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RELAXED CONTROL FOR A CLASS OF SEMILINEAR IMPULSIVE EVOLUTION EQUATIONS

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RELAXED CONTROL FOR A CLASS OF SEMILINEAR IMPULSIVE EVOLUTION EQUATIONS

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วิทยานิพนธ์ฉบับนี้ศึกษาการมีของผลเฉลยชนิดอ่อน และการควบคุมแบบผ่อนคลาย สำหรับชั้นของสมการวิวัฒนาการแบบอิมพัลส์กึ่งเชิงเส้น ในกรณีที่ตัวดำเนินการกณิกนันต์เป็นแบบ C_0- กึ่งกลุ่ม

ในตอนแรก จะเริ่มด้วยการพิสูจน์ทฤษฎีการมีอยู่จริงและมีเพียงหนึ่งเดียวของผลเฉลยชนิดอ่อน ภายใต้สมมติฐานของพจน์กึ่งเชิงเส้น และได้มีการพิสูจน์ความต่อเนื่องของผลเฉลยที่ขึ้นอยู่กับค่าเริ่มต้นและตัวควบคุม

ต่อมาได้พิสูจน์การมีจริงและมีเพียงหนึ่งเดียวของผลเฉลยชนิดอ่อนสำหรับสมการผ่อนคลายแบบอิมพัลส์

ในท้ายสุด ได้พิสูจน์การมีจริงของตัวควบคุมแบบผ่อนคลายที่เหมาะสม และได้เสนอผลของความผ่อนคลายอีกด้วย

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PORNTHIP PONGCHALEE : RELAXED CONTROL FOR A CLASS OF SEMILINEAR IMPULSIVE EVOLUTION EQUATIONS. THESIS ADVISOR : ASSOC. PROF. PAIROTE SATTAYATHAM, Ph.D. 102 PP.

ORIGINAL IMPULSIVE SYSTEMS / RELAXED IMPULSIVE SYSTEM / RELAXATION / SEMIGROUP / BANACH SPACE

This thesis presents a systematic study of the existence of mild solutions and relaxed controls for a class of semilinear impulsive evolution equations where the differential operator involved is the infinitesimal generator of C_0- semigroup.

At first, theorems on existence and uniqueness of mild solutions are obtained under some assumptions on the semilinear term Continuous dependence of solutions on the initial state and control is also proved.

Secondly, the existence and uniqueness of mild solutions for the relaxed impulsive equation are proved.

Finally, the existence of optimal relaxed controls and relaxation results is presented.

School of Mathematics Student's Signature ________________

Academic Year 2006 Advisor's Signature ________________

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CHAPTER I

INTRODUCTION

Differential equations have been found in the study of pure sciences, applied sciences, engineering, and many areas of social sciences extensively. We know that the mathematical model of these systems can be described as differential equations. Depending on the problem, these equations may take various forms: ordinary differential equations, functional differential equations, partial differential equations, etc. The study of ordinary differential equations is mature nowadays. The study of partial differential equations is comparatively more difficult, because of the complexity of its variables. Therefore, under broad assumptions, we try to reformulate the partial differential equations by ordinary differential equations on abstract spaces, for example, Banach spaces. This is where semigroup theory plays a central role and provides a unified and powerful tool for the study of partial differential equations. Semigroup theory concerns the study of existence, uniqueness and continuous dependence of solutions on parameters and their regularity properties. The dynamical system (eg. heat conduction, properties of elastic material, fluid dynamics, etc.) described by partial differential equations (or in combination with ordinary differential equations) involves finding a control policy to minimize or maximize some objective functional subject to a dynamic frame work. This is the *optimal control problem.* In some diffusion reaction processes and emigration problems, we need to use integral differential equations. Furthermore, compared with traditional initial value problems, we know that impulsive conditions can be used to model more physical phenomena than traditional initial value problems.

Many interesting phenomena that cannot be modelled by traditional initial value problems can be done so by impulsive conditions, for example, the dynamics of populations subjected to abrupt changes caused by epidemic, harvesting or immigration. Moreover the study of engineering structures, such as beams, satellites with flexible appendages, suspension bridges, etc: as well as the presence of any form of impact forces also need an impulsive evolution equation. All of these give rise to the study of nonlinear impulsive integral partial differential equations and the optimal control problem.

Impulsive systems have been widely investigated on finite dimensional spaces. The readers can consult the related books written by Lakshmikanthan(1989) and Yang(2001) etc. There is also a great number of papers studying such problems. For example, for general nonlinear impulsive evolution equations:

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t, x(t)) \quad t \in (0, T) \setminus D, \\ x(0) = x_0, \\ \Delta x(t_i) = I_i(x(t_i)), \quad i = 1, 2, \dots, n. \end{cases}$$

Guo and Liu(1995), Liu and Willms(1995) have studied the existence, uniqueness and stability of the solution or other properties respectively on finite dimensional spaces for different cases.

Concerning the study of evolution system on infinite dimensional spaces, the readers are suggested to consult the related books written by Ahmed(1991), Pazy(1983), Li and Yong(1995) etc. Many authors studied evolution systems on infinite dimensional spaces. For example, for Rogovchenks(1997), Liu(1999), Ahmed(2001) have studied the same equations on infinite dimensional spaces for different cases.

Since the end of last century, impulsive equations on infinite dimensional

Banach space have been considered. See Pazy(1983), Lakshmikanthan(1989), Li and Yong(1995) and Liu(1999). Particularly, Ahmed discussed a series of problems for the impulsive system on infinite dimensional space. See also Li and Yong(1995), Liu(1999) and Balder(1987). It is only in recent years that the optimal control of system governed by impulsive evolution equations have been studied. Ahmed for the first time considered the following controlled impulsive system;

$$\begin{cases} dx = (Ax + f(x))dt + g(x)dv, \\ x(0) = x_0, \\ \Delta_l x(t_i) = F_i(x(t_i)), \quad i0 = t_0 < t_1 < t_2 < \ldots < t_n < T. \end{cases}$$

Some authors have studied optimal control theory for infinite dimensional systems governed by impulsive evolution equation in recent years, see Xiang, Peng and Wei(2004).

It is well-known that to guarantee existence of optimal "state-control" pair, we need a convexity hypothesis on a certain orientor field. When this convexity hypothesis is no longer satisfactory to obtain optimal solutions, we need to pass to a longer system, in which the orientor field has been convexified and cohere measure valued controls, so-called r*elaxed control*, have been introduced. Many authors working on variational and optimal control problems have convexified finite-dimensional control system for existence of optimal controls. This problem (called relaxation) has already been studied. See Warga(1962) and Warga(1996).

For infinite-dimensional control systems, Ahmed dealt with this problem and introduced a measure-valued control in which the control space is compact and values of relaxed control are countable additive measures, see Ahmed(1983). Then Papageorgious and other authors continued to discuss this problem. See Papageorgious(1989), Xiang and Ahumed(1993) and Xiang, Sattayatham and Wei(2003). Since 1991, Fattorini has been working with relaxed controls whose values are finitely additive measures. By his approach we can cope with such a control set which is a normal topological space even on arbitrary set. See Fattorini(1999) and Fattorini(1994). However, to our knowledge, there are only a few authors who have studied the problem on relaxed controls of system governed by impulsive evolution equations, in particular, the semilinear impulsive evolution equations on Banach space.

In this thesis, we first study a class of nonlinear impulsive evolution equations as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), \quad t \in I \setminus D, \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad i = 1, 2, \dots, n, \end{cases} \tag{1.1}$$

on a Banach space$(X, \|\cdot\|)$, where $I \equiv [0, T]$, $D = \{t_1, t_2, \cdots, t_n\} \subset (0, T)$, $0 < t_1 < t_2 < \ldots < t_n < T$. A is the infinitesimal generator of a C_0-semigroup and $\Delta x(t_i) = x(t_i^+) - x(t_i)$, representing the jump in the state x at time t_i, with J_i determining the size of the jump at t_i.

We then consider the following controlled system governed by the impulsive differential evolution equation

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)) \\ x(0) = x_0 \\ \Delta x(t_i) = J_i(x(t_i)), \quad u(\cdot) \in U_{ad}. \end{cases} \tag{1.2}$$

As we discuss system (1.1) and the corresponding controlled system (1.2), we first introduce the mild solution, prove the existence and uniqueness of mild solution. We also make a priori estimates of (1.1) and study the continuous dependence on initial states control for the system (1.2). In addition, we generalize Gronwall lemma which plays the important role in the paper.

We consider the following relaxed control system corresponding to (1.2)

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t, x(t))\mu(t), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad \mu(\cdot) \in U_r. \end{cases} \tag{1.3}$$

Before discussing system (1.3), we introduce relaxed control space and discuss system(1.3). We then study the existence and uniqueness of mild solutions, and also introduce original and relaxed trajectories and study their properties.

In addition, we consider the Lagrange problem (P_0) and (P_r) and discuss the existence of optimal relaxed control for Lagrange problem (P_r) and study the relation between problem (P_0) and (P_r).

The thesis is organized as follows. In Chapter II, we give some associated notations and preliminaries. In Chapter III, we begin to discuss the original impulsive systems on Banach space. The existence and uniqueness of mild solution for impulsive differential evolution equations are presented. We also study the continuous dependence on initial state and control of solution for the controlled system. In Chapter IV, we discuss relaxed impulsive systems. We introduce relaxed control space. The existence and uniqueness of mild solution for relaxed impulsive equation are proved, we also introduce original and relaxed trajectories and present properties of relaxed trajectories. In Chapter V, we discuss the existence of optimal relaxed control for Lagrange problem (P_r).

CHAPTER II

MATHEMATICAL PRELIMINARIES

In this chapter we will review some basic concepts and results that are necessary for the presentation of the theories in the later chapters. Most proofs will be omitted as they can be found in standard textbooks.

2.1 Elements of Functional Analysis

In this section, some basic concepts and theorems of functional analysis are collected. We assume that the reader is familiar with basis concepts from topology.

2.1.1 Normed linear spaces

We begin by reviewing the notions of normed linear spaces and inner products.

Definition 2.1. Let X be a linear space over a field $F(F = \mathbb{R}$ or $\mathbb{C})$.
A map $\|\cdot\| : X \to \mathbb{R}$ is called a *norm* on X if it satisfies

$$\begin{aligned} \|x\| &\geq 0 \\ \|x\| &= 0 \Leftrightarrow x = 0 \\ \|\alpha x\| &= \mid \alpha \mid \|x\| \\ \|x+y\| &\leq \|x\| + \|y\|, \end{aligned}$$

for all $x, y \in X$ and $\alpha \in F$.

If the second condition above is omitted from the definition, then $\|\cdot\|$ is called *seminorm*.

A map $(\cdot,\cdot) : X \times X \to F$ is called an inner product on X if it satisfies

$$\begin{aligned}(x,x) &\geq 0\\ (x,x) &= 0 \Leftrightarrow x = 0\\ (x,y) &= \overline{(y,x)}\\ (\alpha x + \beta y, z) &= \alpha(x,z) + \beta(y,z)\end{aligned}$$

for all $x, y, z \in X$ and $\alpha, \beta \in F$, and where $\overline{(y,x)}$ denotes the complex conjugate of (x,y) (If $F = \mathbb{R}$ the bar can be omitted).

Hereafter, we denote a norm on X by $\|\cdot\|_X$. Similarly, we denote an inner product on X by $(\cdot,\cdot)_X$. If X has a norm, then the pair $(X, \|\cdot\|_X)$ is called a normed linear space. The norm $\|\cdot\|_X$ induces a metric d on X by $d(x,y) = \|x-y\|_X$ and thus X become a topological space.

Let us now recall some standard topological concepts in normed linear spaces.

Definition 2.2. Let X be a normed linear space with the norm $\|\cdot\|_X$. We say that a sequence $\{x_n\}$ converges strongly to $x \in X$ if

$$\lim_{n\to\infty} \|x_n - x\|_X = 0.$$

We write $x_n \xrightarrow{s} x$ or $x_n \to x$.

Definition 2.3. A sequence $\{x_n\} \subset X$ is called a *Cauchy* sequence provided that for any $\varepsilon > 0$ there exists $N > 0$ such that

$$\|x_n - x_m\|_X < \varepsilon \text{ for all } n, m \geq N.$$

Definition 2.4. A normed linear space $(X, \|\cdot\|_X)$ is called *complete* if each Cauchy sequence in X converges; that is, whenever $\{x_n\}$ is a Cauchy sequence, then there exists $x \in X$ such that $\{x_n\}$ converges strongly to x.

Definition 2.5. A complete normed linear space $(X, \|\cdot\|_X)$ is called a *Banach space*.

Example 2.1. There are various ways to construct a new Banach space from a given one. For example, let Ω be topological space and Y be a Banach space. The space $C(\Omega, Y) = \{f : \Omega \to Y |\ f \text{ is continuous}\}$ with pointwise addition and scalar multiplication and supremum norm

$$\|f\|_\infty = \sup_{x\in\Omega}\{\|f(x)\|_Y\} \tag{2.1}$$

is Banach space.

Definition 2.6. Let X be a linear space with inner product $(\cdot,\cdot)_X$. The inner product induces a norm on X by $\|\cdot\|_X = \sqrt{(\cdot,\cdot)_X}$. Then X is called a *Hillbert space* if it is complete under the norm $\|\cdot\|_X$.

Definition 2.7. Let X be a Banach space and $G \subset X$.

1. Given $x \in X$ and $r > 0$, the set $O_r(x) = \{y \in X | \|y - x\|_X < r\}$ is called the *open ball* centered at x with radius r.

2. The set Int$G := \{x \in G |\ \exists r > 0$ such that $O_r(x) \subset G\}$ is called the *interior* of G; the smallest closed set containing G is called the *closure* of G and denoted by $\overline{G}$; and $\partial G := \overline{G} \backslash IntG$ is called the *boundary* of G

3. G is *compact* if for any family of open sets $\{G_\alpha, \alpha \in \Lambda\}$ with $G \subset \cup_{\alpha\in\Lambda} G_\alpha$, there exist finitely many G_α, say $G_{\alpha 1,\ldots,G_{\alpha k}}$ in this family, such that $G \subset \cup_{i=1}^k G_{\alpha i}$.

4. G is *relatively compact* if the closure $\overline{G}$ of G compact.

5. G is *totally bounded* if for any $\varepsilon > 0$, there exists a finite set $\{x_1, \ldots, x_k\} \subset G$, such that $G \subset \cup_{i=1}^k O_\varepsilon(x_i)$.

6. G is *separable* if it admits a *countable dense subset*, i.e., there exists a countable set $G_0 \equiv \{x_i, i \geq 1\} \subset G$, such that the closure $\overline{G_0}$ of G_0 contains G. If X is itself separable, then we say that X is a *separable Banach space.*

7. G is *convex* if for any $x, y \in G$ and $\alpha \in [0,1]$ then $\alpha x + (1-\alpha)y \in G$.

Proposition 2.1. *Let X be Banach space and $G \subset X$. Then, the following are equivalent:*

1. *G is relatively compact;*

2. *G is totally bounded;*

3. *Every sequence $\{x_n\} \subset G$ has a (strongly) convergent subsequence.*

2.1.2 Linear Operator

In the following, unless stated otherwise X and Y will denote normed linear spaces over a field F.

Definition 2.8. Let $D(A)$ be not necessarily closed linear subspace of X.

1. A map $A : D(A) \to Y$ is called a *linear operator* if the following holds:

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y), \quad \forall x, y \in D(A), \quad \alpha, \beta \in F.$$

   The set $D(A)$ is called the domain of A.

2. We say that A is *densely defined* if $D(A)$ is dense in X.

3. A is called *closed* if whenever $x_n \to x$ in X, $x_n \in D(A)$ and $Ax_n \to y$ in Y, then

$$x \in D(A) \text{ and } Ax = y.$$

4. We say that A is *closable* if there exists a closed operator $\overline{A} : D(\overline{A}) \subset X \to Y$, such that

$$D(A) \subset D(\overline{A}), \text{ and } \overline{A}x = Ax, \quad \forall x \in D(A).$$

Proposition 2.2. *Let $A : X \to Y$ be a linear operator. The following two conditions are equivalent:*

1. *A is bounded, i.e., there exists a constant $d > 0$ such that*

$$\|Ax\|_Y \leq d\|x\|_X \quad \textit{for all } x \in X.$$

2. *A is continuous, i.e., $x_n \to x$ as $n \to \infty$ implies $Ax_n \to Ax$ as $n \to \infty, \forall \{x_n\} \subseteq X$.*

Given two normed linear spaces X and Y, let $\mathcal{L}(X,Y)$ denote the set of all bounded linear operators from X to Y. $\mathcal{L}(X,Y)$ becomes a normed linear space if we define vector operations in a natural way and define the operator norm $\|A\|_{\mathcal{L}(X,Y)} = \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \|Ax\|_Y$. If $X = Y$, we simply write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$.

Theorem 2.3. *(Uniform Boundedness Principle). Let X and Y be Banach spaces and $\mathcal{A} \subset \mathcal{L}(X,Y)$. Then,*

$$\sup_{A \in \mathcal{A}} \|Ax\|_Y < \infty, \quad \forall x \in X \quad \textit{implies that} \quad \sup_{A \in \mathcal{A}} \|A\|_{\mathcal{L}(X,Y)} < \infty.$$

2.1.3 Linear Functionals and Dual Spaces

Definition 2.9. 1. A bounded linear operator $x^* : X \to F$ is called a *bounded linear functional* on X.

2. We write X^* to denote the collection of all bounded linear functionals on X and call it the dual space of X.

3. If $x \in X, x^* \in X^*$ we write

$$\langle x^*, x\rangle$$

to denote the real number $x^*(x)$.

4. Let $\|\cdot\|_{X^*}$ denote the operator norm on X^*. Then for $x^* \in X^*$,

$$\|x^*\|_{X^*} := \sup\{\langle x^*, x\rangle | \|x\|_X \le 1\}.$$

5. A Banach space is *reflexive* if $(X^*)^* = X$. More precisely, this means that for each $x^{**} \in (X^*)^*$, there exists $x \in X$ such that

$$\langle x^{**}, x^*\rangle = \langle x^*, x\rangle \quad \text{for all } x^* \in X^*.$$

Definition 2.10. A function $f : X \to \mathbb{R}$ is called *convex* provided that for all $x, y \in X$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

Definition 2.11. A *locally convex space* $(X, \{p_j\})$ is a linear space X over F together with a system of seminorm $\{p_j\}_{j\in J}$ such that

$$x = 0 \Leftrightarrow p_j(x) = 0 \text{ for all } j \in J.$$

2.1.4 Closed Operator

Definition 2.12. Let X and Y be normed linear spaces and $T : X \to Y$ a function. The *graph* of T, denote by $\mathcal{G}$, is defined by

$$\mathcal{G}(T) = \{(x, Tx) \mid x \in X\} \subset X \times Y.$$

If T is linear, it is easy to verify that $\mathcal{G}(T)$ is a linear subspace of $X \times Y$. We say that the map $T : X \to Y$ has a *close graph* or T is a *closed operator* if $\mathcal{G}(T)$ is a closed subspace of $X \times Y$.

Lemma 2.4. *Let X and Y be normed linear spaces and $T : X \to Y$ a linear operator. Then T has a closed graph if and only if for every sequence $\{x_n\}$ in X, if $x_n \to x$ and $Tx_n \to y$, then $y = Tx$.*

Theorem 2.5 (Closed Graph Theorem). *Suppose that X and Y are Banach spaces and $T : X \to Y$ a linear operator. Then T is bounded if and only if T has a closed graph.*

2.1.5 Weak Topology and Weak Convergence

The difference between a finite dimensional Banach space and an infinite dimensional Banach space is that in the latter, a bounded sequence need not have a convergent subsequence. This is responsible for many difficulties in the calculus of variations and the theory of partial differential equations. In order to overcome this difficulty, one needs to introduce the concept of weak convergence.

Let X be a Banach space and X^* its dual. Elements of X^* can be used to generate a new topology for X called the *weak topology.* It is obtained by taking as a *base* all sets (neighborhoods of the form)

$$N(x_0, F^*, \varepsilon) := \{x \in X : \langle x^*, x - x_0 \rangle < \varepsilon, x^* \in F^*\},$$

where $x_0 \in X, F^*$ is any finite subset of X^*, and $\varepsilon > 0$. Endowed with this weak topology, X becomes a locally convex linear topological vector space.

The concepts of open or closed sets, compactness, convergence, ect. are topological, hence they must be qualified by referring to the topology involved. In the case of normed linear spaces, when one speaks of open or closed sets, compactness, convergence, etc., one refers to the norm topology, while when referring to the weak topology, one uses the terms weakly open or weakly closed sets, weak compactness, weak convergence, ect.

Definition 2.13. A sequence $\{x_n\} \subset X$ is said to *converge weakly* to $x \in X$, written $x_n \xrightarrow{w} x$, if $\lim_{n\to\infty} \langle x^*, x_n\rangle = \langle x^*, x\rangle$ for all $x^* \in X^*$.

Proposition 2.6. *1. If X is a reflexive Banach space, then every bounded sequence $\{x_n\}$ in X has a weakly convergent subsequence. If in addition, each weakly convergent subsequence of $\{x_n\}$ in X has the same limit x, then*

$$x_n \xrightarrow{w} x \text{ in } X \text{ as } n \to \infty.$$

2. If X is a Banach space, and

$$x_n^* \to x^* \text{ in } X^*, \quad x_n \xrightarrow{w} x \text{ in } X \text{ as } n \to \infty,$$

then

$$x_n^* \xrightarrow{w} x^* \text{ in } X^*, \quad x_n \to x \text{ in } X \text{ as } n \to \infty.$$

Moreover, if X is a reflexive Banach space, then

$$x_n^* \xrightarrow{w} x^* \text{ in } X^*, \quad x_n \to x \text{ in } X \text{ as } n \to \infty,$$

implies

$$\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle \text{ as } n \to \infty.$$

Lemma 2.7. *(Mazur). Let X be Banach space and K be a norm closed convex set in X. Then K is weakly closed in X.*

2.1.6 Compact operators

Definition 2.14. Let X and Y be Banach spaces, and T be a map from some subset $D(T)$ of X into Y. Then T is called *compact* if and only if:

1. T is continuous,

2. T maps bounded sets into relatively compact sets.

Theorem 2.8. *(Arzela-Ascoli). Let X and Y be Banach spaces, $G \subset X$ be compact and $\mathcal{F} \subset C(G, Y)$. Suppose that*

1. *for each $x \in G$, the set $\{F(x) | F \in \mathcal{F}\}$ is relatively compact in Y.*

2. *$\mathcal{F}$ is uniformly bounded, i.e.,*

$$\sup_{F \in \mathcal{F}, x \in G} \|F(x)\|_Y < \infty.$$

3. *$\mathcal{F}$ is equicontinuous, i.e., for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that*

$$\|F(x) - F(x')\|_Y < \varepsilon, \quad \text{whenever } \|x - x'\|_X < \delta, F \in \mathcal{F}, x, x' \in G.$$

Then there exists a sequence $\{F_k\} \subseteq \mathcal{F}$ and $F_0 \in C(G, Y)$, such that

$$\lim_{k \to \infty} \|F_k - F_0\|_{C(G,Y)} = 0$$

where $C(G, Y)$ denotes the supremum norm (2.1).

