LINEARIZATION OF A SYSTEM OF TWO SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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วิทยานิพนธ์นี้ศึกษาระบบสมการเชิงอนุพันธ์สามัญอันดับสอง 2 สมการ โดยแบ่งวิธีการศึกษาออกเป็นสองส่วน ส่วนแรกแนะนำวิธีการใหม่โดยใช้รูปแบบของระบบสมการภาพฉายของระบบสมการที่ศึกษา ซึ่งสามารถทำให้เป็นเชิงเส้นได้ตามลำดับ สำหรับระบบสมการเชิงอนุพันธ์สามัญกึ่งเชิงเส้นกำลังสองอันดับสอง 2 สมการ วิธีการใหม่สามารถทำให้ได้เกณฑ์ของการทำให้เป็นเชิงเส้นที่มีความเป็นทั่วไปกว่าโดยการแปลงแบบจุด ทั้งนี้มีตัวอย่างของระบบสมการที่ไม่สามารถทำให้เป็นเชิงเส้นได้โดยการแปลงแบบจุด แต่สามารถทำให้เป็นเชิงเส้นได้โดยวิธีการใหม่ที่พัฒนาในวิทยานิพนธ์นี้ ผลที่ได้ในการศึกษาส่วนที่สองคือ การสร้างเกณฑ์เพื่อตรวจสอบว่าระบบสมการที่ศึกษานี้จะสมมูลกับระบบสมการเชิงเส้นที่มีสัมประสิทธิ์เป็นค่าคงตัวโดยการแปลงแบบเส้นสงวนจุด เหตุที่เลือกระบบสมการเชิงเส้นนี้เพราะว่าสามารถหาผลเฉลยทั่วไปได้ง่าย นอกจากนี้ยังสร้างเงื่อนไขที่จำเป็นอื่นๆ ภายใต้การแปลงแบบจุด มีการนำเสนอตัวอย่างเพื่ออธิบายกระบวนการใช้ทฤษฎีการทำให้เป็นเชิงเส้นที่พัฒนาขึ้นในวิทยานิพนธ์นี้ด้วย

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LINEARIZATION PROBLEM / FIBER PRESERVING POINT TRANSFORMATION / SEQUENTIAL LINEARIZATION / NONLINEAR SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

The thesis is devoted to the study of a system of two second-order ordinary differential equations. The method of the study is separated into two parts. In the first part, a new method for linearizing this system is introduced, through the definition of a projectable system of such equations which is sequentially linearizable. It is shown that for a system of two second-order quadratically semi-linear ordinary differential equations, the method gives more general linearization criteria than linearization via point transformations. Examples of systems of equations which are not linearizable via point transformations, but linearizable by the new method are given. The main result of second part is to obtain the criteria for this system to be equivalent to a linear system with constant coefficients via fiber preserving point transformations. A linear system with constant coefficients is chosen because of the simplicity of finding its general solution. Some other necessary conditions were also found under point transformations. Examples demonstrating the procedure of using the linearization theorem developed in the thesis are presented.

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CONTENTS

CHAPTER I

INTRODUCTION

1.1 Literature Background

Almost all physical applications of differential equations are based on nonlinear equations, which in general are very difficult to solve explicitly. Ordinary differential equations play a significant role in the theory of differential equations. In the 19th century, one of the most important problems in analysis was the problem of classification of ordinary differential equations, see Lie (1883), Liouville (1889), Tresse (1896) and Cartan (1924).

One type of the classification problem is the equivalence problem. Two systems of differential equations are said to be equivalent if there exists an invertible transformation which transforms any solution of one system to a solution of the other system and vice versa. The linearization problem is a particular case of the equivalence problem, where one of the systems is a linear system. It is one of the essential parts in the study of nonlinear equations.

1.1.1 A Single Ordinary Differential Equation

The analysis of the linearization problem for a single ordinary differential equation was started by Lie (1883). He gave the linearization criterion* for a second-order ordinary differential equation to be transformable into the simplest linear equation† ($\ddot{u} = 0$) by an invertible point transformation of the independent

*See proof in Chapter II.

†See more detail in Appendix A.

and dependent variables,

$$t = \varphi(x, y), \;\; u = \psi(x, y). \tag{1.1}$$

He showed that a second-order ordinary differential equation is linearizable, if and only if it has the form

$$y'' = a(x, y)y'^3 + b(x, y)y'^2 + c(x, y)y' + d(x, y), \tag{1.2}$$

where $y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}$, and the coefficients $a(x, y)$, $b(x, y)$, $c(x, y)$ and $d(x, y)$ satisfy the conditions $H = 0$ and $K = 0$, where

$$\begin{aligned} H &= 2b_{xy} - 3a_{xx} - c_{yy} - 3a_xc + 3a_yd + 2b_xb - 3c_xa - c_yb + 6d_ya, \\ K &= 2c_{xy} - b_{xx} - 3d_{yy} - 6a_xd + b_xc + 3b_yd - 2c_yc - 3d_xa + 3d_yb. \end{aligned} \tag{1.3}$$

Liouville (1889) and Tresse (1896) treated the functions H and K as the relative invariants with respect to invertible[‡] transformation (1.1). Another approach was developed by Cartan (1924), who used differential geometry for solving the linearization problem of a second-order ordinary differential equation.

Later, the linearization problem was also considered with respect to other types of transformations, for example, contact[§] and generalized Sundman[¶] transformations. Lie noted that under contact transformations, all second-order ordinary differential equations are equivalent to another. In 1994, Duarter, Moreira and Santos classified the second-order ordinary differential equations which are equivalent to equation $\ddot{u} = 0$ under generalized Sundman transformations. Transformation methods have also been applied to third-order and fourth-order ordinary differential equations as the following. Chern (1940) used Cartan's approach to obtain linearization criteria for a third-order ordinary differential equation via con-

[‡]The meaning shown in pp. 8.

[§]See definition on pp. 11.

[¶]See definition on pp. 16.

tact transformations. Grebot (1997) studied the linearization problem of third-order ordinary differential equations by fiber preserving point transformations. Some equivalence problems for differential equations under contact transformations were studied in a list of papers Bocharov, Sokolov and Svinolupov (1993); Doubrov (2001); Doubrov, Komrakov and Morimoto (1999); Gusyatnikova and Yumaguzhin (1999). In 2002, Neut and Petitot obtain conditions for equivalence with an arbitrary linear third-order ordinary differential equation. Ibragimov and Meleshko (2004), obtained linearization criteria for third-order ordinary differential equations by point transformations. In 2005, Ibragimov and Meleshko obtained linearization criteria for third-order ordinary differential equations by point and contact transformations. Meleshko (2006), using the Lie linearization test for linearization of a third-order ordinary differential equations. In 2003, Euler, Wolf and Leach, obtained criteria for third-order ordinary differential equations to be equivalent to the equation $X''' = 0$ by generalized Sundman transformations. Euler (2004), studied the symmetries of nonlinear second-order and third order ordinary differential equations by generalized Sundman transformations. Ibragimov, Meleshko and Suksern (2008), obtained linearization criteria for fourth-order ordinary differential equations by point transformations. In 2009, Suksern, Ibragimov and Meleshko, obtained linearization criteria for fourth-order ordinary differential equations by point and contact transformations. Nakpim and Meleshko (2010), obtained linearization criteria for second-order and third-order of ordinary differential equations by generalized Sundman transformations.

It is worth to note that fiber preserving transformations, where the change of the independent variable depends only on the independent variable itself, play a special role: either only such transformations were studied Grebot (1997) or they needed to be studied separately during compatibility analysis Ibragimov and

Meleshko (2004); Ibragimov and Meleshko (2005); Meleshko (2005); Ibragimov, Meleshko and Suksern (2008).

