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$\pi\pi$ REACTION IN NON-RELATIVISTIC QUARK MODEL*

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The reaction $\pi^+\pi^- \to \pi^+\pi^-$ is studied in the non-relativistic quark model with the $^3P_0$ quark–antiquark dynamics. The cross section of the reaction $\pi^+\pi^- \to \pi^+\pi^-$ is well reproduced even for rather high energies.

Keywords: Pion–pion scattering; non-relativistic quark model.

1. Introduction

The study of the low and intermediate pion–pion scattering as well as other strong interaction processes lies within the domain of non-perturbative QCD. Due to the lack of effective methods in obtaining solutions to QCD in the non-perturbative confinement region, we have to resort to the development of effective models. Meson-exchange models, non-relativistic quark models and chiral perturbation theories are among the most successful approaches in studying the strong interaction at low and intermediate energies.

The meson-exchange models have made tremendous successes in the investigation of the nucleon–nucleon, meson–nucleon and meson–meson and nucleon-antinucleon interactions at low and intermediate energies,[1–6] and even in the study of the elastic nucleon-nucleon scattering at high energies.[7,8] The models, however, have many free parameters involved, which is the unavoidable shortcoming of the meson-exchange models.

The chiral perturbation theory, which is the effective field theory of the Standard Model below the scale of spontaneous chiral symmetry breaking, has become a well-established method for describing the low-energy interactions of the pseudoscalar octet. Elastic pion–pion scattering at low energies is a good example of mesonic chiral perturbation theory. A complete analytical calculation of the reaction $\pi\pi \to \pi\pi$ at the two-loop order has been performed.[9] However, it is difficult to use the

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method to describe reactions with higher energies, for example, for the reaction $\pi\pi \to \pi\pi$ at an energy around the $f_2(1270)$ threshold.

In the non-relativistic constituent quark model, quarks and antiquarks are kept as the relevant degrees of freedom whereas the interaction between the quarks, particularly the confinement, is described by effective, QCD inspired potentials. The advantage of the quark model over the meson exchange model is based on the fact that a large number of experimental observables can be understood qualitatively and quantitatively by a low number of free parameters. An overview of the various quark models with a detailed discussion can be found in Ref. 10. The processes of meson decays, baryon decays, meson–baryon reactions and baryon–antibaryon annihilations have been successfully described in the non-relativistic quark models[11–17] in the $^3P_0$ quark–antiquark dynamics which has been proven to be the dominant $\bar{Q}Q$ dynamics in the non-relativistic quark models.

The reaction $\pi\pi \to \pi\pi$ at the isospin $I = 2$ channel has been successfully studied in the non-relativistic quark model,[18] where the $^3P_0$ quark diagrams have no contribution. We will now study the $\pi\pi \to \pi\pi$ reaction in the non-relativistic quark model where the $^3P_0$ quark diagram dominates.

2. Reaction $\pi^+\pi^- \to \pi^+\pi^-$ in $^3P_0$

The success of the $^3P_0$ quark–antiquark dynamics in studying the reactions $e^+e^- \to \pi^+\pi^-$ and $e^+e^- \to \bar{N}N$ suggests that the reactions are completely dominated by the intermediate vector mesons.[19] We may also expect that in the $^3P_0$ quark–antiquark dynamics, the processes shown in Fig. 1 would dominate the reaction $\pi^+\pi^- \to \pi^+\pi^-$, where a $\pi^+\pi^-$ pair annihilates into a virtual time-like meson, then the virtual meson decays into a $\pi^+\pi^-$ pair. The transition amplitude for the two step process takes the form

$$T = \langle\pi\pi|V_{67}^\dagger|\Psi_m\rangle \frac{1}{E - M} \langle\Psi_m|V_{23}|\pi\pi\rangle\,, \tag{1}$$