The proof can be found in Xunjing Li and Jiongmin Yong(1995).

Proposition 2.9. *Let $A : X \to Y$ be linear operator, where X and Y are Banach spaces.*

1. *If A is compact, then A is strongly continuous, i.e.,*

$$u_n \xrightarrow{w} u \text{ as } n \to \infty \text{ implies } Au_n \to Au \text{ as } n \to \infty.$$

2. *Conversely, if A is strongly continuous and X is reflexive, then A is compact.*

The proof can be found in Zeidler(1990).

2.2 Lebesgue Integration Theory

In this section, we review the notions of Lebesgue measure and Lebesgue integral for Banach space valued functions. We then state some standard convergence theorems for integrals and introduce the Lebesgue function spaces in which we will be working. For details and proofs we refer to Zeidler(1990), unless otherwise stated.

2.2.1 The Lebesgue Measure

Let us first give a quick outline of some fundamentals of Lebesgue measure theory. Loosely speaking, the Lebesgue measure provides a way of describing the "size" or "volume" of certain subsets of $\mathbb{R}^n$.

Definition 2.15. A collection $\mathcal{M}$ of subsets of $\mathbb{R}^n$ is called a $\sigma-$*algebra* if

1. $\emptyset \in \mathcal{M}$ and $\mathbb{R}^n \in \mathcal{M}$,
2. $A \in \mathcal{M}$ implies $\mathbb{R}^n \setminus A \in \mathcal{M}$,
3. if $\{A_k\} \subset \mathcal{M}$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$ and $\bigcap_{k=1}^{\infty} A_k \in \mathcal{M}$.

Theorem 2.10 (Existence of Lebesgue measure and Lebesgue measurable sets). *There exists a $\sigma-$algebra $\mathcal{M}$ of subsets of $\mathbb{R}^n$ and a mapping*

$$|\cdot|: \mathcal{M} \to [0, +\infty]$$

with the following properties:

1. *Every open subset of $\mathbb{R}^n$, and thus every closed subset of $\mathbb{R}^n$, belongs to $\mathcal{M}$.*
2. *If B is a ball in $\mathbb{R}^n$, then $\mid B \mid$ equals the $n-$dimensional volume of B.*

3. *If* $\{A_k\} \subset \mathcal{M}$ *and the sets* $\{A_k\}$ *are pairwise disjoint, then*

$$|\bigcup_{k=1}^{\infty} A_k| = \sum_{k=1}^{\infty} |A_k| \quad \textit{(countable additivity).}$$

4. *If* $A \subset B$, *where* $B \in \mathcal{M}$ *and* $|B| = 0$, *then* $A \in \mathcal{M}$ *and* $|A| = 0$.

The sets in $\mathcal{M}$ are called *Lebesgue measurable sets* and $|\cdot|$ is called the $n-$dimensional *Lebesgue measure*. If some property holds everywhere on $\mathbb{R}^n$, except for a measurable set with Lebesgue measure zero, then we say that this property holds *almost everywhere* or for *almost all* $x \in \mathbb{R}^n$, abbreviated "a.e.". In the following, we will simply use the word "measurable" for "Lebesgue measurable".

2.2.2 Measurable Functions

Let $M \subseteq \mathbb{R}^n$ be a measurable set and X a Banach space.

Definition 2.16. 1. A function $f : M \to X$ is called a *step function* if there exist finitely many pairwise disjoint measurable subsets M_i of M such that $|M_i| < \infty$ for all i, and elements a_i of X such that

$$f(x) = \begin{cases} a_i & \text{if } x \in M_i \\ 0 & \text{otherwise.} \end{cases}$$

That is, f is constant on each set M_i.

2. The integral of a step function is defined to be

$$\int_M f dx = \sum_i |M_i| a_i.$$

3. A function $f : M \to X$ is called (strongly) *measurable* if there exists a sequence $\{f_n\}$ of step functions $f_n : M \to X$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for almost all } x \in M.$$

4. (Measurable functions via substitution). Let X, U be real and separable Banach spaces, $M \subseteq \mathbb{R}^n$ be measurable, $f : M \times U \to X$ and $u : M \to U$. Set

$$F(x) = f(x, u(x)).$$

If the function $u : M \to U$ is measurable, then the function $F : M \to X$ is also measurable provided that f satisfies the *Caratheodory condition*:

(a) $x \mapsto f(x, u)$ is measurable on M for all $u \in U$,

(b) $x \mapsto f(x, u)$ is continuous on U for almost all $x \in M$.

2.2.3 The Lebesgue Integral

The Lebesgue theory of integration is especially useful since it provides powerful convergence theorems. Throughout, X will denote a Banach space.

Definition 2.17. Let X be Banach space. A function $f : M \subseteq \mathbb{R}^n \to X$ is called *integrable* if it is strongly measurable and there exists a sequence $\{f_n\}$ of step functions $f_n : M \to X$ such that

1. $f(x) = \lim_{n\to\infty} f_n(x)$ for almost all $x \in M$,

2. given $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\int_M \|f_n(x) - f_m(x)\| dx < \varepsilon \quad \text{for all } n,\ m \geq n_0(\varepsilon).$$

The second condition implies that the sequence$\{ \int_M f_n(x)dx\}$ is *Cauchy* in X, so that we can define the *Lebesgue integral* of f by

$$\int_M f(x)dx = \lim_{n\to\infty} \int_M f_n(x)dx \tag{2.2}$$

One can show that this integral is well defined, i.e., the limit in (2.2) does not depend on the choice of the step functions f_n. Furthermore if $B \in \mathcal{L}(X)$ and the integral of f exists, then the integral of Bf exists, and

$$\int_M Bf(x)dx = B\int_M f(x)dx.$$

Proposition 2.11 (Monotone convergence theorem). *Let* $\{f_n\}$, $f_n : M \subseteq \mathbb{R}^n \to \mathbb{R}$ *be a monotone increasing (monotone decreasing) sequence of integrable functions, with* $\sup_n \mid \int_M f_n dx \mid < \infty$. *Then* $f_n(x)$ *converges a.e. to some integrable function* f, *and*

$$\lim_{n\to\infty}\int_M f_n(x)dx = \int_M f(x)dx.$$

Proposition 2.12 (Majorant criterion). *Let* $f : M \subseteq \mathbb{R}^n \to X$ *be measurable. If there exists* $g : M \to \mathbb{R}$ *such that* $\|f(x)\| \leq g(x)$ *for almost all* $x \in M$, *and* $\int_M g(x)dx$ *exists, then* f *is integrable, and*

$$\|\int_M f(x)dx\| \leq \int_M \|f(x)\|dx \leq \int_M g(x)dx.$$

Proposition 2.13 (Majorant convergence principle). *Let* $f_n : M \subseteq \mathbb{R}^n \to X$ *be measurable for all* n *and suppose that* $f_n(x)$ *converges almost everywhere to some function* $f : M \to X$. *If there exists* $g : M \to \mathbb{R}$ *such that* $\|f_n(x)\| \leq g(x)$ *for almost all* $x \in M$ *and all* $n \in \mathbb{N}$, *and* $\int_M g(x)dx$ *exists, then* f *is integrable, and*

$$\lim_{n\to\infty}\int_M f_n(x)dx = \int_M f(x)dx.$$

2.2.4 Lebesgue Spaces of Vector-valued Functions

We now introduce the function spaces which we will be working in, and discuss some of their properties. Let X be a Banach space and $0 < T < \infty$.

Definition 2.18. By $C^m([0,T],X)$ with $m = 0,1,\ldots$ we denote the set of all continuous functions $u : [0,T] \to X$ that have continuous derivatives up to order

m on $[0, T]$, with the norm

$$\|u\| := \sum_{i=0}^{m} \max_{0 \le t \le T} \mid u^{(i)}(t) \mid . \tag{2.3}$$

Here, only the right-hand and the left-hand derivatives need exist at the end points $t = 0$ and $t = T$, respectively, and $u^{(0)}$ means u. Note that $C^0([0, T], X)$ coincides with $C([0, T], X)$ of example (2.1).

Let $M \subseteq \mathbb{R}$ be measurable.

Definition 2.19. By $L^p(M, X)$ with $1 \le p \le \infty$ we denote the set of all measurable functions $u : M \to X$ for which $\int_M \|u(t)\|^p dt$ exists, endowed with the norm

$$\|u\|_{L^p(M,X)} := \big(\int_M \|u(t)\|_X^p dt \big)^{1/p}. \tag{2.4}$$

If $X = \mathbb{R}$ or $X = \mathbb{C}$, we simply write $L^p(M)$. We note that $L^P((0, T), X)$ can be identified with $L^P([0, T], X)$, and one often writes $L^p(0, T; X)$ for this space.

Definition 2.20. Let X and Y be Banach spaces over F with $X \subseteq Y$. The embedding operator $j : X \to Y$ is defined by $j(u) = u$ for all $u \in X$.

1. The embedding $X \subseteq Y$ is called *continuous* if j is continuous, i.e., if there exists $d > 0$ such that

   $$\|u\|_Y \le d\|u\|_X \qquad \text{for all } u \in X. \tag{2.5}$$

   In this case, write $X \hookrightarrow Y$.

2. The embedding $X \subseteq Y$ is called *compact* if j is compact as a linear map, i.e., (2.5) holds, and each bounded sequence (u_n) in X has a subsequence $(u_{n'})$ which is convergent in Y. In this case, we write $X \hookrightarrow\hookrightarrow Y$.

Proposition 2.14 (Properties of Lebesgue Space). *Let* $m = 0, 1, \dots$ *and* $1 \le p < \infty$. *Let* X *and* Y *be Banach spaces over* F. *Then:*

1. *$C^m([0,T],X)$ with the norm (2.3) is Banach space over F.*

2. *$L^p((0,T),X)$ with the norm (2.4) is Banach space over F if one identifies functions that are equal almost everywhere on $(0,T)$.*

3. *$C([0,T],X)$ is dense in $L^p((0,T),X)$, and the embedding*

$$C([0,T],X) \subseteq L^p((0,T),X)$$

is continuous.

4. *$L^p((0,T),X)$ is separable in the case where X is separable.*

5. *$L^p((0,T),X)$ is reflexive in the case where X is reflexive for $1 < p < \infty$.*

6. *If the embedding $X \subseteq Y$ is continuous, then the embedding*

$$L^r((0,T),X) \subseteq L^q((0,T),Y), \quad 1 \leq q \leq r < \infty$$

is also continuous.

Proposition 2.15 (Hölder's inequality). *Assume $1 < p,\ q < \infty$ with $\frac{1}{p}+\frac{1}{q} = 1$. Then if $u \in L^p(M),\ v \in L^q(M)$, we have $uv \in L^1(M)$ and*

$$\int_M |uv| dx \leq \|u\|_{L^p(M)} \|v\|_{L^q(M)}.$$

2.3 Theory of C_0-semigroups

In this section, we introduce the concept of and present some basic results on C_0-semigroups. For more details and proofs, we refer to Pazy(1983). X and Y will always denote Banach spaces unless otherwise stated.

2.3.1 C_0-semigroups

Let us first introduce the following definition.

Definition 2.21. Let $\{T(t), t \geq 0\}$ be a family of bounded linear operators on X. The family $\{T(t), t \geq 0\}$ is said to be a *semigroup of operators* on X if

1. $T(0) = I$ (I is the identity operator on X),
2. $T(t+s) = T(t)T(s) = T(s)T(t)$ for all $t, s \geq 0$.

The semigroup $\{T(t), t \geq 0\}$ is said to be *uniformly continuous* if $t \mapsto T(t)$ is continuous on $[0, \infty)$ in the norm topology of $\mathcal{L}(X)$. It is easy to show that this is equivalent to

$$\lim_{t \to 0^+} \|T(t) - I\|_{\mathcal{L}(X)} = 0.$$

Definition 2.22. A semigroup $\{T(t), t \geq 0\}$ of bounded linear operators on X is said to be *strongly continuous* if $t \mapsto T(t)x$ is continuous on $[0, \infty)$ for all $x \in X$. It is easy to show that this equivalent to

$$\lim_{t \to 0^+} \|T(t)x - x\|_X = 0 \quad \text{for all } x \in X.$$

A strongly continuous semigroup of bounded linear operators on X is called a C_0-*semigroup.*

Definition 2.23. Let $\{T(t), t \geq 0\}$ be a C_0-semigroup on X. The operator A with domain

$$D(A) := \{x \in X : \lim_{t \to 0^+} \frac{T(t) - I}{t} x \ \text{ exists}\}$$

and defined by

$$Ax := \lim_{t \to 0^+} \frac{T(t) - I}{t} x, \text{ for all } \ x \in D(A),$$

is called the infinitesimal generator of the semigroup. The operator $A : D(A) \subseteq X \to X$ is linear but not necessarily bounded.

Some important properties of strongly continuous semigroups of bounded linear operators and their infinitesimal generators are formulated as follows:

Theorem 2.16. *Let* $\{T(t), t \geq 0\}$ *be a* C_0-*semigroup on* X. *Then there exist constants* $M \geq 1$, *and* $\omega \geq 0$ *such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

If we can choose $\omega = 0$ in Theorem 2.16, then $T(t)$ is called *uniformly bounded* and if in addition, we can choose $M = 1$, then it is called a C_0-*semigroup of contractions.*

Theorem 2.17. *Let* $\{T(t), t \geq 0\}$ *be a* C_0-*semigroup and let* A *be its infinitesimal generator. Then*

1. *For* $x \in X$,
$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x.$$

2. *For* $x \in X$, $\int_0^t T(s)xds \in D(A)$ *and*
$$A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

3. *For* $x \in D(A)$, $T(t)x \in D(A)$ *and the function* $t \to T(t)x$ *is differentiable with*
$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

4. *For* $x \in D(A)$,
$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau.$$

2.3.2 Hille-Yosida's Theorem

In this section we state Hille-Yosida theorem which provides necessary and sufficient conditions for a closed operator to be the generator of a C_0-semigroup. Let A be a linear but not necessarily bounded closed operator on X.

Definition 2.24. 1. We say that a complex number λ belongs to $\rho(A)$, the *resolvent* set of A, provided the operator $\lambda I - A : D(A) \to X$ is one-to-one and onto.

2. If $\lambda \in \rho(A)$, then $\lambda I - A$ is invertible, and its inverse operator $R(\lambda, A) := (\lambda I - A)^{-1}$ is called the *resolvent operator*.

According to the Closed Graph Theorem, the resolvent operator $R(\lambda, A) : X \to D(A) \subset X$ is bounded. Furthermore, $AR(\lambda, A)u = R(\lambda, A)Au$ if $u \in D(A)$.

Theorem 2.18 (Properties of resolvent operators). *1. If $\lambda,\ \mu \in \rho(A)$, we have*

$$R(\lambda, A) - R(\mu, A) = (\lambda - \mu)R(\lambda, A)R(\mu, A)$$

and

$$R(\lambda, A)R(\mu, A) = R(\mu, A)R(\lambda, A).$$

2. Suppose that A is the infinitesimal generator of a C_0-semigroup of contractions $\{T(t), t \geq 0\}$. If $\lambda > 0$, then $\lambda \in \rho(A)$,

$$R(\lambda, A)u = \int_0^\infty e^{-\lambda t} T(t)u dt; \quad u \in X$$

and thus $\|R(\lambda, A)\|_X \leq \frac{1}{\lambda}$.

Theorem 2.19 (Hille-Yosida). *A linear operator A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ if and only if*

1. A is closed, $D(A)$ is dense in X.

2. *For some* $\omega \in \mathbb{R}$ *and* $M \geq 1$, *we have* $\rho(A) \supset \{\lambda \in \mathbb{C} | Re\lambda > \omega\}$ *and*

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(Re\lambda - \omega)^n}, \quad \textit{for all } n \geq 0, \ Re\lambda > \omega.$$

Corollary 2.20. *A linear operator* A *is the infinitesimal generator of a* C_0*−semigroup satisfying* $\|T(t)\| \leq e^{\omega t}$ *if and only if*

1. A *is closed,* $\overline{D(A)} = X$.

2. *The resolvent set* $\rho(A)$ *of* A *contains the ray* $\{\lambda : Im\lambda = 0, \ \lambda > \omega\}$ *and for such* λ,

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda - \omega}.$$

2.3.3 Semigroups of Compact Operators

Definition 2.25. A C_0−semigroup $\{T(t), t \geq 0\}$ on X is called *compact semigroup* for $t > t_0$ if $T(t)$ is compact operator for every $t > t_0$. It is simply called compact if it is compact operator for all $t > 0$.

Note that if $T(0)$ is compact, then X must be a finite dimensional Banach space, since the identity operator is compact if and only if X is finite dimensional. Hence for general Banach space, one can expect $T(t)$ to be compact only for $t > 0$. Note also that if $T(t_0)$ is compact for some $t_0 > 0$, then $T(t)$ is compact for all $t \geq t_0$. This follows from the fact that $T(t) = T(t - t_0)T(t_0), \ t_0 > 0$, and that the composition of a compact operator with a bounded operator is always compact.

Theorem 2.21. *Let* $\{T(t), t \geq 0\}$ *be a* C_0*−semigroup. If* $T(t)$ *is compact for* $t > t_0$, *then* $T(t)$ *is continuous in the uniform operator topology for* $t > t_0$.

For the converse statement one has:

Theorem 2.22. *Let $\{T(t), t \geq 0\}$ be a C_0-semigroup and let A be its infinitesimal generator. $T(t)$ is a compact semigroup for $t > t_0$ if and only if $T(t)$ is continuous in the uniform operator topology for $t > 0$ and $R(\lambda, A)$ is compact for some $\lambda \in \rho(A)$.*

2.4 Linear Evolution Equation

We wish to study both linear and nonlinear evolution differential equations covering only deterministic systems. We shall consider the existence of solutions of evolution differential equations. Let X be a Banach space, $A \in \mathcal{L}(X)$ with $D(A)$ and $R(A) \subset X$ and consider the existence of homogeneous equation in X given by

$$\begin{cases} \dot{x}(t) = Ax(t), \quad \text{for} \quad t > 0, \\ x(0) = x_0, \end{cases} \tag{2.6}$$

where A is the infinitesimal generator of a C_0-semigroup in X.

Definition 2.26 (Classical Solution). The Cauchy problem (2.6) is said to have a classical solution if for each given $x_0 \in D(A)$ there exists a function $x(t) = x(t, x_0), t > 0$, with values in X, satisfying the following properties:

1. $x \in C^1$;
2. $\dot{x}(t) = Ax(t)$ for all $t > 0$;
3. $x(0) = x_0$.

Theorem 2.23. *Let A be a densely defined linear operator on X with $\rho(A) \neq \emptyset$. Then the system (2.6) has a unique classical solution for each $x_0 \in D(A)$ if and only if A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on X.*

Corollary 2.24. *If A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$, on X, then for every $x_0 \in X$, the system (2.6) has a mild solution $x(t) = T(t)x_0$, $t \geq 0$.*

Definition 2.27 (Weak Solution). A function $x \in C([0,a],X)$ is said to be a weak solution of (2.6) if for every $x_0 \in X$ and $t \in [0,a]$ and $x^* \in D(A^*)$

$$(x(t),x^*) = (x_0,x^*) + \int_0^t (x(\tau), A^*x^*)d\tau.$$

Theorem 2.25. *If A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on X, then for every mild solution of (2.6) is weak solution.*

We now consider the inhomogeneous initial value problem

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t), \quad t > 0, \\ x(0) = x_0, \end{cases} \tag{2.7}$$

where A is the infinitesimal generator of a C_0-semigroup on X.

Definition 2.28 (Classical Solution). A function $x : [0,a) \to X$ is said to be a classical solution of the system (2.7) if

1. $x \in C([0,a),X)$;

2. $x \in C^1((0,a),X)$;

3. $x(t) \in D(A)$ for $t \in (0,a)$, and

4. x satisfying (2.7) on $(0,a)$.

Definition 2.29 (Mild Solution). A function $x \in C(I,X)$ for any finite interval $I = [0,a]$, is said to be a mild solution of the system (2.7) corresponding to the initial state $x_0 \in X$ and the input $f \in L^1(I,X)$ if x is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau)d\tau, \quad t > 0.$$

Theorem 2.26. *Let $x_0 \in D(A)$ and $f \in L_1(I,X) \bigcap C((0,a),X)$ and suppose that A is the infinitesimal generator of semigroup $\{T(t), t \geq 0\}$ being the corresponding semigroup and x, given by*

$$x(t) = T(t)x_0 + z(t), \quad t \in [0,a)$$

where

$$z(t) = \int_0^t T(t-\tau)f(\tau)d\tau, \quad t \in I,$$

is the associated mild solution. Then, in order that x be a classical solution, it is necessary and sufficient that any one of the following conditions hold

1. $z \in C^1((0,a), X)$;
2. $z(t) \in D(A)$ *for* $t \in (0,a)$ *and* $Az(t) \in C((0,a), X)$.

Corollary 2.27. *Let A is the infinitesimal generator of semigroup $\{T(t), t \geq 0\}$ being the corresponding semigroup and $f \in L^1(I,X)$ and $x_0 \in X$. Then on any subinterval $[0,b), b < a$ the mild solution x of the system (2.7), is the uniform limit of classical solutions.*

Definition 2.30 (Strong Solution). A function $x \in C(I,X)$ is said to be a strong solution of the system (2.7) if

1. x is differentiable a.e. on I and $\frac{d}{dt}x(t) \in L^1(I,X)$,
2. $x(0) = x_0$ and $\frac{d}{dt}x(t) = Ax(t) + f(t)$ a.e. on I.

Theorem 2.28. *If A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on X, then the system (2.7) has a strong solution if and only if, one of the following conditions hold*

1. *z, as defined by $z(t) = \int_0^t T(t-\tau)f(\tau)d\tau$, is differentiable a.e. on I and $\frac{d}{dt}z(t) \in L^1(I,X)$.*
2. *$z(t) \in D(A)$ a.e. on I and $Az \in L^1(I,X)$.*

2.5 Set-Valued Maps

In this section, we introduce the concept of and present some basic results on set-valued maps. For more details and proofs, we refer to Aubin(1990). Let Z be a Hausdorff topological space (Z should be thought as a metric space or a locally convex space, when appropriate)

The basis notations

2^Z : the collection of all subsets of Z (the power set of Z)

$2^Z \backslash \{\emptyset\}$: the collection of all nonempty subsets of X,

$P_f(Z) = \{A \subset Z :$ nonempty, closed $\}$,

$\hat{P}_f(Z) = P_f(Z) \cup \{\emptyset\}$,

$P_{fc}(Z) = \{A \subset Z :$ nonempty, closed, convex $\}$,

$P_k(Z) = \{A \subset Z :$ nonempty, compact $\}$,

$P_{ck}(Z) = \{A \subset Z :$ nonempty, compact, convex $\}$,

$B(z, \varepsilon) = \{\acute{z} \in Z : d(z, \acute{z}) < \varepsilon\}$ and $\bar{B}(z, \varepsilon) = \{\acute{z} \in Z : d(z, \acute{z}) \leq \varepsilon\}$

$B_\varepsilon = \{z \in Z : \|z\| < \varepsilon\}$ and $\bar{B}_\varepsilon = \{z \in Z : \|z\| \leq \varepsilon\}$,

$[y, z]$ = ordered pair in the product space $Y \times Z$.