1.1.2 System of Ordinary Differential Equations

The linerization problem for systems of second-order ordinary differential equations was studied in Wafo and Mahomed (2001); Sookmee (2005); Mahomed and Qadir (2007); Aminova and Aminov (2006); Neut, Petitot and Dridi (2009) and others[‖]. In Neut, Petitot and Dridi (2009), necessary and sufficient conditions for a system of two second-order ordinary differential equations to be equivalent to the simplest equations were obtained. In Aminova and Aminov (2006), necessary and sufficient conditions for a system of $n \geq 2$ second-order ordinary differential equations to be equivalent to the free particle equations were given. Particular classes of systems of two ($n = 2$) second-order ordinary differential equations were considered in Mahomed and Qadir (2007). In Wafo and Mahomed (2001), criteria for linearization of a system of two second-order ordinary differential equations were related with the existence of an admitted four-dimensional Lie algebra. Some first-order and second-order relative invariants with respect to point transformations for a system of two second-order ordinary differential equations were obtained in Sookmee (2005).

1.2 Accomplishments of the Thesis

This thesis is devoted to the study of the linearization problem of a system of two second-order ordinary differential equations. The method of the study is separated into two parts as follows.

[‖]The references listed are not complete.

1.2.1 Linearization of a Projectable System of Two Second-order ODEs

A new method for linearizing a system of two ordinary differential equations is introduced, through the definition of a projectable system of such equations. This method consists of sequentially reducing the number of dependent variables, and then applying the Lie criterion to the reduced equations. We call systems linearizable by this procedure sequentially linearizable. This method is applied to a system of two second-order ordinary differential equations. Moreover, it is shown that for systems of two second-order quadratically semi-linear ordinary differential equations this new method gives a larger class of linearizable systems than via point transformations. Finally, an example of equations which are not linearizable by point transformations, but do sequentially linearize by the new method, is given.

1.2.2 Linearization of a System of Two Second-order ODEs via Fiber Preserving Point Transformations

The necessary form of a system of two second-order ordinary differential equations which can be linearized via point transformations is obtained. Some additional necessary conditions are also found. Necessary and sufficient conditions for a system of two second-order ordinary differential equations to be transformed to the general form of a linear system with constant coefficients via fiber preserving point transformations** are obtained. A linear system with constant coefficients is chosen because of simplicity of finding its general solution. On the way to establishing the main theorems, we also give an explicit procedure for constructing the linearizing transformation.

**See definition of fiber preserving point transformations on pp. 8.

Since the work in the thesis required a huge amount of analytical calculations, it was necessary to use a computer for these calculations. A brief review of computer systems of symbolic manipulations can be found, for example, in Davenport (1994). In our calculations the system REDUCE (Hearn, 1987) was used.

This thesis is organized systematically as follows. In chapter II, we prepare some information for solving the linearization problem and do a literature review. In chapter III, a new method for linearizing a system of ordinary differential equations is introduced. The first main result of the thesis is also shown in this chapter: conditions for linearization of a projectable system of two second-order ordinary differential equations. The application to a system of second-order quadratically semi-linear ordinary differential equations is demonstrated. Examples of systems of equations which are not linearizable via point transformations, but linearizable by the new method, are given in the subsequent sections. In chapter IV, the necessary form of a linearizable system of two second-order ordinary differential equations is presented. The second main result of the thesis is also exhibited: necessary and sufficient criteria for a system of two second-order ordinary differential equations to be equivalent to a linear system of two second-order ordinary differential with constant coefficients, via fiber preserving point transformations. During this study, we also obtained some necessary conditions for linearizability for a system of two second-order ordinary differential equations, to be equivalent to a linear system of two second-order ordinary differential under point transformations . Examples demonstrating the procedure of using the linearization theorems are presented in the subsequent sections. The thesis conclusions are in the last chapter. Additional information concerning the thesis is shown in the Appendices.

CHAPTER II

FUNDAMENTAL KNOWLEDGE

The material in this chapter constitutes the basic background for solving the linearization problem and constitutes a literature review.

Throughout this thesis, all functions are assumed to be sufficiently many times continuously differentiable.

2.1 Linearization Problem

Definition 2.1. Two differential equations are said to be *equivalent*, if there exists an invertible transformation which transforms any solution of one equation to a solution of the other equation and vice versa.

Definition 2.2. Two systems of differential equations are said to be *equivalent*, if there exists an invertible transformation which transforms any solution of one system to a solution of the other system and vice versa.

Definition 2.3. The *linearization problem* is the problem of finding conditions which guarantee the existence of an invertible transformation mapping a given system of differential equations into a linear system of differential equations.

Remark 2.1. The problem of finding conditions for a differential equation to be equivalent to a given differential equation is called the *equivalence problem.* Thus, in the particular case where the given differential equation is linear differential equations, then the equivalence problem is called the *linearization problem.*

2.2 Point Transformation

Definition 2.4. A transformation of the form

$$t = \varphi(x, y), \quad u = \psi(x, y), \tag{2.1}$$

is called a *point transformation*. Here x is the independent variable and y is the dependent variable; both variables may be vectors. Notice that t and u are the new independent and dependent variables, respectively.

Remark 2.2. If $t = \varphi(x)$, then the point transformation (2.1) is called a fiber preserving point transformation.

If $\Delta \neq 0$, where Δ is the Jacobian of the transformation (2.1)

$$\Delta = \frac{\partial(\varphi, \psi)}{\partial(x, y)} = \varphi_x \psi_y - \varphi_y \psi_x \neq 0,$$

then by virtue of the Inverse Function Theorem*, the transformation (2.1) is an invertible point transformation. That is x and y can locally be written as follows

$$x = \tilde{\varphi}(t, u), \quad y = \tilde{\psi}(t, u). \tag{2.2}$$

For example†, in this thesis, a system of two second-order ordinary differential equations will be considered. The invertible point transformation is defined as follows

$$t = \varphi(x, y_1, y_2), \; u_1 = \psi_1(x, y_1, y_2), \; u_2 = \psi_2(x, y_1, y_2). \tag{2.3}$$

The Jacobian of the change of variables (2.3) is

$$\Delta = (\varphi_x \psi_{1_{y_1}} \psi_{2_{y_2}} - \varphi_x \psi_{1_{y_2}} \psi_{2_{y_1}} - \varphi_{y_1} \psi_{1_x} \psi_{2_{y_2}} + \varphi_{y_1} \psi_{1_{y_2}} \psi_{2_x} + \varphi_{y_2} \psi_{1_x} \psi_{2_{y_1}} - \varphi_{y_2} \psi_{1_{y_1}} \psi_{2_x}).$$

If the function φ of (2.3) depends only on the independent variable x, then the point transformation (2.3) is a fiber preserving point transformation.

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*The statement of the Inverse Function Theorem shown in Appendix B.

†See another example in Appendix C.

2.2.1 Defining Derivatives in Point Transformations

Consider a single[‡] ordinary differential equation of k^{th}-order

$$f_1\left(t, u, \dot{u}, \ddot{u}, ..., u^{(k)}\right) = 0. \tag{2.4}$$

Let us explain how an invertible point transformation (2.1) maps equation (2.4) into another equation

$$g_1\left(x, y, y', y'', ..., y^{(k)}\right) = 0 \tag{2.5}$$

and vice versa.

