MODIFIED MOMENT-AREA METHOD FOR CANTILEVER BEAMS

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Abstract

It is well known that the shearing force effect needs to be included in the determination of the displacements of a deep beam having the length-to-depth ratio less than 10. However, in some loading situations and beam configurations, methods such as Castigliano's theorem are mathematically quite complex and time-consuming. In this paper, a modified moment-area method for cantilevered beams that includes the shearing force effect is presented. This method is based on the Bernoulli-Euler beam theory. It has been shown that the analytical method is easy to apply and the results are in good agreement with the ones obtained by using the Castigliano's theorem.

Introduction

In practice, cantilever beam is sometimes used in a lower floor of structures to support the upper floor for the aesthetic and function of the structure such as in the Charnisala Tower II, Bangkok. This beam is usually very deep since the loads transferring from the upper floor are normally large. Hence, in the determination of the displacement of the beam, the shearing force effects need to be included in the calculation. Usually, the Castigliano's theorem is utilized in this analysis. However, in the complicated situations such as a beam under series of concentrated loads or a beam having segments with different moments of inertia, the Castigliano's theorem is a significantly time-consuming method. The moment-area method is one of the most effective methods for obtaining the displacement in these kind of complicated situations. However, the method possesses a limitation in determining only the displacement due to the bending moment. Therefore, it is not suitable to use in the deep beam. In this paper, the moment-area method is further developed for cantilevered deep beam by including the effect of shearing forces in order to extend the capability of the method.

Theoretical basis

Consider a deep cantilevered beam having length L, modulus of elasticity E, moment of inertia I, cross-sectional area A, shearing modulus of elasticity G, and correction coefficient for strain energy due to shear k. The beam is subjected to a negative arbitrary distributed load $w(x)$ per unit length and deformed as shown in Fig. 1. In this formulation, the deflection of the beam is separated into two parts which are the deflection due to bending moment, $y_{bending}$ and the deflection due to shearing force, y_{shear}. In addition, the following sign conventions are used: the positive deflection is upward and the positive rotation is counterclockwise from the x axis.

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Considering an incremental element of the beam at the distance x from the fixed end of the beam and has a differential length of dx. Let the positive moment tends to bend the element upward and the positive shear tends to rotate the element clockwise. For linearly elastic isotropic homogeneous material and small displacement, when the shearing force is increased dV, the clockwise rotation due to shearing force is increased $d\alpha$. Then, we have $dV = \lambda d\alpha$, where λ is a factor relating the shearing force with the rotation. Hence, the strain energy due to shearing force from the initial position to the final rotation $\beta = V/\lambda$ is

$$U_{shear} = \int_{o}^{L} \lambda\beta^2 dx \qquad (1)$$

Differentiating Eq. 3 twice, we have

$$\frac{d^2y}{dx^2} = \frac{d^2y_{bending}}{dx^2} + \frac{d^2y_{shear}}{dx^2} = \frac{d^2y_{bending}}{dx} + \frac{d\beta}{dx} \qquad (4)$$

According to the Bernoulli-Euler beam theory and the previously mentioned sign conventions, we have $\frac{d^2y_{bending}}{dx^2} = \frac{M}{EI}$ and $dV/dx = -w(x)$. Also, if the factor λ is independent for a given portion of the beam, then

$$\frac{d^2y}{dx^2} = \frac{d\theta}{dx} = \frac{M}{EI} + \frac{d(V/\lambda)}{dx} \qquad (6)$$

$$d\theta = \frac{M}{EI}dx - \frac{w}{\lambda}dx \qquad (7)$$

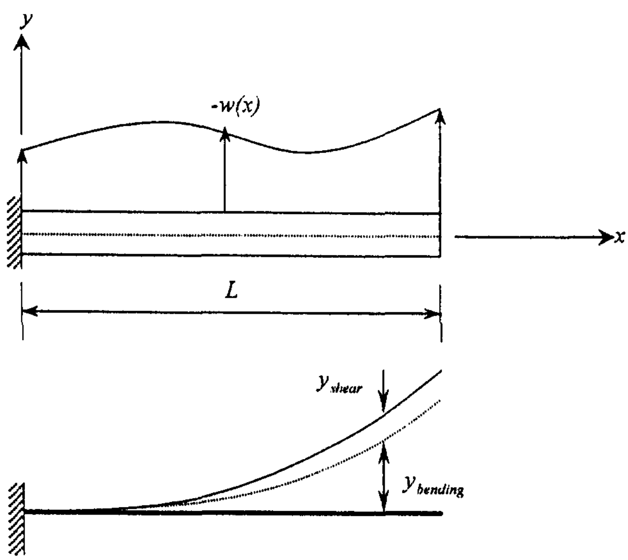

Fig 1. Deep cantilevered beam and its elastic curve with positive sign convention

Comparing Eq. 1 with the well-known strain energy due to shearing force from classical theory, $U_{shear} = \int_{0}^{L} \kappa \frac{V^2}{2GA} dx$, we have

$$\lambda = \frac{GA}{\kappa} \qquad (2)$$

Now, let us consider Fig. 1, the total deflection of the beam can be written as

$$y = y_{bending} + y_{shear} \qquad (3)$$

This equation states that the change in the slope of the tangent on either side of an incremental element dx corresponds to infinitesimal area under the M/EI diagram and the w/λ diagram, respectively. Integrating from point A on the beam to point B,

$$\theta_{B/A} = \int_{A}^{B} \frac{M}{EI}dx - \int_{A}^{B} \frac{w}{\lambda}dx \qquad (8)$$

where $\theta_{B/A}$ is the angle of the tangent at B

measured with respect to the tangent at A and has unit in radian. The positive sign indicates that the angle is in clockwise direction.