Definition 2.31. Let X and Y be sets, $F : X \to 2^Y$ is called *multifunction* or *set-valued map*, if it is characterized by its graph $Graph(F)$, the subset of the product space $X \times Y$ defined by

$$Graph(F) := \{[x, y] \in X \times Y | y \in F(x)\}$$

We shall say that $F(x)$ is the image or the value of F at x.

A set-valued map is said to be *nontrivial* if its graph is not empty, i.e., if there exists at least an element $x \in X$ such that $F(x)$ is not empty.

We say that F is *strict* if all images $F(x)$ are not empty. The domain of F

is the subset of elements $x \in X$ such that $F(x)$ is not empty:

$$Dom(F) := \{x \in x | F(x) \neq \emptyset\}.$$

The image of F is the union of the image (or values) $F(x)$, when x ranges over X:

$$Im(F) := \bigcup_{x \in x} F(x).$$

The inverse F^{-1} of F is the set-valued map from Y to X defined by

$$x \in F^{-1}(y) \Leftrightarrow y \in F(x) \Longleftrightarrow (x, y) \in Graph(F).$$

Definition 2.32. Consider a measurable space $(\Omega, \mathcal{A})$, a complete separable metric space Z and a set-valued map $F : \Omega \to 2^Z$ with closed images. The map F is called *measurable* if the inverse image of each open set is a measurable set: for every open subset $O \subset Z$, we have

$$F^{-1}(O) := \{\omega \in \Omega | F(\omega) \cap O \neq \emptyset\} \in \mathcal{A}.$$

Definition 2.33. Let $(\Omega, \mathcal{A})$ be a measurable space and Z a complete separable metric space. Given a multifunction $F : \Omega \to 2^Z$. The map F is called *graph measurable* if

$$Graph(F) = \{[\omega, x] \in \Omega \times Z : x \in F(\omega)\} \in \mathcal{A} \times B(Z).$$

Proposition 2.29. *Let $(\Omega, \mathcal{A})$ be a measurable space and Z a complete separable metric space, if $F : \Omega \to \hat{P}_f(Z)$ is measurable, then $F(\cdot)$ is graph measurable.*

Definition 2.34. A set-valued map $F : X \to 2^Y \setminus \{\emptyset\}$ is called *upper semicontinuous* at $x \in Dom(F)$ if and only if for any neighborhood $\mathcal{U}$ of $F(x)$,

$$\exists \eta > 0 \quad \text{such that} \quad \forall \acute{x} \in B_X(x, \eta), \quad F(\acute{x}) \subset \mathcal{U}.$$

when $F(x)$ is compact, F is upper semicontinuous at x if and only if

$$\forall \varepsilon > 0, \quad \exists \eta > 0 \quad \text{such that} \quad \forall \acute{x} \in B_X(x, \eta), \quad F(\acute{x}) \subset B_Y(F(x), \varepsilon).$$

Definition 2.35. A set-valued map $F : X \to 2^Y \setminus \{\emptyset\}$ is called *lower semicontinuous* at $x \in Dom(F)$ if and only if for any $y \in F(x)$ and for any sequence of elements $x_n \in Dom(F)$ converging to x, there exists a sequence of elements $y_n \in F(x_n)$ converging to y.

Definition 2.36. We shall say that set-valued map F is *continuous* at x if it is both upper semicontinuous and lower semicontinuous at x, and that it continuous if and only if it is continuous at every point of $Dom(F)$.

Definition 2.37. When X and Y are normed spaces, we shall say that $F : X \to 2^Y \setminus \{\emptyset\}$ is *Lipschitz around* $x \in X$ if there exist a positive constant l and a neighborhood $\mathcal{U} \subset Dom(F)$ of x such that

$$\forall x_1, x_2 \in \mathcal{U}, \quad F(x_1) \subset F(x_2) + l\|x_1 - x_2\|B_Y.$$

In this case F is also called Lipschitz (or $l-$Lipschitz) on $\mathcal{U}$.

Definition 2.38. Consider a set-valued map $F : X \to 2^Y \setminus \{\emptyset\}$. A single valued map $f : X \to Y$ is called a *selection* of F if for every $x \in X$, $f(x) \in F(x)$.

Definition 2.39. Let $(\Omega, \mathcal{A})$ be a measurable space and Z a complete separable metric space. Consider a set-valued map $F : \Omega \to 2^Z \setminus \{\emptyset\}$. A measurable map $f : \Omega \to Z$ satisfying

$$\forall \omega \in \Omega, \quad f(\omega) \in F(\Omega)$$

is called a *measurable selection* of F.

Theorem 2.30 (Measurable Selection). *Let Z be a complete separable metric space, $(\Omega, \mathcal{A})$ a measurable space, F a measurable set-valued map from Ω to closed nonempty subsets of Z. Then there exists a measurable selection of F.*

Theorem 2.31 (Michael's Theorem). *Let F be a lower semicontinuous set-valued map with closed convex values from a compact metric space X to a Banach space Y. It dose have a* continuous selection.

Definition 2.40. Let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma-$finite measure space and Z a separable Banach space. Given a multifunction $F : \Omega \to 2^Z \setminus \{\emptyset\}$, we define the set

$$S_F = \{f \in L^0(\Omega, Z) : f(\omega) \in F(\Omega) \ \ \mu - \text{a.e.}\}$$

For $1 \leq p \leq \infty$, we define

$$S_F^P = \{f \in L^P(\Omega, Z) : f(\omega) \in F(\Omega) \ \ \mu - \text{a.e.}\}.$$

Lemma 2.32. *Let Z be a separable Banach space and $(\Omega, \mathcal{A}, \mu)$ a $\sigma-$finite measure space. Given $F : \Omega \to 2^Z \setminus \{\emptyset\}$ is a multifunction, if $F(\cdot)$ is graph measurable and $1 \leq p \leq \infty$, then $S_F^P \neq \emptyset$ if and only if*

$$\inf\{\|x\| : x \in F(\omega)\} \leq h(\omega) \ \ \mu - \textit{a.e. for some} \ \ h \in L^P(\Omega).$$

CHAPTER III

ORIGINAL IMPULSIVE SYSTEMS

The purpose of this chapter is to establish the existence of solutions for a class of semilinear impulsive equations. The first section will introduce our class of equations and the various assumptions to be used. In addition, we will discuss local existence, global existence, uniqueness of solutions and continuous dependence of the solutions with respect to the initial function. In the remaining section, we will introduce admissible control space and discuss uniqueness of solutions and continuous dependence on the initial and control value.

3.1 The first order nonlinear impulsive equation

We discuss the following impulsive equation

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), \quad t \in (0,T) \setminus D, \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad i = 1, 2, \dots, n, \end{cases} \tag{3.1}$$

in Banach space$(X, \|\cdot\|)$, where $D = \{t_1, t_2, \cdots, t_n\} \subset (0,T)$, $0 < t_1 < t_2 < \cdots < t_n < T$. A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ and $\Delta x(t_i) = x(t_i^+) - x(t_i)$. This system contains the jump in the state x at time t_i with J_i determining the size of the jump at t_i.

In what follows, let the Banach space $(X, \|\cdot\|_X)$ be the state space, $I \equiv [0,T]$ be a closed and bounded interval of real line, $C(I,X)$ denote the space of continuous functions and $C^1(I,X)$ denote the space of one order continuous differentiable functions. Let $L(X,Y)$ denote the space of bounded linear operators from X to Y and $L(X)$ denote the space of bounded linear operators from X to X. We denote the ball $\{x \in X : \|x\| \leq r\}$ by B_r.

Define $PC(I, X) \equiv \{x : I \to X | x(t)$ is continuous at $t \neq t_i$, left continuous at $t = t_i$ and right hand limit $x(t_i^+)$ exists $\}$. Equipped with the supremum norm topology, it is a Banach space.

We introduce the following assumptions.

[A] : The operator A is the infinitesimal generator of a C_0-semigroup $\Big\{T(t), t \geq 0\Big\}$ on X.

[F] : $f : I \times X \to X$ is an operator such that

1. $t \to f(t, \xi)$ is measurable and locally Lipschitz continuous with respect to the last variable, i.e. for any finite number $\rho > 0$ there exists constant $L_1(\rho) > 0$ such that

$$\|f(t, x_1) - f(t, x_2)\|_X \leq L_1(\rho)\|x_1 - x_2\|_X,$$

   $\forall x_1,\ x_2 \in B_\rho$.

2. There exists a constant $k > 0$, such that

$$\|f(t, x)\|_X \leq k(1 + \|x\|_X).$$

[J] : $J_i : X \to X$ is an operator such that

1. J_i maps bounded set to bounded set.

2. There exist constants $h_i > 0, \quad i = 1, 2, \ldots, n$ such that

$$\|J_i(x) - J_i(y)\| \leq h_i\|x - y\|, \qquad x, y \in X.$$

3.1.1 Nonlinear equation without impulsive

Now, we consider nonlinear evolution equations

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)) & t > 0 \\ x(0) = x_0 \end{cases} \tag{3.2}$$

Definition 3.1 (Mild Solution). A function $x \in C([0,T], X)$ is said to be a *mild solution* of the (3.2) if x satisfies the integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau, \quad t \in I \equiv [0,T].$$

Theorem 3.1. *Suppose the assumptions **[A]** and **[F]** hold, then the problem (3.2) has a unique mild solution on* $[0,T]$.

Proof: Step 1: Local existence

Define a closed ball $\bar{B}(x_0, 1)$ as follows.

$$\bar{B}(x_0, 1) = \{x \in C([0,T_1], X),\ \|x(t) - x_0\| \leq 1,\ 0 \leq t \leq T_1\},$$

where T_1 would be chosen later and $\|x(t)\| \leq 1 + \|x_0\| = \rho,\ 0 \leq t \leq T_1,\ \bar{B}(x_0, 1) \subseteq C([0,T_1], X)$ is a closed set.

Define a map P on $\bar{B}(x_0, 1)$ by

$$(Px)(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau$$

and define $M \equiv \sup\limits_{t\in[0,T]} \|T(t)\|$.

Using assumption**[F]**, one can verify that P maps $\bar{B}(x_0, 1)$ to $\bar{B}(x_0, 1)$. To prove this, we note that

$$\begin{aligned} \|(Px)(t) - x_0\| &\leq \|T(t)x_0 - x_0\| + \int_0^t \|T(t-\tau)\|\|f(\tau, x(\tau))\|d\tau \\ &\leq \|T(t)x_0 - x_0\| + \int_0^t \|T(t-\tau)k(1 + \|x(\tau)\|)d\tau \\ &\leq \|T(t)x_0 - x_0\| + Mkt + Mk\rho t \\ &= Mk(1+\rho)t + \|T(t)x_0 - x_0\|. \end{aligned}$$

Since $T(t)$ is the strongly continuous C_0-semigroup, there exists $T_{11} > 0$ s.t. for all $t \in [0, T_{11}]$, $\|T(t)x_0 - x_0\| \leq \frac{1}{2}$. Now, let $0 < T_{22} < \frac{1}{2Mk(1+\rho)}$. Set $T_1' = \min\{T_{11}, T_{22}\}$ hence for all $t \in [0, T_1']$ we have $\|(Px)(t) - x_0\| \leq 1$. This means that $(Px)(t) \in \bar{B}(x_0, 1)$. Hence $P : \bar{B}(x_0, 1) \to \bar{B}(x_0, 1)$.

Next, to show that P is a contraction map on $\bar{B}(x_0, 1)$.

Let $x_1, x_2 \in \bar{B}(x_0, 1)$. By assumption **[F](1)**, we have

$$\begin{aligned}\|(Px_1)(t) - (Px_2)(t)\| &\leq \int_0^t \|T(t-\tau)\|\|f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))\|d\tau \\ &\leq MtL_1(\rho)\|x_1 - x_2\|.\end{aligned}$$

Now, let $0 < T'' = \frac{1}{2ML_1(\rho)}$, then $\|(Px_1)(t) - (Px_2)(t)\| \leq \frac{1}{2}\|x_1 - x_2\|$. This means that the map P is contraction map. This implies that we can choose $T_1 = \min\{T_1', T_1''\} > 0$ (small enough) such that P is a contraction map on $\bar{B}(x_0, 1)$. By contraction map principle, there exists a unique fixed point, i.e. equation (3.2) has a unique mild solution on $[0, T_1]$.

Step 2: An (*a priori*) Estimate

Suppose $x(\cdot)$ is a mild solution of (3.2), then we have

$$\begin{aligned}\|x(t)\| &\leq \|T(t)x_0\| + \int_0^t \|T(t-\tau)\|\|f(\tau, x(\tau))\|d\tau \\ &\leq M\|x_0\| + Mk\int_0^t (1 + \|x(\tau)\|)d\tau \\ &\leq M\|x_0\| + MkT + Mk\int_0^t \|x(\tau)\|d\tau.\end{aligned}$$

By Gronwall inequality, we obtain

$$\begin{aligned}\|x(t)\| &\leq (M\|x_0\| + MkT)e^{Mk\int_0^t d\tau} \\ &\leq (M\|x_0\| + MkT)e^{MkT} \equiv \overline{M}.\end{aligned}$$

That is, there exists a constant $\overline{M} = (M\|x_0\| + MkT)e^{MkT} > 0$ such that for $t \in [0, T]$ we have $\|x_(t)\| \leq \overline{M}$.

Step 3: Global Existence

We give the result on existence of mild solution on $[0, T]$.

First, by step2 we get a priori estimate of the solutions. That is, there exists $\overline{M} > 0$ such that for $t \in [0, T]$ we have $\|x(t)\| \leq \overline{M}$.

Second, we have to find the δ and extend solution for $[0, T_1]$ to $[0, T]$.

We start by showing that for every $t_0 \geq 0,\ x_0 \in X$, the initial value problem(3.2) has, under our assumptions, a unique mild solution x on an interval $[t_0, T_1]$.

Indeed, let $T_1 = t_0 + \delta(t_0, \|x_0\|), \quad t_0 \geq 0$

where

$$\delta = \min\{1, \frac{\|x_0\|}{k + L_1(\rho)\overline{M}}\}.$$

The mapping P defined by

$$(Px)(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau$$

maps the ball of radius $2M\|x_0\|$ centered at 0 of $C([t_0, T_1], X)$ into itself. This follows from the estimate, for $0 \leq t \leq t_0 + 1$.

$$\begin{aligned}
\|(Px)(t)\| &\leq \|T(t)x_0\| + \int_{t_0}^t \|T(t-\tau)\|\|f(\tau, x(\tau))\|d\tau \\
&\leq M\|x_0\| + M\int_{t_0}^t \|f(\tau, 0)\|d\tau + M\int_{t_0}^t \|f(\tau, x(\tau)) - f(\tau, 0)\|d\tau \\
&\leq M\|x_0\| + Mk(t-t_0) + ML_1(\rho)\int_{t_0}^t \|x(\tau)\|d\tau \\
&\leq M\|x_0\| + Mk(t-t_0) + ML_1(\rho)\overline{M}(t-t_0) \\
&\leq M(\|x_0\| + (k(t-t_0) + ML_1(\rho)\overline{M})(t-t_0)) \\
&\leq 2M\|x_0\|
\end{aligned}$$

where the last inequality follows from the definition of T_1. In this ball, f satisfies a uniform Lipschitz condition with constant $L_1(\rho)$, so it is a contraction map on this ball. By the contraction map principle, there exists a unique fixed point. i.e. equation (3.2) has a unique mild solution $x(t)$ on $[t_0, T_1]$. Since δ only depends on

$(t_0, \|x_0\|)$ and $\|x(t)\| \leq \overline{M},\ x(t)$ can be extended to $[t_0, T_1 + \delta]$.

Especially, if $t_0 = 0$, then $x(t)$ on $[0, \delta]$ can be extended to $[0, 2\delta]$ with $\delta(\|x_0\|) > 0$.

By defining on $[\delta, 2\delta]$, $x(t) = y(t)$ where $y(t)$ is the solution of the integral equation

$$y(t) = T(t-\delta)x(\delta) + \int_\delta^t T(t-\tau)f(\tau, x(\tau))d\tau, \qquad \delta \leq t \leq 2\delta.$$

Moreover, δ depends on $\|x(\delta)\|$.

Similarly, one can verify that the (3.2) has a unique mild solution on $[2\delta, 3\delta], [3\delta, 4\delta], \ldots$

This implies the global existence of mild solution of (3.2).$\square$

3.1.2 Impulsive Evolution Equation

We consider the following impulsive equation

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), \quad t \in [0, T] \setminus D, \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad i = 1, 2, \ldots, n. \end{cases} \tag{3.3}$$

Definition 3.2 (Mild Solution). A function $x \in PC(I, X)$ is called a *mild solution* of system (2.1) if it satisfies the following integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x(t_i))$$

for all $t \in [0, T]$.

Theorem 3.2. *Suppose the assumptions **[A]**,**[F]** and **[J]** hold, then the system (2.1) has a unique mild solution on* $[0, T]$.

Proof: First, we study the following equation

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), \quad 0 \leq t \leq t_1, \\ x(0) = x_0. \end{cases} \tag{3.4}$$

By Theorem 3.1, we have

$$x_1(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x_1(\tau))d\tau \quad , t \in [0, t_1)$$

is a unique mild solution of (3.4).

Now, define

$$x_1(t_1) = T(t_1)x_0 + \int_0^{t_1} T(t_1-\tau)f(\tau, x_1(\tau))d\tau$$

so that $x_1(\cdot)$ is left continuous at t_1.

Secondly, consider $t \in (t_1, t_2]$. We study the following equation

$$\begin{cases} \dot{x}(t) + Ax(t) = f(t, x(t)), \quad t_1 < t \leq t_2 \\ x(t_1) = x_1 \end{cases}$$

where

$$\begin{aligned} x_1 &= x(t_1 - 0) + J_1(x(t_1 - 0)) \\ &= x_1(t_1) + J_1(x(t_1)). \end{aligned}$$

By Theorem 3.1 again, we have a unique mild solution $x_2(\cdot)$ satisfying for $t \in (t_1, t_2]$:

$$\begin{aligned} x_2(t) &= T(t-t_1)x(t_1) + \int_{t_1}^t T(t-\tau)f(\tau, x_2(\tau))d\tau \\ &= T(t-t_1)[x_1(t_1) + J_1(x(t_1))] + \int_{t_1}^t T(t-\tau)f(\tau, x_2(\tau))d\tau \\ &= T(t-t_1)[T(t_1)x_0 + \int_0^{t_1} T(t_1-\tau)f(\tau, x_1(\tau))d\tau + J_1(x(t_1))] \\ &\quad + \int_{t_1}^t T(t-\tau)f(\tau, x_2(\tau))d\tau \\ &= T(t)x_0 + \int_0^{t_1} T(t_1-\tau)f(\tau, x_1(\tau))d\tau + T(t-t_1)J_1(x(t_1)) \\ &\quad + \int_{t_1}^t T(t-\tau)f(\tau, x_2(\tau))d\tau. \end{aligned}$$

Define

$$x(t) = \begin{cases} x_1(t) & \text{if} \quad 0 \leq t \leq t_1, \\ x_2(t) & \text{if} \quad t_1 < t \leq t_2. \end{cases}$$

Hence

$$x_2(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau + T(t-t_1)J_1(x(t_1)).$$

Now define

$$x_2(t_2) = T(t_2)x_0 + \int_0^{t_2} T(t_2-\tau)f(\tau, x(\tau))d\tau + T(t_2-t_1)J_1(x(t_1))$$

so that $x_2(\cdot)$ is left continuous at t_2.

Third, consider $t \in (t_2, t_3]$. We study the following equation

$$\begin{cases} \dot{x}(t) + Ax(t) = f(t, x(t)), \quad t_2 < t \leq t_3, \\ \qquad x(t_2) = x_2, \end{cases}$$

where

$$\begin{aligned} x_2 &= x(t_2 - 0) + J_2(x(t_2 - 0)) \\ &= x_2(t_2) + J_2(x(t_2)). \end{aligned}$$

By Theorem 3.1 again, we have a unique mild solution $x_3(\cdot)$ satisfying for $t \in (t_2, t_3]$

$$\begin{aligned} x_3(t) &= T(t-t_2)x(t_2) + \int_{t_2}^t T(t-\tau)f(\tau, x_3(\tau))d\tau \\ &= T(t-t_2)[x_2(t_2) + J_2(x(t_2))] + \int_{t_2}^t T(t-\tau)f(\tau, x_3(\tau))d\tau \\ &= T(t-t_2)[T(t_2)x_0 + \int_0^{t_2} T(t_2-\tau)f(\tau, x(\tau))d\tau + T(t_2-t_1)J_1(x(t_1)) + J_2(x(t_2))] \\ &\quad + \int_{t_2}^t T(t-\tau)f(\tau, x_3(\tau))d\tau \\ &= T(t)x_0 + \int_0^{t_2} T(t_1-\tau)f(\tau, x(\tau))d\tau + T(t-t_1)J_1(x(t_1)) + T(t-t_2)J_2(x(t_2)) \\ &\quad + \int_{t_2}^t T(t-\tau)f(\tau, x_3(\tau))d\tau \\ &= T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau + T(t-t_1)J_1(x(t_1)) + T(t-t_2)J_2(x(t_2)) \end{aligned}$$

where $x(t) = x_3(t)$ if $t_2 < t \leq t_3$.

Similarly, this procedure can be repeated on $t \in (t_3, t_4],\ (t_4, t_5], \ldots,\ (t_n, T]$, Thus we obtain a unique mild solution of problem (2.1) on $[0, T]$ given by

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x(t_i)), \quad 0 \le t \le T.$$

Next, we consider the continuous dependence of solutions on the initial values.

