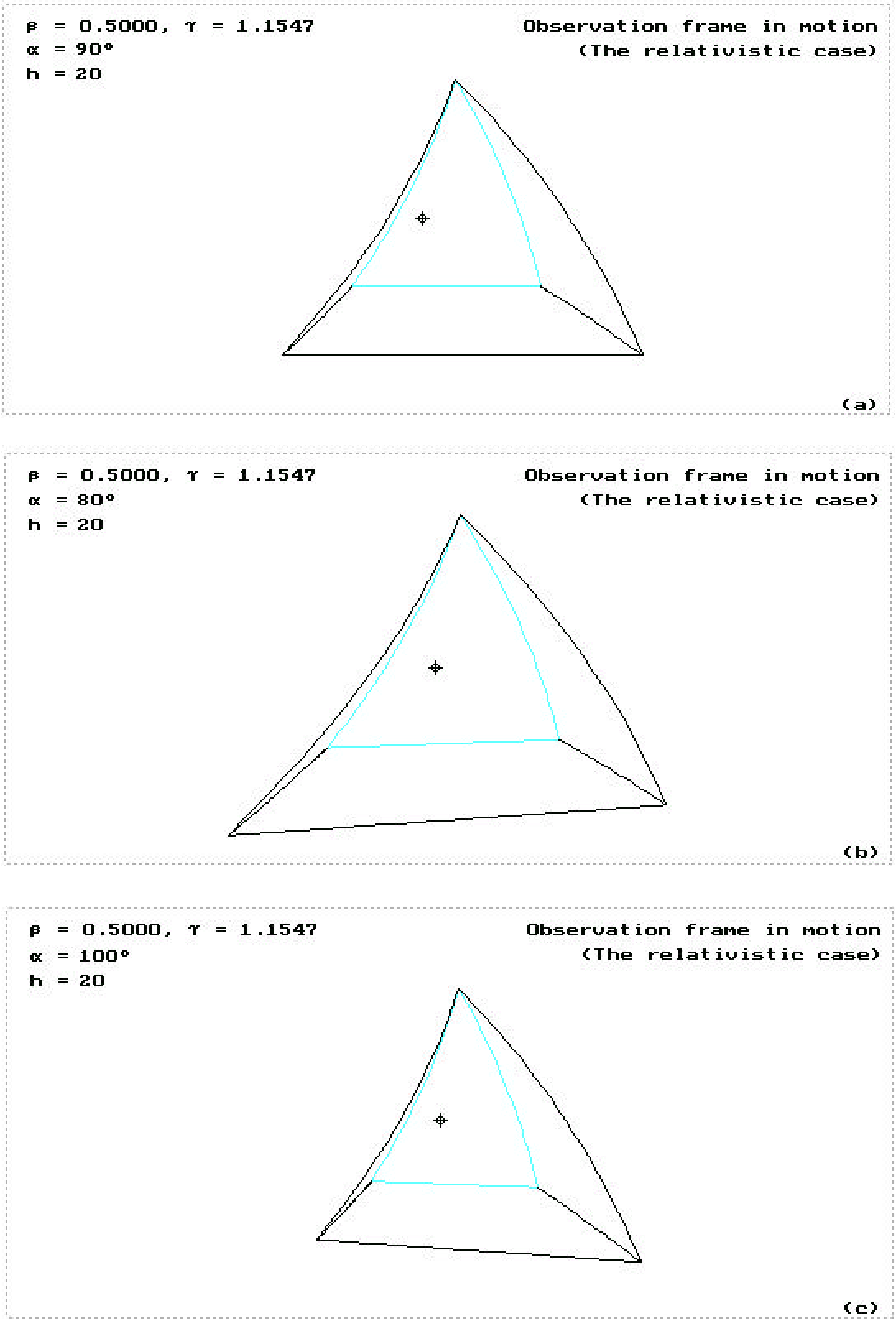


Fig. 4.22. The relativistic case: $\boldsymbol{b} = 0.5$. Pyramid for: (a) $\boldsymbol{a} = 90^o$. (b) $\boldsymbol{a} = 80^o$. (c) $\boldsymbol{a} = 100^o$, where $\mathrm{n} = (\cos \boldsymbol{a}, \sin \boldsymbol{a}, 0)$.

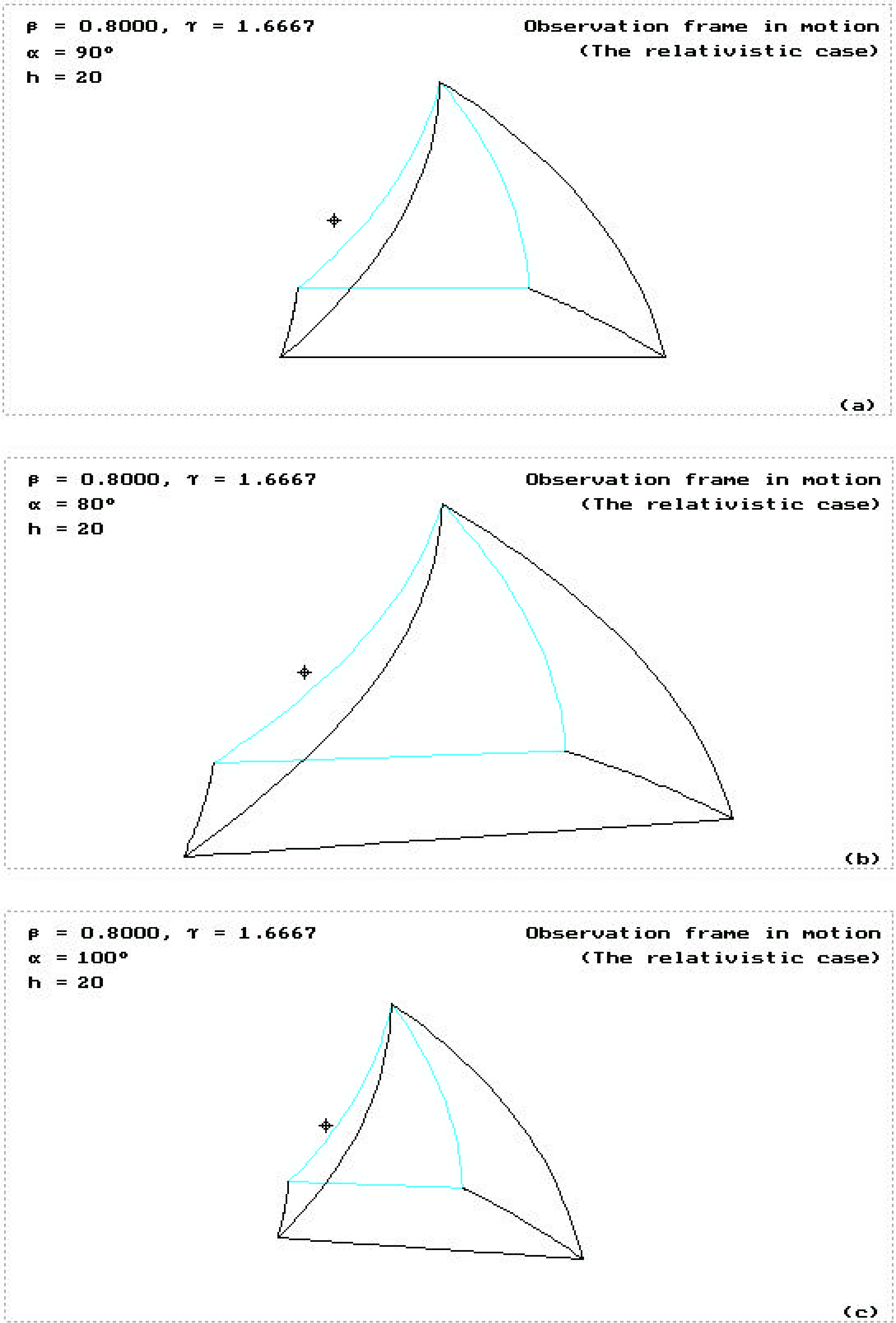


Fig 4.23. The relativistic case: $\beta = 0.8$. Pyramid for: (a) $\alpha = 90^o$. (b) $\alpha = 80^o$. (c) $\alpha = 100^o$, where n = $(\cos \alpha, \sin \alpha, 0)$.

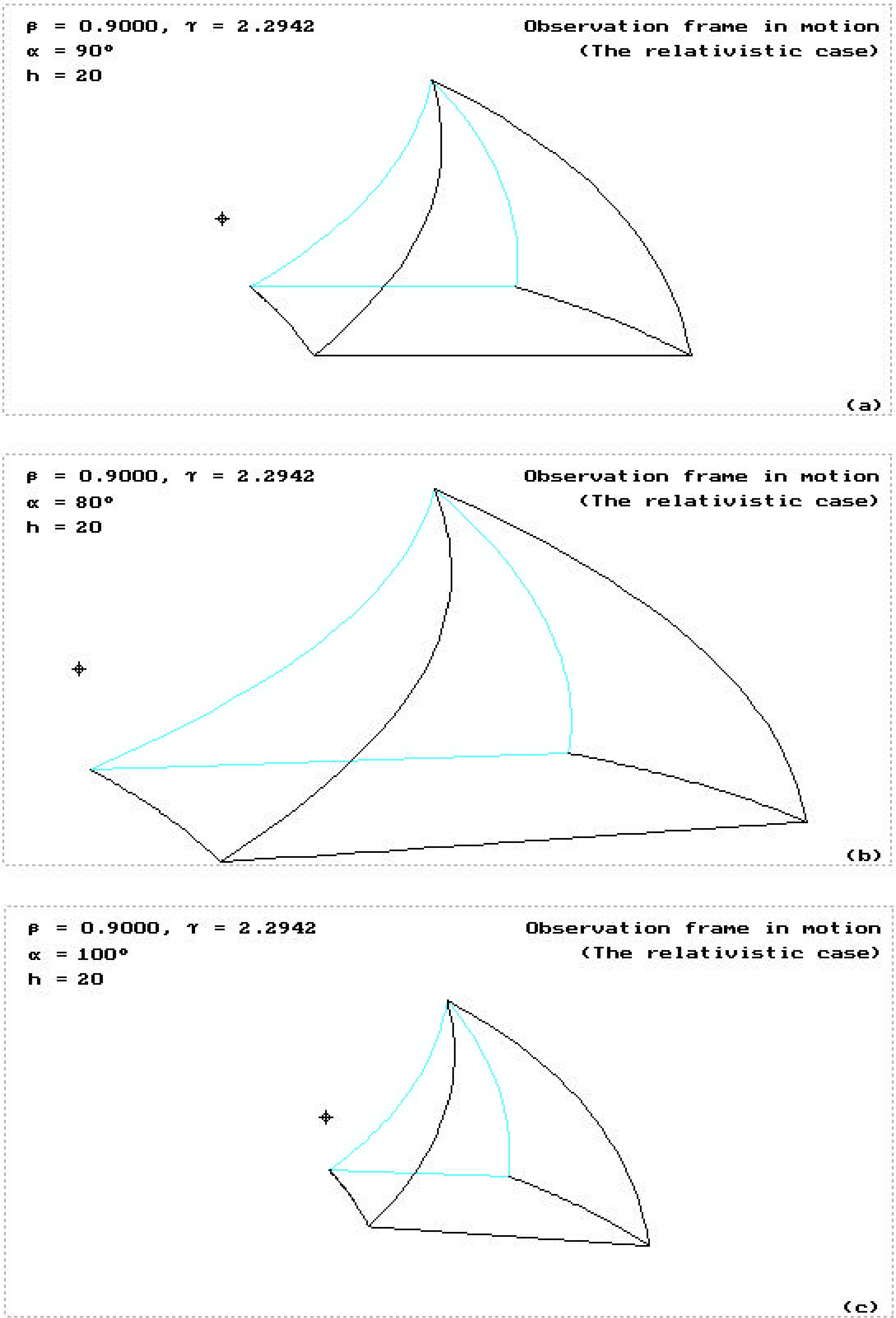


Fig. 4.24. The relativistic case: $b = 0.9$. Pyramid for: (a) $a = 90^\circ$. (b) $a = 80^\circ$. (c) $a = 100^\circ$, where $n = (\cos a, \sin a, 0)$.

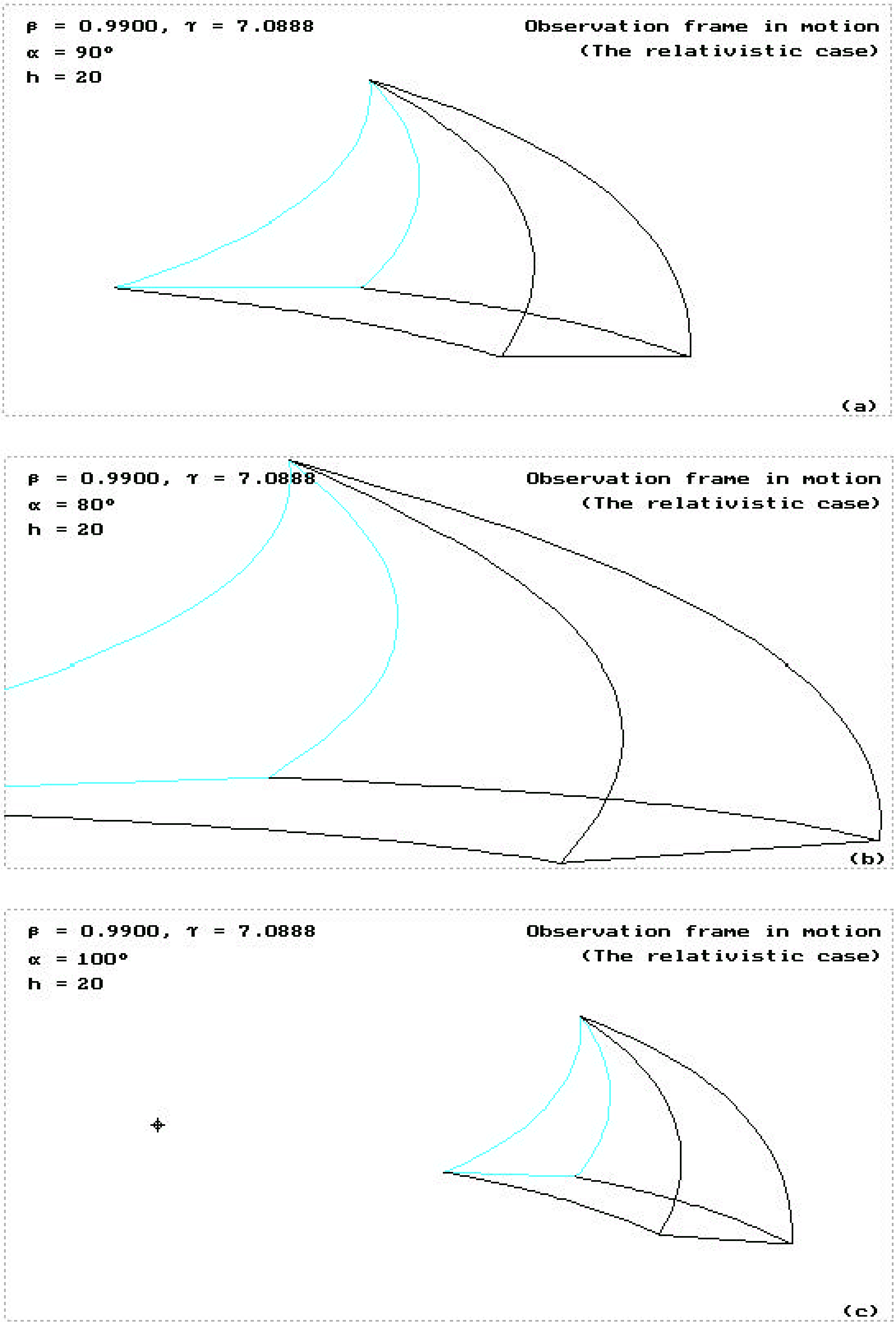


Fig. 4.25. The relativistic case: $\boldsymbol{b} = 0.99$. Pyramid for: (a) $\boldsymbol{a} = 90^{o}$. (b) $\boldsymbol{a} = 80^{o}$. (c) $\boldsymbol{a} = 100^{o}$, where $\mathrm{n} = (\cos \boldsymbol{a}, \sin \boldsymbol{a}, 0)$.