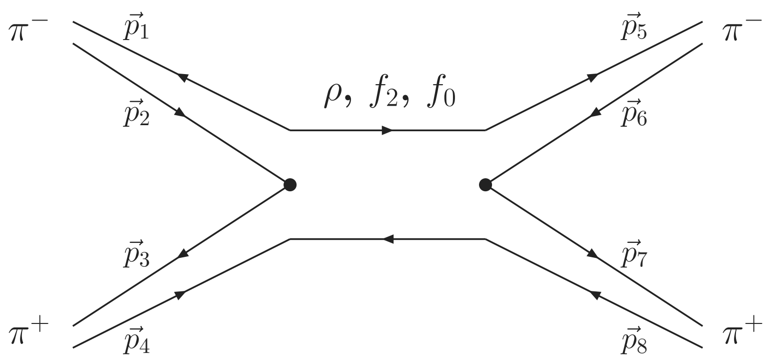

Fig. 1. $\pi^+\pi^- \to \pi^+\pi^-$ in the $^3P_0$ quark model.

where E is the center-of-mass energy of the two π system. Ψ_m and M are respectively the wave function and mass of the intermediate mesons. $\langle\pi\pi|V_{67}^{\dagger}|\Psi_m\rangle$ and $\langle\Psi_m|V_{23}|\pi\pi\rangle$ are respectively the transition amplitude of the intermediate meson annihilation into two pions and the one of two pions annihilation into a virtual time-like meson. V_{ij} is the quark–antiquark $^3P_0$ vertex defined as

$$
\begin{aligned}
V_{ij} &= \lambda \boldsymbol{\sigma}_{ij} \cdot (\boldsymbol{p}_i - \boldsymbol{p}_j)\hat{F}_{ij}\hat{C}_{ij}\delta(\boldsymbol{p}_i + \boldsymbol{p}_j) \\
&= \lambda \sum_{\mu} \sqrt{\frac{4\pi}{3}}(-1)^{\mu}\sigma_{ij}^{\mu} y_{1\mu}(\boldsymbol{p}_i - \boldsymbol{p}_j)\hat{F}_{ij}\hat{C}_{ij}\delta(\boldsymbol{p}_i + \boldsymbol{p}_j)
\end{aligned} \tag{2}
$$

where $y_{1\mu}(\boldsymbol{q}) = |\boldsymbol{q}|Y_{1\mu}(\hat{q})$, $\boldsymbol{\sigma}_{ij} = (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j)/2$, $\mathbf{p}_i$ and $\mathbf{p}_j$ are the momenta of quark and antiquark created out of the vacuum. $\hat{F}_{ij}$ and $\hat{C}_{ij}$ are the flavor and color operators projecting a quark–antiquark pair to the respective vacuum quantum numbers. The derivation and interpretation of the quark–antiquark $^3P_0$ dynamics may be found in the literature.[11,12]

The evaluation of the transition amplitudes of one meson to two mesons in the quark–antiquark $^3P_0$ dynamics is straightforward (see details in Appendix A). There are two free parameters, the size parameter of the mesons and the effective strength parameter λ in the quark–antiquark $^3P_0$ vertex. The size parameter b may be nailed down by the reaction $\rho^0 \to e^+e^-$, as done in Ref. 19, where we get $b = 3.847\ \mathrm{GeV}^{-1}$.

The effective strength parameter λ may be determined by the reaction $\rho^0 \to \pi^+\pi^-$. The decay width of the reaction takes the form

$$
\Gamma = \frac{\pi}{2} M_\rho k \left(\frac{M_\pi}{E_\pi}\right)^2 |T_{\rho\to\pi^+\pi^-}|^2 \,, \tag{3}
$$

where $T_{\rho\to\pi^+\pi^-}$ is the transition amplitude given in Eq. (A.5) in Appendix A. k is the momentum of the final pion mesons in the center-of-mass system. We consider the final pions to be rather relativistic. We associate each pion with a "minimal relativity" factor $(M_\pi/E_\pi)^{1/2}$.[2] With the size parameter $b = 3.847\ \mathrm{GeV}^{-1}$, determined from the reaction $\rho^0 \to e^+e^-$ in Ref. 19, the experimental value $\Gamma = 150$ MeV for the decay width of $\rho^0 \to \pi^+\pi^-$ requires the effective strength parameter λ to take the value $\lambda = 2.73$.