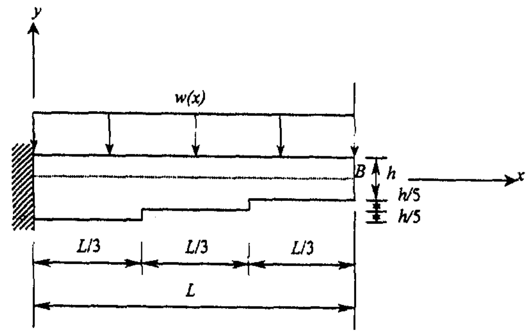

Fig. 2 Deep cantilevered beam having rectangular cross-section with a constant width.

For a small displacement, the vertical deviation dt of the tangent on each side of the incremental element dx can be determined by using the circular arch formula $dt = xd\theta$. Then, the deviation of the tangent at point A to point B can be found as

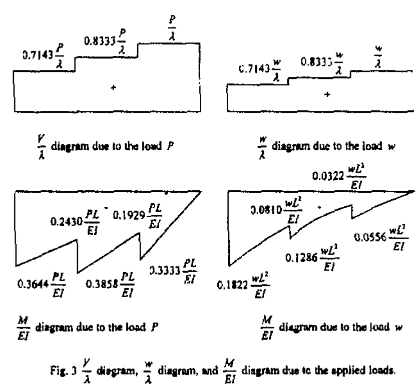

Fig.3 $\frac{V}{\lambda}$ diagram, $\frac{w}{\lambda}$ diagram, and $\frac{M}{EI}$ diagram due to the applied loads.

$$t_{B/A} = \int_A^B \frac{M}{EI} x dx - \int_A^B \frac{w}{\lambda} x dx \qquad (9)$$

where positive sign indicates the upward vertical deviation. It should be noted that the Eq. 9 must be modified in the case of concentrated load due to $dV/dx = 0$. Since we have $d(Vx)/dx = [V + x(dV/dx)] = V$, then

$$t_{B/A} = \int_A^B \frac{M}{EI} x dx - \int_A^B \frac{V}{\lambda} x dx \qquad (10)$$

Example: Determine the deflection and rotation at the end B of the deep cantilevered concrete beam as shown in Fig. 2 by using the modified moment area method. The beam has a rectangular cross-section with a constant width, b.

1. Draw the $\frac{V}{\lambda}$ diagram, $\frac{w}{\lambda}$ diagram, and $\frac{M}{EI}$ diagram due to the loads as shown in Fig.3.

2. Determine the rotation at end B.

θ_B is equal to the total area between point A and B under diagram due to the load P plus M/EI diagram due to the load w plus $w(x)/\lambda$ diagram due to the load w.

$$\theta_B = -0.2532\frac{PL^2}{EI} - 0.0659\frac{wL^3}{EI} - 0.8492k\frac{wL}{GA}$$

The negative sign means that the rotation at B is in the clockwise direction.

3. Determine the deflection at end B.

$t_{B/A}$ is equal to the total moment about point B of the area between point A and B under M/EI diagram due to the load P plus M/EI diagram due to the load w plus $w(x)/\lambda$ diagram due to the load w and plus the area under V/λ diagram between point A and B .

$$t_{B/A} = -0.1478\frac{PL^3}{EI} - 0.0437\frac{wL^4}{EI} - 0.3929\kappa\frac{wL^2}{GA} - 0.4892\kappa\frac{PL}{GA}$$

The negative sign means that the deflection at B is in the downward direction.

It should be noted that the obtained displacements are identical to the ones obtained by using the Castigliano's theorem when the shear force effect is included. However, it was found that the Castigliano's theorem is very lengthy and time-consuming.

Let the beam has $E = 20.3$ GPa, $G = 7.80$ GPa, $k = 1.2$, $b = 0.3$ m, $h = 1.0$ m, and $L = 3.0$ m, and is subjected to the loads $w(x) = 200$ kN/m and $P = 200$ kN. The numerical values of the rotation and the deflection at the end B of the beam can be determined by using different calculation methods as shown in Table 1.

However, since the shear force effect is included in the program and in the modified moment-area method, the difference of the result is merely 1.4%. Thus, from the example, it can be seen that the shear force can contribute partially to the displacements of the beam and should not be neglected.

Conclusions

The moment-area method was modified by including the effect of shearing force for determining the displacements of the cantilevered beams. The calculation procedures are easy to apply as shown in the example, and the results are identical to ones obtained by using the Castigliano's theorem and agree well with the results from the well- known structural analysis computer programs. This method is very effective for the beam having segments with different constants and moments of inertia. It is believed that the method should be useful to engineering designers because they can avoid the more complex flexural-shear-displacement analysis.

Table 1. Comparison of the analytical results from the modified moment-area method and the structural analysis computer programs.

Methods of Analysis	Rotations	Deflection
Modified Moment-Area	$1.860(10^{-3})$ rad	3.592 *mm*
Grasp	$1.685(10^{-3})$ rad	3.217 *mm*
Ansys	$1.685(10^{-3})$ rad	3.641 *mm*

From Table 1., it can be seen that the rotations from the Grasp and Ansys program are identical and less than the result obtained by using the modified moment-area by 9.4% respectively. This is because the beam elements in the Grasp and Ansys program do not include the shear force effect in the calculation for the rotation. Also, the beam deflection obtained by using the modified moment-area is larger than the result from the Grasp program by 10.4%.

References

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