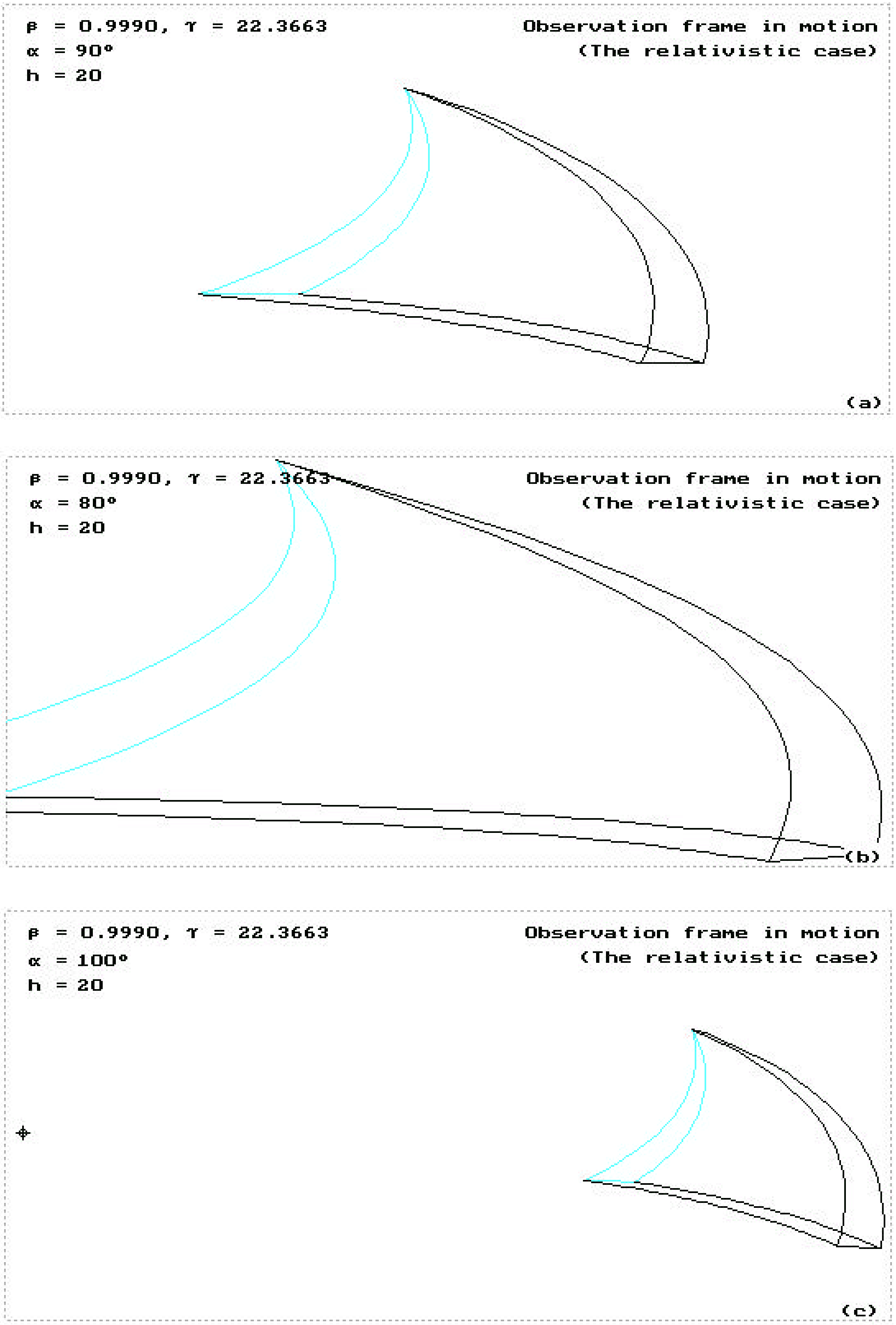


Fig. 4.26. The relativistic case: $b = 0.999$. Pyramid for: (a) $a = 90^o$. (b) $a = 80^o$. (c) $a = 100^o$, where n = $(\cos a, \sin a, 0)$.

4.3 Application III

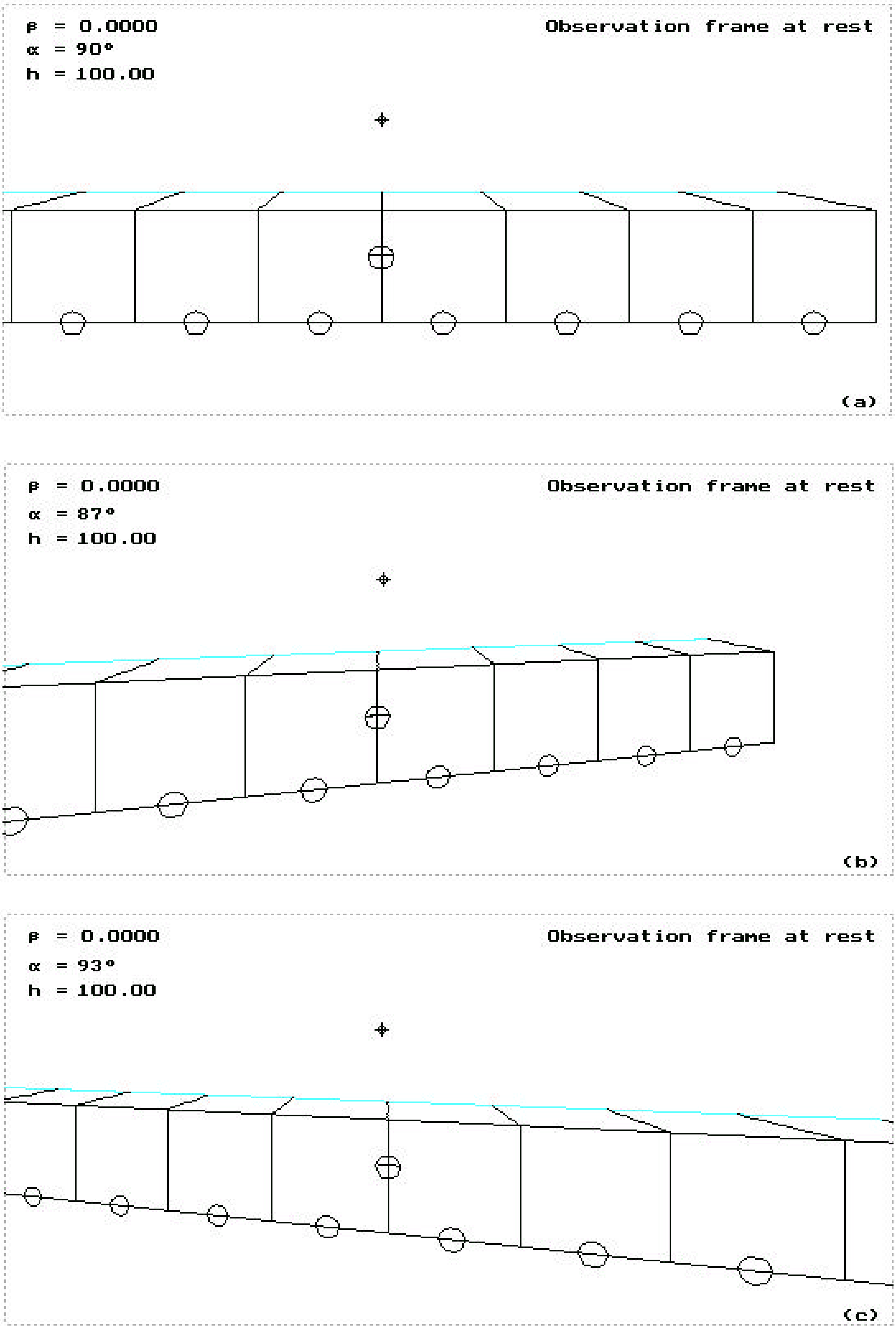


Fig. 4.27. Train for : (a) $\alpha = 90^o$. (b) $\alpha = 87^o$. (c) $\alpha = 93^o$, where n = (cos α, sin α,0) .

4.3.1 Galilean Treatment with Time Delay

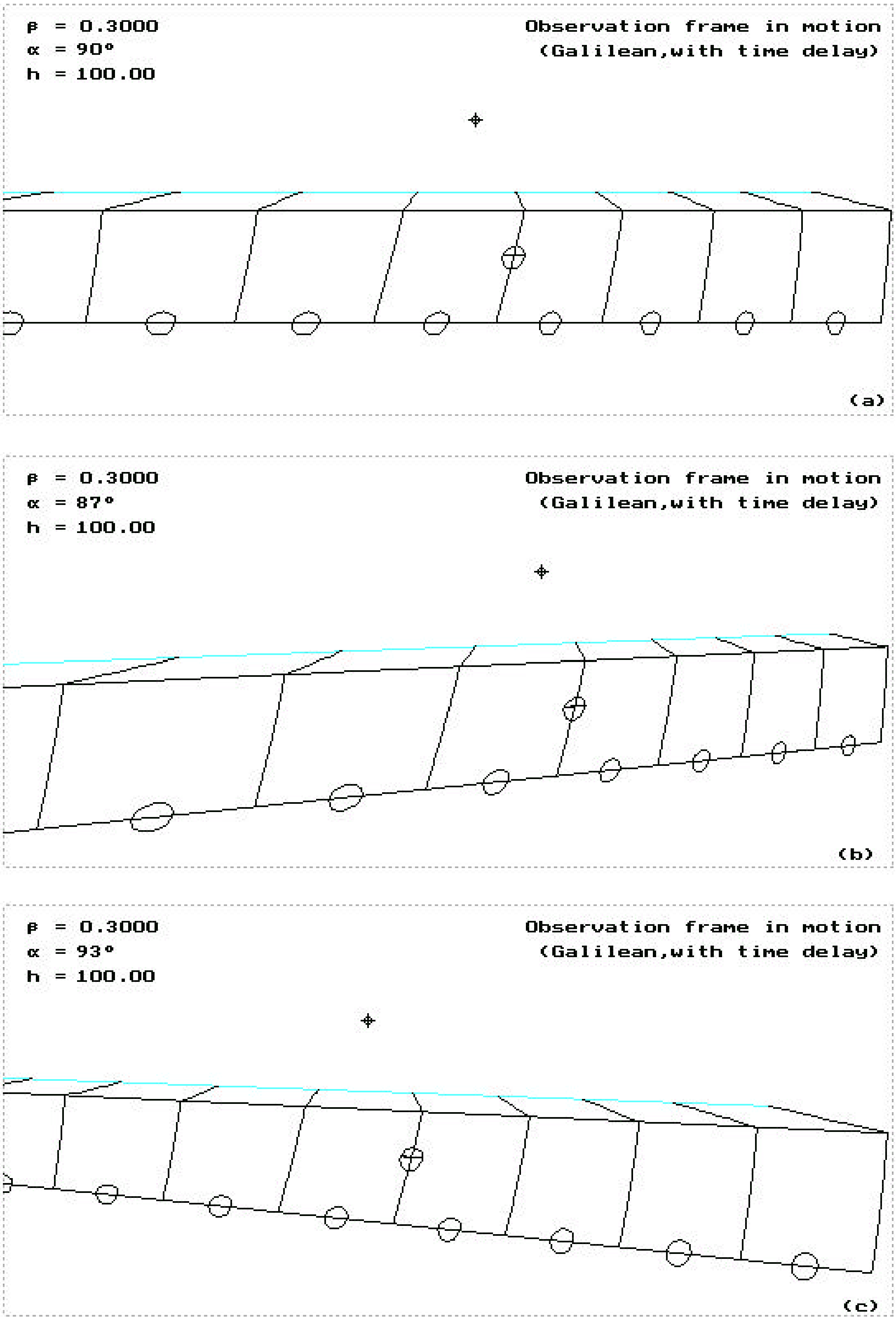


Fig. 4.28. Galilean, with time delay: $\boldsymbol{b} = 0.3$. Train for: (a) $\boldsymbol{a} = 90^o$. (b) $\boldsymbol{a} = 87^o$. (c) $\boldsymbol{a} = 93^o$, where $\mathrm{n} = (\cos \boldsymbol{a}, \sin \boldsymbol{a}, 0)$.