Theorem 3.3. *Assuming that the hypotheses of Theorem 3.2 are satisfied, if $x_0,\ y_0 \in X$ and if $x(t),\ y(t)$ are mild solutions of equation (2.1) which satisfy $x(0) = x_0$ and $y(0) = y_0$. Then there exists a constant $C > 0$ s.t.*

$$\sup_{t\in[0,T]} \|x(t) - y(t)\| \le C\|x_0 - y_0\|.$$

Proof: For $t \in [0, t_1]$

$$\begin{aligned}
\|x(t)\| &\le \|T(t)x_0\| + \int_0^t \|T(t-\tau)\|\|f(\tau, x(\tau))\|d\tau \\
&\le M\|x_0\| + M\int_0^t k(1 + \|x(\tau)\|)d\tau \\
&\le M\|x_0\| + Mkt + Mk\int_0^t \|x(\tau)\|d\tau.
\end{aligned}$$

By Gronwall inequality, we have

$$\|x(t)\| \le M(\|x_0\| + kt)\exp\int_0^t Mkd\tau.$$

Hence, there is a finite number $\rho_1 > 0$ such that $\|x(t)\| \le \rho_1$. For the same reason, $\|y(t)\| \le \rho_1$. Then

$$\begin{aligned}
\|x(t) - y(t)\| &\le \|T(t)x_0 - T(t)y_0\| + \int_0^t \|T(t-\tau)\|\|f(\tau, x(\tau)) - f(\tau, y(\tau))\|d\tau \\
&\le M\|x_0 - y_0\| + ML(\rho_1)\int_0^t \|x(\tau) - y(\tau)\|d\tau.
\end{aligned}$$

By the Gronwall inequality, we have

$$\|x(t) - y(t)\| \le M\|x_0 - y_0\|\exp\int_0^t ML(\rho_1)d\tau.$$

Hence

$$\|x(t) - y(t)\| \le C_1\|x_0 - y_0\|,$$

where $C_1 \equiv M \exp \int_0^t ML(\rho_1)d\tau$.

For $t \in (t_1, t_2]$;

$$\begin{aligned}
\|x(t)\| &\leq \|T(t)x_0\| + \int_0^t \|T(t-\tau)\|\|f(\tau, x(\tau))\|d\tau + \|T(t-t_1)\|\|J_1(x(t_1))\| \\
&\leq M(\|x_0\| + h_1\rho_1 + \|J_1(0)\|) + M\int_0^t k(1 + \|x(\tau)\|)d\tau \\
&\leq M(\|x_0\| + h_1\rho_1 + \|J_1(0)\| + kt) + Mk\int_0^t \|x(\tau)\|d\tau.
\end{aligned}$$

By the Gronwall inequality, we have

$$\|x(t)\| \leq M(\|x_0\| + h_1\rho_1 + kt + \|J_1(0)\|) \exp \int_0^{t_2} Mkd\tau.$$

Hence, there is a finite number $\rho_2 > 0$, such that $\|x(t)\| \leq \rho_2$. For the same reason, $\|y(t)\| \leq \rho_2$. Then

$$\begin{aligned}
\|x(t) - y(t)\| &\leq \|T(t)x_0 - T(t)y_0\| + \int_0^t \|T(t-\tau)\|\|f(\tau, x(\tau)) - f(\tau, y(\tau))\|d\tau \\
&\quad + \|T(t-t_1)\|\|J_1(x(t_1)) - J_1(y(t_1))\| \\
&\leq M\|x_0 - y_0\| + ML(\rho_2)\int_0^t \|x(\tau) - y(\tau)\|d\tau + Mh_1C_1\|x_0 - y_0\| \\
&= M(1 + h_1C_1)\|x_0 - y_0\| + ML(\rho_2)\int_0^t \|x(\tau) - y(\tau)\|d\tau.
\end{aligned}$$

By the Gronwall inequality, we have

$$\|x(t) - y(t)\| \leq M(1 + h_1C_1)\|x_0 - y_0\| \exp \int_0^t ML(\rho_2)d\tau.$$

Hence

$$\|x(t) - y(t)\| \leq C_2\|x_0 - y_0\|,$$

where $C_2 \equiv M(1 + h_1C_1) \exp \int_0^t ML(\rho_1)d\tau$.

Repeating the procedure above, we can deduce that there exist finite $\rho,\ C > 0$, such that for $t \in [0, T]$

$$\|x(t)\| \leq \rho, \quad \|y(t)\| \leq \rho$$

and

$$\|x(t) - y(t)\| \leq C\|x_0 - y_0\|.$$

3.2 Original Control Systems

In this section we introduce three different of admissible controls space U_{ad}.

Case 1: Let Γ be a compact Polish space (i.e. separable complete metric space). We define

$$U_{ad}^1 = \{u : [0,T] \to \Gamma |\ u \text{ is strongly measurable function}\}.$$

Case 2: Let Γ be a compact Polish space, $U : [0,T] \to P_{fc}(\Gamma)$ a measurable multifunction.

We define

$$U_{ad}^2 = \{u : [0,T] \to \Gamma |\ u \text{ is strongly measurable function and } u(t) \in U(t) \text{ a.e. in } 0 \leq t \leq T\}.$$

Case 3: Let Γ be a separable Banach space, $U : [0,T] \to P_{fc}(\Gamma)$ a measurable multifunction.

We define

$$U_{ad}^3 = \{u : [0,T] \to \Gamma |\ u \in L^1([0,T],\Gamma) \text{ and } u(t) \in U(t) \text{ a.e. in } 0 \leq t \leq T\}.$$

By the Measurable Selection theorem, the three definitions $U_{ad} \neq \emptyset$.(see Aubin and Frakowska (1990)).

We make the following assumptions for our control systems.

Assumption:

[F1] $f : I \times X \times \Gamma \to X$ is an operator such that

1. $t \mapsto f(t,\xi,\eta)$ is measurable, and
   $(\xi,\eta) \mapsto f(t,\xi,\eta)$ is continuous on $X \times \Gamma$.

2. For any finite number $\rho > 0$ there exists a constant $L(\rho) > 0$ such that

   $$\|f(t,x_1,\sigma) - f(t,x_2,\sigma)\|_X \leq L(\rho)\|x_1 - x_2\|_X,$$

   for all $\|x_1\|_X \leq \rho,\ \|x_2\|_X \leq \rho$, and $t \in [0,T],\ \sigma \in \Gamma$.

3. There exists a constant $k_F > 0$, such that

$$\|f(t,x,\sigma)\|_X \leq k_F(1+\|x\|_X) \ \ (\forall \sigma \in \Gamma, \ t \in I).$$

We consider the following original control system

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t,x(t),u(t)), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad u(\cdot) \in U_{ad}. \end{cases} \tag{3.5}$$

Definition 3.3 (Mild Solution). A function $x \in PC(I,X)$ is called a *mild solution* of system (2.2) with respect to $u \in U_{ad}$ if x satisfies the following integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau,x(\tau),u(\tau))d\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x(t_i))$$

for all $t \in [0,T]$.

Theorem 3.4. *Suppose the assumption [A], [J] and [F1] hold. For every $x_0 \in X$ and $u \in U_{ad}$, the system (2.2) has a unique mild solution.*

Proof: Let $u \in U_{ad}$ and define $g_u(t,x) = f(t,x,u)$.
Since f is measurable, then $g_u : I \times X \to X$ is measurable on $[0,T]$ for each fixed $x \in X$.
Hence g_u satisfies assumption [F]. By theorem 3.2, the system (2.2) has a unique mild solution $x \in PC(I,X)$.

Next, we consider continuous dependence of solutions on parameters such as initial states and controls for the controlled system (2.2).

Theorem 3.5. *Assuming that the hypotheses of Theorem 3.4 are satisfied, and that there exists a nonnegative $L_2(\rho_1)$ for any finite number $\rho_1 > 0$, provided*

$\|x_1\|_X,\ \|x_2\|_X \le \rho_1$, *such that for every* $\sigma_1, \sigma_2 \in \Gamma$ *(in case 3):*

$$\|f(t,x_1,\sigma_1) - f(t,x_2,\sigma_2)\|_X \le L_2(\rho_1)(\|x_1 - x_2\|_X + \|\sigma_1 - \sigma_2\|_\Gamma).$$

Then the mild solution of (2.2) is continuously dependent on the initial value and control with respect to the strong topology $X \times L^1([0,T],\Gamma)$, *i.e., let* x_{ξ,u_1} *and* x_{η,u_2} *denote the mild solution of (2.2) corresponding to the initial value and control* $\{\xi, u_1\}$ *and* $\{\eta, u_2\}$ *respectively. Then there is a constant* $C > 0$ *such that*

$$\sup_{t\in[0,T]} \|x_{\xi,u_1}(t) - x_{\eta,u_2}(t)\| \le C(\|\xi - \eta\|_X + \|u_1 - u_2\|_{L^1([0,T],\Gamma)}).$$

Proof: It suffices to prove the inequality above, since it also implies continuity, For $t \in [0, t_1]$

$$\begin{aligned}\|x_{\xi,u_1}(t)\| &\le \|T(t)\|\|\xi\| + \int_0^t \|T(t-\tau)\|\|f(\tau, x_{\xi,u_1}(\tau), u_1(\tau))\|d\tau \\ &\le M\|\xi\| + M\int_0^t k_F(1 + \|x_{\xi,u_1}(\tau)\|)d\tau \\ &\le M\|\xi\| + Mk_F t + Mk_F \int_0^t \|x_{\xi,u_1}(\tau)\|d\tau.\end{aligned}$$

By the Gronwall inequality, we have

$$\|x_{\xi,u_1}(t)\| \le M(\|\xi\| + k_F t)\exp(Mk_F \int_0^t d\tau).$$

Thus there is a finite number $\rho_1 > 0$, such that $\|x_{\xi,u_1}(t)\| \le \rho_1$, For the same reason, $\|x_{\eta,u_2}(t)\| \le \rho_1$.

Then,

$$\begin{aligned}\|x_{\xi,u_1}(t) - x_{\eta,u_2}(t)\| &\le \int_0^t \|T(t-\tau)\|\|f(\tau, x_{\xi,u_1}(\tau), u_1(\tau)) - f(\tau, x_{\eta,u_2}(\tau), u_2(\tau))\|d\tau \\ &\quad +\|T(t)\|\|\xi - \eta\| \\ &\le M\int_0^t L_2(\rho_1)(\|x_{\xi,u_1}(\tau)) - x_{\eta,u_2}(\tau)\| + \|u_1(\tau) - u_2(\tau)\|)d\tau \\ &\quad +M\|\xi - \eta\| \\ &\le M\|\xi - \eta\| + ML_2(\rho_1)\|u_1 - u_2\|_{L^1([0,T],\Gamma)} \\ &\quad +ML_2(\rho_1)\int_0^t \|x_{\xi,u_1}(\tau) - x_{\eta,u_2}(\tau)\|d\tau.\end{aligned}$$

By the Gronwall inequality, we have

$$\|x_{\xi,u_1}(t)-x_{\eta,u_2}(t)\| \leq M(\|\xi-\eta\|+L_2(\rho_1)\|u_1-u_2\|_{L^1([0,T],\Gamma)})\exp(ML_2(\rho_1)\int_0^{t_1} d\tau).$$

Hence,

$$\|x_{\xi,u_1}(t)-x_{\eta,u_2}(t)\| \leq C_1(\|\xi-\eta\|_X+\|u_1-u_2\|_{L^1([0,T],\Gamma)})$$

where $C_1 = M\max(1, L_2(\rho_1))\exp\int_0^{t_1} ML_2(\rho)d\tau$.

For $t \in (t_1, t_2)$:

$$\begin{aligned}
\|x_{\xi,u_1}(t)\| &\leq \|T(t)\|\|\xi\| + \int_0^t \|T(t-\tau)\|\|f(\tau, x_{\xi,u_1}(\tau), u_1(\tau))\|d\tau \\
&\quad +\|T(t-t_1)\|\|J_1(x_{\xi,u_1}(t_1))\| \\
&\leq M(\|\xi\|+h_1\rho_1+\|J_1(0)\|)+M\int_0^t k_F(1+\|x_{\xi,u_1}(\tau)\|)d\tau \\
&\leq M(\|\xi\|+h_1\rho_1+\|J_1(0)\|+k_Ft)+Mk_F\int_0^t \|x_{\xi,u_1}(\tau)\|d\tau.
\end{aligned}$$

By the Gronwall inequality, we have

$$\|x_{\xi,u_1}(t)\| \leq M(\|\xi\|+h_1\rho_1+\|J_1(0)\|+k_Ft)\exp\int_0^{t_2} Mk_Fd\tau.$$

Thus, there is a finite number $\rho_2 > 0$, such that $\|x_{\xi,u_1}(t)\| \leq \rho_2$. For the same reason, $\|x_{\eta,u_2}(t)\| \leq \rho_2$.

Then,

$$\begin{aligned}
\|x_{\xi,u_1}(t)-x_{\eta,u_2}(t)\| &\leq \int_0^t \|T(t-\tau)\|\|f(\tau, x_{\xi,u_1}(\tau), u_1(\tau))-f(\tau, x_{\eta,u_2}(\tau), u_2(\tau))\|d\tau \\
&\quad \|T(t-t_1)\|\|J_1(x_{\xi,u_1}(t_1))-J_1(x_{\eta,u_1}(t_1))\|+\|T(t)\|\|\xi-\eta\| \\
&\leq M\int_0^t L_2(\rho_2)(\|x_{\xi,u_1}(\tau))-x_{\eta,u_2}(\tau)\|+\|u_1(\tau)-u_2(\tau)\|)d\tau \\
&\quad +M\|\xi-\eta\|+Mh_1\|x_{\xi,u_1}(t_1)-x_{\eta,u_1}(t_1)\| \\
&\leq M(C_1h_1+L_2(\rho_2))\|u_1-u_2\|_{L^1([0,T],\Gamma)}+M(1+C_1h_1)\|\xi-\eta\| \\
&\quad +M\int_0^t L_2(\rho_2)\|x_{\xi,u_1}(\tau)-x_{\eta,u_2}(\tau)\|d\tau \\
&\leq Mw(|\xi-\eta\|_X+\|u_1-u_2\|_{L^1([0,T],\Gamma)}) \\
&\quad +M\int_0^t L_2(\rho_2)\|x_{\xi,u_1}(\tau)-x_{\eta,u_2}(\tau)\|d\tau,
\end{aligned}$$

where $w = \max(1, L_2(\rho_2)) + C_1 h_1$.

By Gronwall inequality, we have

$$\|x_{\xi,u_1}(t) - x_{\eta,u_2}(t)\| \leq C_2(\|\xi - \eta\|_X + \|u_1 - u_2\|_{L^1([0,T],\Gamma)})$$

where $C_2 = M\{\max(1, L_2(\rho_2)) + C_1 h_1\} \exp \int_0^{t_2} M L_2(\rho_2) d\tau$.

Repeating the procedure above, we can deduce that there exist finite number ρ and $C > 0$, such that for $t \in [0, T]$:

$$\|x_{\xi,u_1}(t)\| \leq \rho, \quad \|x_{\eta,u_2}(t)\| \leq \rho,$$

and

$$\sup_{t \in [0,T]} \|x_{x_0,u_1}(t) - x_{y_0,u_2}(t)\| \leq C(\|x_0 - y_0\|_X + \|u_1 - u_2\|_{L^1([0,T],\Gamma)}).$$

CHAPTER IV

RELAXED IMPULSIVE SYSTEMS

In this chapter, we introduce the space of relaxed control and study the existence and uniqueness of relaxed control systems.

4.1 Relaxed Control Spaces

In this section we introduce some basic concepts and results that are necessary for the theories in this chapter. For details and proofs we refer to Fattorini (1999).

Let Γ be compact polish space (i.e separable complete metrics space)

Define

$$Y = C(\Gamma) = \{u : \Gamma \to \mathbb{R} \mid u \text{ is continuous}\}$$

with norm

$$\|u\|_Y = \sup_{\sigma \in \Gamma} |u(\sigma)| < +\infty.$$

First, to show $(Y, \|\cdot\|_Y)$ is normed space. Let $u, v \in Y$ and α be a scalar.

(N1) Since $|u(\sigma)| \geq 0$ for all $\sigma \in \Gamma$ so $\sup\limits_{\sigma \in \Gamma} |u(\sigma)| \geq 0$. Hence $\|u\|_Y \geq 0$.

(N2)

$$\begin{aligned}
\|u\|_Y = 0 &\Leftrightarrow \sup_{\sigma \in \Gamma} |u(\sigma)| = 0 \\
&\Leftrightarrow |u(\sigma)| = 0 \quad \forall \sigma \in \Gamma \\
&\Leftrightarrow u(\sigma) = 0 \quad \forall \sigma \in \Gamma \\
&\Leftrightarrow u = 0.
\end{aligned}$$

Hence $\|u\|_Y = 0 \Leftrightarrow u = 0$.

(N3)

$$\begin{aligned}\|\alpha u\|_Y &= \sup_{\sigma\in\Gamma} |\alpha u(\sigma)| \\ &= \sup_{\sigma\in\Gamma} |\alpha||u(\sigma)| \\ &= |\alpha| \sup_{\sigma\in\Gamma} |u(\sigma)| \\ &= |\alpha|\|u\|_Y.\end{aligned}$$

Hence $\|\alpha u\|_Y = |\alpha|\|u\|_Y$.

(N4)

$$\begin{aligned}\|u+v\|_Y &= \sup_{\sigma\in\Gamma} |(u+v)(\sigma)| \\ &= \sup_{\sigma\in\Gamma} |u(\sigma)+v(\sigma)| \\ &\le \sup_{\sigma\in\Gamma}(|u(\sigma)|+|v(\sigma)|) \\ &= \sup_{\sigma\in\Gamma} |u(\sigma)| + \sup_{\sigma\in\Gamma} |v(\sigma)| \\ &= \|u\|_Y + \|v\|_Y.\end{aligned}$$

Hence $\|u+v\|_Y \le \|u\|_Y + \|v\|_Y$.

So $(Y, \|\cdot\|_Y)$ is normed space.

Next, to show $(Y, \|\cdot\|_Y)$ is Banach space.

Let $\{u_n\} \subseteq Y$ be a Cauchy sequence. Then for every $\sigma \in \Gamma$, we have

$$|u_n(\sigma) - u_m(\sigma)| \to 0 \text{ as } n, m \to \infty.$$

Since $\mathbb{R}$ is Banach space, $u_n(\sigma) \to u(\sigma)$ as $n \to \infty$. It implies that u is bounded and continuous, we have $u(\cdot) \in Y$. Hence $(Y, \|\cdot\|_Y)$ is Banach space.□

Let $\Phi(\mathbf{C})$ be a $\sigma-$field generated by the collection $\mathbf{C}$ of all closed subsets of Γ. Then we have $(\Gamma, \Phi(\mathbf{C}))$ is measurable space.

Definition 4.1. Let X be a Hausdorff topological space. A Borel measure on X is a measure whose domain is $\mathcal{B}(X)$. Suppose that $\mathcal{A}$ is a $\sigma-$algebra on X such that $\mathcal{B}(X) \subset \mathcal{A}$. A positive measure μ on $\mathcal{A}$ is *regular* if

1. each compact subset K of X satisfies $\mu(K) < +\infty$,

2. each set A in $\mathcal{A}$ satisfies

$$\mu(A) = \inf\{\mu(U) \mid A \subset U \text{ and } U \text{ is open}\}, \text{ and}$$

3. each open subset U of X satisfies

$$\mu(U) = \sup\{\mu(K) \mid K \subset U \text{ and } K \text{ is compact}\}.$$

Let $\Sigma_{rca}(\Gamma)$ be the space of all regular countably additive measures on the measurable space $(\Gamma, \Phi(\mathbf{C}))$. For μ in $\Sigma_{rca}(\Gamma)$, $|\mu|$ denotes the total variation of μ.

Lemma 4.1. *The dual space $C(\Gamma)^*$ can be identified algebraically and metrically $\Sigma_{rca}(\Gamma)$ with the norm $\|\mu\|_{\Sigma_{rca}(\Gamma)} = |\mu|(\Gamma)$. The duality pairing of $C(\Gamma)$ and $\Sigma_{rca}(\Gamma)$ is given by*

$$\langle f, \mu \rangle = \int_\Gamma f(\sigma)\mu(d\sigma).$$

Definition 4.2. Given a Banach space X, we say that an X^*-valued function $f(\cdot)$ is $X-$*weakly measurable* if

$$t \mapsto \langle f(t), y \rangle$$

is measurable function for each $y \in X$.

We denote by $L^\infty_w(I, C(\Gamma)^*)$ the linear space of all $C(\Gamma)^*-$valued $C(\Gamma)-$weakly measurable functions $g(\cdot)$ such that there exists $C > 0$ with

$$|\langle g(t), y \rangle| \leq C\|y\|_{C(\Gamma)} \quad \text{a.e. in } 0 \leq t \leq T, \tag{4.1}$$

for each $y \in C(\Gamma)$, (the null set where (4.1) fails to hold may depend on y). Two functions $g(\cdot),\ h(\cdot)$ are said to be equivalent in $L^\infty_w(I, C(\Gamma)^*)$ (in symbols, $g \approx h$) if $\langle g(t), y\rangle = \langle h(t), y\rangle$ a.e. in $0 \le t \le T$ for each $y \in C(\Gamma)$.

Let $L^1(I, C(\Gamma))$ be the space of all (equivalence class of) strongly measurable $C(\Gamma)-$valued functions $u(\cdot)$ defined on I such that

$$\|u\| = \int_I \|u(t)\| dt < +\infty.$$

We have $L^1(I, C(\Gamma))$ is Banach space.

Lemma 4.2. *The dual $L^1(I, C(\Gamma))^*$ is isometrically isomorphic to $L^\infty_w(I, C(\Gamma)^*)$. The duality pairing between both spaces is given by*

$$\langle\langle g, f\rangle\rangle = \int_0^T \langle g(t), f(t)\rangle dt. \tag{4.2}$$

Where $g \in L^\infty_w(I, C(\Gamma)^*)$ and $f \in L^1(I, C(\Gamma))$.

Proof: If $g(\cdot) \in L^\infty_w(I, C(\Gamma)^*)$, the integral (4.2) makes sense for every $f(\cdot) \in L^1(I, C(\Gamma))$ and defines a bounded linear functional in $L^1(I, C(\Gamma))$.
In fact, if $f(\cdot)$ is countably valued, the function $t \to \langle g(t), f(t)\rangle$ is measurable in each set where $f(t)$ is constant, thus it is measurable, and

$$|\langle g(t), f(t)\rangle| \le \|g\|_{L^\infty_w(I,C(\Gamma)^*)} \|f(t)\|_{C(\Gamma)} \quad \text{a.e. in } 0 \le t \le T. \tag{4.3}$$

That $t \to \langle g(t), f(t)\rangle$ is measurable for any $f(\cdot) \in L^1(I, C(\Gamma))$ and (4.3) holds follows approximating $f(\cdot)$ uniformly a.e. by countably valued functions.
Inequality (4.3) also shows that, if $\|g\|_{L^1(I,C(\Gamma))^*}$ is the norm of $g(\cdot)$ as a linear functional in $L^1(I, C(\Gamma))$ then $\|g\|_{L^1(I,C(\Gamma))^*} \le \|g\|_{L^\infty_w(I,C(\Gamma)^*)}$.
To prove the opposite inequality, let $\varepsilon > 0$. By definition of $\|g\|_{L^\infty_w}(I, C(\Gamma)^*)$ there exists $y \in C(\Gamma)$, $\|y\| = 1$ and a set e of positive measure $0 \le t \le T$ such that $\langle g(t), f(t)\rangle \ge \|g\|_{L^\infty_w(I,C(\Gamma)^*)} - \varepsilon(t \in e)$. Hence, if $\chi(\cdot)$ is the characteristic

function of e we have $\langle g, \chi y\rangle \geq |e|(\|g\|_{L^\infty_w(I,C(\Gamma)^*)} - \varepsilon)$ so that $\|g\|_{L^1(I,C(\Gamma))^*} \geq \|g\|_{L^\infty_w(I,C(\Gamma)^*)} - \varepsilon$.