The differential cross section for the reaction $a + b \to c + d$ takes, in the center-of-mass system, the form[21]

$$
\frac{d\sigma}{d\Omega} = \frac{v_f}{v_i}|M(\mathbf{p},\mathbf{k})|^2 \,, \tag{4}
$$

with

$$
M(\mathbf{p},\mathbf{k}) = -(2\pi)^2 \frac{E_c E_d}{E_{cm}} T(\mathbf{p},\mathbf{k}) \,, \tag{5}
$$

where $v_f \equiv E_f/dp$ and $v_i \equiv dE_i/dk$ are the final and initial speeds of the pions, respectively. $\mathbf{k}$ and $\mathbf{p}$ are the incoming and outgoing momenta, respectively. The total cross section for the reaction $\pi^+\pi^- \to \pi^+\pi^-$, in terms of the partial wave transition amplitudes, is

$$\sigma = (2\pi)^4 \frac{E_{cm}^2}{16} \sum_L |T_L(k)|^2 \,, \tag{6}$$

with the partial wave transition amplitudes $T_L(p,k)$ defined as

$$T_L(p,k) = 2\pi \int_0^\pi T(\mathbf{p},\mathbf{k}) \sqrt{\frac{2l+1}{4\pi}} P_L(\cos\theta) \sin\theta \, d\theta \,, \tag{7}$$

where θ is the angle between the momenta $\mathbf{k}$ and $\mathbf{p}$. For the reaction $\pi^+\pi^- \to \pi^+\pi^-$, the partial wave transition amplitudes T_L in Eq. (6) are linear combinations of the amplitudes in the isospin basis, that is

$$\begin{aligned} T_{2n} &= \frac{2}{3} T_{2n}(I=0) + \frac{1}{3} T_{2n}(I=2) \,, \\ T_{2n+1} &= T_{2n+1}(I=1) \,, \end{aligned} \tag{8}$$

where I is the isospin of the $\pi\pi$ system.

Figure 2 shows the predictions for the cross section of the reaction $\pi^+\pi^- \to \pi^+\pi^-$ in the diagram in Fig. 1. The dashed line is the prediction for which only the ρ and $f_2(1270)$ mesons[20] are involved as the intermediate states. There is no free parameter in the calculation. The length parameter b of the ρ meson is fixed in the reaction $\rho \to e^+e^-$ and for simplicity we assign the meson $f_2(1270)$ the same length parameter. The effective strength parameter λ of the $^3P_0$ quark–antiquark

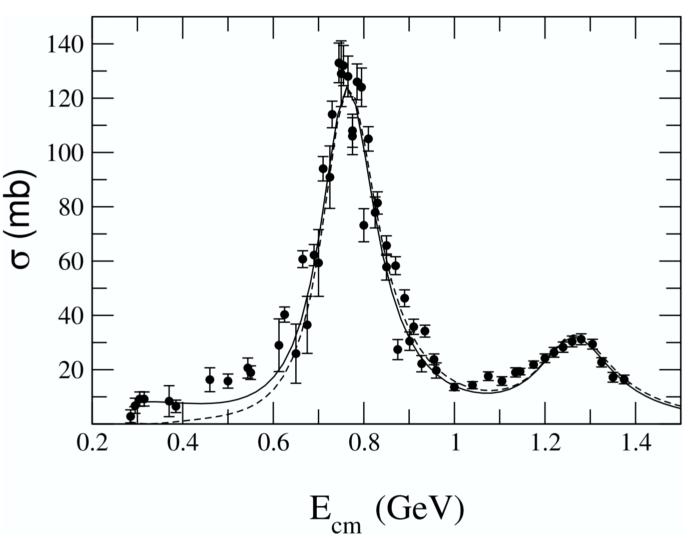

Fig. 2. Predictions for the cross section of the reaction $\pi^+\pi^- \to \pi^+\pi^-$ in the $^3P_0$ quark model with the ρ and $f_2(1270)$ mesons as the intermediate states (dashed line) and with the ρ, $f_2(1270)$ and $f_0(600)$ mesons as the intermediate states (solid line). Experimental data (solid circles) are taken from Refs. 22 and 23.