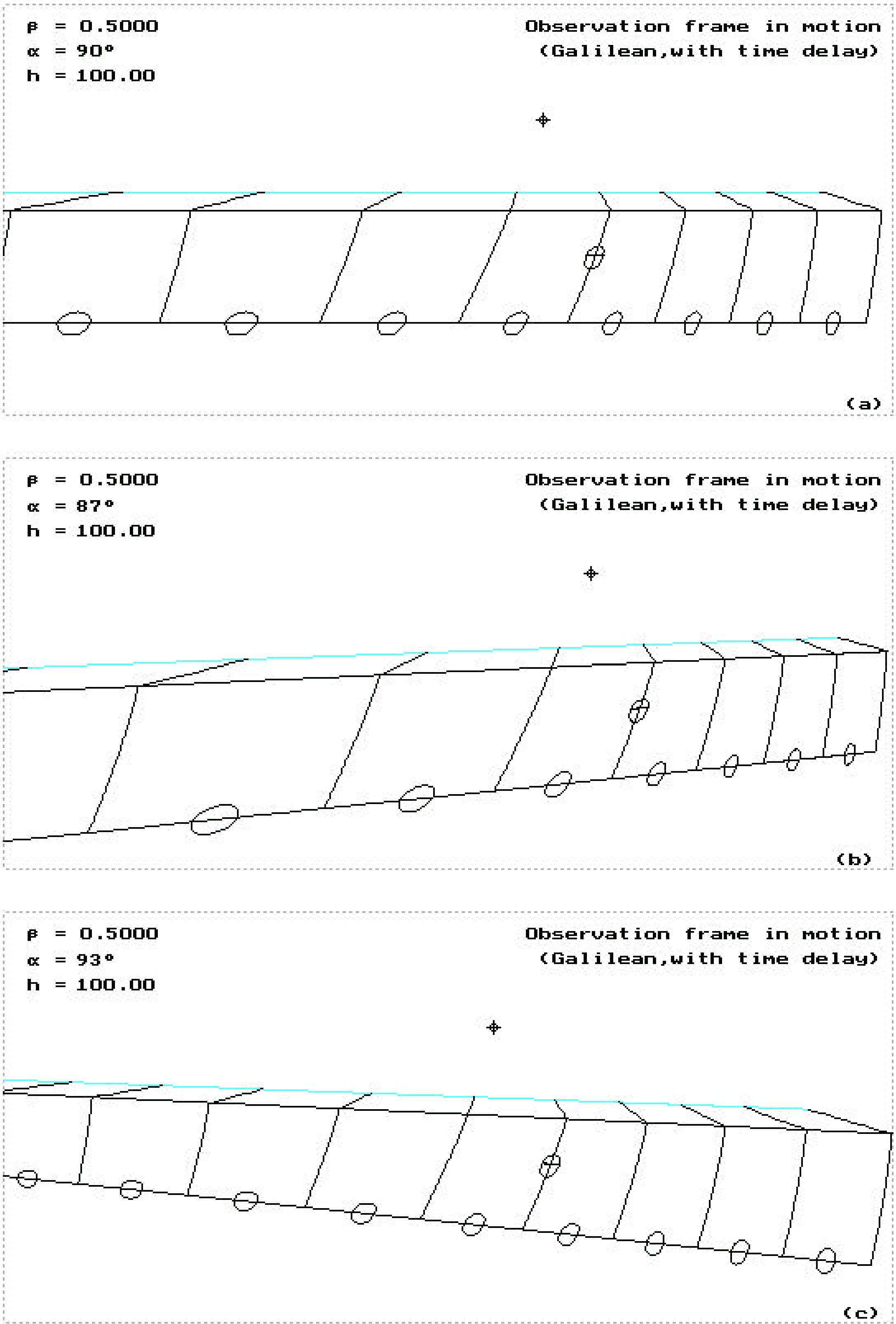


Fig. 4.29. Galilean, with time delay: $\beta = 0.5$. Train for: (a) $\alpha = 90^o$. (b) $\alpha = 87^o$. (c) $\alpha = 93^o$, where n $= (\cos\alpha, \sin\alpha, 0)$.

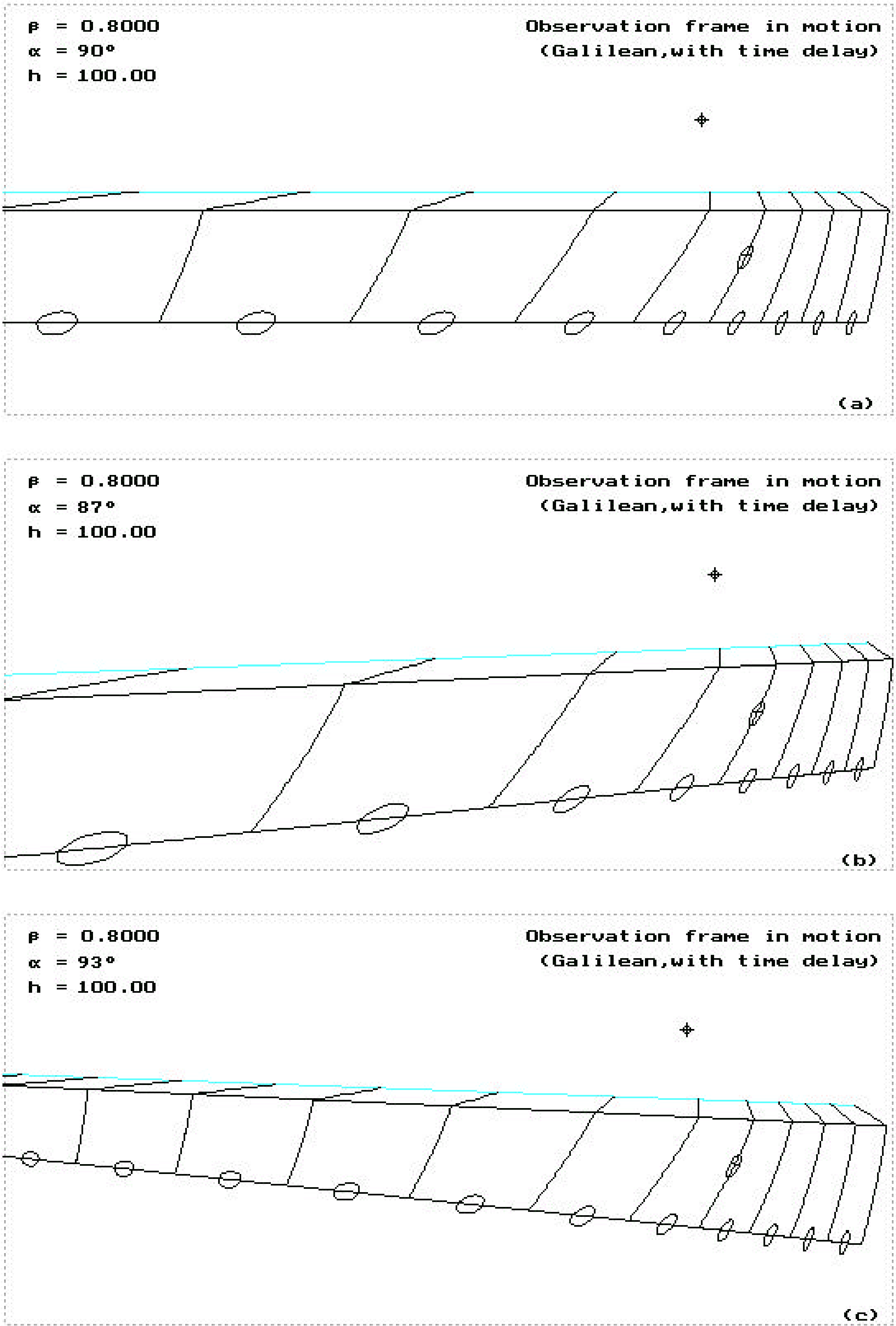


Fig. 4.30. Galilean, with time delay: $\beta = 0.8$. Train for: (a) $\alpha = 90^o$. (b) $\alpha = 87^o$. (c) $\alpha = 93^o$, where $\mathrm{n} = (\cos\alpha, \sin\alpha, 0)$.

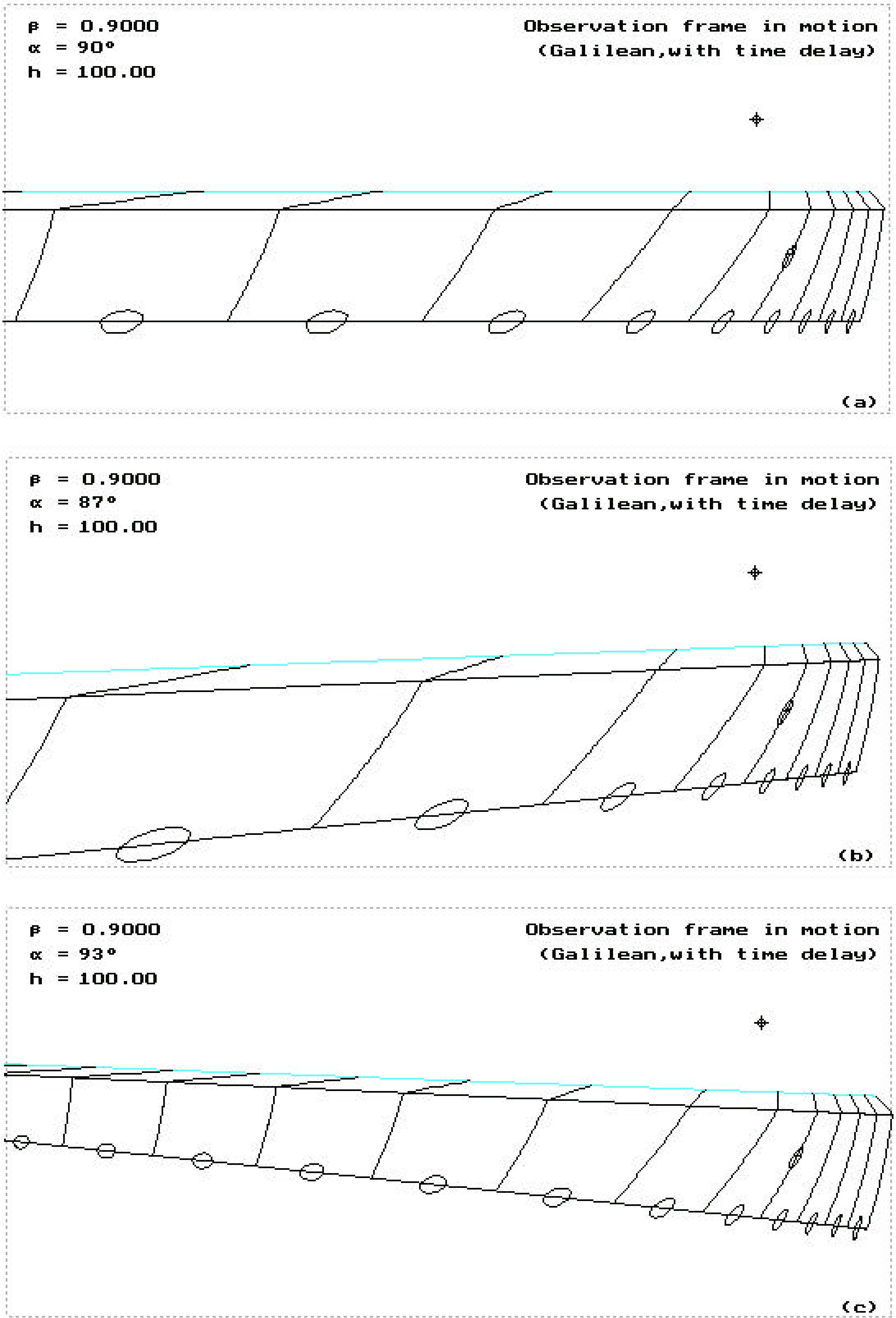


Fig. 4.31. Galilean, with time delay: $\beta = 0.9$. Train for: (a) $\alpha = 90^o$. (b) $\alpha = 87^o$. (c) $\alpha = 93^o$, where $\mathrm{n} = (\cos\alpha, \sin\alpha, 0)$.

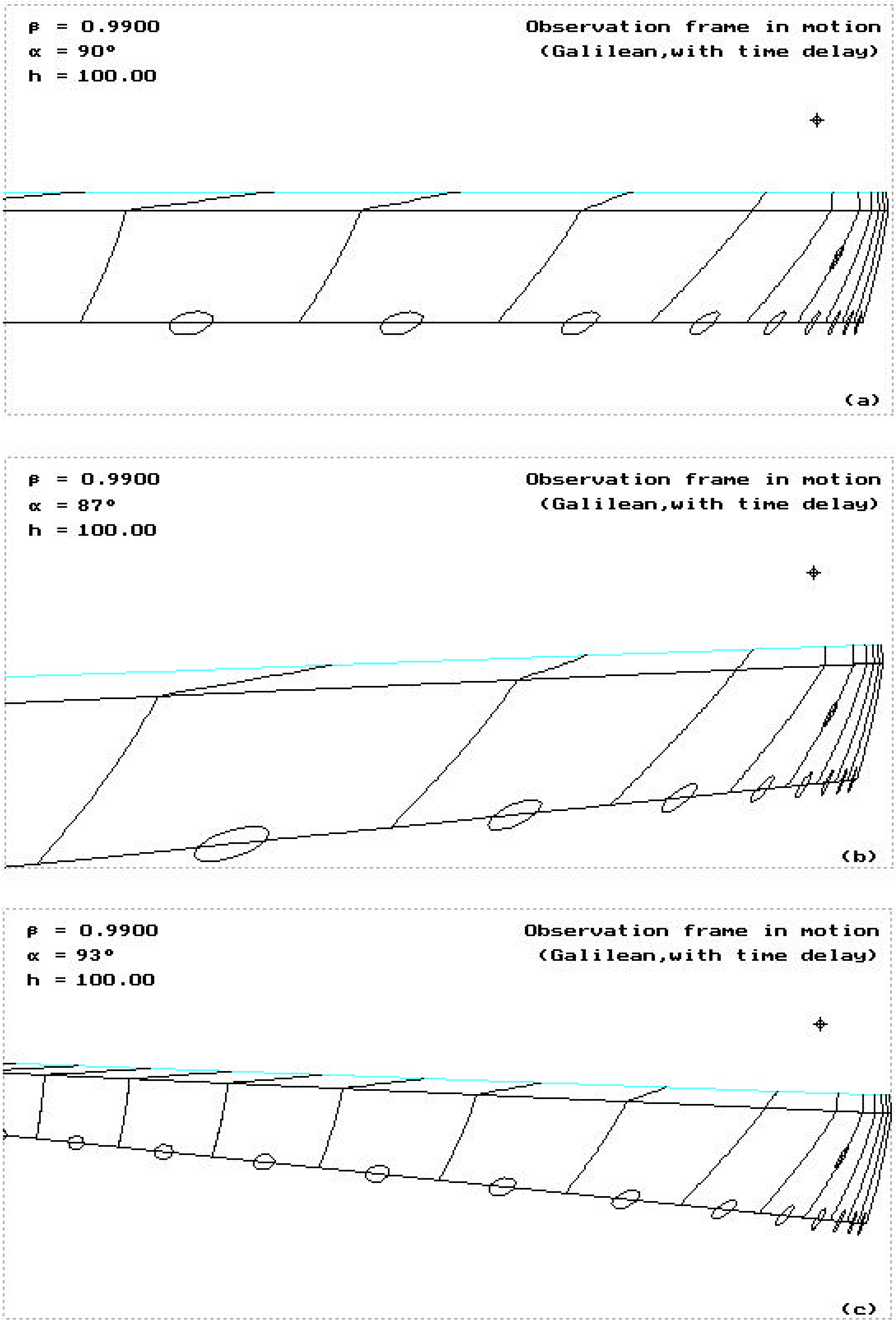


Fig. 4.32. Galilean, with time delay: $\beta = 0.99$. Train for: (a) $\alpha = 90^\circ$. (b) $\alpha = 87^\circ$. (c) $\alpha = 93^o$, where $n = (\cos\alpha, \sin\alpha, 0)$.