We now show that every bounded linear functional in $L^1(I, C(\Gamma))$ admits the representation (4.2). Let ϕ be such a functional. Given $f(\cdot) \in L^1(I)$, the map $y \to \phi(f(\cdot)y)$ is a linear functional in $C(\Gamma)$. Since

$$|\phi(f(\cdot)y)| \leq \|\phi\|_{L^1(I,C(\Gamma))^*}\|f\|_{L^1(I)}\|y\|_{C(\Gamma)}, \tag{4.4}$$

this functional is bounded (with norm $\leq \|\phi\|_{L^1(I,C(\Gamma))^*}\|f\|_{L^1(I)}$), thus there exists $y_f^* \in X^*$ with $\phi(f(\cdot)y) \leq \langle y_f^*, y\rangle$ and $\|y_f^*\|_{C(\Gamma)^*} \leq \|\phi\|_{L^1(I,C(\Gamma))^*}\|f\|_{L^1(I)}$.

Consider the map

$$B_f = y_f^* \tag{4.5}$$

from L^1 into $C(\Gamma)^*$. Obviously, B is a linear bounded operator from $L^1(I)$ into $C(\Gamma)^*$ with norm $\|B\|_{(L^1(I),C(\Gamma)^*)} \leq \|\phi\|_{L^1(I,C(\Gamma))^*}$. By Theorem 12.2.4 (Dunford-Pettis Part 1) in Fattorini(1999), there exists a representing function $g(\cdot) \in L^\infty_w(I, C(\Gamma)^*)$ satisfying $\|g\|_{L^\infty_w(I,C(\Gamma)^*)} = \|B\|_{(L^1(I),C(\Gamma)^*)}$ and such that

$$\phi(f(\cdot)y) = \langle y_f^*, y\rangle = \int_0^T f(t)\langle g(t), y\rangle dt \tag{4.6}$$

for $f(\cdot) \in L^1(I)$ and $y \in C(\Gamma)$. The argument after (4.3) shows that $\|g\|_{L^\infty_w(I,C(\Gamma)^*)} = \|\phi\|_{L^1(I,C(\Gamma))^*}$. Moreover, taking $f_1(\cdot), \ldots, f_n(\cdot) \in L^1(I)$ and $y_1, \ldots, y_n \in C(\Gamma)$ we have

$$\phi(\sum f_k(\cdot)y_k) = \sum \int_0^T f_k(t)\langle g(t), y_k\rangle dt = \int_0^T \langle g(t), \sum f_k(t)y_k\rangle dt.$$

The proof of Lemma (4.2) ends observing that elements of the form $\sum f_k(\cdot)y_k$ with $f_k(\cdot) \in L^1(I)$ and $y_k \in C(\Gamma)$ are dense in Ł$^1(I)$.

Definition 4.3 (Radon-Nikodym Property). A Banach space X has the *Radon-Nikodym Property with respect to* (Ω, Σ, μ) if for each $\mu-$continuous vector measure $G : \Sigma \to X$ of bounded variation there exists $g \in L^1(\mu, X)$ such that $G(E) = \int_E g d\mu$ for all $E \in \Sigma$.

A Banach space X has the *Radon-Nikodym Property* if X has the Radon-Nikodym Property with respect to every finite measure space.

Definition 4.4. A Banach space E is *called a Gelfand space* if every $E-$valued absolutely continuous function is differentiable almost everywhere.

Theorem 4.3. *Let E be either (a) reflexive or (b) separable and the dual of another Banach space. Then E is a Gelfand space.*

The result above identifies some Gelfand space through the equivalent Radon-Nikodym property.

Since Γ is a compact metric space, $C(\Gamma)^*$ is a separable Banach space and hence has the Radon-Nikodym property which tells us that $L^1(I, C(\Gamma))^* = L^\infty(I, \Sigma_{rca}(\Gamma)$.

Definition 4.5. The space $R(I, \Gamma)$ of relaxed controls consists of all $\mu(\cdot)$ in $L^\infty(I, \Sigma_{rca}(\Gamma)) = L^1(I, C(\Gamma))^*$ that satisfy

1. if $f(\cdot, \cdot) \in L^1(I, C(\Gamma))$ is such that $f(t, \sigma) \geq 0$ for $\sigma \in \Gamma$ a.e. in $t \in I$ then
$$\int_0^T \int_\Gamma f(t, \sigma)\mu(t, d\sigma)dt \geq 0,$$
2. if $\chi(t)$ is the characteristic function of a measurable set $e \subseteq [0, T]$ and $\mathbf{1} \in C(\Gamma)$ is the function $\mathbf{1}(\sigma) = 1$, then
$$\int_0^T \int_\Gamma (\chi(t) \otimes \mathbf{1}(\sigma))\mu(t, d\sigma)dt = |e|.$$

Note that $\chi(\cdot) \otimes \mathbf{1}(\cdot) \in L^1(I, C(\Gamma))$.

We note that (2) can be generalized to

$$\int_0^T \int_\Gamma (\phi(t) \otimes \mathbf{1}(\sigma))\mu(t, d\sigma)dt = \int_0^T \phi(t)dt$$

for any $\phi(\cdot) \in L^1(I)$.

In fact, for $\mu(\cdot) \in R(I, \Gamma)$, we have

$$\|\mu\|_{L^\infty(I, \Sigma_{rca}(\Gamma))} \leq 1, \; \mu(t) \geq 0, \text{ and } \mu(t, \Gamma) = 1 \text{ a.e. in } 0 \leq t \leq T.$$

In particular,

$$\|\mu(t)\|_{\Sigma_{rca}(\Gamma)} = 1 \quad \text{a.e. in } 0 \leq t \leq T.$$

Lemma 4.4. *Let $\{\mu_n(\cdot)\}$ be a sequence in $R(I, \Gamma)$. Then there exists a subsequence $L^1(I, C(\Gamma))-$weakly convergent in $L^\infty(I, \Sigma_{rca}(\Gamma))$ to $\mu(\cdot) \in R_c(I, \Gamma)$.*

Sometimes, using another equivalent definition of $R(I, \Gamma)$ is more convenient. We denote by $\Pi_{rca}(\Gamma)$ the set of all probability measures μ in $\Sigma_{rca}(\Gamma)$. We denote the Dirac measure with mass at u by the functional notation $\delta(\cdot - u)$ or by δ_u. The set $D = \{\delta_u : u \in \Gamma\}$ of all Dirac measures is a subset of $\Pi_{rca}(\Gamma)$.

Lemma 4.5. $\Pi_{rca}(\Gamma)$ *is $C(\Gamma)-$weakly compact, also $C(\Gamma)-$weakly closed in* $\Sigma_{rac}(\Gamma)$.

Let $\overline{conv}$ denote closed convex hull (closure taken in the weak $C(\Gamma)-$topology). Then

$$\Pi_{rca}(\Gamma) = \overline{conv}(D).$$

Since $C(\Gamma)$ be a Banach space $\Pi_{rca}(\Gamma) \subseteq \Sigma_{rca}(\Gamma)$ convex, bounded and $C(\Gamma)-$weakly closed. A set $D \subseteq \Pi_{rca}(\Gamma)$ is *total* in $\Pi_{rca}(\Gamma)$ if $\overline{conv}(D) = \Pi_{rca}(\Gamma)$ where $\overline{conv}(D)$ is closed convex hull in the $C(\Gamma)-$weak topology of $\Sigma_{rca}(\Gamma)$. This means that for every $u \in \Pi_{rca}(\Gamma)$ there exists a generalized sequence $\{u_k\} \subseteq$

$conv(D)$ with $u_k \to u$ $C(\Gamma)-$weakly. Since $C(\Gamma)$ is separable, the equivalent relation in $L^\infty(I, \Sigma_{rca}(\Gamma))$ is equality almost everywhere. Let us denote the set

$$R(I, \Pi_{rca}(\Gamma)) = \{u \in Ł^\infty(I, \Sigma_{rca}(\Gamma)),\ \exists v \text{ s.t. } v \approx u \text{ and } v(t) \in \Pi_{rca}(\Gamma) \text{ a.e. in } 0 \le t \le T\}.$$

If $u(\cdot) \in U_{ad}$ then one can check that the Dirac delta with mass at $u(t)$ (written $\delta(\cdot - u(t))$) is an element of $R(I, \Pi_{rca}(\Gamma))$. Hence we can identify U_{ad} as a subset of $R(I, \Pi_{rca}(\Gamma))$. We note further that $R(I, \Pi_{rca}(\Gamma)) = R(I, \Gamma)$

4.2 Original Control Systems and Relaxed Control Systems

4.2.1 Original Control Systems

Let $U_0 = U_{ad}$ be the set of all strongly measurable $\Gamma-$valued functions defined in $0 \le t \le T$. The **original** control system is

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad u(\cdot) \in U_0. \end{cases} \tag{4.7}$$

We will go back to assumption **[F1]**.

[F1] $f : I \times X \times \Gamma \to X$ is an operator such that

1. $t \mapsto f(t, \xi, \eta)$ is measurable, and
   $(\xi, \eta) \mapsto f(t, \xi, \eta)$ is continuous on $X \times \Gamma$.

2. For any finite number $\rho > 0$ there exists a constant $L(\rho) > 0$ such that

$$\|f(t, x_1, \sigma) - f(t, x_2, \sigma)\|_X \le L(\rho)\|x_1 - x_2\|_X,$$

   for all $\|x_1\|_X \le \rho$, $\|x_2\|_X \le \rho$, and $t \in [0, T]$, $\sigma \in \Gamma$.

3. There exists a constant $k_F > 0$, such that

$$\|f(t,x,\sigma)\|_X \le k_F(1+\|x\|_X) \ \ (\forall \sigma \in \Gamma,\ t \in I).$$

Theorem 4.6. *Assume that assumptions [A], [J] and [F1] hold. For every $u(\cdot) \in U_0$, the original control system (4.7) has a unique mild solution.*

Proof: For any given $u(t) \in U_0$ fixed, we have that $f(t,x,u(t))$ satisfies assumption[F]. By theorem 3.2, the original system (4.7) has a unique mild solution $x \in PC([0,T],X)$:

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau,x(\tau),u(\tau))d\tau + \sum_{0<t_i<t} J_i(x(t_i)).$$

Details can be found in Theorem 3.4.

4.2.2 Relaxed Control Systems

Let $U_r = R(I,\Gamma) = R(I,\Pi_{rca}(\Gamma))$ be the space of relaxed controls. Given any relaxed control $\mu(\cdot) \in U_r$, the function $F : I \times X \times \Sigma_{rca}(\Gamma) \to X$ is defined by

$$F(t,x)\mu = \int_\Gamma f(t,x,\sigma)\mu(d\sigma),$$

the integral interpreted coordinatewise. For each $t,\ x,\ F(t,x)$ is a linear operator from $\Sigma_{rca}(\Gamma)$ into X.

For fixed $x \in X$, by assumption [F1], each $t \mapsto f(t,x,\cdot)$ is measurable and $(t,\sigma) \mapsto f(t,x,\sigma)$ belongs to $L^1(I,C(\Gamma))$.

For $\mu \in \Pi_{rac}(\Gamma)$ fixed, hence

$$F(t,x)\mu = \int_\Gamma f(t,x,\sigma)\mu(d\sigma)$$

is meaningful.

Now, let us consider the new larger system known as *relaxed impulsive system* corresponding to (4.7)

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t, x(t))\mu(t), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad \mu(\cdot) \in U_r. \end{cases} \tag{4.8}$$

The admissible control space is U_r.

Theorem 4.7. *Assume that assumptions [A], [J] and [F1] hold. For every $\mu(\cdot) \in U_r$, the relaxed control system (4.8) has a unique solution.*

Proof: For $\mu_t \in U_r$ be fixed, by assumption [F1], $(t, \sigma) \mapsto f(t, x, \sigma) \in L^1(I, C(\Gamma))$ and on the other hand $\mu_t \in L^\infty(I, \Sigma_{rca}(\Gamma))$. Hence $F(t, x)\mu_t = \int_\Gamma f(t, x, \sigma)\mu_t(d\sigma)$ is meaningful and denoted by $\tilde{F}(t, x)$.

By definition of the duality pairing between $L^1(I, C(\Gamma))$ and its dual $L^\infty(I, \Sigma_{rca}(\Gamma))$, the function

$$t \to \langle f(t, x, \cdot), \mu(t) \rangle = \int_\Gamma f(t, x, \sigma)\mu(t)(d\sigma)$$

belongs to $L^1(0, T)$. In particular, the function $\tilde{F}(t, x)$ is measurable.

Next, we show $\tilde{F}(t, x)$ satisfies the Lipschitz continuity property.

Let x_1 and x_2 belong to X, s.t. for any finite $\rho > 0$, provided $\|x_1\|_X, \ \|x_2\|_X \le \rho$, we have

$$\begin{aligned} \|\tilde{F}(t, x_1) - \tilde{F}(t, x_2)\| &\le \int_\Gamma \|f(t, x_1, \sigma) - f(t, x_2, \sigma)\|\mu_t(d\sigma) \\ &\le L(\rho)\|x_1 - x_2\|\|\mu_t\|_{\Sigma_{rca}(\Gamma)}. \end{aligned}$$

If $\mu(\cdot) \in U_r$ we may assume that $\|\mu(t)\| \equiv 1$, thus $\tilde{F}(t, x)$ satisfies Lipschitz continuity property.

Finally, we show the linearly growth condition for $\tilde{F}(t, x)$.

Let x belongs to X, we have

$$\begin{aligned}\|\tilde{F}(t,x)\| &= \|\int_\Gamma f(t,x,\sigma)\mu_t(d\sigma)\| \\ &\leq \int_\Gamma \|f(t,x,\sigma)\|\mu_t(d\sigma) \\ &\leq k_F(1+\|x\|)\|\mu_t\|_{\Sigma_{rca}(\Gamma)}\end{aligned}$$

and it clearly holds independently of $\mu(\cdot) \in U_r$. Hence, $\tilde{F}(t,x)$ satisfies assumption [F]. By theorem 3.2, the relaxed control system (4.8) has a unique solution.

4.2.3 Original and Relaxed trajectory

We will denote the set of original trajectories and relaxed trajectories of system (4.7) by X_0 and system (4.8) by X_r, respectively, i.e.

$$X_0 = \{x \in PC([0,T];X)|\ x \text{ is a solution of (4.7)corresponding to } u(\cdot) \in U_0\}$$

and $X_r = \{x \in PC([0,T];X)|\ x \text{ is a solution of (4.8) corresponding to } \mu(\cdot) \in U_r\}$.

By Theorem 4.6 and Theorem 4.7 show that $X_0 \neq \emptyset$ implies $X_r \neq \emptyset$ Moreover, since $U_0 \subseteq U_r$ we have $X_0 \subseteq X_r$.

4.3 Properties of relaxed trajectories

We introduce one more hypothesis concerning the operator A.

[A1] An operator A is the infinitesimal generator of a compact C_0-semigroup $\{T(t), t \geq 0\}$.

Lemma 4.8. *Let A satisfy assumption [A1] on some Banach space X. Let $p > 1$ and define a linear operator S by*

$$S(g(t)) = \int_0^t T(t-s)g(s)ds \qquad \forall g \in L^p((0,T),X).$$

Then $S : L^p((0,T),X) \to C([0,T],X)$ is compact.

Proof: Let $M = \sup_{0\le t\le T}\|T(t)\|$ and let $\|\cdot\|_\infty$ and $\|\cdot\|_{L^p((0,T),X)}$ denote the norms on $C([0,T],X)$ and $L^p((0,T),X)$ respectively. Also let $g_k \in L^p((0,T),X)$ with $\|g_k\|_{L^p((0,T),X)} \le 1,\ \forall k\ge 1$. We need to prove that $\{S(g_k)\}_{k\ge1}\subseteq C([0,T],X)$ and that this sequence possesses a convergent subsequence. To this end, we show that the assumptions of Arzela-Ascoli's theorem are satisfied.

We first prove that $\{S(g_k)(t)\}_{k\ge1}$ is relatively compact in X for each $t\in[0,T]$. In fact, for $t=0$, this trivial. So let $t\in(0,T]$ be fixed, and let $\varepsilon>0$ be given. If $0<\delta<t$, then

$$\begin{aligned} S(g_k)(t) &= \int_0^t T(t-s)g_k(s)ds \\ &= \int_0^{t-\delta} T(t-s)g_k(s)ds + \int_{t-\delta}^t T(t-s)g_k(s)ds \qquad (4.9)\\ &= T(\delta)\int_0^{t-\delta} T(t-\delta-s)g_k(s)ds + \int_{t-\delta}^t T(t-s)g_k(s)ds. \end{aligned}$$

Now set $x_k = \int_{t-\delta}^t T(t-s)g_k(s)ds$ and $y_k = \int_0^{t-\delta} T(t-\delta-s)g_k(s)ds$. We estimate the norms of x_k and y_k. Note that for $0\le t_1\le t_2\le t\le T$,

$$\begin{aligned} \Big\|\int_{t_1}^{t_2} T(t-s)g_k(s)ds\Big\|_X &\le \int_{t_1}^{t_2}\|T(t-s)\|\|g_k(s)\|_X ds \\ &\le \Big(\int_{t_1}^{t_2}\|T(t-s)\|^q ds\Big)^{1/q}\Big(\int_{t_1}^{t_2}\|g_k(s)\|_X^p ds\Big)^{1/p} \qquad (4.10)\\ &\le M(t_2-t_1)^{1/q}\|g_k\|_p \le M(t_2-t_1)^1/q. \end{aligned}$$

Here we have used Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$.

Thus, reducing δ so that $\delta < (\frac{\varepsilon}{2M})^q$, we obtain from (4.10) that

$$\|x_k\|_X \le M\delta^{1/q} < \frac{\varepsilon}{2},$$

and also that

$$\|y_k\|_X \le M(t-\delta)^{1/q} \le MT^{1/q}$$

for all k. This shows that the set $\{y_k\}_{k\ge1}$ is bounded in X. Thus by the compactness of the operator $T(\delta)$, we can find a finite set $\{z_i : 1\le i\le m\}$ in X such that

$$\{T(\delta)y_k\}_{k\geq 1} \subset \bigcup_{i=1}^{m} \mathcal{O}_{\frac{\varepsilon}{2}}(z_i).$$

This means that for each $k \geq 1$, there exists an element z_i so that

$$\|T(\delta)y_k - z_i\|_X < \frac{\varepsilon}{2}.$$

Then for these values of k and i we have by (4.9),

$$\begin{aligned} \|S(g_k)(t) - z_i\|_X &= \|(T(\delta)y_k + x_k) - z_i\|_X \\ &\leq \|T(\delta)y_k - z_i\|_X + \|x_k\|_X \\ &< \varepsilon. \end{aligned}$$

This show that $\{S(g_k)(t)\} \subset \bigcup_{i=1}^{m} \mathcal{O}_\varepsilon(z_i)$, so that $\{S(g_k)(t)\}_{k\geq 1}$ is totally bounded in X. Thus $\{S(g_k)(t)\}_{k\geq 1}$ is relatively compact in X.

Next, we show that the family $\{S(g_k)(t)\}_{k\geq 1}$ is equicontinuous on $[0, T]$. Let $\varepsilon > 0$ be given. Set $\varepsilon' = (\frac{\varepsilon}{6M})^q$, and choose $\delta > 0$ so that $\delta < \varepsilon'$. Let $t',\ t \in [0, T]$ with $0 < t' - t < \delta$. We consider two cases, $t' \leq \varepsilon'$ and $t' > \varepsilon'$. If $t' \leq \varepsilon'$ then by (4.10),

$$\begin{aligned} \|S(g_k)(t') - S(g_k)(t)\|_X &\leq \|S(g_k)(t')\|_X + \|S(g_k)(t)\|_X \\ &\leq \int_0^{t'} \|T(t'-s)\|\|g_k(s)\|_X ds + \int_0^{t} \|T(t-s)\|\|g_k(s)\|_X ds \\ &< 2M(t')^{1/q} \leq 2M(\varepsilon')^{1/q} \leq \varepsilon. \end{aligned} \tag{4.11}$$

If $t \geq \varepsilon'$ then

$$\begin{aligned}
S(g_k)(t') - S(g_k)(t) &= \int_0^{t'} T(t'-s)g_k(s)ds - \int_0^t T(t-s)g_k(s)ds \\
&= \int_0^t T(t'-s)g_k(s)ds + \int_t^{t'} T(t'-s)g_k(s)ds - \int_0^t T(t-s)g_k(s)ds \\
&= \int_t^{t'} T(t'-s)g_k(s)ds + \int_0^t \{T(t'-s) - T(t-s)\}g_k(s)ds \\
&= \int_t^{t'} T(t'-s)g_k(s)ds + \int_0^{t-\varepsilon'} \{T(t'-s) - T(t-s)\}g_k(s)ds \\
&\quad + \int_{t-\varepsilon'}^t \{T(t'-s) - T(t-s)\}g_k(s)ds \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

We now estimate each of the integrals. By (4.10),

$$\|I_1\|_X = \Big\| \int_t^{t'} T(t'-s)g_k(s)ds \Big\|_X \leq M(t'-t)^{1/q} < M(\varepsilon')^{1/q} = \frac{\varepsilon}{6}$$

and,

$$\begin{aligned}
\|I_3\|_X &= \Big\| \int_{t-\varepsilon'}^t \{T(t'-s) - T(t-s)\}g_k(s)ds \Big\|_X \\
&\leq \int_{t-\varepsilon'}^t T(t'-s)\|\|g_k(s)\|_X ds + \int_{t-\varepsilon'}^t T(t-s)\|\|g_k(s)\|_X ds \\
&\leq 2M(\varepsilon')^{1/q} < \frac{\varepsilon}{3}.
\end{aligned}$$

Finally, applying Hölder's inequality and a change of variables,

$$\begin{aligned}
\|I_2\|_X &= \Big\| \int_0^{t-\varepsilon'} \{T(t'-s) - T(t-s)\}g_k(s)ds \Big\|_X \\
&\leq \int_0^{t-\varepsilon'} \|T(t'-s) - T(t-s)\|\|g_k(s)\|_X ds \\
&\leq \Big(\int_0^{t-\varepsilon'} \|T(t'-s) - T(t-s)\|^q ds \Big)^{1/q} \Big(\int_0^{t-\varepsilon'} \|g_k(s)\|_X^p ds \Big)^{1/p} \\
&\leq \Big(\int_{\varepsilon'}^t \|T(t'-t+s) - T(s)\|^q ds \Big)^{1/q} \|g_k\|_{L^p((0,T),X)} \\
&\leq \Big(\int_{\varepsilon'}^t \|T(t'-t+s) - T(s)\|^q ds \Big)^{1/q}.
\end{aligned}$$

Since $T(t)$ is a compact semigroup, it is uniformly continuous on $[\varepsilon', T]$, and thus the integrand can be made arbitrarily small by choosing $t' - t$ sufficient small.