vertex is fixed in the reaction $\rho \to \pi\pi$. It is found that the prediction is reasonable at the resonance region of the reaction $\pi^+\pi^- \to \pi^+\pi^-$, that is the center-of-mass energies from 0.6 to 1.4 GeV. However, it is noticed that the prediction for the low energy region (lower than 0.6 GeV), with only the ρ and $f_2(1270)$ mesons involved as the intermediate states, is much lower than the experimental data.[22,23]

The solid line in Fig. 2 stands for the model prediction where the ρ, $f_2(1270)$ and $f_0(600)$ mesons[20] are considered as the intermediate states in Fig. 1. In this work, all the three mesons are assigned the same length parameter $b = 3.847\ \text{GeV}^{-1}$, as determined in the process $\rho \to e^+e^-$. We employ the effective strength parameter $\lambda = 2.73$ for the processes $\pi^+\pi^- \to \rho \to \pi^+\pi^-$ and $\pi^+\pi^- \to f_2(1270) \to \pi^+\pi^-$, and $\lambda = 2.0$ for the process $\pi^+\pi^- \to f_0(600) \to \pi^+\pi^-$. It is noticed that the application of the same strength parameter $\lambda = 2.73$ to all the three processes leads to poor predictions for the low energy region. Although the contribution of the $f_0(600)$ intermediate state is negligible for the higher energy region (over 0.6 GeV), the involvement of the f_0 meson is very much necessary for understanding the reaction $\pi^+\pi^- \to \pi^+\pi^-$ at the low energy region (lower than 0.6 GeV).

3. Discussions and Conclusions

The cross section of the reaction $\pi^+\pi^- \to \pi^+\pi^-$ is well reproduced in the ${}^3P_0$ quark model in which there is only one free parameter involved. The reaction $\pi^+\pi^- \to \pi^+\pi^-$ at higher energies is dominated by the processes $\pi^+\pi^- \to \rho \to \pi^+\pi^-$ and $\pi^+\pi^- \to f_2(1270) \to \pi^+\pi^-$, while the process $\pi^+\pi^- \to f_0(600) \to \pi^+\pi^-$, is the dominant one at lower energies.

The parameters for the processes $\rho^0 \to e^+e^-$ and $\rho^0 \to \pi^+\pi^-$ work well with the meson f_2 but are not applicable to the meson $f_0(600)$. This may indicate that ρ and $f_2(1270)$ are mesons of the same kind while $f_0(600)$ is something else.

Appendix A. One Meson Annihilation Into Two Mesons in the ${}^3P_0$ Model

We study the reaction of one meson annihilation into two mesons shown in Fig. 3 in the quark–antiquark ${}^3P_0$ vertex of Eq. (2). The $\boldsymbol{\sigma}_{ij}$ in the vertex can be understood as an operator projecting a quark–antiquark pair onto a spin-1 state. It can be easily proven that

$$\langle 0,0|\sigma^{\mu}_{ij}|[\bar{\chi}_i \otimes \chi_j]_{JM}\rangle = (-1)^M\sqrt{2}\delta_{J,1}\delta_{M,-\mu}\,. \tag{A.1}$$

Concerning SU(2) flavor a quark–antiquark pair which annihilates into the vacuum must have zero isospin. So the operator $\hat{F}_{ij}$ has the similar property $\langle 0,0|\hat{F}_{ij}|T,T_z\rangle = \sqrt{2}\delta_{T,0}\delta_{T_z,0}$. For the color part, we simply have $\langle 0,0|\hat{C}_{ij}|q^i_\alpha \bar{q}^j_\beta\rangle = \delta_{\alpha\beta}$, where α and β are color indices. The transition amplitude for a meson decay into two mesons in the ${}^3P_0$ model is defined as $T = \langle\Psi_i|V^\dagger_{45}|\Psi_f\rangle$, where $|\Psi_i\rangle$ and $|\Psi_f\rangle$ are the initial and final states, respectively. The initial state is simply the one