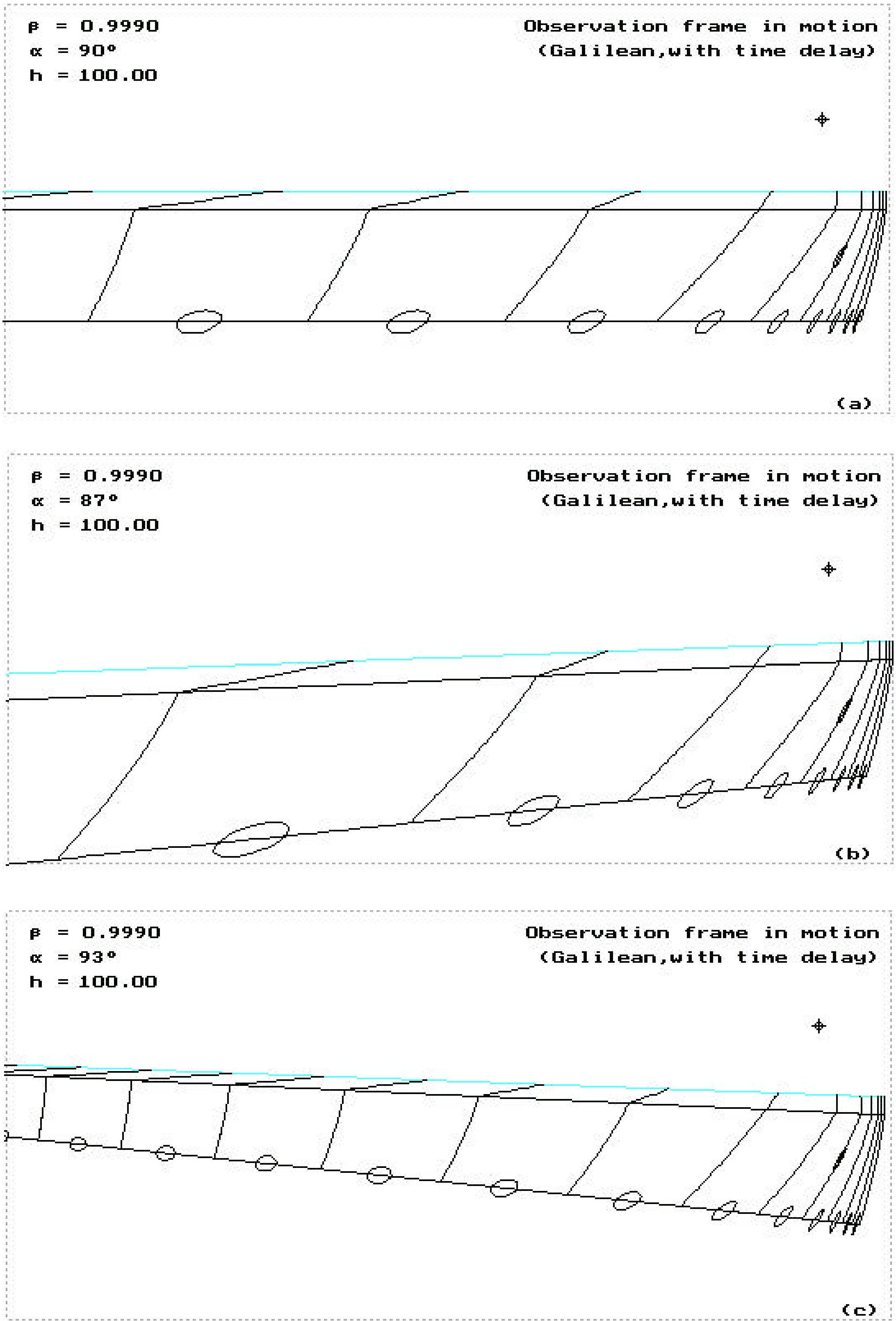


Fig. 4.33. Galilean, with time delay: $\beta = 0.999$. Train for: (a) $\alpha = 90^o$. (b) $\alpha = 87^o$. (c) $\alpha = 93^o$, where $\mathrm{n} = (\cos\alpha, \sin\alpha, 0)$.

4.3.2 Relativistic Treatment

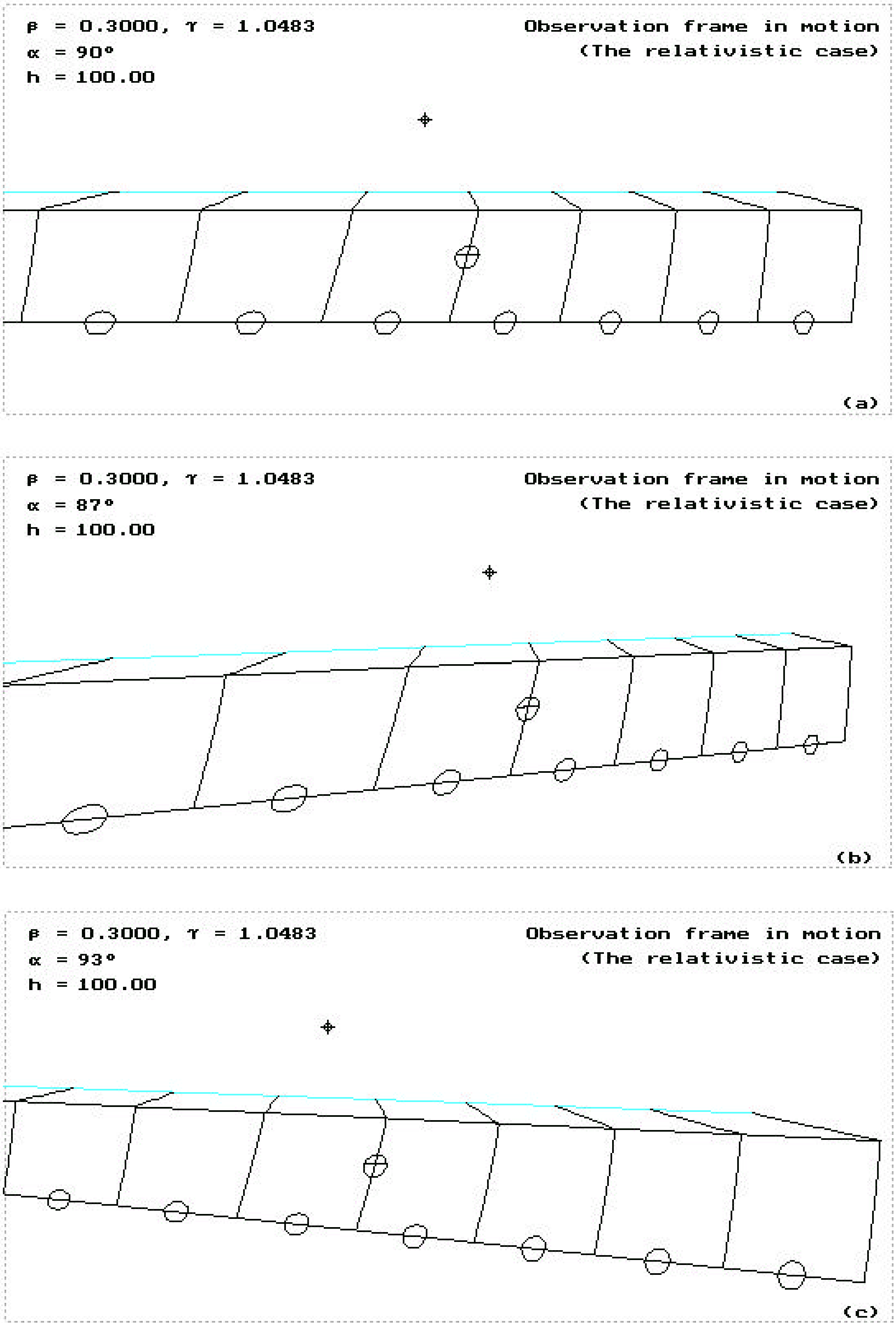


Fig. 4.34. The relativistic case: $\boldsymbol{b} = 0.3$. Train for: (a) $\boldsymbol{a} = 90^o$. (b) $\boldsymbol{a} = 87^o$.
(c) $\boldsymbol{a} = 93^o$, where $\mathrm{n} = (\cos \boldsymbol{a}, \sin \boldsymbol{a}, 0)$.

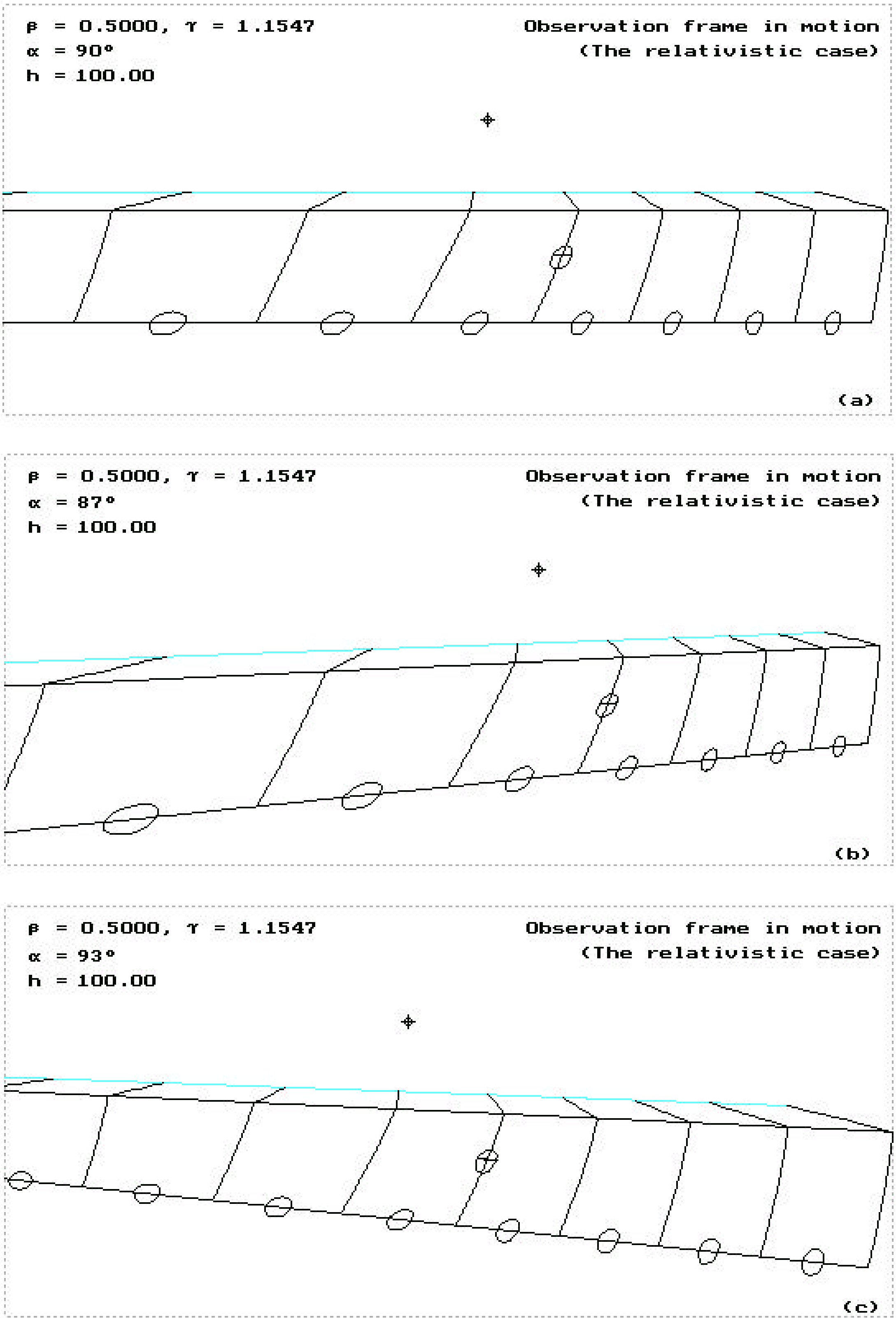


Fig. 4.35. The relativistic case: $\beta = 0.5$. Train for: (a) $\alpha = 90^{o}$. (b) $\alpha = 87^{o}$. (c) $\alpha = 93^{o}$, where $\mathrm{n} = (\cos\alpha, \sin\alpha, 0)$.

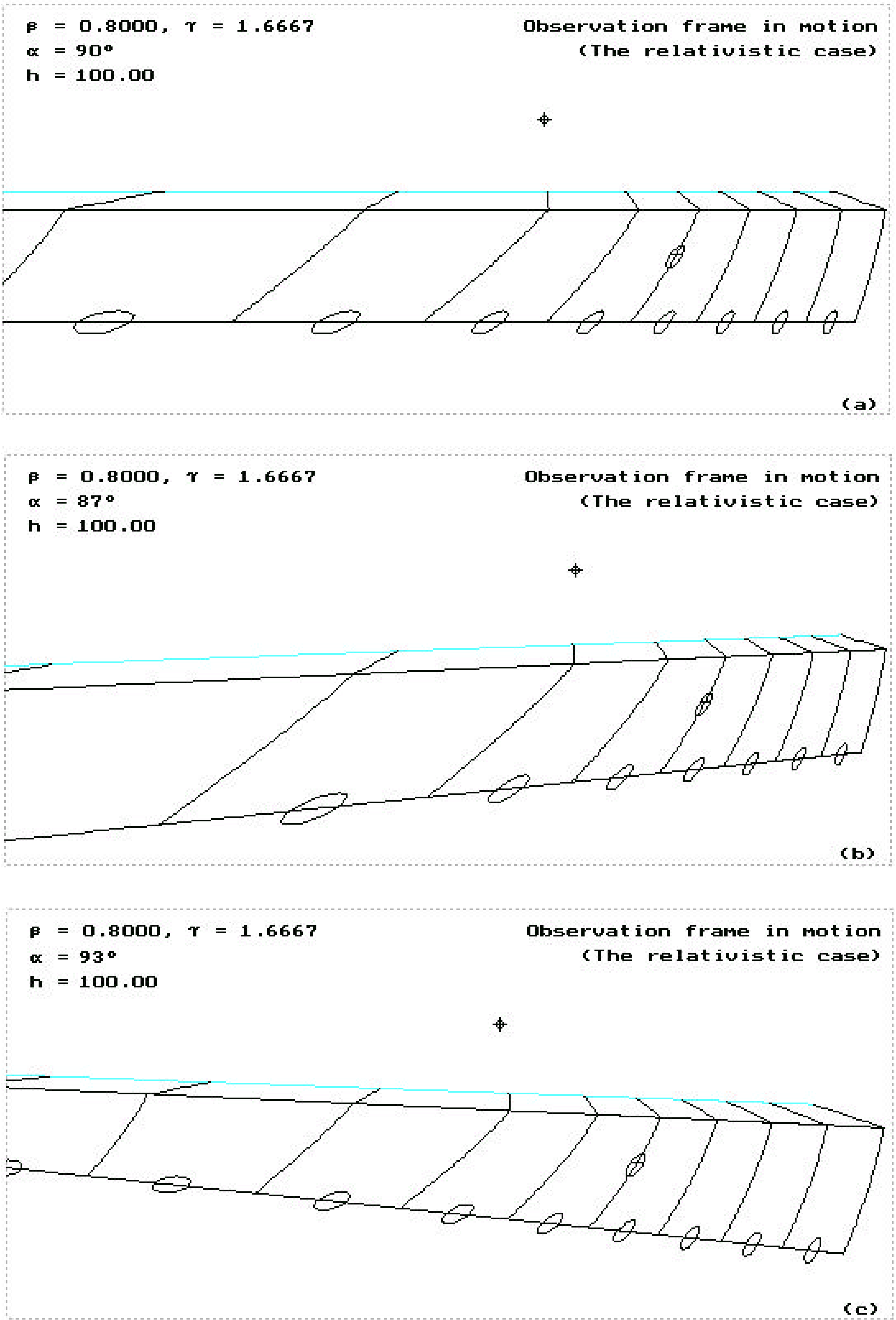


Fig. 4.36. The relativistic case: $\beta = 0.8$. Train for: (a) $\alpha = 90^{o}$. (b) $\alpha = 87^{o}$. (c) $\alpha = 93^{o}$, where $\mathrm{n} = (\cos \alpha, \sin \alpha, 0)$.