Reducing δ if necessary, it follows that $\|I_2\| < \frac{\varepsilon}{2}$ provided that $t' - t < \delta$. Thus,

$$\|S(g_k)(t') - S(g_k)(t)\|_X \leq \|I_1\|_X + \|I_2\|_X + \|I_3\|_X < \varepsilon.$$

which shows that $\{S(g_k)\}_{k\geq 1}$ is equicontinuous on $[\varepsilon', T]$. Combining with (4.11), we have shown that $\{S(g_k)\}_{k\geq 1}$ is equicontinuous on $[0, T]$. In particular, each $S(g_k)$ is an element of $C([0,T], X)$.

Finally, $\{S_g\}$ is uniformly bounded, as by (4.10)

$$S(g_k)(t)\| \leq Mt^{1/q} \leq MT^{1/q}$$

for all $t \in [0, T]$. Arzela-Ascoli's Theorem now establishes that $\{S(g_k)\}$ possesses a convergent subsequence and thus the compactness of the operator S.

Lemma 4.9. *Let X be reflexive and separable. Suppose the assumptions [A1] and [F1] hold. If $\{\mu^n(\cdot)\}$ be a sequence in $L^\infty(I, \Sigma_{rca}(\Gamma))$ with $\mu^n(\cdot) \to \mu(\cdot)$ $L^1(I, C(\Gamma))-$ weakly as $n \to \infty$ then*

$$\rho_n(\cdot) = \int_0^{\cdot} T(\cdot - \tau) \int_\Gamma f(\tau, x(\tau), \sigma)(\mu^n(\tau) - \mu(\tau))(d\sigma)d\tau \to 0 \ \ in\ C(I, X)\ as\ n \to \infty,$$

where $x \in C([0,T], X)$.

Proof: Due to reflexivity of X, $T^*(t), t \geq 0$ is a C_0-semigroup in Banach space X^* (Ahmed, 1991)

Define $\quad g_n(\tau) = \int_\Gamma f(\tau, x(\tau), \sigma)(\mu^n(\tau) - \mu(\tau))(d\sigma)$

then

$$\begin{aligned}\|g_n(\tau)\| &\leq \int_\Gamma \|f(\tau, x(\tau), \sigma)\|(\mu^n(\tau) - \mu(\tau))(d\sigma) \\ &\leq k_F(1 + \|x(\tau)\|)\|\mu^n(\tau) - \mu(\tau)\|_{\Sigma_{rca}(\Gamma)} \\ &\leq 2k_F(1 + \|x(\tau)\|).\end{aligned}$$

Since $x(t)$ is the solution of (4.8), then it is bounded by $\bar{M}$. This implies that $\{g_n(\cdot)\}$ is bounded in $L^p(I, X)$, $1 < p < \infty$. Hence there exists a subsequence (denote with the same symbol) with $g_n(\cdot) \xrightarrow{w} g(\cdot)$ in $L^p(I, X)$.

By lemma 4.8, we have

$$\rho_n(\cdot) = \int_0^{\cdot} T(\cdot - \tau) g_n(\tau) d\tau \xrightarrow{s} \int_0^{\cdot} T(\cdot - \tau) g(\tau) d\tau \equiv \rho(\cdot) \quad \text{in} \quad C(I, X).$$

For fixed $0 \le t \le T, \quad h^* \in X^*$, we have

$$\begin{aligned}
\langle \rho_n(t), h^* \rangle &= \int_0^t \langle T(t-\tau) g_n(\tau), h^* \rangle d\tau \\
&= \int_0^t \langle g_n(\tau), T^*(t-\tau) h^* \rangle d\tau \\
&= \int_0^t \int_\Gamma \langle f(\tau, x(\tau), \sigma), T^*(t-\tau) h^* \rangle (\mu^n(\tau) - \mu(\tau))(d\sigma) d\tau \\
&= \int_0^T \int_\Gamma \xi(\tau, \sigma)(\mu^n(\tau) - \mu(\tau))(d\sigma) d\tau
\end{aligned}$$

where $\quad \xi(\tau, \sigma) = \langle f(\tau, x(\tau), \sigma), T^*(t-\tau) h^* \rangle$.

By assumption [F1], for τ fixed, the map $\sigma \mapsto \xi(\tau, \sigma)$ is continuous. This implies that $\xi(\tau, \sigma) \in C(\Gamma)$ and

$$|\xi(\tau, \sigma)| \le k(1 + \|x(\tau)\|).$$

Hence $\xi(\cdot, \cdot) \in L^1(I, C(\Gamma))$.

Since $\mu^n(\cdot) \to \mu(\cdot) \quad L^1(I, C(\Gamma))-$weakly, then

$$\int_0^t \int_\Gamma \xi(\tau, \sigma)(\mu^n(\tau) - \mu(\tau))(d\sigma) dt \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

This implies that, for fixed $t \in I$,

$$\langle \rho_n(t), h^* \rangle \longrightarrow 0 \qquad \forall h^* \in X^*.$$

Hence $\rho_n(t) \xrightarrow{w} 0$ as $n \to \infty$. Thus $\rho(t) \equiv 0$. This means that $\rho_n(\cdot) \longrightarrow 0$ as $n \to \infty$ in $C(I, X)$.□

Remark: Using the same proof, one can see that the result of Lemma 4.9 is also true when $x \in PC([0, T], X)$.

Theorem 4.10. *Let X be reflexive and separable. Suppose the assumptions [A1], [J] and [F1] hold. If $x(\cdot,\mu)$ be the solution of (4.8) corresponding to μ then, for every $\varepsilon > 0$, there exists $u(\cdot) \in U_0$ such that $x(\cdot,u)$ is solution of (4.7) corresponding to u and satisfying*

$$\|x(\cdot,\mu) - x(\cdot,u)\|_{PC(I,X)} < \varepsilon, \qquad t \in I.$$

Proof: Let $\mu(\cdot) \in U_r$, since $U_0 \subseteq U_r$ and U_0 is dense in U_r. Thus there exists a sequence $\{u_n\} \subseteq U_0$ such that $u_n \xrightarrow{w^*} \mu$.

Let $x_n(\cdot) = x(\cdot,u_n)$ be the solution of (4.7)

and $x(\cdot) = x(\cdot,\mu)$ be the solution of (4.8) corresponding to μ.

Since

$$\begin{aligned} x_n(t) &= T(t)x_0 + \int_0^t T(t-\tau)f(\tau,x_n(\tau),u_n(\tau))d\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x_n(t_i)) \\ &= T(t)x_o + \int_0^t T(t-\tau)[\int_\Gamma f(\tau,x_n(\tau),\sigma)\delta_{u_n}(\tau)(d\sigma)]d\tau \\ &\quad + \sum_{0<t_i<t} T(t-t_i)J_i(x_n(t_i)) \end{aligned}$$

and

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)[\int_\Gamma f(\tau,x(\tau),\sigma)\mu(\tau)(d\sigma)]d\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x(t_i))$$

we have

$$\begin{aligned} x_n(t) - x(t) &= \int_0^t T(t-\tau)[\int_\Gamma (f(\tau,x_n(\tau),\sigma)\delta_{u_n}(\tau) - f(\tau,x(\tau),\sigma)\delta_{u_n}(\tau))(d\sigma)]d\tau \\ &\quad + \int_0^t T(t-\tau)[\int_\Gamma f(\tau,x(\tau),\sigma)(\delta_{u_n}(\tau) - \mu(\tau))(d\sigma)]d\tau \\ &\quad + \sum_{0<t_i<t} T(t-t_i)[J_i(x_n(t_i)) - J_i(x(t_i))] \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

By the Lipschitz condition [F1], we get

$$|I_1| \le M\int_0^t L(\rho)\|x_n(\tau) - x(\tau)\|,$$

where $I_1 \equiv \int_0^t T(t-\tau)[\int_\Gamma (f(\tau, x_n(\tau), \sigma)\delta_{u_n}(\tau) - f(\tau, x(\tau), \sigma)\delta_{u_n}(\tau))(d\sigma)]d\tau$ and M is a bound for $\|T(t)\|$ in $0 \leq t \leq T$.

Using assumption [J](2), we have

$$|I_3| \leq \sum_{0<t_i<t} Mh_i \|x_n(t_i) - x(t_i)\|$$

where $I_3 \equiv \sum_{0<t_i<t} T(t-t_i)[J_i(x_n(t_i)) - J_i(x(t_i))]$.

We denote the second integral I_2 by $\rho_n(t)$, i.e.,

$$\rho_n(t) \equiv I_2 \equiv \int_0^t T(t-\tau)[\int_\Gamma f(\tau, x(\tau), \sigma)(\delta_{u_n}(\tau) - \mu(\tau))(d\sigma)]d\tau.$$

Thus

$$\|x_n(t)-x(t)\| \leq M \int_0^t L(\rho)\|x_n(\tau)-x(\tau)\|d\tau + \|\rho_n(t)\| + \sum_{0<t_i<t} Mh_i\|x_n(t_i)-x(t_i)\|.$$

By the impulsive Gronwall inequality, we obtain

$$\|x_n(t) - x(t)\| \leq C\|\rho_n(t)\|$$

where $C \equiv \prod_{0<t_i<t} (1 + Mh_i)exp(ML(\rho)t)$.

By using lemma 4.9, we have that $\rho_n(\cdot) \to 0$ as $n \to \infty$ in $PC([0, T], X)$. Hence $x_n(\cdot) \to x(\cdot)$ as $n \to \infty$ in $PC([0, T], X)$. The proof is now complete.$\square$

CHAPTER V

RELAXED OPTIMAL CONTROL

In this chapter, we now study the existence of optimal Lagrange-type controls.

5.1 Relaxed optimal control

5.1.1 Original optimal control problem

Let U_{ad} be the set of all strongly measurable $\Gamma-$ valued functions defined in $0 \leq t \leq T$.

Consider the following original control system

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad u(\cdot) \in U_{ad}. \end{cases} \tag{5.1}$$

The cost functional J is given by

$$J(u) = \int_I l(t, x_u(t), u(t))dt,$$

where $x_u(\cdot)$ is solution of system (5.1) corresponding to the control $u \in U_{ad}$. Consider the following Lagrange optimal control problem (P_0):

Find a control policy $u_0 \in U_{ad}$ such that it imparts a minimum to the cost functional J, i.e.

$$J(u_0) \leq J(u), \qquad \forall u \in U_{ad}.$$

5.1.2 Relaxed optimal control problem

Let $U_r = R(I, \Gamma)$ be the space of relaxed controls.

Consider the following relaxed control system

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t, x(t))\mu(t), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad \mu(\cdot) \in U_r. \end{cases} \tag{5.2}$$

The cost functional J is given by

$$J(\mu) = \int_I \int_\Gamma l(t, x_\mu(t), \sigma)\mu(t)(d\sigma)dt.$$

where $x_\mu(\cdot)$ is solution of (5.2) corresponding to the control $\mu \in U_r$.

Consider the following Lagrange optimal control problem (P_r):

Find a control policy $\mu_0 \in U_r$ such that it imparts a minimum to the cost functional J, i.e.

$$J(\mu_0) \leq J(\mu) \qquad \forall \mu \in U_r.$$

5.2 Existence of relaxed optimal control

We make the following hypotheses concerning the integrand $l(\cdot, \cdot, \cdot)$.

[L] $l : I \times X \times \Gamma \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is an operator such that

1. $(t, \xi, \sigma) \to l(t, \xi, \sigma)$ is measurable,
2. $(\xi, \sigma) \to l(t, \xi, \sigma)$ is lower semicontinuous,
3. $|l(t, \xi, \sigma)| \leq \theta_R(t)$ for almost all $t \in I$ provided that $\|\xi\|_X \leq R, \sigma \in \Gamma$ and $\theta_R(t) \in L^1(I)$.

Before proving the existence of the relaxed control, we need a lemma.

Lemma 5.1. *Suppose* $h : I \times X \times \Gamma \to \mathbb{R}$ *satisfying*

(1) $t \mapsto h(t,\xi,\sigma)$ *is measurable,* $(\xi,\sigma) \mapsto h(t,\xi,\sigma)$ *is continuous,*

(2) $|h(t,\xi,\sigma)| \le \psi_R(t) \in L^1(I)$ *provided that* $\|\xi\|_X \le R$ *and* $\sigma \in \Gamma$.

If $x_n \to x$ *in* $C(I,X)$ *then* $h_n(\cdot,\cdot) \to h(\cdot,\cdot)$ *in* $L^1(I, C(\Gamma))$ *as* $n \to \infty$,

where $h_n(t,\sigma) = h(t,x_n(t),\sigma)$ *and* $h(t,\sigma) = h(t,x(t),\sigma)$.

Proof: It follows immediately from the first hypothesis of this lemma that

$$h_n, h \in L^1(I, C(\Gamma)).$$

For each fixed $t \in I$, we shall show that $h_n(t,\cdot) \to h(t,\cdot)$ in $C(\Gamma)$ as $n \to \infty$.

By definition, we have

$$\sup_{\sigma\in\Gamma} |h_n(t,\sigma) - h(t,\sigma)| = \|h_n(t,\cdot) - h(t,\cdot)\|_{C(\Gamma)}.$$

Since Γ is compact, there exists $\sigma_n \in \Gamma$ such that

$$|h_n(t,\sigma_n) - h(t,\sigma_n)| = \|h_n(t,\cdot) - h(t,\cdot)\|_{C(\Gamma)}$$

and we can assume $\sigma_n \to \sigma^*$ as $n \to \infty$.We note that

$$\begin{aligned}\sup_{\sigma\in\Gamma} |h_n(t,\sigma) - h(t,\sigma)| &= |h_n(t,\sigma_n) - h(t,\sigma_n)| \\ &\le |h_n(t,\sigma_n) - h_n(t,\sigma^*)| + |h_n(t,\sigma^*) - h(t,\sigma^*)| \\ &\quad + |h(t,\sigma^*) - h(t,\sigma_n)|.\end{aligned}$$

Then, by continuity of h, we have $|h_n(t,\sigma_n) - h(t,\sigma_n)| \to 0$ as $n \to \infty$.

This means

$$\|h_n(t,\cdot) - h(t,\cdot)\|_{C(\Gamma)} \to 0 \quad \text{as} \quad n \to \infty.$$

Assuming that $x_n \to x$ in $C(I,X)$ as $n \to \infty$ then there exists R such that $\|x_n(t)\|,\ \|x(t)\| \le R$.

Hence, by the second hypothesis of this lemma, we have

$$\begin{aligned}\|h_n(t,\cdot) - h(t,\cdot)\|_{C(\Gamma)} &\le \|h_n(t,\cdot)\|_{C(\Gamma)} + \|h(t,\cdot)\|_{C(\Gamma)} \\ &\le \psi_R(t) + \psi_R(t) = 2\psi_R(t).\end{aligned}$$

We have

$$\lim_{n\to\infty} \|h_n(t,\cdot) - h(t,\cdot)\|_{C(\Gamma)} = 0,$$

hence

$$\int_I (\lim_{n\to\infty} \|h_n(t,\cdot) - h(t,\cdot)\|_{C(\Gamma)})dt = 0$$

and

$$\lim_{n\to\infty} [\int_I \|h_n(t,\cdot) - h(t,\cdot)\|_{C(\Gamma)}dt] = 0.$$

This implies

$$\int_I \|h_n(t,\cdot) - h(t,\cdot)\|_{C(\Gamma)}dt \to 0 \;\text{ as }\; n\to\infty.$$

We have

$$h_n(\cdot,\cdot) \to h(\cdot,\cdot) \;\text{ in }\; L^1(I, C(\Gamma)) \;\text{ as }\; n\to\infty.$$

This proves the lemma.□

Let $m_r = \inf\{J(\mu) : \mu \in U_r\}$. We have the following existence of relaxed optimal control.

Theorem 5.2. *Suppose assumptions [A1], [F1], [J] and [L] hold. Then there exists* $\mu^* \in U_r$ *such that* $J(\mu^*) = m_r$.

Proof: Let $\{\mu_n\}$ be a minimizing sequence so that $\lim_{n\to\infty} J(\mu_n) = m_r$.
Recall that U_r is w^*-compact in $L^\infty(I, \Sigma_{rca}(\Gamma))$, by passing to a subsequence if necessary, we may assume $\mu_n \overset{w^*}{\to} \mu^*$ in $L^\infty(I, \Sigma_{rca}(\Gamma))$ as $n\to\infty$.
Next, we shall prove that (x, μ^*) is an optimal pair, where x is the solution of (5.2) corresponding to μ^*.

Since every lower semicontinuous measurable integrand is the limit of an increasing sequence of Caratheodory integrands, there exists an increasing sequence of Caratheodory integrands $\{l_k\}$ such that

$$l_k(t,\xi,\sigma) \uparrow l(t,\xi,\sigma) \;\text{ as }\; k\to\infty \;\text{ for all }\; t\in I, \sigma\in\Gamma.$$

Invoking the definition of the weak topology and applying Lemma 5.1 on each subinterval of $[0,T]$, $l_k(t,x_n(t),\sigma) \to l_k(t,x(t),\sigma)$ as $n \to \infty$ for almost all $t \in I$ and all $\sigma \in \Gamma$. Then

$$\begin{aligned} J(x,\mu^*) = J(\mu^*) &= \int_I \int_\Gamma l(t,x(t),\sigma)\mu^*(t)(d\sigma)dt \\ &= \lim_{k\to\infty} \int_I \int_\Gamma l_k(t,x(t),\sigma)\mu^*(t)(d\sigma)dt \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \int_I \int_\Gamma l_k(t,x_n(t),\sigma)\mu_n(t)(d\sigma)dt \\ &\le \lim_{n\to\infty} \int_I \int_\Gamma l(t,x_n(t),\sigma)\mu_n(t)(d\sigma)dt \\ &= m_r. \end{aligned}$$

However, by definition of m_r, it is obvious that $J(x,\mu^*) \ge m_r$. Hence $J(x,\mu^*) = m_r$. This implies that (x,μ^*) is an optimal pair.$\square$

5.3 Relaxation Theorem

Suppose that problem (P_0) has a solution i.e. If $J(u) = \int_I l(t,x(t),u(t))dt$ is the cost function for the original problem, then $\exists u_0 \in U_{ad}$ s.t.

$$J(u_0) \le J(u) \qquad \forall u \in U_{ad}$$

or

$$J(u_0) = \inf\{J(u),\ u \in U_{ad}\} = m_0.$$

In general, since $U_{ad} \subseteq U_r$, we have $m_r \le m_0$. It is desirable that $m_r = m_0$, i.e. our relaxation is reasonable. We have the following relaxation theorem. For this, we need stronger hypotheses on l than [L]:

[L1] $l : I \times X \times \Gamma \to \mathbb{R}$ is an integrality such that

1. $(t,\xi,\sigma) \to l(t,\xi,\sigma)$ is measurable,

2. $(\xi,\sigma) \to l(t,\xi,\sigma)$ is continuous,

3. $|l(t,\xi,\sigma)| \leq \theta_R(t)$ for almost all $t \in I$, provided $\|\xi\|_X \leq R$, $\sigma \in \Gamma$ and $\theta_R \in L^1(I)$.

Theorem 5.3. *If assumptions [A1], [J], [F1] and [L1] hold and* Γ *is compact, then* $m_0 = m_r$.

Proof: Let (x,μ^*) be the optimal pair (the existence was guaranteed by the previous theorem) that is $m_r = J(x,\mu^*)$.

By Theorem 6.6 there exists $\{u^n\} \subseteq U_{ad}$ and $\{x_n\} \subseteq PC(I,X)$ such that

$$\delta_{u_n}(\cdot) \to \mu^*(\cdot)\ L^1(I,C(\Gamma)) - \text{weakly in } L^\infty(I,\Sigma_{rca}(\Gamma)),$$

and

$$x_n \to x \text{ in } PC(I,X) \text{ as } n \to \infty.$$

Applying Lemma 5.1 to each subinterval of $[0,T]$, one can verify that

$$l(\cdot,x_n(\cdot),\cdot) \to l(\cdot,x(\cdot),\cdot) \ \text{ in } \ L^1(I,C(\Gamma)).$$

By definition of weak topology on U_r, we have

$$\begin{aligned} J(x_n,u_n) = J(x_n,\delta_{u_n}) &= \int_I\int_\Gamma l(t,x_n(t),\sigma)\delta_{u_n}(t)(d\sigma)dt \\ &\to \int_I\int_\Gamma l(t,x(t),\sigma)\mu^*(t)(d\sigma)dt = J(x,\mu^*) = m_r. \end{aligned}$$

But, by definition of m_0, $J(u_n) \geq m_0$.

Hence $m_r = \lim\limits_{n\to\infty} J(u_n) \geq m_0$.This implies $m_0 = m_r$.

The proof is now complete.

CHAPTER 6

CONCLUSIONS

6.1 Thesis Summary

In this thesis, we have studied the existence of both, solution for a class of semilinear impulsive evolution equations and relaxed controls, in the case where the operator involved is the infinitesimal generator of C_0-semigroup.

6.1.1 Problems Considered

This thesis has considered the following problems:

1. Existence and uniqueness of mild solution for a class of nonlinear impulsive evolution equations as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), \quad t \in I \setminus D, \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad i = 1, 2, \dots, n. \end{cases} \tag{6.1}$$

2. Existence and uniqueness of mild solution for the following controlled system governed by impulsive differential evolution equation

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad u(\cdot) \in U_{ad}. \end{cases} \tag{6.2}$$

3. Existence and uniqueness of mild solution for the following relaxed control system corresponding to (6.2) is

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t, x(t))\mu(t), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad \mu(\cdot) \in U_r. \end{cases} \tag{6.3}$$

and discuss the existence of optimal relaxed control for Lagrange problem (P_r) and study relation between problem (P_0) and (P_r). We found that in order to guarantee that these properties hold, we required some of the following assumptions:

[A] : The operator A is the infinitesimal generator of a C_0-semigroup $\Big\{T(t), t \geq 0\Big\}$ on a Banach space X.