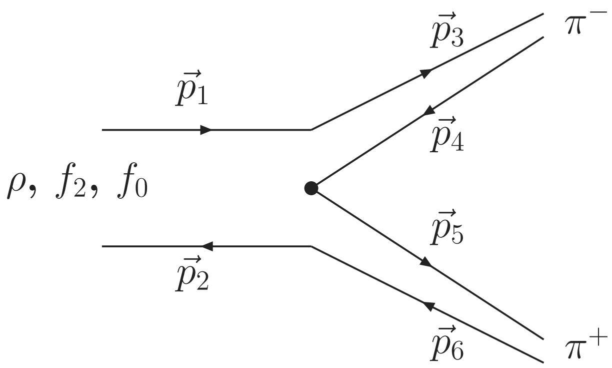

Fig. 3. A meson annihilation into a $\pi^+\pi^-$ pair in the ${}^3P_0$ quark model.

meson wave function (WF) having the form

$$|\Psi_i\rangle_S = N_S\, e^{-\frac{1}{8}b^2(\mathbf{p}_1-\mathbf{p}_2)^2} \left[\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right]_{S_i} \left[\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right]_{T_i}, \tag{A.2}$$

for the S-wave meson (for example, the ρ meson), and

$$|\Psi_i\rangle_P = N_P\, e^{-\frac{1}{8}b^2(\mathbf{p}_1-\mathbf{p}_2)^2} \left[y_{1\mu}(\mathbf{p}_1-\mathbf{p}_2) \otimes \left[\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right]_{S'}\right]_{S_i} \left[\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right]_{T_i}, \tag{A.3}$$

for the P-wave mesons (for example, the $f_2(1270)$ meson), where $y_{1\mu}(\mathbf{q}) = |\mathbf{q}|Y_{l\mu}(\hat{q})$.

We have spin $S_i = 1$ and isospin $T_i = 1$ for the ρ meson (the isospin projection $T_z = 0$ for ρ^0), spin $S_i = 2$ and isospin $T_i = 0$ for the $f_2(1270)$ meson, and spin $S_i = 0$ and isospin $T_i = 0$ for the $f_0(600)$ meson. Here we have employed the harmonic oscillator interaction between quark and antiquark. The final state $|\Psi_f\rangle$ is formed by coupling the WFs of the two final mesons. For two S-wave mesons we have

$$|\Psi_f\rangle = N_s N_s e^{-\frac{1}{8}b^2(\mathbf{p}_3-\mathbf{p}_4)^2} e^{-\frac{1}{8}b^2(\mathbf{p}_5-\mathbf{p}_6)^2} \left[\left[\frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)}\right]_{S_1} \otimes \left[\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right]_{S_2}\right]_{S_f,M_f}$$
$$\times \left[\left[\frac{1}{2}^{(3)} \otimes \frac{1}{2}^{(4)}\right]_{T_1} \otimes \left[\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right]_{T_2}\right]_{T,T_z}. \tag{A.4}$$

The transition amplitude is derived as

$$
\begin{aligned}
T_{\rho\to\pi^+\pi^-} &= \lambda\frac{2^4}{3^3\sqrt{3}\pi^{1/4}}b^{3/2}ke^{-\frac{1}{12}b^2k^2}(-1)^m Y_{1m}(\hat{k})\,,\\
T_{f_2(1270)\to\pi^+\pi^-} &= \lambda\frac{2^4\sqrt{3}}{3^4\sqrt{5}\pi^{1/4}}b^{5/2}k^2e^{-\frac{1}{12}b^2k^2}(-1)^m Y_{2m}(\hat{k})\,,\\
T_{f_0(600)\to\pi^+\pi^-} &= \lambda\frac{2^3}{3^4\pi^{1/4}}b^{1/2}e^{-\frac{1}{12}b^2k^2}(2b^2k^2-9)\,,
\end{aligned}
\tag{A.5}
$$

where $\mathbf{k}$ is the momentum of the outgoing π mesons in the center-of-mass system. Note that we have, for simplicity, set the ρ, $f_2(1270)$, $f_0(600)$ and π mesons to have the same size parameter b, that is $N_S = (b^2/\pi)^{3/4}$ and $N_P = (2b^5/3\pi^{1/2})^{1/2}$.

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