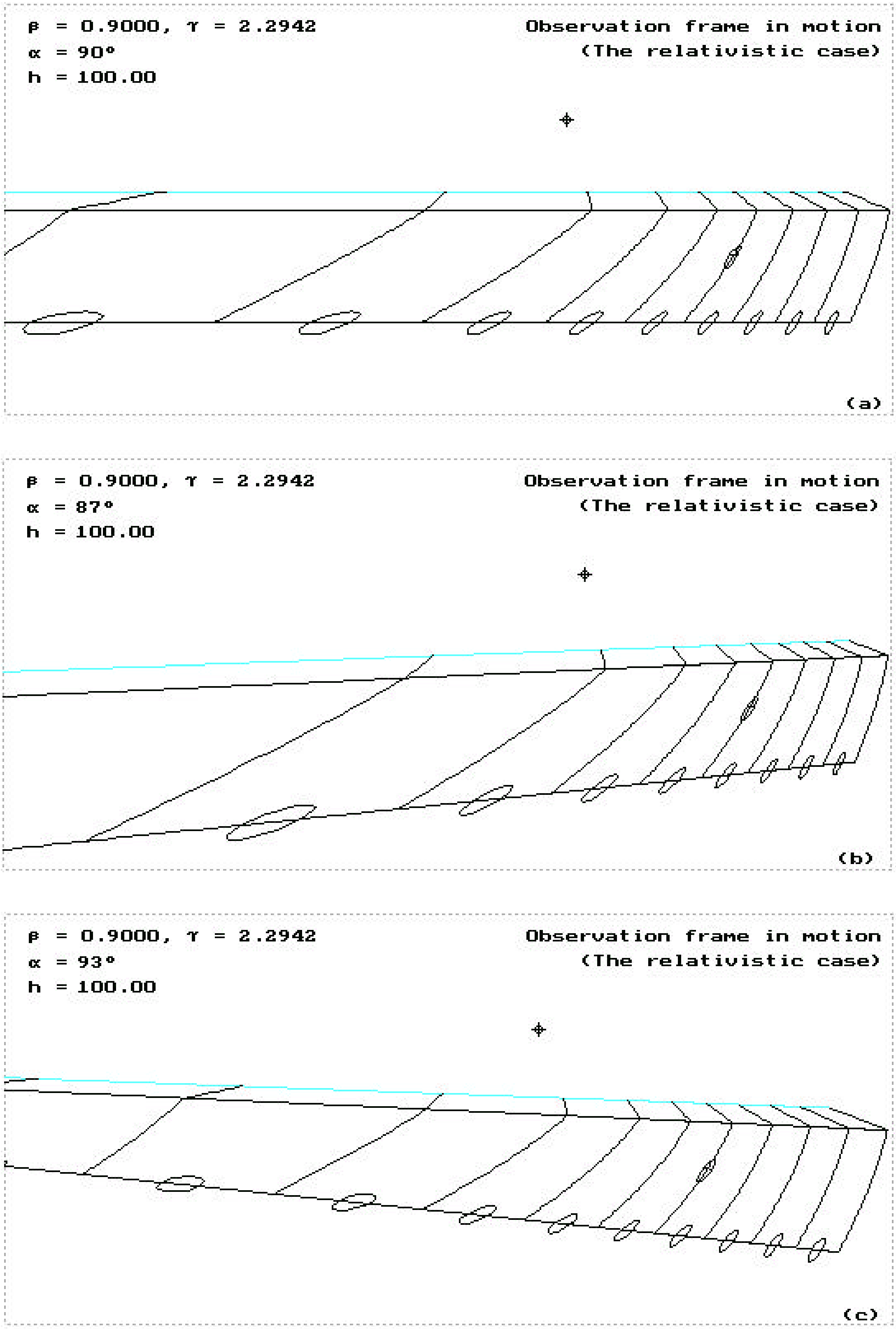


Fig. 4.37. The relativistic case: $\boldsymbol{b} = 0.9$. Train for: (a) $\boldsymbol{a} = 90^{o}$. (b) $\boldsymbol{a} = 87^{o}$. (c) $\boldsymbol{a} = 93^{o}$, where n $= (\cos \boldsymbol{a}, \sin \boldsymbol{a}, 0)$.

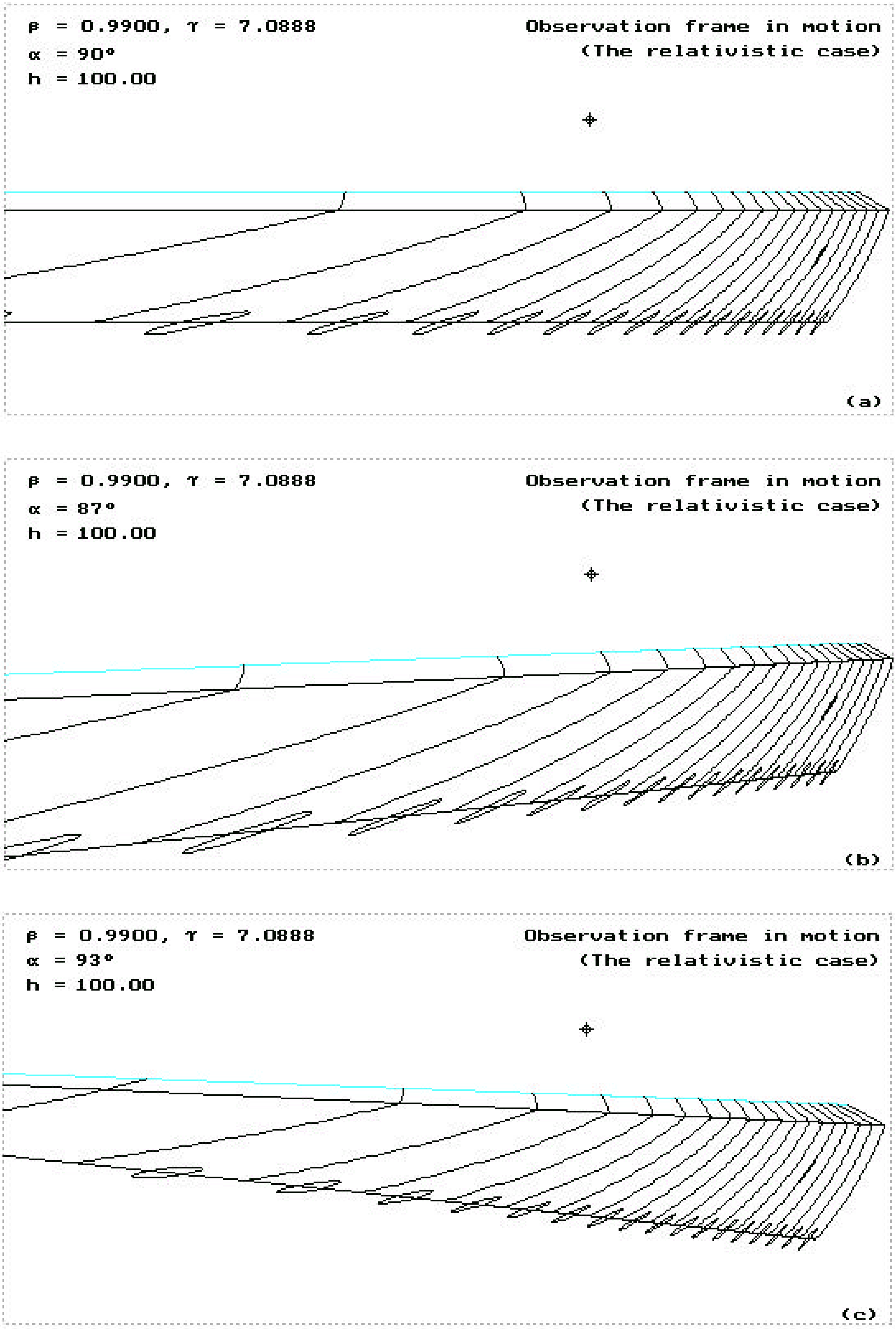


Fig. 4.38. The relativistic case: $\beta = 0.99$. Train for: (a) $\alpha = 90^{o}$. (b) $\alpha = 87^{o}$. (c) $\alpha = 93^{o}$, where $\mathrm{n} = (\cos\alpha, \sin\alpha, 0)$.

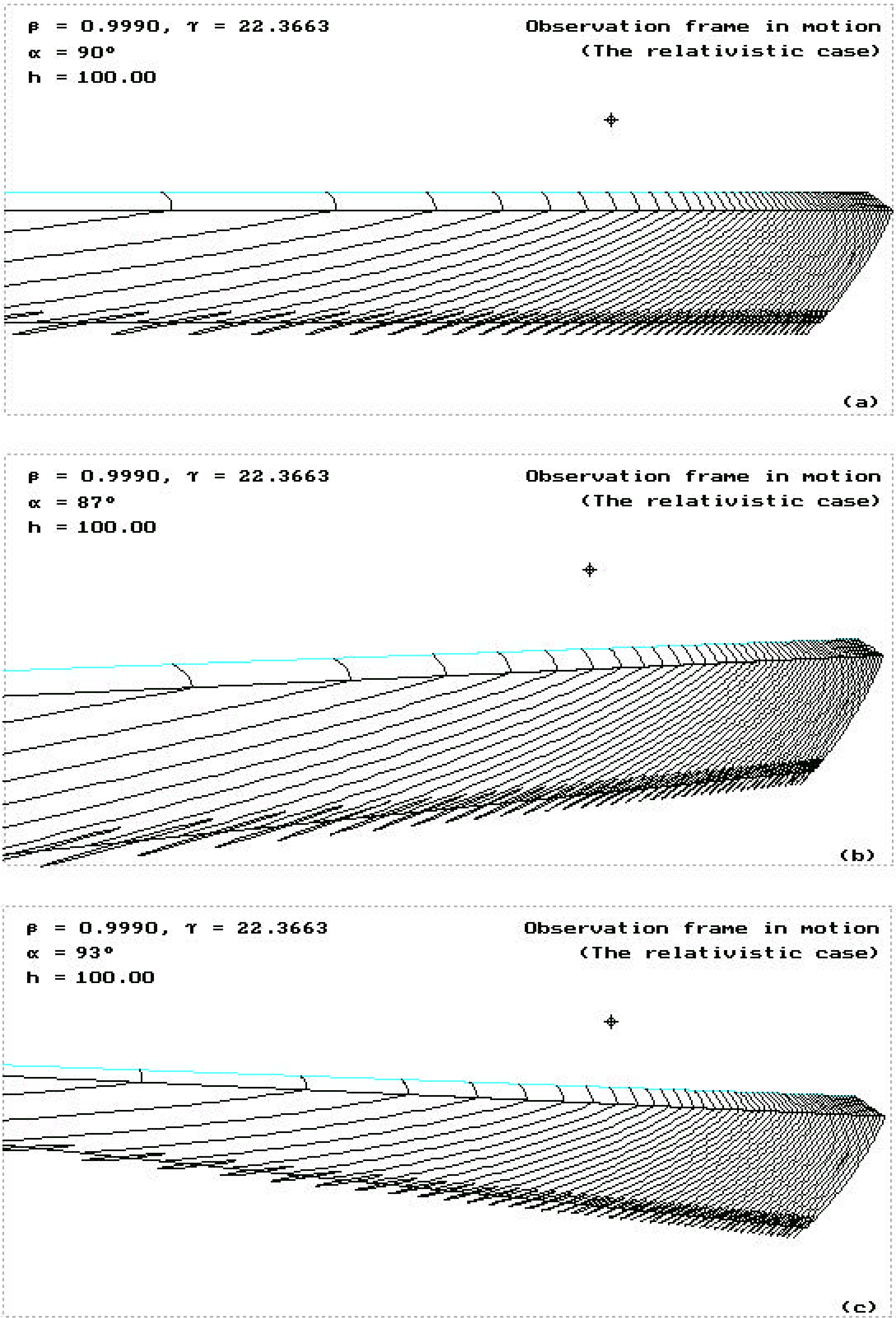


Fig. 4.39. The relativistic case: $\boldsymbol{b} = 0.999$. Train for: (a) $\boldsymbol{a} = 90^o$. (b) $\boldsymbol{a} = 87^o$. (c) $\boldsymbol{a} = 93^o$, where $\mathrm{n} = (\cos \boldsymbol{a}, \sin \boldsymbol{a}, 0)$.

Chapter V
Resolution of the Long Standing "Train" Paradox and Some Pertinent Analytical Properties of the Non – Linear Terrell Transformations

This chapter involves in the resolution of the long standing "train" paradox already mentioned in our introduction. This paradox has its roots in the early work of Terrell (1959) and Weisskopf (1960, 1961), and was emphasized almost thirty years ago by Mathews and Lakshmanan (1972). The "train" paradox is resolved by proving that any point on the object at rest $(\boldsymbol{b} = 0)$, which touches <u>any</u> horizontal line parallel to the *x*-axis (direction of motion) remains in contact with this same line when also the object is in motion $(\boldsymbol{b} \neq 0)$. This chapter is also involved with some pertinent analytical properties of the non-linear Terrell transformations, which give further insight into the applications carried out in Chapter IV and show why some lines are deformed and where the so-called Lorentz contraction is hiding in these figures.

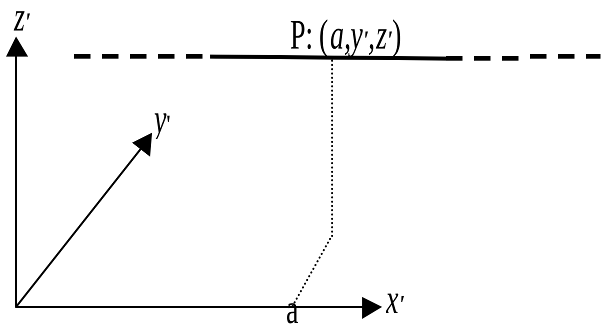


Fig. 5.1. A given horizontal line parallel to the x'-axis. The point (a, y', z') on the object, for $\boldsymbol{b} = 0$, touches the line at P.

5.1 Resolution of the Long Standing "Train" Paradox

To bring out the physics of the resolution of the "train" paradox we consider first the situation of no relative motion of the object and the observation frame: $\boldsymbol{b} = 0$. The situation with $\boldsymbol{b} \neq 0$ is considered afterwards.