Assume that $y(x)$ is a given function[§]. The first equation of (2.1) becomes

$$t = \varphi\left(x, y\left(x\right)\right) =: \bar{\varphi}(x).$$

Suppose that $\bar{\varphi}'(x) = 0$, then $t = \varphi = constant$. This contradicts that t is the independent variable. That is $\bar{\varphi}'(x) = \varphi_x + y'\varphi_y \neq 0$, and then by virtue of the Inverse Function Theorem, one finds

$$x = \beta\left(t\right). \tag{2.6}$$

Substituting x into the second equation of (2.1), one obtains

$$u\left(t\right) = \psi\left(\beta\left(t\right), y\left(\beta\left(t\right)\right)\right). \tag{2.7}$$

Thus, the first-order derivative of u with respect to t is defined by the formula

$$\dot{u} = \frac{du}{dt} = \frac{\partial\psi}{\partial x}\frac{d\beta}{dt} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}\frac{d\beta}{dt} = \left(\psi_x + y'\psi_y\right)\frac{d\beta}{dt}. \tag{2.8}$$

For finding $\frac{d\beta}{dt}$ let us consider the identity

$$t = \varphi\left(\beta\left(t\right), y\left(\beta\left(t\right)\right)\right). \tag{2.9}$$

[‡]The case of a system of two-order ODEs is presented in in Appendix D.

[§]This function at this stage need not to be the solution of equation (2.5).

Differentiating the equation (2.9) with respect to t, one obtains

$$1 = (\varphi_x + y'\varphi_y)\,\frac{d\beta}{dt}$$

or

$$\frac{d\beta}{dt} = \frac{1}{(\varphi_x + y'\varphi_y)}. \tag{2.10}$$

Substituting $\frac{d\beta}{dt}$ into equation (2.8), one obtains

$$\dot{u} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} = \frac{D\psi}{D\varphi} = h_1\left(x, y, y'\right),$$

where

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots + y^{(k)}\frac{\partial}{\partial y^{(k-1)}}$$

is the total derivative with respect to x.

The second-order derivative of u with respect to t is defined by the formula

$$\begin{aligned}
\ddot{u} &= \frac{d^2u}{dt^2} = \frac{d\dot{u}}{dt} = \frac{dh_1\left(\beta\left(t\right), y\left(\beta\left(t\right)\right), y'\left(\beta\left(t\right)\right)\right)}{dt} \\
&= \frac{\partial h_1}{\partial x}\frac{d\beta}{dt} + \frac{\partial h_1}{\partial y}\frac{dy}{dx}\frac{d\beta}{dt} + \frac{\partial h_1}{\partial y'}\frac{dy'}{dx}\frac{d\beta}{dt} \\
&= \left(h_{1x} + y'h_{1y} + y''h_{1y'}\right)\frac{d\beta}{dt} \\
&= \frac{h_{1x} + y'h_{1y} + y''h_{1y'}}{\varphi_x + y'\varphi_y} \\
&= \frac{Dh_1}{D\varphi} \\
&= h_2(x, y, y', y'').
\end{aligned}$$

Repeating by the same process, one obtains the higher order derivatives

$$u^{(n)} = \frac{du^{(n-1)}}{dt} = \frac{Dh_{n-1}}{D\varphi} = h_n(x, y, y', y'', ..., y^{(n)}), \quad (n = 1, 2, ..., k).$$

Here $u^{(0)} = u$ and $h_0 = \psi$.

Assume that $y_0(x)$ is a given function. Using equation (2.7), one can convert the function $y_0(x)$ into the function $u_0(t)$. Conversely, if one has the function $u_0(t)$, then by applying the Inverse Function Theorem to the first equation of (2.2), one obtains

$$t = \gamma(x).$$

Substitution of this t into the second equation of (2.2), gives the function

$$y_0(x) := \tilde{\psi}\left(\gamma(x), u_0(\gamma(x))\right).$$

Observes that the order of the given ordinary differential equation (2.4) is preserved under the invertible point transformation (2.1).

2.3 Contact Transformation

Definition 2.5. A transformation

$$t = \varphi_1\left(x, y, p\right), \quad u = \varphi_2\left(x, y, p\right), \quad w = \varphi_3\left(x, y, p\right), \tag{2.11}$$

where $p = y' = \frac{dy}{dx}$ is called a *contact transformation* if it satisfies the *contact condition*

$$du - wdt = 0.$$

Notice that the variables x, y and p can also be expressed as vectors.

2.3.1 Defining Derivatives in Contact Transformations

Let us consider how the given k^{th}-order ordinary differential equation

$$f_2\left(t, u, \dot{u}, \ddot{u}, ..., u^{(k)}\right) = 0. \tag{2.12}$$

is transformed into

$$g_2\left(x, y, y', y'', ..., y^{(k)}\right) = 0. \tag{2.13}$$

by an invertible contact transformation.

Assume that $y(x)$ is a given function[¶]. The first equation of (2.11) becomes

$$t = \varphi_1\left(x, y\left(x\right), p(x)\right) =: \bar{\varphi}(x).$$

Suppose that $\bar{\varphi}^{'}(x) = 0$, then $t = \varphi_1 = constant$. This contradicts to t being the independent variable. That is $\bar{\varphi}^{'}(x) = \varphi_{1x} + p\varphi_{1y} + y''\varphi_{1p} \neq 0$. By virtue of the Inverse Function Theorem, one finds

$$x = \alpha(t). \tag{2.14}$$

Substituting x into the second equation of (2.11), one gets

$$u\left(t\right) = \varphi_2\left(\alpha\left(t\right), y\left(\alpha\left(t\right)\right), p\left(\alpha\left(t\right)\right)\right). \tag{2.15}$$

Thus the first-order derivative of u with respect to t, is defined by the formula

$$\begin{aligned} \dot{u} &= \frac{du}{dt} \\ &= \frac{\partial\varphi_2}{\partial x}\frac{d\alpha}{dt} + \frac{\partial\varphi_2}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt} + \frac{\partial\varphi_2}{\partial p}\frac{dp}{dx}\frac{d\alpha}{dt} \\ &= \left(\varphi_{2x} + p\varphi_{2y} + y''\varphi_{2p}\right)\frac{d\alpha}{dt}. \end{aligned} \tag{2.16}$$

For finding $\frac{d\alpha}{dt}$, let us consider the identity

$$t = \varphi_1\left(\alpha\left(t\right), y\left(\alpha\left(t\right)\right), p\left(\alpha\left(t\right)\right)\right). \tag{2.17}$$

Differentiating equation (2.17) with respect to t, one obtains

$$1 = \left(\varphi_{1x} + p\varphi_{1y} + y''\varphi_{1p}\right)\frac{d\alpha}{dt}$$

or

$$\frac{d\alpha}{dt} = \frac{1}{\left(\varphi_{1x} + p\varphi_{1y} + y''\varphi_{1p}\right)}. \tag{2.18}$$

Substituting a $\frac{d\alpha}{dt}$ into equation (2.16), one obtains

$$\dot{u} = \frac{\varphi_{2x} + p\varphi_{2y} + y''\varphi_{2p}}{\varphi_{1x} + p\varphi_{1y} + y''\varphi_{1p}} = \frac{D\varphi_2}{D\varphi_1}.$$

[¶]This function at this place need not to be a solution of equation (2.13).