[F] : $f : I \times X \to X$ is an operator such that

1. $t \to f(t, \xi)$ is measurable and locally Lipschitz continuous with respect to the last variable, i.e. for any finite number $\rho > 0$ there exists constant $L_1(\rho) > 0$ such that

$$\|f(t, x_1) - f(t, x_2)\|_X \leq L_1(\rho)\|x_1 - x_2\|_X,$$

$\forall x_1,\ x_2 \in B_\rho$.

2. There exists a constant $k > 0$, such that

$$\|f(t, x)\|_X \leq k(1 + \|x\|_X).$$

[J] : $J_i : X \to X$ is an operator such that

1. J_i maps bounded set to bounded set.

2. There exist constant $h_i > 0, \quad i = 1, 2, \ldots, n$ such that

$$\|J_i(x) - J_i(y)\| \leq h_i\|x - y\|, \qquad x, y \in X.$$

[F1] $f : I \times X \times \Gamma \to X$ is an operator such that

1. $t \mapsto f(t, \xi, \eta)$ is measurable, and
   $(\xi, \eta) \mapsto f(t, \xi, \eta)$ is continuous on $X \times \Gamma$.

2. For any finite number $\rho > 0$ there exists a constant $L(\rho) > 0$ such that

$$\|f(t, x_1, \sigma) - f(t, x_2, \sigma)\|_X \le L(\rho)\|x_1 - x_2\|_X,$$

   for all $\|x_1\|_X \le \rho,\ \|x_2\|_X \le \rho$, and $t \in [0, T],\ \sigma \in \Gamma$.

3. There exists a constant $k_F > 0$, such that

$$\|f(t, x, \sigma)\|_X \le k_F(1 + \|x\|_X) \ \ (\forall \sigma \in \Gamma,\ t \in I).$$

[A1] An operator A is the infinitesimal generator of a compact C_0-semigroup $\{T(t), t \ge 0\}$.

[L] $l : I \times X \times \Gamma \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is an operator such that

1. $(t, \xi, \sigma) \to l(t, \xi, \sigma)$ is measurable,

2. $(\xi, \sigma) \to l(t, \xi, \sigma)$ is lower semicontinuous,

3. $|l(t, \xi, \sigma)| \le \theta_R(t)$ for almost all $t \in I$ provided that $\|\xi\|_X \le R, \sigma \in \Gamma$ and $\theta_R(t) \in L^1(I)$.

[L1] $l : I \times X \times \Gamma \to \mathbb{R}$ is an integrality such that

1. $(t, \xi, \sigma) \to l(t, \xi, \sigma)$ is measurable,

2. $(\xi, \sigma) \to l(t, \xi, \sigma)$ is continuous,

3. $|l(t, \xi, \sigma)| \le \theta_R(t)$ for almost all $t \in I$, provided $\|\xi\|_X \le R,\ \sigma \in \Gamma$ and $\theta_R \in L^1(I)$.

6.1.2 Results

The main results of this thesis are summarized as follows:

Theorem 6.1. *Suppose the assumption [A], [F] and [J] hold, then the system (6.1) has a unique mild solution on* $[0, T]$.

Theorem 6.2. *Assume that the hypotheses of Theorem 6.1 are satisfied, if* $x_0,\ y_0 \in X$ *and if* $x(t)$, $y(t)$ *are mild solutions of equation (6.1)which satisfy* $x(0) = x_0$ *and* $y(0) = y_0$. *Then there exists a constant* $C > 0$ *s.t.*

$$\sup_{t\in[0,T]} \|x(t) - y(t)\| \leq C\|x_0 - y_0\|.$$

Theorem 6.3. *Suppose the assumption [A], [J] and [F1] hold. For every* $x_0 \in X$ *and* $u \in U_{ad}$, *the system (6.2) has a unique mild solution.*

Theorem 6.4. *Assume that the hypotheses of Theorem 6.3 are satisfied, and there exists a nonnegative* $L_2(\rho_1)$ *for any finite number* $\rho_1 > 0$, *provided* $\|x_1\|_X,\ \|x_2\|_X \leq \rho_1$, *such that for every* $\sigma_1, \sigma_2 \in \Gamma$ *(in case 3).*

$$\|f(t, x_1, \sigma_1 - f(t, x_2, \sigma_2))\|_X \leq L_2(\rho_1)(\|x_1 - x_2\|_X + \|\sigma_1 - \sigma_2\|_\Gamma)$$

Then the mild solution of (6.2) is continuous dependence on the initial value and control with respect to the strong topology $X \times L^1([0, T], \Gamma)$, *i.e., let* x_{ξ,u_1} *and* x_{η,u_2} *denote the mild solution of (6.2) corresponding to the initial value and control* $\{\xi, u_1\}$ *and* $\{\eta, u_2\}$ *respectively, there is a constant* $C > 0$ *such that*

$$\sup_{t\in[0,T]} \|x_{\xi,u_1}(t) - x_{\eta,u_2}(t)\| \leq C(\|\xi - \eta\|_X + \|u_1 - u_2\|_{L^1([0,T],\Gamma)}).$$

Theorem 6.5. *Assume that assumption [A], [J] and [F1] hold. For every* $\mu(\cdot) \in U_r$, *the relaxed control system (6.3) has a unique solution.*

Theorem 6.6. *Let X be reflexive and separable. Suppose the assumptions [A1], [J] and [F1] hold.If $x(\cdot,\mu)$ be the solution of (6.3) corresponding to μ then, for every $\varepsilon > 0$, there exists $u(\cdot) \in U_0$ such that $x(\cdot, u)$ is solution of (6.2) corresponding to u and satisfying*

$$\|x(\cdot,\mu) - x(\cdot,u)\|_{PC(I,X)} < \varepsilon \qquad , t \in I.$$

Theorem 6.7. *Suppose assumptions [A1], [F1], [J] and [L] hold. Then there exists $\mu^* \in U_r$ such that $J(\mu^*) = m_r$.*

Theorem 6.8. *If assumptions [A1], [J], [F1] and [L1] hold and Γ is compact, then $m_0 = m_r$.*

6.2 Discussion and Recommendations

We studied a class of semilinear impulsive evolution equation associated with C_0-semigroups and discussed the existence, uniqueness and continuous dependence of mild solutions. Further more we investigated the existence, uniqueness and continuous dependence of mild solutions for controlled system governed by impulsive evolution equation.

For the relaxed control, we discussed the existence and uniqueness of mild solution and studied properties of relaxed trajectories.

In addition, we discussed the existence of optimal relaxed control for Lagrange problem (P_r) and studied relation between problem (P_0) and (P_r).

Based on the results and the approach of our thesis, we can continue to discuss related problems, such as:

1. relaxed optimal control of this problem but in the case that Γ is a normal topological space;

2. relaxed optimal control of this problem but in the case of analytic semigroups;

3. time relaxed control problem for semilinear impulsive evolution equations;

4. semilinear impulsive inclusion;

5. relaxed optimal control of this problem for integro-differential impulsive equations.

Furthermore, we can also consider applications to practical problems and algorithms for computing relaxed optimal controls.

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APPENDICES

APPENDIX A

PUBLICATION

Relaxation of Nonlinear Impulsive Controlled Systems on Banach Spaces *

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ABSTRACT

Relaxation control for a class of semilinear impulsive controlled systems is investigated. Existence of mild solutions for semilinear impulsive controlled systems is proved. By introducing a regular countably additive measure, we convexify the original control systems and obtain the corresponding relaxed control systems. The existence of optimal relaxed controls and relaxation results is also proved.

Keywords: Impulsive systems, Banach spaces, semilinear evolution equations, relaxation.

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A.1 Introduction

Let $I \equiv [0, T]$ be a closed and bounded interval of the real line. Let $D \equiv \{t_1, t_2, \cdots, t_n\}$ be a partition on $(0, T)$ such that $0 < t_1 < t_2 < \cdots < t_n < T$. A semilinear impulsive controlled system can be described by the following evolution equation

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)) \quad t \in (0, T) \setminus D, \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad i = 1, 2, \cdots, n, \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in a Banach space X, the functions $f,\ J_i,\ i = 1, 2, ...,$ are continuous nonlinear operators from X to X, and $\Delta x(t_i) \equiv x(t_i + 0) - x(t_i - 0) = x(t_i + 0) - x(t_i)$. This system contains the jump in the state x at time t_i with J_i determining the size of the jump at t_i. In this paper, we aim to prove the existence of state-control pairs of the system (1.1). Moreover, by defining the objective functional $J(x, u) = \int_0^T L(t, x(t), u(t))dt$, we shall find sufficient conditions to guarantee the existence of optimal state-control pairs when convexity conditions on a certain orientor field are not assumed. This is the relaxation problem. By introducing regular countable additive measures, we convexify the original control systems and obtain the corresponding relaxed control systems. Under some reasonable assumptions, we prove that the set of original trajectories is dense in the set of relaxed trajectories in an appropriate space. The existence of optimal relaxed controls is obtained under some regularity hypotheses concerning the cost functional.

In recent years, relaxed systems have attracted much attention since some orientor fields do not satisfy the convexity condition. See, for instance, [1,8,10]. Ahmed [1] dealt with this problem and introduced measure-valued controls in which the control space is compact and values of relaxed control are countable additive measures, while Papageorgious [8] and other authors including us continue to discuss this problem in another direction. However, to our knowledge, there are few authors who have studied the problem of relaxed controls of system governed by impulsive evolution equations, particularly, relaxation on semilinear impulsive evolution equations. We organize the paper as follows. In section 2, we describe the original control systems and the corresponding relaxed control systems. The properties of relaxed trajectories are given in section 3. Section 4 is devoted to the existence of relaxed optimal controls and relaxation theorems.

A.2 Original and Relaxed Controlled Systems

In what follows, let the Banach space $(X, \|\cdot\|_X)$be the state space, $I \equiv [0,T]$ be a closed and bounded interval of the real line, $C(I,X)$ denote the space of continuous functions, and $C^1(I,X)$ denote the space of one order continuous differentiable functions. Let $L(X,Y)$ denote the space of bounded linear operators from X to Y and $L(X)$ denote the space of bounded linear operators from X to X.

We denote the ball $\{x \in X : \ \|x\| \leq r\}$ by B_r. Define $PC(I,X) \equiv \{x : I \to X :$ $x(t)$ is continuous at $t \neq t_i$, left continuous at $t = t_i$, and right hand limit $x(t_i^+)$ exists$\}$. Equipped with the supremum norm topology, it is a Banach space.

We introduce the following assumptions.

[**A**] : The operator A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on X.

[**F**] : $f : I \times X \to X$ is an operator such that

1. $t \to f(t,\xi)$ is measurable and locally Lipschitz continuous with respect to the last variable, i.e., for any finite number $\rho > 0$, there exists constant

$L_1(\rho) > 0$ such that

$$\|f(t,x_1) - f(t,x_2)\|_X \le L_1(\rho)\|x_1 - x_2\|_X,$$

$\forall x_1, x_2 \in B_{\rho.}$

2. There exists a constant $k > 0$, such that $\|f(t,x)\|_X \le k(1 + \|x\|_X)$.

[**J**] : $J_i : X \to X$ is an operator such that

1. J_i maps bounded set to bounded set.

2. There exist constants $h_i > 0, \quad i = 1, 2, \ldots, n$ such that

$$\|J_i(x) - J_i(y)\| \le h_i\|x - y\|, \qquad x, y \in X.$$

Consider the following impulsive systems

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)) \quad t \in [0,T] \setminus D, \\ x(0) = x_{0,} \\ \Delta x(t_i) = J_i(x(t_i)), \quad i = 1, 2, \cdots, n. \end{cases} \tag{2.1}$$

By a mild solution of (2.1), we shall mean that a function $x \in PC(I, X)$ satisfies the following integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x(t_i)).$$

Theorem 1. Suppose the assumptions [A], [F] and [J] hold, then for every $x_0 \in X$ the system (2.1) has a unique mild solution $x \in PC(I, X)$ and the mild solution dependends continuously on the initial conditions- that is, if x_0, $y_0 \in X$ and if $x(t)$, $y(t)$ are mild solutions of equation (2.1) which satisfy $x(0) = x_0$ and $y(0) = y_0$. Then there exists a constant $C > 0$ s.t.

$$\sup_{t\in[0,T]} \|x(t) - y(t)\| \le C\|x_0 - y_0\|_X.$$

Proof: Firstly, we consider the following general differential equation without impulse

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)) \quad t > 0, \\ x(0) = x_0. \end{cases} \tag{2.1.1}$$

Define a closed ball $\bar{B}(x_0, 1)$ as follows.

$$\bar{B}(x_0, 1) = \{x \in C([0, T_1], X),\ \|x(t) - x_0\| \leq 1,\ 0 \leq t \leq T_1\},$$

where T_1 will be chosen later. Define a map P on $\bar{B}(x_0, 1)$ by

$$(Px)(t) = T(t)x_0 + \int_0^t T(t - \tau)f(\tau, x(\tau))d\tau$$

and let $M \equiv \sup\limits_{t \in [0,T]} \|T(t)\|$. Using assumption **[F]**, one can verify that P maps $\bar{B}(x_0, 1)$ to $\bar{B}(x_0, 1)$. To prove this, we note that

$$\begin{aligned} \|(Px)(t) - x_0\| &\leq \|T(t)x_0 - x_0\| + \int_0^t \|T(t - \tau)\| \|f(\tau, x(\tau))\| d\tau \\ &\leq Mk(1 + \rho)t + \|T(t)x_0 - x_0\|. \end{aligned}$$

Since $T(t)$ is the strongly continuous C_0-semigroup, there exists $T_{11} > 0$ such that for all $t \in [0, T_{11}]$, $\|T(t)x_0 - x_0\| \leq \frac{1}{2}$. Now, let $0 < T_{22} < \frac{1}{2Mk(1+\rho)}$. Set $T_1' = \min\{T_{11}, T_{22}\}$ hence for all $t \in [0, T_1']$ we have $\|(Px)(t) - x_0\| \leq 1$. Hence $P : \bar{B}(x_0, 1) \to \bar{B}(x_0, 1)$.

Let $x_1, x_2 \in \bar{B}(x_0, 1)$. By assumption **[F](1)**, we have

$$\begin{aligned} \|(Px_1)(t) - (Px_2)(t)\| &\leq \int_0^t \|T(t - \tau)\| \|f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))\| d\tau \\ &\leq MtL_1(\rho)\|x_1 - x_2\|. \end{aligned}$$

Now, let $0 < T_1'' = \frac{1}{2ML_1(\rho)}$, then $\|(Px_1)(t) - (Px_2)(t)\| \leq \frac{1}{2}\|x_1 - x_2\|$. Hence, we shall choose $T_1 = \min\{T_1', T_1''\}$ to guarantee that P is a contraction map on $\bar{B}(x_0, 1)$. This implies that (2.1.1) has a unique mild solution on $[0, T_1]$. Again, using the assumption **[F]**, we can obtain the a priori estimate of mild solutions of equation (2.1.1). To see this, we note that

$$\begin{aligned}\|x(t)\| &\leq \|T(t)x_0\| + \int_0^t \|T(t-\tau)\|\|f(\tau, x(\tau))\|d\tau \\ &\leq M\|x_0\| + MkT + Mk\int_0^t \|x(\tau)\|d\tau.\end{aligned}$$

By Gronwall inequality, we obtain

$$\begin{aligned}\|x(t)\| &\leq (M\|x_0\| + MkT)e^{Mk\int_0^t d\tau} \\ &\leq (M\|x_0\| + MkT)e^{MkT} \equiv \overline{M}.\end{aligned}$$

That is, there exists a constant $\overline{M} = (M\|x_0\| + MkT)e^{MkT} > 0$ such that for $t \in [0, T]$ we have $\|x_(t)\| \leq \overline{M}$. Then we can prove the global existence of the mild solution of system (2.1.1) on $[0, T]$.

Now, we are ready to construct a mild solution for the impulsive system (2.1). For $t \in [0, t_1)$, the above result imples that $x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau$ is the unique mild solution of the system (2.1) on $[0, t_1]$. Clearly the solution is continuous on $[0, t_1)$ and since $T(t)$ is a continuous semigroup, then $x(t)$ can be extended continuously until the point of time t_1 which is denoted by $x(t_1)$. It is easy to see that $x(t_1) \in X$. Since J_1 maps bounded sets to bounded subsets of X, the jump is uniquely determined by the expression

$$x(t_1 + 0) = x(t_1 - 0) + J_1(x(t_1 - 0)) \equiv x(t_1) + J_1(x(t_1)) \equiv x_1.$$

Consider the time $t \in (t_1, t_2)$. We have

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau + T(t - t_1)J_1(x(t_1)).$$

Again, $x \in C((t_1, t_2), X)$ and can be extended continuously until the point of time t_2 which is denoted by $x(t_2) \in X$. By the previous result, $x(\cdot)$ is a mild solution of equation (2.1) on $(t_1, t_2]$. Because J_2 maps bounded sets to bounded sets, the jump is uniquely determined by

$$x(t_2+0) = x(t_2-0) + J_2(x(t_2-0)) \equiv x(t_2) + J_2(x(t_2)) \equiv x_2.$$

This procedure can be repeated on $t \in (t_2, t_3],\ (t_3, t_4], \ldots, (t_n, T]$. Thus we obtain a unique mild solution of problem (2.1) on$[0, T]$ and it is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x(\tau))d\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x(t_i)), \quad 0 \le t \le T.$$

For the proof of continuous dependence on the initial value, one can use Gronwall inequality to find a constant C such that $\|x(t) - y(t)\| \le C\|x_0 - y_0\|_X$ for all $t \in [0, T]$. The proof is now complete.

Now, we introduce admissible controls space U_{ad}.

Let Γ be a compact Polish space (i.e., a separable complete metric space).

We define

$$U_{ad} = \{u : [0, T] \to \Gamma | u \text{ is strongly measurable } \}.$$

By the measurable selection theorem, $U_{ad} \neq \phi$ (see Aubin, 1990). We make the following assumptions for our control systems.

Assumptions:

[**F1**] $f : I \times X \times \Gamma \to X$ is an operator such that

1. $t \mapsto f(t, \xi, \eta)$ is measurable, and

   $(\xi, \eta) \mapsto f(t, \xi, \eta)$ is continuous on $X \times \Gamma$.

2. For any finite number $\rho > 0$, there exists a constant $L(\rho) > 0$ such that

$$\|f(t, x_1, \sigma) - f(t, x_2, \sigma)\|_X \le L(\rho)\|x_1 - x_2\|_X,$$

   for all $||x_1|| < \rho, ||x_2|| < \rho$, and $t \in I, \sigma \in \Gamma$.

3. There exists a constant $k_F > 0$ such that

$$\|f(t,x,\sigma)\|_X \leq k_F(1+\|x\|_X) \ \ (t \in I, \sigma \in \Gamma).$$

Consider the following original control system

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t,x(t),u(t)), \\ x(0) = x_{0,} \\ \Delta x(t_i) = J_i(x(t_i)), \quad u(\cdot) \in U_{ad}. \end{cases} \tag{2.2}$$

Theorem 2. Suppose the assumptions [A], [J] and [F1] hold. Then for every $x_0 \in X$ and $u \in U_{ad}$, the system (2.2) has a unique mild solution $x \in PC(I,X)$ which satisfies

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau,x(\tau),u(\tau))d\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x(t_i)).$$

Proof: Let $u \in U_{ad}$ and define $g_u(t,x) = f(t,x,u)$. Since f is measurable, then $g_u : I \times X \to X$ is measurable on $[0,T]$ for each fixed $x \in X$. Hence g_u satisfies the assumption [F]. By Theorem 1, the system (2.2) has a unique mild solution $x \in PC(I,X)$.

In order to introduce the relaxed control system corresponding to (2.2), we need some preparations which are drawn from (Fattorini, 1999. page 618-650). Let Γ be a compact Polish space, and $C(\Gamma)$ consists of all continuous real valued functions. Endowed with the supremum norm, $C(\Gamma)$ is a Banach space. Let $\Phi(\mathbf{C})$ be a $\sigma-$field generated by the collection $\mathbf{C}$ of all closed sets of Γ and let $\Sigma_{rca}(\Gamma)$ be the space of all regular countably additive measures on the measurable space $(\Gamma, \Phi(\mathbf{C}))$. For $\mu \in \Sigma_{rca}(\Gamma)$, $|\mu|$ denotes the total variation of μ.

Lemma 3. The dual space $C(\Gamma)^*$ can be identified algebraically and metrically with $\Sigma_{rca}(\Gamma)$ with the norm

$$\|\mu\|_{\Sigma_{rca}(\Gamma)} = |\mu|(\Gamma).$$

The duality pairing of $C(\Gamma)$ and $\Sigma_{rca}(\Gamma)$ is given by

$$\langle f, \mu \rangle = \int_{\Gamma} f(\sigma)\mu(d\sigma)$$

for $f \in C(\Gamma),\ \mu \in \Sigma_{rca}(\Gamma)$.

Let $L^1(I, C(\Gamma))$ be the space of all (equivalence class of) strongly measurable $C(\Gamma)-$valued functions $u(\cdot)$ defined on I such that

$$\|u\| = \int_I \|u(t)\| dt < +\infty.$$

$L^1(I, C(\Gamma))$ is a Banach space. $L^\infty_w(I, C(\Gamma)^*)$ is the space of all $C(\Gamma)^*-$valued $C(\Gamma)-$weakly measurable functions $g(\cdot)$ such that there exists $C > 0$ with

$$|\langle g(t), y \rangle| \le C\|y\|_{C(\Gamma)} \quad \text{a.e. in} \ \ 0 \le t \le T, \tag{2.1.2}$$

for each $y \in C(\Gamma)$ (the null set where (2.1.2) fails to hold may depend on y). Two functions $g(\cdot),\ h(\cdot)$ are said to be equivalent in $L^\infty_w(I, C(\Gamma)^*)$ (in symbols, $g \approx h$) if $\langle g(t), y \rangle = \langle f(t), y \rangle$ a.e. in $0 \le t \le T$ for each $y \in C(\Gamma)$.

Lemma 4. The dual $L^1(I, C(\Gamma))^*$ is isometrically isomorphic to $L^\infty_w(I, C(\Gamma)^*)$. The duality pairing between both spaces is given by

$$\langle\langle g, f \rangle\rangle = \int_0^T \langle g(t), f(t) \rangle dt,$$

where $g \in L^\infty_w(I, C(\Gamma)^*)$ and $f \in L^1(I, C(\Gamma))$.

Since Γ is a compact metric space, $C(\Gamma)^*$ is a separable Banach space (see [12], p.265) and hence has the Radon-Nokodym property which tells us that $L^1(I, C(\Gamma))^* = L^\infty(I, \Sigma_{rca}(\Gamma))$.