According to Fig.5.1 and Eq.(3.49), we can write the equation of U at points (x', y', z') and (a, y', z'), respectively, as

$$U(x', y', z') = d \cdot \frac{x'n_2 - y'n_1}{x'n_1 + y'n_2} \equiv U\,, \tag{5.1}$$

$$U(a, y', z') = d \cdot \frac{an_2 - y'n_1}{an_1 + y'n_2} \equiv U_a\,, \tag{5.2}$$

or

$$U - U_a = d\left[\frac{x'n_2 - y'n_1}{x'n_1 + y'n_2} - \frac{an_2 - y'n_1}{an_1 + y'n_2}\right]$$

$$= d\left[\frac{(x'n_2 - y'n_1)(an_1 + y'n_2) - (x'n_1 + y'n_2)(an_2 - y'n_{1)}}{(x'n_1 + y'n_2)(an_1 + y'n_2)}\right]$$

$$= d\left[\frac{x'y'n_2{}^2 - y'an_1{}^2 + x'y'n_1{}^2 - y'an_2{}^2}{(x'n_1 + y'n_2)(an_1 + y'n_2)}\right]$$

$$= d\left[\frac{x'y' - y'a}{(x'n_1 + y'n_2)(an_1 + y'n_2)}\right]$$

$$= \frac{dy'(x' - a)}{(x'n_1 + y'n_2)(an_1 + y'n_2)}\,. \tag{5.3}$$

According to Eq.(3.50), we can also write the equation for V at points (x', y', z') and (a, y', z'), respectively, as

$$V(x', y', z') = \frac{d(z'-h)}{x'n_1 + y'n_2} \equiv V\,, \tag{5.4}$$

and

$$V(a, y', z') = \frac{d(z'-h)}{an_1 + y'n_2} \equiv V_a\,, \tag{5.5}$$

thus

$$V - V_a = d(z'-h)\left[\frac{1}{x'n_1 + y'n_2} - \frac{1}{an_1 + y'n_2}\right], \tag{5.6}$$

$$= d(z'-h)\left[\frac{an_1 + y'n_2 - x'n_1 - y'n_2}{(x'n_1 + y'n_2)(an_1 + y'n_2)}\right]$$

$$= d(z'-h)\left[\frac{n_1(a - x')}{(x'n_1 + y'n_2)(an_1 + y'n_2)}\right]$$

$$= \frac{-d(z'-h)n_1(x'-a)}{(x'n_1 + y'n_2)(an_1 + y'n_2)}\,. \tag{5.7}$$

Upon comparison of Eq.(5.3) and Eq.(5.7) we see that

$$V - V_a = -\frac{(z'-h)n_1}{y'}(U - U_a), \tag{5.8}$$

or

$$V = -\frac{(z'-h)n_1}{y'}U + V_a + \frac{(z'-h)n_1}{y'}U_a\,. \tag{5.9}$$

Consider the term $V_a + \frac{(z'-h)n_1}{y'}U_a$. From Eqs.(5.2) and (5.5) we get

$$V_a + \frac{(z'-h)n_1}{y'}U_a = \frac{(z'-h)d}{an_1 + y'n_2} + \frac{(z'-h)n_1}{y'}\left(\frac{an_2 - y'n_1}{an_1 + y'n_2}\right)d$$

$$= (z'-h)d\left[\frac{1}{an_1 + y'n_2} + \frac{n_1(an_2 - y'n_1)}{y'(an_1 + y'n_2)}\right]$$

$$= \frac{(z'-h)d}{y'}\left[\frac{y'(an_1 + y'n_2) + n_1(an_2 - y'n_1)(an_1 + y'n_2)}{(an_1 + y'n_2)(an_1 + y'n_2)}\right]$$

$$= \frac{(z'-h)d}{y'}\left[\frac{y' + n_1(an_2 - y'n_1)}{an_1 + y'n_2}\right]$$

$$= \frac{(z'-h)d}{y'}\left[\frac{y'(1 - n_1^2) + an_1n_2}{an_1 + y'n_2}\right]$$

$$= \frac{(z'-h)d}{y'}\left[\frac{y'n_2^2 + an_1n_2}{an_1 + y'n_2}\right]$$

$$= \frac{(z'-h)d}{y'}\left[\frac{n_2(an_1 + y'n_2)}{an_1 + y'n_2}\right].$$

Thus we get

$$V_a + \frac{(z'-h)}{y'}U_a = \frac{n_2(z'-h)d}{y'}. \tag{5.10}$$

Insert Eq.(5.10) into Eq.(5.9), to obtain

$$V = -\frac{n_1(z'-h)}{y'}U + \frac{n_2(z'-h)d}{y'}, \tag{5.11}$$

which is the equation of a <u>straight line</u> in the *UV*-plane with slope $-\frac{n_1(z'-h)}{y'}$.

Now we consider the cases with $b \neq 0$. To this end we evaluate the expression:

$$-\frac{n_1(z'-h)}{y'}U + \frac{n_2(z'-h)d}{y'}, \tag{5.11a}$$

occurring on the right-hand side of Eq.(5.11), with U as given in Eq.(3.49) with $b \neq 0$. According to Eqs.(3.49) and (3.50) we may write U_a and V_a as

$$U_a = \left(\frac{g\left[(a+gbh) - b\sqrt{(a+gbh)^2 + y'^2 + (z'-h)^2}\right] n_2 - y' n_1}{g\left[(a+gbh) - b\sqrt{(a+gbh)^2 + y'^2 + (z'-h)^2}\right] n_1 + y' n_2} \right) d, \tag{5.12}$$

and

$$V_a = \frac{d(z'-h)}{g\left[(a+gbh) - b\sqrt{(a+gbh)^2 + y'^2 + (z'-h)^2}\right] n_1 + y' n_2}, \tag{5.13}$$

set

$$X_a = g\left[(a+gbh) - b\sqrt{(a+gbh)^2 + y'^2 + (z'-h)^2}\right].$$

We may write in short U_a and V_a as:

$$U_a = \frac{X_a n_2 - y' n_1}{X_a n_1 + y' n_2} d, \tag{5.14}$$

and

$$V_a = \frac{d(z'-h)}{X_a n_1 + y' n_2}\,. \tag{5.15}$$

Upon substituting U_a into the expression (5.11a) we get

$$= -\frac{(z'-h)n_1}{y'}U_a + \frac{n_2 d(z'-h)}{y'}$$

$$= -\frac{(z'-h)n_1}{y'}\frac{X_a n_2 - y' n_1}{X_a n_1 + y' n_2}d + \frac{n_2 d(z'-h)}{y'}$$

$$= \frac{d(z'-h)}{y'}\left[\frac{-X_a n_1 n_2 + y' n_1{}^2}{X_a n_1 + y' n_2} + n_2\right]$$

$$= \frac{d(z'-h)}{y'}\left[\frac{-X_a n_1 n_2 + y' n_1{}^2 + X_a n_1 n_2 + y' n_2{}^2}{X_a n_1 + y' n_2}\right]$$

$$= \frac{d(z'-h)\,y'}{y'\left[X_a n_1 + y' n_2\right]}$$

$$= \frac{d(z'-h)}{X_a n_1 + y' n_2}$$

$$= \frac{d(z'-h)}{g\left[(a+gbh) - b\sqrt{(a+gbh)^2 + y'^2 + (z'-h)^2}\right]n_1 + y' n_2}\,. \tag{5.16}$$

Upon comparing Eq.(5.16) with Eq.(5.15) and using the definition of X_a below Eq.(5.13), we conclude that the point (U_a,V_a) lies on the <u>same</u> line in Eq.(5.11) for $b \neq 0$ as well.

For example, refer to Fig.4.1(b) (also Fig.4.1(a)), and consider an imaginary straight line joining the tips of the three roofs as drawn in the observation frame. Now consider the corresponding case for $\beta = 0.8$ in Fig.4.4(b). Here one has the impression that the tip of the roof of the first house on the left has cut through and passed through this line. The proof provided above shows that the tips of the roofs of all the three houses remain always in contact, for $\beta \neq 0$, as well, with the straight line but at different points for $\beta = 0$ and $\beta \neq 0$ due to the relative motion.

5.2 Expansion and Contraction of Approaching and Receding Objects – A Doppler-Like Effect for Scale

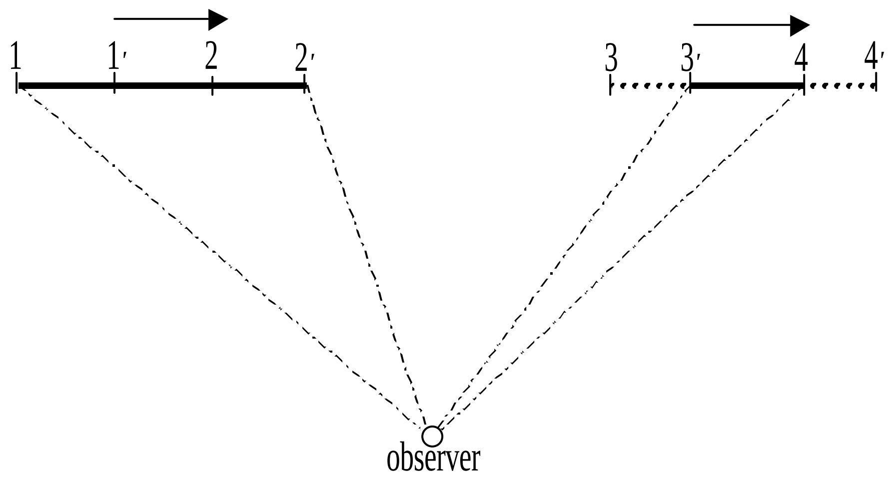


Fig. 5.2. Approaching and receding objects.

Consider two rulers of equal proper lengths, each moving to the right with speed v, with one approaching and one receding from an observer. The end points of the ruler on the left are labeled by 1, 2, and the end points of the one on the right are labeled by 3, 4. Due to the time delay mechanism light must be "emitted" from end point 1 first <u>then</u> from point 2, which in the mean time has moved to point 2′, in order to reach simultaneously the observer. That is, the ruler on the left appears to correspond to the extended object (1,2′) rather than the object (1,2). This phenomenon works against the so-called Lorentz contraction due to relativity. On the other hand, the object on the right appears to correspond to the contracted object (3′,4) rather than to the object (3,4). This works together with the Lorentz contraction. To get the full net contribution, in general, one, however, has to use the exact transformation in Eq. (3.47).

To get further insight into the above Doppler-like effect for scale it is worth reconsidering the transformation rule in Eq.(3.47). Consider an infinitesimal partitioning of a very long ruler of parts equal each in length to $\Delta x'$ in the ruler's proper frame. Consider parts of the moving ruler to the left of and parts to the right of the observer

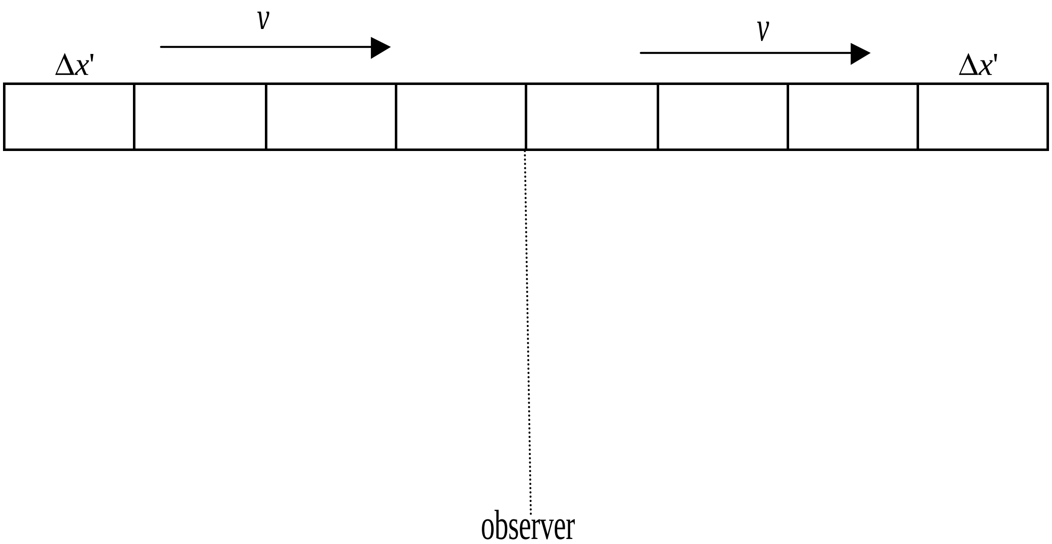


Fig. 5.3. Portions of a ruler approaching to and receding from an observer.

If Δx_L and Δx_R are the infinitesimal portions on the left hand side and the right hand side, respectively, <u>to the observer</u>, then according to Eq.(3.47) we can write for an infinitesimal portion Δx in the observation frame by differentiating that equation to obtain

$$\Delta x = g\Delta x' - \frac{gb \cdot \Delta x'(x'+gbh)}{\sqrt{(x'+gbh)^2 + y'^2 + (z'-h)^2}}. \qquad (5.17)$$

The latter may be rewritten as

$$\Delta x = \frac{\Delta x'}{g} + \left(g - \frac{1}{g}\right)\Delta x' - \frac{gb \cdot \Delta x'(x'+gbh)}{\sqrt{(x'+gbh)^2 + y'^2 + (z'-h)^2}}. \qquad (5.18)$$

Using the identity $g - \frac{1}{g} = gb^2$, Eq.(5.18) can be rewritten as

$$\Delta x = \frac{\Delta x'}{g} + gb^2 \Delta x' - \frac{gb \cdot \Delta x'(x'+gbh)}{\sqrt{(x'+gbh)^2 + y'^2 + (z'-h)^2}}, \qquad (5.19)$$

$$\Delta x = \frac{\Delta x'}{g} + gb\Delta x'\left(b - \frac{\Delta x'(x'+gbh)}{\sqrt{(x'+gbh)^2 + y'^2 + (z'-h)^2}}\right), \qquad (5.20)$$

$$\Delta x = \Delta x'\left\{\frac{1}{g} + gb\left(b - \frac{x'+gbh}{\sqrt{(x'+gbh)^2 + y'^2 + (z'-h)^2}}\right)\right\}. \qquad (5.21)$$

We see in Eq.(5.20) that the first term is the Lorentz contraction and the last term is a term which incorporates time delay. Next we can write the expressions for Δx_L and Δx_R which obviously depend on the sign of the x' where the sign will be negative for x' on the left-hand side and positive for x' the right-hand side of the observer:

$$\Delta x_L = \frac{\Delta x'}{g} + gb\left(\frac{(|x'| - gbh)}{\sqrt{(|x'| - gbh)^2 + y'^2 + (z'-h)^2}} + b\right)\Delta x', \qquad (5.22)$$

and

$$\Delta x_R = \frac{\Delta x'}{\gamma} - \gamma\beta\left(\frac{(|x'| + \gamma\beta h)}{\sqrt{(|x'| + \gamma\beta h)^2 + y'^2 + (z' - h)^2}} - \beta\right)\Delta x' . \qquad (5.23)$$

We now consider two extreme points of the very long ruler, that is, for which $|x'|$ is very large. In this case Eqs.(5.22) and (5.23) may be, respectively, rewritten approximately as

$$\Delta x_L \doteq \frac{\Delta x'}{\gamma} + \Delta x' \beta \sqrt{\frac{1+\beta}{1-\beta}} , \qquad (5.24)$$

$$\Delta x_R \doteq \frac{\Delta x'}{\gamma} - \Delta x' \beta \sqrt{\frac{1-\beta}{1+\beta}} . \qquad (5.25)$$

Eq.(5.24) clearly shows how an additional expansion occurs to the left which works against the Lorentz contraction. Similarly, Eq.(5.25) clearly shows how an additional contraction occurs that works with the Lorentz contraction. In the latter case it is worth noting that we may rewrite

$$\Delta x_R \doteq \frac{\Delta x'}{\gamma}[1 - \beta(1-\beta)] > 0 , \qquad (5.26)$$

for $\beta < 1$. We also note, in particular, that for the extreme points

$$\Delta x_L - \Delta x_R = 2\Delta x' \beta\gamma > 0 . \qquad (5.27)$$

For other points of $|x'|$ one has to rely on the exact expressions Eqs.(5.22) and (5.23).