According to the contact condition, we have the relation

$$\varphi_3 = \frac{D\varphi_2}{D\varphi_1}. \tag{2.19}$$

Then equation (2.19) can be represented as follows,

$$\varphi_3\left(\varphi_{1x} + p\varphi_{1y} + y''\varphi_{1p}\right) = \varphi_{2x} + p\varphi_{2y} + y''\varphi_{2p}. \tag{2.20}$$

Since the contact condition (2.20) is satisfied for any y'', one has

$$\varphi_3\left(\varphi_{1x} + p\varphi_{1y}\right) = \varphi_{2x} + p\varphi_{2y}, \qquad \varphi_3\varphi_{1p} = \varphi_{2p}. \tag{2.21}$$

The second-order derivative of u with respect to t, is defined by the formula

$$\begin{aligned} \ddot{u} &= \frac{d^2u}{dt^2} = \frac{d\dot{u}}{dt} \\ &= \frac{\partial\varphi_3}{\partial x}\frac{d\alpha}{dt} + \frac{\partial\varphi_3}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt} + \frac{\partial\varphi_3}{\partial p}\frac{dp}{dx}\frac{d\alpha}{dt} \\ &= \left(\varphi_{3x} + p\varphi_{3y} + y''\varphi_{3p}\right)\frac{d\alpha}{dt} \\ &= \frac{\varphi_{3x} + p\varphi_{3y} + y''\varphi_{3p}}{\varphi_{1x} + p\varphi_{1y} + y''\varphi_{3p}} \\ &= \frac{D\varphi_3}{D\varphi_1} \\ &= \varphi_4(x, y, p, y''). \end{aligned}$$

Repeating, one obtains the higher order derivatives by the formulae

$$u^{(n)} = \frac{du^{(n-1)}}{dt} = \frac{D\varphi_{n+1}}{D\varphi} = \varphi_{n+2}(x, y, p, y'', y''', ..., y^{(n)}), \quad (n = 1, 2, ..., k).$$

Here $u^{(0)} = u$.

Assume that $y_0(x)$ is a solution of (2.13) and $u_0(t)$ is a solution of (2.12). Using equation (2.15), one can convert the function $y_0(x)$ to the function $u_0(t)$. Conversely, if one has the function $u_0(t)$, since $\Delta \neq 0$ where $\Delta = \frac{\partial(\varphi_1,\varphi_2,\varphi_3)}{\partial(x,y,p)}$, the Inverse Function Theorem gives

$$x = \tilde{\varphi}_1(t, u, w), \quad y = \tilde{\varphi}_2(t, u, w), \quad p = \tilde{\varphi}_3(t, u, w), \tag{2.22}$$

and then the first equation of (2.2) becomes

$$x = \tilde{\varphi}_1 \left(t, u_0(t), \dot{u}_0(t)\right). \tag{2.23}$$

Thus applying the Inverse Function Theorem to (2.23), one obtains

$$t = \Omega(x).$$

Substituting this t into the second equation of (2.22), one obtains the solution

$$y_0(x) := \tilde{\varphi}_2 \left(\Omega(x), u_0(\Omega(x)), \dot{u}_0(\Omega(x))\right).$$

Observe that the order of the given ordinary differential equation (2.12) is conserved by the invertible contact transformation (2.11). Notice also that according to the contact condition (2.21), if $\varphi_{1_p} = 0$ then $\varphi_{2_p} = 0$. Hence, a contact transformations is also a point transformation.

Remark 2.3. It is worth noting that the application of contact transformations is more complicated than the application of point transformations, for example, see in Ibragimov and Meleshko (2005); Suksern, Ibragimov and Meleshko (2009).

2.4 Tangent Transformation

A tangent transformation is a transformation of the independent, dependent variables and their derivatives. Let $x = (x_1, x_2, ..., x_n)$ be the independent variables, $y = (y^1, y^2, ..., y^m)$ be the dependent variables and p be the vector of the partial derivatives:

$$p^k_\alpha := \frac{\partial^{|\alpha|} y^k}{\partial x^\alpha} := \frac{\partial^{|\alpha|} y^k}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index and $k \in \{1, 2, ..., m\}$. Here $|\alpha| := \alpha_1 + \alpha_2 + ... + \alpha_n$ and $\alpha_i \in \{0 \cup N\}$, $(i = 1, 2, .., n)$.

Definition 2.6. The transformation

$$t_j = \varphi_j(x, y, p), \quad u^k = \psi^k(x, y, p), \quad \bar{p}_j^k = \phi_j^k(x, y, p),$$

is called a *tangent transformation* if it satisfies the *tangent conditions*

$$du^k - \bar{p}_j^k dt_j = 0, \quad d\bar{p}_\alpha^k - \bar{p}_{\alpha,j}^k dt_j = 0, \tag{2.24}$$

where $j = 1, 2, 3, ..., n$, $k = 1, 2, 3, ..., m$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha, j :=$ $(\alpha_1, \alpha_2, ..., \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, ..., \alpha_n)$, $\bar{p}_\alpha^k = \frac{\partial^{|\alpha|} u^k}{\partial t_1^{\alpha_1} \partial t_2^{\alpha_2} ... \partial t_n^{\alpha_n}}$ and $0 \leq |\alpha| \leq \tau - 1$. Here τ is the maximum order of the partial derivatives appearing in the vector p.

For example, let us consider the case $n = m = 1$ and the functions ψ_1, ψ_2 and ψ_3 of the variables x, y, y' and y''. Define the mapping

$$t = \psi_1 (x, y, y'), \quad u = \psi_2 (x, y, y'), \quad \xi = \psi_3 (x, y, y', y''). \tag{2.25}$$

Let $y(x)$ be a given function. Substituting $y(x)$ into the first equation of (2.25), one yields

$$t = \psi_1 (x, y(x), y'(x)).$$

By the virtue of the Inverse Function Theorem, one gets $x = \alpha(t)$. Substituting this x into the second equation of (2.25), one obtains the transformed function

$$u(t) = \psi_2 (\alpha(t), y(\alpha(t)), y'(\alpha(t))).$$

The derivatives are changed by the formulae

$$\dot{u} = \frac{\psi_{2x} + p\psi_{2y} + y''\psi_{2p}}{\psi_{1x} + p\psi_{1y} + y''\psi_{1p}} = \frac{D_x\psi_2}{D_x\psi_1}.$$

where

$$D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + y''\frac{\partial}{\partial p} + y'''\frac{\partial}{\partial y''} + \cdots$$

is the total derivative with respect to x. Here $p = y'$.

According to the tangent condition, we have the relation

$$\psi_3 = \frac{D_x\psi_2}{D_x\psi_1}. \tag{2.26}$$

Thus the second-order derivative of u with respect to t is defined by the formula

$$\ddot{u} = \psi_4(x, y, p, y'', y''') := \frac{D_x\psi_3}{D_x\psi_1} = \frac{\psi_{3x} + p\psi_{3y} + y''\psi_{3p} + y'''\psi_{3y''}}{\psi_{1x} + p\psi_{1y} + y''\psi_{1p}}. \tag{2.27}$$

Note that tangent condition (2.26) need not satisfy for any y'' like the contact condition. Equation (2.27) shows that a tangent transformations need not to preserve the order of the given ordinary differential equation.

Remark 2.4. The contact transformation forms a particular case of tangent transformations.

2.5 Generalized Sundman Transformation

The generalized Sundman transformation is a transformation defined by the formulae

$$u(t) = P(x, y), \quad dt = S(x, y)dx, \quad (P_yS \neq 0). \tag{2.28}$$

Let us explain how the generalized Sundman transformation maps one function into another.

Assume that $y\,(x)$ is a given function. Integrating the second equation of (2.28)

$$\frac{dt}{dx} = S(x, y(x)),$$

one obtains $t = Q\,(x)$. Since the function $S \neq 0$, thus $Q' \neq 0$. By virtue of the Inverse Function Theorem, one finds $x = \beta(t)$. Substituting this x into the function $P(x, y(x))$, one gets the transformed function

$$u(t) = P(\beta(t), y(\beta(t)).$$

Conversely, let $u(t)$ be a given function. Since $P_y \neq 0$, applying the Inverse Function Theorem to the equation

$$u(t) = P(x, y),$$

one gets $y = \phi(x, t)$. Integrating the ordinary differential equation

$$\frac{dt}{dx} = S(x, \phi(x, t)),$$

one finds $t = H(x)$. Substituting $t = H(x)$ into the function $\phi(x, t)$, the transformed function $y(x) = \phi(x, H(x))$ is obtained.