Definition 1. The space $R(I, \Gamma)$ of relaxed controls consists of all $\mu(\cdot)$ in $L^\infty(I, \Sigma_{rca}(\Gamma)) = L^1(I, C(\Gamma))^*$ that satisfy

(i) if $f(\cdot,\cdot) \in L^1(I, C(\Gamma))$ is such that $f(t,\sigma) \geq 0$ for $\sigma \in \Gamma$ a.e. in $0 \leq t \leq T$ then

$$\int_0^T \int_\Gamma f(t,\sigma)\mu(t, d\sigma)dt \geq 0,$$

(ii) if $\chi(t)$ is the characteristic function of a measurable set $e \subseteq [0,T]$, and $\mathbf{1} \in C(\Gamma)$ is the function $\mathbf{1}(\sigma) = 1$, then

$$\int_0^T \int_\Gamma (\chi(t) \otimes \mathbf{1}(\sigma))\mu(t, d\sigma)dt = |e|.$$

Note that $\chi(\cdot) \otimes \mathbf{1}(\cdot) \in L^1(I, C(\Gamma))$.

We note that (ii) can be generalized to

$$\int_0^T \int_\Gamma (\phi(t) \otimes \mathbf{1}(\sigma))\mu(t, d\sigma)dt = \int_0^T \phi(t)dt$$

for any $\phi(\cdot) \in L^1(I)$.

In fact, for $\mu(\cdot) \in R(I, \Gamma)$, we have

$$\|\mu\|_{L^\infty(I,\Sigma_{rca}(\Gamma))} \leq 1, \mu(t) \geq 0, \text{and } \mu(t,\Gamma) = 1 \quad \text{a.e. in } 0 \leq t \leq T.$$

In particular,

$$\|\mu(t)\|_{\Sigma_{rca}(\Gamma)} = 1 \quad \text{a.e. in } 0 \leq t \leq T.$$

Lemma 5. Let $\{\mu_n(\cdot)\}$ be a sequence in $R(I, \Gamma)$. Then there exists a subsequence which is $L^1(I, C(\Gamma))-$weakly convergent in $L^\infty(I, \Sigma_{rca}(\Gamma))$ to $\mu(\cdot) \in R(I, \Gamma)$.

Sometimes, using another equivalent definition of $R(I, \Gamma)$ is more convenient. We denote by $\Pi_{rca}(\Gamma)$ the set of all probability measures μ in $\Sigma_{rca}(\Gamma)$. We denote the Dirac measure with mass at u by the functional notation $\delta(\cdot - u)$ or by δ_u. The set $D = \{\delta_u : u \in \Gamma\}$ of all Dirac measures is a subset of $\Pi_{rca}(\Gamma)$.

Lemma 6. $\Pi_{rca}(\Gamma)$ is $C(\Gamma)-$weakly compact, also $C(\Gamma)-$weakly closed in $\Sigma_{rac}(\Gamma)$. Let $\overline{conv}$ denote closed convex hull (closure taken in the weak $C(\Gamma)-$topology). Then

$$\Pi_{rca}(\Gamma) = \overline{conv}(D).$$

Since $C(\Gamma)$ is separable, the equivalent relation in $L^\infty(I, \Sigma_{rca}(\Gamma))$ is equality almost everywhere. Let us denote the set

$$\begin{aligned} R(I, \Pi_{rca}(\Gamma)) &= \{u \in L^\infty(I, \Sigma_{rca}),\ \exists v \ \text{ s.t.} \\ &\qquad v \approx u \text{ and } \ v(t) \in \Pi_{rca}(\Gamma) \ \text{ a.e. in } \ 0 \le t \le T\}. \end{aligned}$$

If $u(\cdot) \in U_{ad}$ then one can check that the Dirac delta with mass at $u(t)$ (written $\delta(\cdot - u(t))$ is an element of $R(I, \Pi_{rca}(\Gamma))$. Hence we can identify U_{ad} as a subset of $R(I, \Pi_{rca}(\Gamma))$. We note further that $R(I, \Pi_{rca}(\Gamma)) = R(I, \Gamma)$ (see Fattorini, 1999. Theorem 12.6.7).

Now, let us consider the new larger system known as "**relaxed impulsive system**"

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t, x(t))\mu(t), \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), \quad \mu(\cdot) \in U_r. \end{cases} \tag{2.3}$$

The admissible control space is $U_r = R(I, \Pi_{rca}(\Gamma))$.The function $F : I \times X \times \Sigma_{rca}(\Gamma) \to X$ is defined by

$$F(t, x)\mu = \int_\Gamma f(t, x, \sigma)\mu(d\sigma).$$

The following Theorem is an immediate consequence of Theorem 2.

Theorem 7. Assume that assumptions [A], [J] and [F1] hold. For every $\mu(\cdot) \in U_r$, the relaxed control system (2.3) has a unique solution.

A.3 Properties of relaxed trajectories

In this section, we will denote the set of original trajectories and relaxed trajectories of the system (2.2) by X_0 and the system (2.3) by X_r, respectively, i.e.,

$X_0 = \{x \in PC([0, T]; X)|\ x \text{ is a solution of (2.2) corresponding to } u(\cdot) \in U_{ad}\}$

and $X_r = \{x \in PC([0, T]; X)|\ x \text{ is a solution of (2.3) corresponding to } \mu(\cdot) \in U_r\}$.

Theorem 2 and 7 show that $X_0 \neq \emptyset$ implies $X_r \neq \emptyset$. Moreover, since $U_{ad} \subseteq U_r$ we have $X_0 \subseteq X_r$.

Next, we introduce one more hypothesis concerning the operator A.

[A1] An operator A is the infinitesimal generator of a compact C_0-semigroup $\{T(t), t \geq 0\}$.

Lemma 8. Let A satisfy assumption [A1] on Banach space X. Let $1 < p$ and define

$$S(g(\cdot)) = \int_0^{\cdot} T(\cdot - s)g(s)ds \qquad \forall g(\cdot) \in L^p(I, X).$$

Then $S : L^p(I, X) \rightarrow C(I, X)$ is compact.

Proof: See lemma 3.2 in Li-Yong(1995).

Lemma 9. Let X be reflexive and separable. Suppose the assumptions [A1] and [F1] hold. If $\{\mu^n(\cdot)\}$ is a sequence in $L^\infty(I, \Sigma_{rca}(\Gamma))$ with $\mu^n(\cdot) \rightarrow \mu(\cdot)$ $L^1(I, C(\Gamma))-$ weakly as $n \rightarrow \infty$ then

$$\rho_n(\cdot) = \int_0^{\cdot} T(\cdot - \tau) \int_\Gamma f(\tau, x(\tau), \sigma)(\mu^n(\tau) - \mu(\tau))(d\sigma)d\tau \rightarrow 0 \text{ in } C(I, X) \text{ as } n \rightarrow \infty,$$

where $x \in C([0, T], X)$.

Proof: Due to reflexivity of X, $\{T^*(t), t \geq 0\}$ is a C_0-semigroup in Banach space X^* (see [2], p.47). Define $g_n(\tau) = \int_\Gamma f(\tau, x(\tau), \sigma)(\mu^n(\tau) - \mu(\tau))(d\sigma)$ then

$$\begin{aligned}
\|g_n(\tau)\| &\leq \int_\Gamma \|f(\tau, x(\tau), \sigma)\|(\mu^n(\tau) - \mu(\tau))(d\sigma) \\
&\leq k_F(1 + \|x(\tau)\|)\|\mu^n(\tau) - \mu(\tau)\|_{\Sigma_{rca}(\Gamma)} \\
&\leq 2k_F(1 + \|x(\tau)\|.
\end{aligned}$$

Since $x(t)$ is the solution of (2.3), then it is bounded by $\bar{M}$. This implies that $\{g_n(\cdot)\}$ is bounded in $L^p(I, X)$, $1 < p < +\infty$. Hence there exists a subsequence (denoted with the same symbol) with $g_n(\cdot) \xrightarrow{w} g(\cdot)$ in $L^p(I, X)$.

By lemma 8, we have

$$\rho_n(\cdot) = \int_0^{\cdot} T(\cdot - \tau) g_n(\tau) d\tau \xrightarrow{s} \int_0^{\cdot} T(\cdot - \tau) g(\tau) d\tau \equiv \rho(\cdot) \quad \text{in} \quad C(I, X).$$

For fixed $0 \leq t \leq T, \quad h^* \in X^*$, we have

$$\begin{aligned}
\langle \rho_n(t), h^* \rangle &= \int_0^t \langle T(t-\tau) g_n(\tau), h^* \rangle d\tau \\
&= \int_0^t \langle g_n(\tau), T^*(t-\tau) h^* \rangle d\tau \\
&= \int_0^t \int_\Gamma \langle f(\tau, x(\tau), \sigma), T^*(t-\tau) h^* \rangle (\mu^n(\tau) - \mu(\tau))(d\sigma) d\tau \\
&= \int_0^t \int_\Gamma \xi(\tau, \sigma)(\mu^n(\tau) - \mu(\tau))(d\sigma) d\tau
\end{aligned}$$

where $\quad \xi(\tau, \sigma) = \langle f(\tau, x(\tau), \sigma), T^*(t-\tau) h^* \rangle$.

By assumption [F1], for τ fixed, the map $\sigma \mapsto \xi(\tau, \sigma)$ is continuous. It implies that $\xi(\tau, \sigma) \in C(\Gamma)$ and

$$|\xi(\tau, \sigma)| \leq k(1 + \|x(\tau)\|).$$

Hence $\xi(\cdot, \cdot) \in L^1(I, C(\Gamma))$.

Since $\mu^n(\cdot) \to \mu(\cdot) \quad L^1(I, C(\Gamma))-$weakly in $L^\infty(I, \Sigma_{rca}(\Gamma))$ then

$$\int_0^t \int_\Gamma \xi(\tau, \sigma)(\mu^n(\tau) - \mu(\tau))(d\sigma) dt \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

This implies that, for fixed $t \in I$,

$$\langle \rho_n(t), h^* \rangle \longrightarrow 0 \qquad \forall h^* \in X^*.$$

Hence $\rho_n(t) \xrightarrow{w} 0$ as $n \to \infty$. Thus $\rho(t) \equiv 0$. This means that $\rho_n(\cdot) \longrightarrow 0$ as $n \to \infty$ in $C(I, X)$.

Remark: Using the same proof, one can see that the result of Lemma 9 is also true when $x \in PC([0, T], X)$.

Theorem 10. Let X be reflexive and separable. Suppose the assumptions [A1] [J], and [F1] hold. If $x(\cdot,\mu)$ is the solution of (2.3) corresponding to μ then, for every $\varepsilon > 0$, there exists $u(\cdot) \in U_{ad}$ such that $x(\cdot, u)$ is solution of (2.2) corresponding to u and satisfing

$$\|x(\cdot,\mu) - x(\cdot,u)\|_{PC(I,X)} < \varepsilon, \qquad t \in I.$$

Proof: Let $\mu(\cdot) \in U_r$, since $U_{ad} \subseteq U_r$ and U_{ad} is dense in U_r. Thus there exists a sequence $\{u_n\} \subseteq U_{ad}$ such that $u_n \xrightarrow{w^*} \mu$. Let $x_n(\cdot) = x(\cdot, u_n)$ be the solution of (2.2) corresponding to u_n and $x(\cdot) = x(\cdot,\mu)$ be the solution of (2.3) corresponding to μ. Since

$$\begin{aligned} x_n(t) &= T(t)x_0 + \int_0^t T(t-\tau)f(\tau, x_n(\tau), u_n(\tau))d\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x_n(t_i)) \\ &= T(t)x_o + \int_0^t T(t-\tau)[\int_\Gamma f(\tau, x_n(\tau),\sigma)\delta_{u_n}(\tau)(d\sigma)]d\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x_n(t_i)) \end{aligned}$$

and

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)[\int_\Gamma f(\tau, x(\tau),\sigma)\mu(\tau)(d\sigma)]d\tau + \sum_{0<t_i<t} T(t-t_i)J_i(x(t_i)).$$

We have

$$\begin{aligned} x_n(t) - x(t) &= \int_0^t T(t-\tau)[\int_\Gamma (f(\tau, x_n(\tau),\sigma)\delta_{u_n}(\tau) - f(\tau, x(\tau),\sigma)\delta_{u_n}(\tau))(d\sigma)]d\tau \\ &\quad + \int_0^t T(t-\tau)[\int_\Gamma f(\tau, x(\tau),\sigma)(\delta_{u_n}(\tau) - \mu(\tau))(d\sigma)]d\tau \\ &\quad + \sum_{0<t_i<t} T(t-t_i)[J_i(x_n(t_i)) - J_i(x(t_i))] \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

By the Lipschitz condition [F1], we get

$$|I_1| \leq M \int_0^t L(\rho)\|x_n(\tau) - x(\tau)\|,$$

where $I_1 \equiv \int_0^t T(t-\tau)[\int_\Gamma (f(\tau, x_n(\tau),\sigma)\delta_{u_n}(\tau) - f(\tau, x(\tau),\sigma)\delta_{u_n}(\tau))(d\sigma)]d\tau$, and M is a bound for $\|T(t)\|$ in $0 \leq t \leq T$.

Using assumption [J](2), we have

$$|I_3| \leq \sum_{0<t_i<t} Mh_i \|x_n(t_i) - x(t_i)\|,$$

where $I_3 \equiv \sum\limits_{0<t_i<t} T(t-t_i)[J_i(x_n(t_i)) - J_i(x(t_i))]$.
We denote the second integral I_2 by $\rho_n(t), i.e.,$

$$\rho_n(t) \equiv I_2 \equiv \int_0^t T(t-\tau)[\int_\Gamma f(\tau, x(\tau), \sigma)(\delta_{u_n}(\tau) - \mu(\tau))(d\sigma)]d\tau.$$

Thus

$$\|x_n(t)-x(t)\| \leq M\int_0^t L(\rho)\|x_n(\tau)-x(\tau)\|d\tau + \|\rho_n(t)\| + \sum_{0<t_i<t} Mh_i\|x_n(t_i)-x(t_i)\|.$$

By impulsive Gronwall inequality, we get

$$\|x_n(t) - x(t)\| \leq C\|\rho_n(t)\|,$$

where $C \equiv \prod\limits_{0<t_i<t} (1 + Mh_i)exp(ML(\rho)t)$.
By using lemma 9, we show that $\rho_n(\cdot) \to 0$ as $n \to \infty$ in $PC([0,T], X)$. Hence $x_n(\cdot) \to x(\cdot)$ as $n \to \infty$ in $PC([0,T], X)$. The proof is complete.

A.4 Relaxed Optimal Controls and Relaxation Theorems

Consider the following Lagrange optimal control (P_r): Find a control policy $\mu_0 \in U_r$ such that it imparts a minimum to the cost functional J given by

$$J(\mu) \equiv J(x_\mu, \mu) \equiv \int_I \int_\Gamma l(t, x_\mu(t), \sigma)\mu(t)(d\sigma)dt, \tag{Pr}$$

where x_μ is solution of the system (2.3)corresponding to the control $\mu \in U_r$.

We make the following hypotheses concerning the integrand $l(\cdot,\cdot,\cdot)$.

[L] $l : I \times X \times \Gamma \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is an operator such that

(1) $(t, \xi, \sigma) \mapsto l(t, \xi, \sigma)$ is measurable,

(2) $(\xi, \sigma) \mapsto l(t, \xi, \sigma)$ is lower semicontinuous,

(3) $|l(t, \xi, \sigma)| \leq \theta_R(t)$ for almost all $t \in I$ provided that $\|\xi\|_X \leq R, \sigma \in \Gamma$ and $\theta_R(t) \in L^1(I)$.

Before proving the existence of the relaxed control, we need a lemma.

Lemma 11. Suppose $h : I \times X \times \Gamma \to \mathbb{R}$ satisfying

(1) $t \mapsto h(t, \xi, \sigma)$ is measurable, $(\xi, \sigma) \mapsto h(t, \xi, \sigma)$ is continuous,

(2) $|h(t, \xi, \sigma)| \leq \psi_R(t) \in L^1(I)$ provided that $\|\xi\|_X \leq R$ and $\sigma \in \Gamma$.

If $x_n \to x$ in $C(I, X)$ then $h_n(\cdot, \cdot) \to h(\cdot, \cdot)$ in $L^1(I, C(\Gamma))$ as $n \to \infty$,

where $h_n(t, \sigma) = h(t, x_n(t), \sigma)$ and $h(t, \sigma) = h(t, x(t), \sigma)$.

Proof: It follows immediately from the first hypothesis of this lemma that

$$h_n, h \in L^1(I, C(\Gamma)).$$

For each fixed $t \in I$, we shall show that $h_n(t, \cdot) \to h(t, \cdot)$ in $C(\Gamma)$ as $n \to \infty$.

By definition, we have

$$\sup_{\sigma \in \Gamma} |h_n(t, \sigma) - h(t, \sigma)| = \|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)}.$$

Since Γ is compact, there exists $\sigma_n \in \Gamma$ such that

$$|h_n(t, \sigma_n) - h(t, \sigma_n)| = \|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)}$$

and we can assume $\sigma_n \to \sigma^*$ as $n \to \infty$. We note that

$$\begin{aligned} \sup_{\sigma \in \Gamma} |h_n(t, \sigma) - h(t, \sigma)| &= |h_n(t, \sigma_n) - h(t, \sigma_n)| \\ &\leq |h_n(t, \sigma_n) - h_n(t, \sigma^*)| + |h_n(t, \sigma^*) - h(t, \sigma^*)| + |h(t, \sigma^*) - h(t, \sigma_n)|. \end{aligned}$$

Then, by continuity of h, we have $|h_n(t, \sigma_n) - h(t, \sigma_n)| \to 0$ as $n \to \infty$.

This means

$$\|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)} \to 0 \quad \text{as} \quad n \to \infty.$$

Assuming that $x_n \to x$ in $C(I, X)$ as $n \to \infty$ then there exists R such that $\|x_n(t)\|, \|x(t)\| \leq R$.

Hence, by the second hypothesis of this lemma, we have

$$\|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)} \leq \psi_R(t).$$

This implies

$$\int_I \|h_n(t, \cdot) - h(t, \cdot)\|_{C(\Gamma)} dt \to 0 \quad \text{as} \quad n \to \infty.$$

We have

$$h_n(\cdot, \cdot) \to h(\cdot, \cdot) \quad \text{in} \quad L^1(I, C(\Gamma)) \quad \text{as} \quad n \to \infty.$$

This proves the lemma.

Let $m_r = \inf\{J(\mu) : \mu \in U_r\}$. We have the following existence of relaxed optimal control.

Theorem 12. Suppose assumptions [A1], [F1], [J] and [L] hold. Then there exists $\mu^* \in U_r$ such that $J(\mu^*) = m_r$.

Proof: Let $\{\mu_n\}$ be a minimizing sequence so that $\lim\limits_{n\to\infty} J(\mu_n) = m_r$. Recall that U_r is w^*-compact in $L^\infty(I, \Sigma_{rca}(\Gamma))$, by passing to a subsequence if necessary, we may assume $\mu_n \stackrel{w^*}{\to} \mu^*$ in $L^\infty(I, \Sigma_{rca}(\Gamma))$ as $n \to \infty$. Next, we shall prove that (x, μ^*) is an optimal pair, where x is the solution of (2.3) corresponding to μ^*.

Since every lower semicontinuous measurable integrand is the limit of an increasing sequence of Caratheodory integrands, there exists an increasing sequence of Caratheodory integrands $\{l_k\}$ such that

$$l_k(t, \xi, \sigma) \uparrow l(t, \xi, \sigma) \quad \text{as} \quad k \to \infty \quad \text{for all} \quad t \in I, \sigma \in \Gamma.$$

Invoking the definition of weak topology and applying Lemma 11 on each subinterval of $[0, T]$, $l_k(t, x_n(t), \sigma) \to l_k(t, x(t), \sigma)$ as $n \to \infty$ for almost all $t \in I$ and all $\sigma \in \Gamma$. Then

$$\begin{aligned} J(x,\mu^*) = J(\mu^*) &= \int_I \int_\Gamma l(t,x(t),\sigma)\mu^*(t)(d\sigma)dt \\ &= \lim_{k\to\infty} \int_I \int_\Gamma l_k(t,x(t),\sigma)\mu^*(t)(d\sigma)dt \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \int_I \int_\Gamma l_k(t,x_n(t),\sigma)\mu_n(t)(d\sigma)dt \\ &\le \lim_{n\to\infty} \int_I \int_\Gamma l(t,x_n(t),\sigma)\mu_n(t)(d\sigma)dt \\ &= m_r. \end{aligned}$$

However, by definition of m_r, it is obvious that $J(x,\mu^*) \ge m_r$. Hence $J(x,\mu^*) = m_r$.

This implies that (x,μ^*) is an optimal pair.□

If $J(u) = \int_I l(t,x(t),u(t))dt$ is the cost function for the original problem, and $J(u_0) = \inf\{J(u),\ u \in U_{ad}\} = m_0$. In general, since $U_{ad} \subseteq U_r$, we have $m_r \le m_0$. It is desirable that $m_r = m_0$, i.e., our relaxation is reasonable. We have the following relaxation theorem. For this, we need stronger hypotheses on l than [L]:

[L1] $l : I \times X \times \Gamma \to \mathbb{R}$ is an operator such that

(1) $(t,\xi,\sigma) \to l(t,\xi,\sigma)$ is measurable,

(2) $(\xi,\sigma) \to l(t,\xi,\sigma)$ is continuous,

(3) $|l(t,\xi,\sigma)| \le \theta_R(t)$ for almost all $t \in I$, provided $\|\xi\|_X \le R,\ \sigma \in \Gamma$ and $\theta_R \in L^1(I)$.

Theorem 13. If assumptions [A1], [J], [F1] and [L1] hold and Γ is compact then $m_0 = m_r$.

Proof: Let (x,μ^*) be the optimal pair (the existence was guaranteed by the previous theorem) that is $m_r = J(x,\mu^*)$. By Theorem 10, there exists $\{u^n\} \subseteq U_{ad}$ and $\{x_n\} \subseteq PC(I,X)$ such that

$$\delta_{u_n}(\cdot) \to \mu^*(\cdot)\ L^1(I, C(\Gamma)) - \textit{weakly} \text{ in } L^\infty(I, \Sigma_{rca}(\Gamma)),$$

and $$x_n \to x \text{ in } PC(I, X) \text{ as } n \to \infty.$$

Applying Lemma 11 to each subinterval of $[0, T]$, one can verify that

$$l(\cdot, x_n(\cdot), \cdot) \to l(\cdot, x(\cdot), \cdot) \text{ in } L^1(I, C(\Gamma)).$$

By definition of the weak topology on U_r, we have

$$\begin{aligned} J(u_n) &= J(\delta_{u_n}) = \int_I \int_\Gamma l(t, x_n(t), \sigma)\delta_{u_n}(t)(d\sigma)dt \\ &\to \int_I \int_\Gamma l(t, x(t), \sigma)\mu^*(t)(d\sigma)dt = J(x, \mu^*) = m_r. \end{aligned}$$

But, by definition of m_0, $J(u_n) \geq m_0$. Hence $m_r = \lim_{n\to\infty} J(u_n) \geqslant m_{0.}$ This implies $m_0 = m_r$. The proof is now complete.

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