A beautiful demonstration of this Doppler-like effect for scale is given in the application to the train compartments (c.f., Fig.4.38).

The old fundamental and critical question now comes to haunt us: Can we photograph the Lorentz contraction? To answer this question explicitly we set the observer at the origin O of his coordinate frame, i.e., set $h = 0$, and consider a ruler moving to the right of the *x*-axis with speed such that at its bottom edge $z'=0$ and y' is arbitrary but fixed. To this end, Eq.(3.55) gives to the corresponding *U*-values of its bottom edge:

$$\frac{U}{d}=\frac{\boldsymbol{g}\left[x'-\boldsymbol{b}\sqrt{x'^2+y'^2}\right]n_2-y'n_1}{\boldsymbol{g}\left[x'-\boldsymbol{b}\sqrt{x'^2+y'^2}\right]n_1+y'n_2}. \tag{5.28}$$

(Also $V=0$). Accordingly,

$$\frac{\Delta U}{d}=\frac{\boldsymbol{g}\left[\Delta x'-\boldsymbol{b}\frac{x'\Delta x'}{\sqrt{x'^2+y'^2}}\right]n_2}{\boldsymbol{g}\left[x'-\boldsymbol{b}\sqrt{x'^2+y'^2}\right]n_1+y'n_2}$$

$$-\frac{\boldsymbol{g}\left[x'-\boldsymbol{b}\sqrt{x'^2+y'^2}\right]n_2-y'n_1}{\left(\boldsymbol{g}\left[x'-\boldsymbol{b}\sqrt{x'^2+y'^2}\right]n_1+y'n_2\right)^2}\boldsymbol{g}\left(\Delta x'-\boldsymbol{b}\frac{x'\Delta x'}{\sqrt{x'^2+y'^2}}\right)n_1, \tag{5.29}$$

$$\frac{\Delta U}{d}=\frac{\boldsymbol{g}\left[\Delta x'-\boldsymbol{b}\frac{x'\Delta x'}{\sqrt{x'^2+y'^2}}\right]}{\left(\boldsymbol{g}\left[x'-\boldsymbol{b}\sqrt{x'^2+y'^2}\right]n_1+y'n_2\right)^2}\Big[n_2\left[\boldsymbol{g}\left(x'-\boldsymbol{b}\sqrt{x'^2+y'^2}\right)n_1+y'n_2\right]$$

$$-n_1\left[\boldsymbol{g}\left(x'-\boldsymbol{b}\sqrt{x'^2+y'^2}\right)n_2-y'n_1\right]\Big], \tag{5.30}$$

$$\frac{\Delta U}{d}=\frac{y'\boldsymbol{g}\left[1-\boldsymbol{b}\frac{x'}{\sqrt{x'^2+y'^2}}\right]\Delta x'}{\left(\boldsymbol{g}\left[x'-\boldsymbol{b}\sqrt{x'^2+y'^2}\right]n_1+y'n_2\right)^2}, \tag{5.31}$$

For $n_1 = 0, \quad n_2 = 1$

$$\frac{\Delta U}{d} = \frac{g}{y'}\left[1 - b\frac{x'}{\sqrt{x'^2 + y'^2}}\right]\Delta x', \tag{5.32}$$

$$\frac{\Delta U}{d} = \frac{1}{y'}\left\{\left[g - \frac{1}{g} + \frac{1}{g}\right]\Delta x' - \frac{\Delta x' g b x'}{\sqrt{x'^2 + y'^2}}\right\}, \tag{5.33}$$

$$\frac{\Delta U}{d} = \frac{1}{y'}\left\{\left[\frac{g^2 - 1}{g} + \frac{1}{g}\right]\Delta x' - \frac{\Delta x' g b x'}{\sqrt{x'^2 + y'^2}}\right\}, \tag{5.34}$$

$$\frac{\Delta U}{d} = \frac{1}{y'}\left\{\left[\frac{g^2 b^2}{g} + \frac{1}{g}\right]\Delta x' - \frac{\Delta x' g b x'}{\sqrt{x'^2 + y'^2}}\right\}, \tag{5.35}$$

$$\frac{\Delta U}{d} = \frac{1}{y'}\left[\frac{\Delta x'}{g} + \Delta x' g b^2 - \frac{\Delta x' g b x'}{\sqrt{x'^2 + y'^2}}\right], \tag{5.36}$$

or

$$y'\frac{\Delta U}{d} = \frac{\Delta x'}{g} + g b\left(b - \frac{x'}{\sqrt{x'^2 + y'^2}}\right)\Delta x'. \tag{5.37}$$

Since $\left|\frac{x'}{\sqrt{x'^2 + y'^2}}\right| < 1$, we infer that about the point $\frac{x'}{\sqrt{x'^2 + y'^2}} = b$, for a given b,

$$\Delta U = \frac{d}{y'}\frac{\Delta x'}{g}, \tag{5.38}$$

which, apart from the trivial scaling factor d/y', is the famous Lorentz contraction. That is, about a certain point of the object the Lorentz contraction is visible on our *UV*-plane.

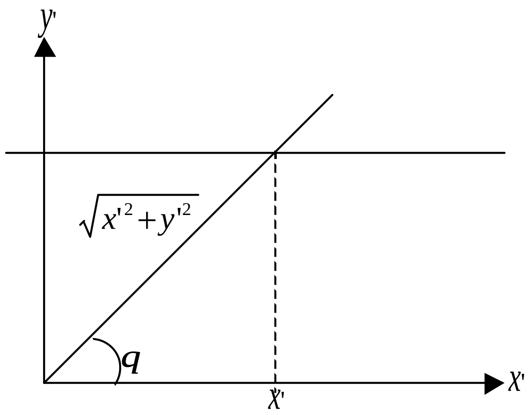


Fig. 5.4. Line making an angle $\boldsymbol{q} = \cos^{-1}(\boldsymbol{b})$ with the x'-axis in the proper frame.

To find this critical point on the ruler, at which point the Lorentz contraction is visible, we draw a line making an angle $\boldsymbol{q} = \cos^{-1}(\boldsymbol{b})$ with the x'-axis before setting the ruler to move with a given speed $\boldsymbol{b}$ (see Fig.5.4). This line will cross a point on the lower side of the ruler and defines this critical point. By partitioning the ruler with small intervals of lengths $\Delta x'$, Eq.(5.38) shows that the Lorentz contraction is visible about the critical point upon comparison with the stationary case

$$\Delta U = \frac{d}{y'}\Delta x', \quad \text{for} \quad \boldsymbol{b} = 0 . \qquad (5.39)$$

This also provides a test for the comparison of the Galilean case to the relativistic. The Galilean case coincides with that of Eq.(5.39) even for $\boldsymbol{b} \neq 0$ since $\boldsymbol{g}$ is effectively set equal to one in Eq.(5.38).

5.3 The Curving-Up of Lines Perpendicular to the Direction of Motion

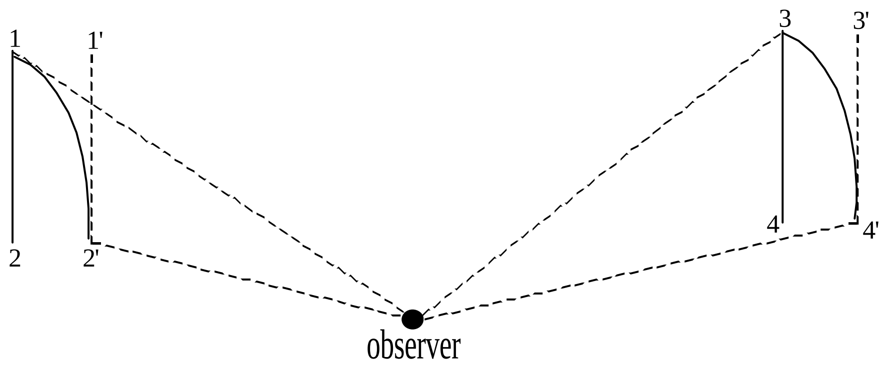


Fig. 5.5. Lines perpendicular to the direction of motion.

As a consequence of time-delay resulting from the fact that light has to be "emitted" from end point 1 before the end point 2, a line perpendicular to the direction of motion necessarily appears curved with end points 1, $2'$ rather than 1, 2. The same analysis applies to the line with end points 3, 4 on the right of the observer. This curving up of lines perpendicular to the direction of motion is well illustrated in our applications carried out in Chapter IV. Compare, for example, the illustrations in Fig.4.10 with the corresponding ones in Fig.4.1. It is precisely because point 1 appears ahead of point 2 for any two lines perpendicular to the x' axis along the y' and z' axes, that an object, due to its relative motion and time delay, appears to be rotated (with deformations) about the latter two axes and hence the "train" paradox.

Now we discuss in detail the curving up of these lines analytically. According to Eqs.(3.49) and (3.50), with $n_1 = 0, \quad n_2 = 1$, we may write

$$\frac{U}{d} = \frac{\gamma\left[(x'+\gamma\beta h) - \beta\sqrt{(x'+\gamma\beta h)^2 + y'^2 + (z'-h)^2}\right]}{y'}, \qquad (5.40)$$

and

$$\frac{V}{d}=\frac{[z'-h]}{y'}. \tag{5.41}$$

For a line parallel to the z'-axis and hence, in particular, perpendicular to the x'-axis (direction of motion) we have that x' and y' are some constants, and z' is a variable. Thus from Eq.(5.40) we have

$$\left(\frac{U}{d}-\frac{g(x'+gbh)}{y'}\right)^2=\frac{g^2b^2}{y'^2}\left[(x'+gbh)^2+y'^2+(z'-h)^2\right]. \tag{5.42}$$

Upon inserting Eq.(5.41) into Eq.(5.42) we get

$$\left(\frac{U}{d}-\frac{g(x'+gbh)}{y'}\right)^2=\frac{g^2b^2\left[(x'+gbh)^2+y'^2\right]}{y'^2}+\frac{g^2b^2V^2}{d^2}. \tag{5.43}$$

We rewrite Eq.(5.43) as

$$\frac{\left(\frac{U}{d}-\frac{g(x'+gbh)}{y'}\right)^2}{\frac{g^2b^2\left[(x'+gbh)^2+y'^2\right]}{y'^2}}-\frac{V^2}{d^2\left[\frac{\left[(x'+gbh)^2+y'^2\right]}{y'^2}\right]}=1,$$

$$\frac{\left(U-\frac{dg(x'+gbh)}{y'}\right)^2}{\frac{d^2g^2b^2\left[(x'+gbh)^2+y'^2\right]}{y'^2}}-\frac{V^2}{d^2\left[\frac{\left[(x'+gbh)^2+y'^2\right]}{y'^2}\right]}=1,$$

or

$$\frac{(U-U_0)^2}{a^2}-\frac{V^2}{b^2}=1, \tag{5.44}$$

where

$$U_0=\frac{dg(x'+gbh)}{y'},\quad a^2=\frac{d^2g^2b^2\left[(x'+gbh)^2+y'^2\right]}{y'^2},\quad b^2=d^2\left[\frac{\left[(x'+gbh)^2+y'^2\right]}{y'^2}\right].$$

Eq.(5.44) is the equation of a hyperbola with the focal points lying on the horizontal *U*-axis. Thus we conclude that a line parallel to the z'-axis and hence perpendicular to the direction of motion becomes a portion of a hyperbola in the *U*-*V* plane as shown in Eq.(5.44)

Now we consider the other case of a line perpendicular parallel to the z'-axis and hence, as before, perpendicular to the direction of motion (the x'-axis). In this case x' and z' are some constants and y' is a variable. From Eq.(5.41) we may solve for y' as follows:

$$y' = \frac{[z' - h]}{V} d \,. \tag{5.45}$$

Upon inserting Eq.(5.45) into Eq.(5.40) to obtain

$$U = \frac{V}{(z' - h)} \left[g(x' + gbh) - gb\sqrt{(x' + gbh)^2 + \frac{(z' - h)^2 d^2}{V^2} + (z' - h)^2} \right], \tag{5.46}$$

$$U = \frac{g(x' + gbh)V}{(z' - h)} - \frac{gb}{(z' - h)}\sqrt{V^2 (x' + gbh)^2 + (z' - h)^2 d^2 + V^2 (z' - h)^2} \,, \tag{5.47}$$

$$U = \frac{g(x' + gbh)V}{(z' - h)} - \frac{gb}{(z' - h)}\sqrt{V^2 \left[(x' + gbh)^2 + (z' - h)^2\right] + (z' - h)^2 d^2} \,, \tag{5.48}$$

or

$$U = AV - \sqrt{C^2 V^2 + D^2} \,, \tag{5.49}$$

where $A = \frac{g(x' + gbh)}{(z' - h)}$, $C^2 = \frac{g^2 b^2}{(z' - h)^2}\left[(x' + gbh)^2 - (z' - h)^2\right]$, $D^2 = g^2 b^2 d^2$.