Observe that the formula (2.28) not only allows us to obtain the derivatives of $u(t)$ through the derivatives of the function $y(x)$ and vice versa, but also relate the solutions of the two differential equations $y^{(n)} = N_1(x, y, y', y'', \ldots, y^{(n-1)})$ and $u^{(n)} = N_2(t, u, \dot{u}, \ddot{u}, \ldots, u^{(n-1)})$.

Remark 2.5. The generalized Sundman transformation conserves the order of particular ordinary differential equations only, for example, see in Euler, Wolf and Leach (2003); Nakpim and Meleshko (2010).

2.6 Canonical Form of a Single Linear Second-order ODE

Theorem 2.1. *Every linear second-order ordinary differential equation*

$$y''(x) + a(x)\, y'(x) + b(x)\, y(x) = c(x),$$

can be transformed into the simplest equation

$$\ddot{u} = 0.$$

Note that the proof of this theorem is shown in Appendix A.

Remark 2.6. A linear ordinary differential equation of order $m \geq 3$ need not be equivalent to the simplest equation. In fact, the generalization of Theorem 2.1 to higher order equations is discussed in the next section.

Remark 2.7. A linear system of n second-order ordinary differential equations need not to be equivalent to the simplest system of equations.

2.7 Canonical Form of a Single Linear i^{th}-order ODE

Laguerre showed that given any linear ordinary differential equation of form

$$y^{(i)} + a_1(x)y^{(i-1)} + a_2(x)y^{(i-2)} + ... + a_{i-1}(x)y' + a_i(x)y = c(x), \; i \geqslant 3, \tag{2.29}$$

the two terms following the highest-order term can be eliminated by point transformations. Therefore, the general linear i^{th}-order ordinary differential equation in *Laguerre*'s form is defined by the following theorem.

Theorem 2.2. *(Laguerre Canonical Form).*
A linear i^{th}-order ordinary differential equation (2.29) can be reduced to the equation

$$u^{(i)} + b_3(t)u^{(i-3)} + b_4(t)u^{(i-4)} + ... + b_{i-1}(t)\dot{u} + b_i(t)u = 0, \tag{2.30}$$

by point transformations.

2.8 Canonical Forms of a Linear System of n Second-order ODEs

The general form of a linear system of n second-order ordinary differential equations is

$$\ddot{v} + C\dot{v} + Dv + E = 0, \tag{2.31}$$

where $v = v(t)$ and $E = E(t)$ are vectors, $C = C(t)$ and $D = D(t)$ are $n \times n$ square matrices. It can be shown (Wafo and Mahomed, 2001) that there exists a change $u = Uv$ such that system (2.31) is reduced to one of the following forms[‖]

[‖]The proof of this statement is given in Appendix E.

either

$$\ddot{u} + K_1\dot{u} = 0$$

or

$$\ddot{u} + Ku = 0. \tag{2.32}$$

Here $U(t)$, $K = K(t)$ and $K_1 = K_1(t)$ are $n \times n$ square matrices.

Thus the linearization problem via point transformations consists of solving the problem of reducibility of a system of second-order ordinary differential equations to one of these forms. In this thesis, the second canonical form (2.32) is used.

One of the main motivations for studying the linearization problem is the possibility of finding the general solution. Notice that even after finding the linearizing transformation one has to solve a linear system of second-order ordinary differential equations. The simplest case is where $K = 0$. More general and also not complicated is the case where the matrix K is constant. For example, for $n = 2$ this case leads to solving either a simple linear fourth-order ordinary differential equation with constant coefficients or two simple linear second-order ordinary differential equations. Indeed, for $n = 2$ system (2.32) is

$$\ddot{u}_1 + k_1u_1 + k_3u_2 = 0, \quad \ddot{u}_2 + k_4u_1 + k_2u_2 = 0, \tag{2.33}$$

where k_i, $(i = 1, 2, 3, 4)$ are constant. If $k_3 \neq 0$, then finding u_2 from the first equation of (2.33) and substituting it into the second equation of (2.33), one obtains a fourth-order ordinary differential equation

$$u_1^{(4)} + (k_1 + k_2)\ddot{u}_1 + (k_1k_2 - k_3k_4)u_1 = 0.$$

Here

$$K = \begin{pmatrix} k_1 & k_3 \\ k_4 & k_2 \end{pmatrix}.$$

The general solution of the last equation depends on the roots λ of the characteristic equation

$$\lambda^4 + (k_1 + k_2)\lambda^2 + (k_1k_2 - k_3k_4) = 0.$$

The solution is similar for $k_4 \neq 0$. On the other hand, if $k_3 = 0$ and $k_4 = 0$, then system (2.31) is decoupled:

$$\ddot{u}_1 + k_1u_1 = 0, \quad \ddot{u}_2 + k_2u_2 = 0. \tag{2.34}$$

Notice also that if in this case $k_1 = k_2$, then the last system of equations is equivalent to the system of two trivial equations $z'' = 0$.

2.9 Theory of Compatibility

This section gives some knowledge on compatibility theory used in the thesis. Compatibility theory analyzes the existence of a solution of an overdetermined system of equations. An overdetermined system is a system with the number of equations greater than the number of unknown functions. Since this theory is a special subject of mathematical analysis, the statements are given without proofs**.

There are two approaches for studying compatibility. These approaches are related to the works of E. Cartan and C. H. Riquier.

The Cartan approach is based on the calculus of exterior differential forms. The problem of the compatibility of a system of partial differential equations is reduced to the problem of the compatibility of a system of exterior differential forms. E. Cartan studied the formal algebraic properties of systems of exterior forms. For their description he introduced special integer numbers, called characters. With the help of the characters he formulated a criterion for a given system of partial

**A review of the theory of compatibility can be found in Meleshko (2005).

differential equations to be involutive. Detailed theory of involutive systems can be found in Cartan (1946), Finikov (1948), Kuranashi (1967) and Pommaret (1978).

The Riquier approach has a different theory of establishing the involution. This method can be found in Kuranashi (1967) and Pommaret (1978). The main advantage of this approach is that there is no necessity to reduce the system of partial differential equations under study to exterior differential forms. The calculations in the Riquier approach are shorter than in the Cartan approach. The main operations of the study of compatibility in the Riquier approach are prolongations of a system of partial differential equations and the study of the ranks of some matrices. The Riquier approach is used in this thesis.

2.9.1 Completely Integrable Systems

One class of overdetermined systems, for which the problem of compatibility is solved, is the class of completely integrable systems.

Definition 2.7. A system

$$\frac{\partial y^i}{\partial x_j} = \phi_j^i(x, y), \ (i = 1, 2, ..., m; \ j = 1, 2, ..., n) \tag{2.35}$$

is called *completely integrable* if it has a solution for any initial values x_0, y_0 in some open domain D.

Theorem 2.3. *A system of the type (2.35) is completely integrable if and only if all of the mixed derivatives equalities*

$$\frac{\partial \phi_j^i}{\partial x_k} + \sum_{\gamma=1}^{N} \phi_k^\gamma \frac{\partial \phi_j^i}{\partial y^\gamma} = \frac{\partial \phi_k^i}{\partial x_j} + \sum_{\gamma=1}^{N} \phi_j^\gamma \frac{\partial \phi_k^i}{\partial y^\gamma}, \ (i = 1, 2, ..., n; \ \ k, j = 1, 2, ..., m) \tag{2.36}$$

are identically satisfied with respect to the variables $(x, y) \in D$.

In this thesis, the following corollary of the above theorem is used.

Corollary 2.4. *If in an overdetermined system of partial differential equations all derivatives of order n are defined and comparison of all mixed derivatives of order $n+1$ does not produce new equations of order less or equal to n, then this system is compatible.*

To demonstrate the importance of Corollary 2.4, let us apply it for obtaining Lie's criterion.

2.10 Lie Criterion

Theorem 2.5. *(Lie criterion).*
A second-order ordinary differential equation is equivalent to the simplest equation ($\ddot{u}=0$) if and only if it has the form

$$y'' + a(x,y)\,y'^3 + b(x,y)\,y'^2 + c(x,y)\,y' + d(x,y) = 0, \tag{2.37}$$

with the coefficients satisfying the conditions:

$$3a_{xx} - 2b_{xy} + c_{yy} - 3a_x c + 3a_y d + 2b_x b - 3c_x a - c_y b + 6d_y a = 0,$$

$$b_{xx} - 2c_{xy} + 3d_{yy} - 6a_x d + b_x c + 3b_y d - 2c_y c - 3d_x a + 3d_y b = 0.$$

Proof.

Notice that the canonical form of a second-order linear ordinary equation with independent variable t and dependent variable u is

$$\ddot{u} = 0. \tag{2.38}$$

Assume that the equation $y'' = F(x,y,y')$ is obtained from the linear ordinary differential equation (2.38) by the change of the variables

$$t = \varphi(x,y), \quad u = \psi(x,y). \tag{2.39}$$

The derivatives are changed by the formulae

$$\dot{u} = g(x, y, y') = \frac{D_x\psi}{D_x\varphi} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} \ ,$$

$$\ddot{u} = p(x, y, y', y'') = \frac{D_x g}{D_x\varphi} = \frac{g_x + y'g_y + y''g_{y'}}{\varphi_x + y'\varphi_y}$$

$$= \Delta(\varphi_x + y'\varphi_y)^{-3}[y'' + a(x, y)y'^3 + b(x, y)y'^2 + c(x, y)y' + d(x, y)],$$

where

$$\begin{aligned} a &= \Delta^{-1}(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y), \\ b &= \Delta^{-1}(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y)), \\ c &= \Delta^{-1}(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x)), \\ d &= \Delta^{-1}(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x), \end{aligned} \tag{2.40}$$

and

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'}$$

is the total derivative with respect to x. Here $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$ is the Jacobian of the change of variables (2.39).

Since $\Delta \neq 0$ and $\varphi_x + y'\varphi_y \neq 0$, substitution of $\ddot{u}$ into (2.38) gives the equation (2.37).

Therefore, if a second-order ordinary differential equation is linearizable, then it has the form (2.37). This is the necessary condition for all second-order ordinary differential equations to be linearized. The mapping of this equation into a linear equation is reconstituted by finding the functions $\varphi(x, y)$ and $\psi(x, y)$ that satisfy the relations (2.40) with given coefficients a, b, c and d. Since for a given differential equation there are only two unknown functions $\varphi(x, y)$ and $\psi(x, y)$, equations (2.40) form an overdetermined system of partial differential equations.

Let us analyze the compatibility of system (2.40).

2.10.1 case $\varphi_y = 0$

Since $\Delta \neq 0$, assuming that $\varphi_y = 0$, it implies that $\varphi_x \psi_y \neq 0$. From relations (2.40) one has

$$a = 0, \quad \psi_{yy} = \psi_y b, \quad \psi_{xy} = (\varphi_x^{-1} \psi_y \varphi_{xx} + c\psi_y)/2, \quad \psi_{xx} = \varphi_x^{-1} \psi_x \varphi_{xx} + \psi_y d. \tag{2.41}$$

Comparing the mixed derivatives $(\psi_{xy})_y = (\psi_{yy})_x$ and $(\psi_{xy})_x = (\psi_{xx})_y$, one gets

$$c_y = 2b_x, \quad \varphi_x^{-2} \left(2\varphi_x \varphi_{xxx} - 3\varphi_{xx}^2\right) = 4\left(d_y + bd\right) - \left(2c_x + c^2\right). \tag{2.42}$$

From $\varphi_y = 0$ and the second equation of (2.42), one obtains

$$\varphi_{xxy} = 0, \ \varphi_{xyy} = 0, \ \varphi_{yyy} = 0,$$

$$\varphi_{xxx} = \left({\varphi_x}^2 \left(4\left(d_y + bd\right) - \left(2c_x + c^2\right)\right) + 3\varphi_{xx}^2\right)/2\varphi_x.$$

Comparing the mixed derivatives

$$\left(\varphi_{xxy}\right)_y = \left(\varphi_{xyy}\right)_x, \ \left(\varphi_{xyy}\right)_y = \left(\varphi_{yyy}\right)_x, \ \left(\varphi_{xxx}\right)_y = \left(\varphi_{xxy}\right)_x, \tag{2.43}$$

one finds that the first and second equations of (2.43) are identically satisfied. The third equation of (2.43) gives the condition

$$d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0.$$

In brief, a second-order ordinary differential equation of the form (2.37) is equivalent to the simplest equation (2.38) under fiber preserving point transformation $t = \varphi(x)$ and $u = \psi(x, y)$, if and only if the coefficients of this equation satisfy the conditions

$$a = 0, \quad c_y = 2b_x, \quad d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0. \tag{2.44}$$

Note that conditions (2.44) guarantee the existence of functions $\varphi(x)$ and $\psi(x, y)$ satisfying the overdetermined system of equations (2.40).

2.10.2 case $\varphi_y \neq 0$

Considering the relations (2.40), one has

$$\psi_{yy} = (\varphi_{yy}\psi_y + a\Delta)/\varphi_y, \tag{2.45}$$

$$\psi_{xy} = (2\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\Delta - (a\varphi_x - b\varphi_y)\Delta)/2\varphi_y^2, \tag{2.46}$$

$$\psi_{xx} = (2\varphi_{xy}\varphi_y\psi_x - \varphi_x\varphi_{yy}\psi_x - \varphi_x^2\psi_x a + \varphi_x\varphi_y\psi_x b + \varphi_y^2(\psi_y d - \psi_x c))/\varphi_y^2, \tag{2.47}$$

$$\varphi_{xx} = (2\varphi_{xy}\varphi_x\varphi_y - \varphi_x^2\varphi_{yy} - \varphi_x^3 a + \varphi_x^2\varphi_y b - \varphi_x\varphi_y^2 c + \varphi_y^3 d)/\varphi_y^2. \tag{2.48}$$

From the equations (2.45)-(2.47), one can compare the mixed derivatives $(\psi_{xy})_y = (\psi_{yy})_x$ and $(\psi_{xy})_x = (\psi_{xx})_y$. These conditions give

$$\varphi_{yyy} = (3(\varphi_{yy}^2 - 2\varphi_{xy}\varphi_y a + 2\varphi_x\varphi_{yy} a + \varphi_x^2 a^2) - 2\varphi_x\varphi_y(a_y + ab) + \varphi_y^2(2b_y - 4a_x + 4ac - b^2))/2\varphi_y, \tag{2.49}$$

$$\varphi_{xyy} = (3(4\varphi_{xy}\varphi_{yy}\varphi_y - \varphi_x\varphi_{yy}^2 + 2\varphi_x\varphi_{yy}\varphi_y b - 2\varphi_{xy}\varphi_y^2 b) + 3\varphi_x^3 a^2 + 3\varphi_x\varphi_y^2(-2a_x + 2ac - b^2) + 2\varphi_y^3(-b_x + 2c_y + 3ad))/6\varphi_y^2. \tag{2.50}$$

Using the equation (2.48), one has

$$\varphi_{xxx} = \frac{\partial(\varphi_{xx})}{\partial x}, \quad \varphi_{xxy} = \frac{\partial(\varphi_{xx})}{\partial y}. \tag{2.51}$$

Considering the equations (2.49)-(2.51), one generates the conditions

$$(\varphi_{xxy})_x = (\varphi_{xxx})_y\,, \ (\varphi_{xyy})_y = (\varphi_{yyy})_x\,, \ (\varphi_{xxy})_y = (\varphi_{xyy})_x\,. \tag{2.52}$$

The first equation of (2.52) is satisfied. The second and third equations of (2.52) give the conditions

$$\begin{aligned} &3a_{xx} - 2b_{xy} + c_{yy} - 3a_x c + 3a_y d + 2b_x b - 3c_x a - c_y b + 6d_y a = 0, \\ &b_{xx} - 2c_{xy} + 3d_{yy} - 6a_x d + b_x c + 3b_y d - 2c_y c - 3d_x a + 3d_y b = 0. \end{aligned} \tag{2.53}$$

The condition (2.53) obtained, guarantees the existence of the functions $\varphi(x, y)$ and $\psi(x, y)$ satisfying the overdetermined system of equations (2.40). Observe that the conditions (2.44) make the conditions (2.53) vanish. That is the conditions (2.44) form a particular case of the conditions (2.53).