To study the analytical structure of Eq.(5.49), we consider a rotation of the U,V axes by some angle α. That is, we write

$$U = U'\cos\alpha + V'\sin\alpha\,, \tag{5.50}$$

$$V = -U'\sin\alpha + V'\cos\alpha\,, \tag{5.51}$$

Insert Eqs.(5.50) and (5.51) into Eq.(5.49) and solve for α self consistently to obtain

$$U'\cos\alpha + V'\sin\alpha = A[-U'\sin\alpha + V'\cos\alpha]$$

$$-\sqrt{C^2(U'^2\sin^2\alpha + V'^2\cos^2\alpha - 2U'V'\sin\alpha\cos\alpha) + D^2}\,, \tag{5.52}$$

$$U'[\cos\alpha + A\sin\alpha] + V'[\sin\alpha - A\cos\alpha] =$$

$$-\sqrt{C^2(U'^2\sin^2\alpha + V'^2\cos^2\alpha - 2U'V'\sin\alpha\cos\alpha) + D^2}\,, \tag{5.53}$$

Upon squaring the above equation we have

$$U'^2[\cos\alpha + A\sin\alpha]^2 + V'^2[\sin\alpha - A\cos\alpha]^2 + 2U'V'[\cos\alpha + A\sin\alpha][\sin\alpha - A\cos\alpha]$$

$$= C^2(U'^2\sin^2\alpha + V'^2\cos^2\alpha - 2U'V'\sin\alpha\cos\alpha) + D^2\,, \tag{5.54}$$

$$U'^2\left[(\cos\alpha + A\sin\alpha)^2 - C^2\sin^2\alpha\right] + V'^2\left[(\sin\alpha - A\cos\alpha)^2 - C^2\cos^2\alpha\right]$$

$$+ 2U'V'\left[(\cos\alpha + A\sin\alpha)(\sin\alpha - A\cos\alpha) + C^2\sin\alpha\cos\alpha\right] = D^2\,, \tag{5.55}$$

To solve for a, we set the coefficient of the $U'V'$ term equal to zero and obtain a conic section:

$$U'^2 S_1 + V'^2 S_2 = D^2, \qquad (5.56)$$

where

$$S_1 = (\cos a + A\sin a)^2 - C^2 \sin^2 a \quad \text{and} \quad S_2 = (\sin a - A\cos a)^2 - C^2 \cos^2 a,$$

and for the coefficient of the $U'V'$ term we have

$$(\cos a + A\sin a)(\sin a - A\cos a) + C^2 \sin a \cos a = 0. \qquad (5.57)$$

Solving for the angle a we get

$$\cos a \sin a - A\cos^2 a + A\sin^2 a - A^2 \sin a \cos a + C^2 \sin a \cos a = 0$$

$$\cos a \sin a(1 - A^2 + C^2) - A(\cos^2 a - \sin^2 a) = 0$$

$$\frac{\sin 2a}{2}(1 - A^2 + C^2) - A\cos 2a = 0$$

$$\frac{\sin 2a}{2}(1 - A^2 + C^2) = A\cos 2a$$

$$\frac{\sin 2a}{\cos 2a} = \frac{2A}{1 - A^2 + C^2},$$

or

$$\tan 2a = \frac{2A}{1 - A^2 + C^2}. \qquad (5.58)$$

Now we use Eq.(5.58) to simplify the expressions for S_1 and S_2 in Eq.(5.56). To this end

$$S_1 = \cos^2 a + A^2 \sin^2 a + 2A\sin a \cos a - C^2 \sin^2 a, \qquad (5.59)$$

or

$$S_1 = \cos^2 a + (A^2 - C^2)\sin^2 a + 2A\sin a \cos a. \qquad (5.60)$$

On the other hand, from Eq.(5.58) we obtain $A^2 - C^2 = 1 - \frac{2A}{\tan 2\alpha}$.
Thus Eq.(5.58) can be rewritten as

$$
\begin{aligned}
S_1 &= \cos^2\alpha + \left[1 - \frac{2A}{\tan 2\alpha}\right]\sin^2\alpha + 2A\sin\alpha\cos\alpha \\
&= 1 - 2A\left[\frac{\sin^2\alpha}{\tan 2\alpha} - \sin\alpha\cos\alpha\right] \\
&= 1 - 2A\sin\alpha\left[\frac{\sin\alpha\cos 2\alpha}{\sin 2\alpha} - \cos\alpha\right] \\
&= 1 - 2A\sin\alpha\left[\frac{\sin\alpha(\cos^2\alpha - \sin^2\alpha)}{2\sin\alpha\cos\alpha} - \cos\alpha\right] \\
&= 1 - 2A\sin\alpha\left[\frac{\cos^2\alpha - \sin^2\alpha - 2\cos^2\alpha}{2\cos\alpha}\right] \\
&= 1 - 2A\sin\alpha\left[\frac{-\cos^2\alpha - \sin^2\alpha}{2\cos\alpha}\right] \\
&= 1 + \frac{A\sin\alpha}{\cos\alpha}.
\end{aligned}
$$

Hence

$$S_1 = 1 + A\tan\alpha. \tag{5.61}$$

Also

$$
\begin{aligned}
S_2 &= \sin^2\alpha + A^2\cos^2\alpha - 2A\sin\alpha\cos\alpha - C^2\cos^2\alpha \\
&= \sin^2\alpha + (A^2 - C^2)\cos^2\alpha - 2A\sin\alpha\cos\alpha \\
&= \sin^2\alpha + \left[1 - \frac{2A}{\tan 2\alpha}\right]\cos^2\alpha - 2A\sin\alpha\cos\alpha \\
&= 1 - 2A\left[\frac{\cos^2\alpha}{\tan 2\alpha} + \sin\alpha\cos\alpha\right] \\
&= 1 - 2A\cos\alpha\left[\frac{\cos\alpha(\cos^2\alpha - \sin^2\alpha)}{2\sin\alpha\cos\alpha} + \sin\alpha\right] \\
&= 1 - 2A\cos\alpha\left[\frac{\cos^2\alpha - \sin^2\alpha}{2\sin\alpha} + \sin\alpha\right] \\
&= 1 - 2A\cos\alpha\left[\frac{\cos^2\alpha - \sin^2\alpha + 2\sin^2\alpha}{2\sin\alpha}\right] \\
&= 1 - \frac{2A\cos\alpha}{2\sin\alpha},
\end{aligned}
$$

$$S_2 = 1 - A\cot \alpha \,. \tag{5.62}$$

Finally Eq.(5.56) can be written as

$$U'^2 (1 + A\tan \alpha) + V'^2 (1 - A\cot \alpha) = D^2 \,. \tag{5.63}$$

If

$$1 + A\tan \alpha < 0 \,,$$

that is,

$$A\tan \alpha < -1 ,$$

$$-A\tan \alpha > 1 ,$$

then

$$-A\cot \alpha > 0 \,,$$

and we have

$$1 - A\cot \alpha > 0 \,,$$

and Eq.(5.63) specifies the equation of a hyperbola. On the other hand if

$$1 + A\tan \alpha > 0 \,,$$

$$A\tan \alpha > -1 ,$$

$$-A\tan \alpha > 1 ,$$

then

$$A\cot \alpha > \frac{1}{\tan^2 \alpha} \,,$$

$$-A\cot \alpha < \frac{1}{\tan^2 \alpha} \,,$$

$$1 - A\cot \boldsymbol{a} < 1 + \frac{1}{\tan^2 \boldsymbol{a}},$$

and we obtain quite generally

$$1 - A\cot \boldsymbol{a} < \frac{1}{\sin^2 \boldsymbol{a}}.$$

Accordingly, Eq.(5.63) will specify portions of a rotated hyperbola or a rotated ellipse as the case may be.

5.4 Critical Speeds for Expansions Versus Contractions

By comparing Figs.4.8 - 4.13 we infer that some critical speed occurs below which expansion occurs in the direction of motion and above which the situation is reversed and contraction occurs. These figures seem also to indicate that such a critical speed occurs as a common critical speed, simultaneously, for both *U* and *V*. That is

$$\left.\frac{dU}{d\boldsymbol{b}}\right|_{b=b_{critical}} = 0, \qquad (5.64)$$

implies that

$$\left.\frac{dV}{d\boldsymbol{b}}\right|_{b=b_{critical}} = 0, \qquad (5.65)$$

for the same critical value of $\boldsymbol{b} = \boldsymbol{b}_{critical}$. To prove this we carry out the derivatives $dU/d\boldsymbol{b}$, $dV/d\boldsymbol{b}$ explicitly.

$$U = \left(\frac{g\left[(x'+g\boldsymbol{b}h) - \boldsymbol{b}\sqrt{(x'+g\boldsymbol{b}h)^2 + y'^2 + (z'-h)^2}\right] n_2 - y'n_1}{g\left[(x'+g\boldsymbol{b}h) - \boldsymbol{b}\sqrt{(x'+g\boldsymbol{b}h)^2 + y'^2 + (z'-h)^2}\right] n_1 + y'n_2} \right) d\,. \qquad (5.66)$$

Set $F = g\left[(x'+g\boldsymbol{b}h) - \boldsymbol{b}\sqrt{(x'+g\boldsymbol{b}h)^2 + y'^2 + (z'-h)^2}\,\right].$

Then we may rewrite Eq.(5.66) as

$$U = \frac{Fn_2 - y'n_1}{Fn_1 + y'n_2}d\,, \tag{5.67}$$

$$\frac{dU}{d\boldsymbol{b}} = d\,\frac{(Fn_1 + y'n_2)n_2\dfrac{dF}{d\boldsymbol{b}} - (Fn_2 - y'n_1)n_1\dfrac{dF}{d\boldsymbol{b}}}{(Fn_1 + y'n_2)^2}\,, \tag{5.68}$$

$$\frac{dU}{d\boldsymbol{b}} = \frac{y'd}{(Fn_1 + y'n_2)^2}\frac{dF}{d\boldsymbol{b}}. \tag{5.69}$$

Similarly, we may rewrite

$$V = \frac{d(z'-h)}{g\left[(x'+g\boldsymbol{b}h) - \boldsymbol{b}\sqrt{(x'+g\boldsymbol{b}h)^2 + y'^2 + (z'-h)^2}\,\right]n_1 + y'n_2}\,, \tag{5.70}$$

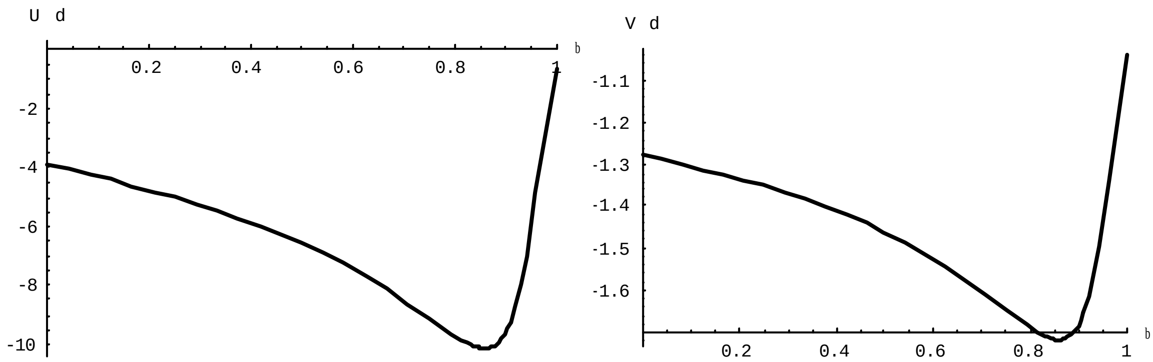
$$V = \frac{(z'-h)d}{Fn_1 + y'n_2}\,, \tag{5.71}$$

Then

$$\frac{dV}{d\boldsymbol{b}} = \frac{-d(z'-h)n_1}{(Fn_1 + y'n_2)^2}\frac{dF}{d\boldsymbol{b}}\,, \tag{5.72}$$

which establishes the statements given through Eqs.(5.64) and (5.65).

For example consider the lowest point (-150,50,0) on the left-hand side of the house on the left in Fig.4.1(b). A plot of *U/d* and *V/d* versus b are given, respectively, in Fig.5.5(a) and Fig.5.5(b).



(a) (b)

Fig. 5.6. Critical speed at point (-150,50,0) is equal to 0.857493. (a) Graph of *U/d* versus b. (b) Graph of *V/d* versus b.

Chapter VI
Conclusions

By taking into account the following three points:

(i) Terrell's basic observation that different points on an object must "emit" light at different times in order to reach an observation point simultaneously,

(ii) the Lorentz transformations of relativity, and

(iii) the piercing of these light rays an appropriate 2D plane (the *UV*-plane) in the observation frame,

the mapping onto such a 2D plane was derived, which may be applied to any object no matter how complicated, which is in relative motion to the observation frame at arbitrary speeds ($\boldsymbol{b}c$) including extreme relativistic ones. These transformations are given explicitly by

$$U = \left(\frac{g\left[(x'+\boldsymbol{g}\boldsymbol{b}h) - \boldsymbol{b}\sqrt{(x'+\boldsymbol{g}\boldsymbol{b}h)^2 + y'^2 + (z'-h)^2}\right] n_2 - y' n_1}{g\left[(x'+\boldsymbol{g}\boldsymbol{b}h) - \boldsymbol{b}\sqrt{(x'+\boldsymbol{g}\boldsymbol{b}h)^2 + y'^2 + (z'-h)^2}\right] n_1 + y' n_2} \right) d\,, \qquad (3.49)$$

$$V = \frac{d(z'-h)}{g\left[(x'+\boldsymbol{g}\boldsymbol{b}h) - \boldsymbol{b}\sqrt{(x'+\boldsymbol{g}\boldsymbol{b}h)^2 + y'^2 + (z'-h)^2}\right] n_1 + y' n_2}\,. \qquad (3.50)$$

Here (x', y', z') denotes any point on the object in its proper frame. The motion is along the positive *x*-axis for the object relative to the observation frame. The *U* and *V* axes specify the axes of the 2D plane at rest in the observation frame. The orientation of this plane, at a distance *d* from the observation point, is specified by a unit vector n $= (n_1, n_2, 0)$, perpendicular to the *UV*-plane, and gives the direction of the so-called optic axis. The optic axis is taken to lie parallel to the *xy*-plane. *h* denotes the position, along the *z*-axis, of the observation point above the origin (see Fig.3.3). Unlike the Lorentz transformations $(t', x', y', z') \to (t, x, y, z)$, the present transformations $(x', y', z') \to (U, V)$ are necessarily non-linear. We appropriately refer to these transformations as non-linear Terrell transformations. Several applications of these transformations were provided in Chapter IV. By formally setting $\gamma \to 1$ in these transformations we obtain the corresponding Galilean ones which, however, take into account the finite propagation speed of light. In particular, it was shown in Chapter V that any straight line parallel to the *x*-axis (specifying the direction of motion) remains necessarily a straight line in the *UV*-plane. This property was used to resolve the so-called "train" paradox, emphasized in the literature almost thirty years ago. By rigorously establishing the fact that any point of the object in contact with any given line parallel to the *x*-axis for $\beta = 0$ necessarily stays in contact with this line for $\beta \neq 0$ as well in the *UV*-plane. This point is non-trivial due to the fact that lines perpendicular to the x'-axis necessarily curve-up due to the time-delay mechanism (the Terrell effect), and thus give the impression of an object to be rotated (with deformation), and off of a track stationary relative to the observation frame. The curving up of straight lines perpendicular to the direction of motion was first established intuitively, by using the time-delay mechanism and then analytically providing rigorously conic-sections (Eqs.(5.44), (5.63)). Due to the time-delay mechanism, a Doppler-like effect was established for scale showing that the partitionings of a ruler, for example, become, in general, expanded (!) when approaching the observer and contracting (!), in general, when receding from the observer. The former case works against the Lorentz contraction, and the latter one works together with the Lorentz contraction. This Doppler-like effect thus masks, in general, the visibility of the Lorentz contraction. We were able to show, in a rather direct way, that upon marking a specific critical point on a moving ruler, depending simply on its speed βc, the Lorentz transformations may be actually visible about a small interval around this critical point which would provide a discrimination between the Galilean and the relativistic transformations at high speeds for which $g >> 1$ as observed on our projection plane.

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Biography

Biography

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