CHAPTER III

CONDITIONS FOR LINEARIZATION OF A PROJECTABLE SYSTEM OF TWO SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

3.1 Establishment of the First Main Problem

A system of two second-order ordinary differential equations with two dependent variables x and y and one independent variable t:

$$\ddot{x} = G(t, x, y, \dot{x}, \dot{y}), \quad \ddot{y} = F(t, x, y, \dot{x}, \dot{y}), \tag{3.1}$$

is considered in this chapter. Here a dot denotes the derivative with respect to t

$$\dot{x} = \frac{dx}{dt}\ , \quad \dot{y} = \frac{dy}{dt}\ , \quad \ddot{x} = \frac{d^2x}{dt^2}\ , \quad \ddot{y} = \frac{d^2y}{dt^2}\ .$$

A new method for linearization of two second-order ordinary differential equations (3.1) with two dependent variables x and y and one independent variable t is proposed here.

Assume that $\dot{x} \neq 0$. By virtue of the Inverse Function Theorem one can consider $y = y(x)$. Substituting the derivatives

$$\dot{y} = y'\dot{x}, \ \ddot{y} = y''\dot{x}^2 + y'\ddot{x}$$

into the first equation of (3.1), and using the second equation of (3.1), one obtains

$$\dot{x}^2 y'' + y'\tilde{G} - \tilde{F} = 0,$$

where

$$\tilde{G}(t,x,y,\dot{x},y') = G(t,x,y,\dot{x},\dot{x}y'),\ \ \tilde{F}(t,x,y,\dot{x},y') = F(t,x,y,\dot{x},\dot{x}y'),$$

$$y' = \frac{dy}{dx}\ ,\ y'' = \frac{d^2y}{dx^2}\ .$$

Suppose that

$$\tilde{F}(t,x,y,\dot{x},y') - y'\tilde{G}(t,x,y,\dot{x},y') = \dot{x}^2\lambda(x,y,y'), \tag{3.2}$$

where $x, y, \dot{x}$ and y' are considered as the independent variables of the functions $\tilde{G}, \tilde{F}$ and λ. We call a system (3.1) satisfying the condition (3.2) a projectable system. This definition of a projectable system of equations can be extended to any normal system of ordinary differential equations. Another extension of the definition can be given as follows. A system of equations (3.1) is called projectable if there exists an invertible change of the independent and dependent variables $\bar{x} = g_1(t,x,y),\ \bar{y} = g_2(t,x,y)$ and $\bar{t} = g_3(t,x,y)$ such that the equivalent system possesses the property (3.2). In this thesis we consider the simple case of a projectable system, where $g_1 = x,\ g_2 = y$ and $g_3 = t$.

Equation (3.2) requires that the function λ defined by the formula

$$\lambda(x,y,z) = \frac{1}{\dot{x}^2}(F(t,x,y,\dot{x},z\dot{x}) - zG(t,x,y,\dot{x},z\dot{x})), \tag{3.3}$$

only depends on x, y and $z = \frac{\dot{y}}{\dot{x}}$. The function $y(x)$ thus satisfies the second-order ordinary differential equation

$$y'' = \lambda(x,y,y')\ . \tag{3.4}$$

A solution of a projectable system (3.1) can be found in two sequential steps: in the first step one solves equation (3.4); in the second step one finds a solution $x(t)$ of the first equation of (3.1) with substituted $y = y(x)$ and $\dot{y} = y'(x)\dot{x}$:

$$\ddot{x} = G(t,x,y(x),\dot{x},\dot{x}y'(x))\ . \tag{3.5}$$

If at each step one has a linearizable second-order ordinary differential equation, then we call system (3.1) a sequentially linearizable system of equations. In this thesis we give necessary and sufficient conditions for system (3.1) to be sequentially linearizable.

3.2 Sequentially Linearizable System (3.1)

Since at each step the equations are second-order ordinary differential equations, one can consecutively apply the Lie criterion to equations (3.4) and (3.5).

Theorem 3.1. A projectable system (3.1) is sequentially linearizable if and only if the functions $\lambda(x, y, z)$ and $G(t, x, y, \dot{x}, \dot{y})$ have the representations

$$\lambda(x, y, z) = b_1(x, y)z^3 + b_2(x, y)z^2 + b_3(x, y)z + b_4(x, y), \tag{3.6}$$

$$\begin{gathered} G(t, x, y, \dot{x}, \dot{y}) = a_1(t, x, y, z)\dot{x}^3 + a_2(t, x, y, z)\dot{x}^2 \\ +a_3(t, x, y, z)\dot{x} + a_4(t, x, y, z), \end{gathered} \tag{3.7}$$

where the coefficients $b_i(x, y)$ and $a_i(t, x, y, z)$, $(i = 1, 2, 3, 4)$ satisfy the equations

$$\begin{gathered} 2b_{2xy} - 3b_{1xx} - b_{3yy} - 3b_{1x}b_3 + 3b_{1y}b_4 + 2b_{2x}b_2 \\ -3b_{3x}b_1 - b_{3y}b_2 + 6b_{4y}b_1 = 0, \end{gathered} \tag{3.8}$$

$$\begin{gathered} 2b_{3xy} - b_{2xx} - 3b_{4yy} - 6b_{1x}b_4 + b_{2x}b_3 + 3b_{2y}b_4 \\ -2b_{3y}b_3 - 3b_{4x}b_1 + 3b_{4y}b_2 = 0, \end{gathered} \tag{3.9}$$

$$\sum_{i=1}^{6} \beta_i(z)^{7-i} + \beta_7 = 0, \ \sum_{i=1}^{6} \beta_{i+7}(z)^{7-i} + \beta_{14} = 0. \tag{3.10}$$

Here

$\beta_1 = -a_{3zz}{b_1}^2, \ \beta_2 = b_1(-2a_{3zz}b_2 - 3a_{3z}b_1),$

$\beta_3 = -2a_{3yz}b_1 - 2a_{3zz}b_1b_3 - a_{3zz}{b_2}^2 - a_{3z}b_{1y} - 5a_{3z}b_1b_2,$

$\beta_4 = 3a_{1z}a_4b_1 + 2a_{2tz}b_1 - 2a_{3xz}b_1 - 2a_{3yz}b_2 - a_{3y}b_1 - 2a_{3zz}b_1b_4 - 2a_{3zz}b_2b_3$

$$-a_{3z}b_{1x} - a_{3z}b_{2y} - a_{3z}a_2b_1 - 4a_{3z}b_1b_3 - 2a_{3z}{b_2}^2 + 6a_{4z}a_1b_1,$$

$$\beta_5 = 3a_{1z}a_4b_2 + 2a_{2tz}b_2 - 2a_{3xz}b_2 - 2a_{3yz}b_3 - a_{3yy} - a_{3y}b_2 - 2a_{3zz}b_2b_4$$
$$-a_{3zz}{b_3}^2 - a_{3z}b_{2x} - a_{3z}b_{3y} - a_{3z}a_2b_2 - 3a_{3z}b_1b_4 - 3a_{3z}b_2b_3 + 6a_{4z}a_1b_2,$$

$$\beta_6 = 3a_{1y}a_4 + 3a_{1z}a_4b_3 + 2a_{2ty} + 2a_{2tz}b_3 - 2a_{3xy} - 2a_{3xz}b_3 - 2a_{3yz}b_4 - a_{3y}a_2$$
$$-a_{3y}b_3 - 2a_{3zz}b_3b_4 - a_{3z}b_{3x} - a_{3z}b_{4y} - a_{3z}a_2b_3 - 2a_{3z}b_2b_4 - a_{3z}{b_3}^2$$
$$+6a_{4y}a_1 + 6a_{4z}a_1b_3,$$

$$\beta_7 = 3a_{1x}a_4 - 3a_{1tt} - 3a_{1t}a_3 + 3a_{1z}a_4b_4 + 2a_{2xt} + 2a_{2tz}b_4 + 2a_{2t}a_2 - 2a_{3xz}b_4$$
$$-a_{3xx} - a_{3x}a_2 - 3a_{3t}a_1 - a_{3y}b_4 - a_{3zz}{b_4}^2 - a_{3z}b_{4x} - a_{3z}a_2b_4 - a_{3z}b_3b_4$$
$$+6a_{4x}a_1 + 6a_{4z}a_1b_4,$$

$$\beta_8 = -3a_{4zz}{b_1}^2,\ \beta_9 = 3b_1(-2a_{4zz}b_2 - 3a_{4z}b_1),$$

$$\beta_{10} = 3(-2a_{4yz}b_1 - 2a_{4zz}b_1b_3 - a_{4zz}{b_2}^2 - a_{4z}b_{1y} - 5a_{4z}b_1b_2),$$

$$\beta_{11} = 3a_{2z}a_4b_1 + 2a_{3tz}b_1 - 2a_{3z}a_3b_1 - 6a_{4xz}b_1 - 6a_{4yz}b_2 - 3a_{4y}b_1 - 6a_{4zz}b_1b_4$$
$$-6a_{4zz}b_2b_3 - 3a_{4z}b_{1x} - 3a_{4z}b_{2y} + 3a_{4z}a_2b_1 - 12a_{4z}b_1b_3 - 6a_{4z}{b_2}^2,$$

$$\beta_{12} = 3a_{2z}a_4b_2 + 2a_{3tz}b_2 - 2a_{3z}a_3b_2 - 6a_{4xz}b_2 - 6a_{4yz}b_3 - 3a_{4yy} - 3a_{4y}b_2$$
$$-6a_{4zz}b_2b_4 - 3a_{4zz}{b_3}^2 - 3a_{4z}b_{2x} - 3a_{4z}b_{3y} + 3a_{4z}a_2b_2 - 9a_{4z}b_1b_4 - 9a_{4z}b_2b_3,$$

$$\beta_{13} = 3a_{2y}a_4 + 3a_{2z}a_4b_3 + 2a_{3ty} + 2a_{3tz}b_3 - 2a_{3y}a_3 - 2a_{3z}a_3b_3 - 6a_{4xy}$$
$$-6a_{4xz}b_3 - 6a_{4yz}b_4 + 3a_{4y}a_2 - 3a_{4y}b_3 - 6a_{4zz}b_3b_4 - 3a_{4z}b_{3x} - 3a_{4z}b_{4y}$$
$$+3a_{4z}a_2b_3 - 6a_{4z}b_2b_4 - 3a_{4z}{b_3}^2,$$

$$\beta_{14} = -6a_{1t}a_4 + 3a_{2x}a_4 - a_{2tt} + a_{2t}a_3 + 3a_{2z}a_4b_4 + 2a_{3xt} - 2a_{3x}a_3 + 2a_{3tz}b_4$$
$$-2a_{3z}a_3b_4 - 6a_{4xz}b_4 - 3a_{4xx} + 3a_{4x}a_2 - 3a_{4t}a_1 - 3a_{4y}b_4 - 3a_{4zz}{b_4}^2$$
$$-3a_{4z}b_{4x} + 3a_{4z}a_2b_4 - 3a_{4z}b_3b_4.$$

3.2.1 Proof of the Theorem 3.1

The second-order ordinary differential equation $y'' = \lambda(x, y, y')$ is linearizable if and only if the function $\lambda(x, y, y')$ has the form (1.2)

$$\lambda(x, y, y') = b_1(x, y)y'^3 + b_2(x, y)y'^2 + b_3(x, y)y' + b_4(x, y), \tag{3.11}$$

where the coefficients $b_i(x,y)$, $(i=1,2,3,4)$ satisfy the conditions

$$\begin{gathered}2b_{2xy}-3b_{1xx}-b_{3yy}-3b_{1x}b_3+3b_{1y}b_4+2b_{2x}b_2\\ -3b_{3x}b_1-b_{3y}b_2+6b_{4y}b_1=0,\end{gathered} \tag{3.12}$$

$$\begin{gathered}2b_{3xy}-b_{2xx}-3b_{4yy}-6b_{1x}b_4+b_{2x}b_3+3b_{2y}b_4\\ -2b_{3y}b_3-3b_{4x}b_1+3b_{4y}b_2=0.\end{gathered} \tag{3.13}$$

Assuming that a solution $y(x)$ of the equation $y''=\lambda(x,y,y')$ with the function λ as in (3.11) is given, the first equation of (3.1) becomes

$$\ddot{x}=G(t,x,y(x),\dot{x},y'(x)\dot{x})\ . \tag{3.14}$$

According to the Lie criterion, equation (3.14) is linearizable if and only if

$$G(t,x,y(x),\dot{x},y'(x)\dot{x})=h_1(t,x)\dot{x}^3+h_2(t,x)\dot{x}^2+h_3(t,x)\dot{x}+h_4(t,x), \tag{3.15}$$

where the coefficients $h_i(t,x)=a_i(t,x,y(x),y'(x))$, $(i=1,2,3,4)$ satisfy the condition (1.3), with $a=h_1$, $b=h_2$, $c=h_3$, $d=h_4$. These conditions become

$$\begin{gathered}2D_xa_{2t}-3a_{1tt}-D_x^2a_3-3a_{1t}a_3+3(D_xa_1)a_4+2a_{2t}a_2\\ -3a_{3t}a_1-(D_xa_3)a_2+6(D_xa_4)a_1=0,\end{gathered} \tag{3.16}$$

$$\begin{gathered}2D_xa_{3t}-a_{2tt}-3D_x^2a_4-6a_{1t}a_4+a_{2t}a_3+3(D_xa_2)a_4\\ -2(D_xa_3)a_3-3a_{4t}a_1+3(D_xa_4)a_2=0.\end{gathered} \tag{3.17}$$

Here the operator D_x is the operator of the total derivative with respect to x

$$D_x=\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+y''\frac{\partial}{\partial y'}+y'''\frac{\partial}{\partial y''}\ .$$

Substituting $y''=\lambda$, $y'''=D_x\lambda$ into equations (3.16) and (3.17), one obtains equations (3.10).

3.3 Other Sequentially Linearizable System

There are other systems of two second-order ordinary differential equations which might neither be linearizable via point transformations nor projectable, but in some sense they are sequentially linearizable.

Consider a system of two second-order ordinary differential equations, where one of these equations can be considered independently

$$\ddot{y} = f(t, x, y, \dot{x}, \dot{y}), \;\; \ddot{x} = g(t, x, \dot{x}). \tag{3.18}$$

The second equation of (3.18) is linearizable if it has the form

$$g(t, x, \dot{x}) = c_1(t, x)\dot{x}^3 + c_2(t, x)\dot{x}^2 + c_3(t, x)\dot{x} + c_4(t, x), \tag{3.19}$$

where the coefficients $c_i(t, x)$, $(i = 1, 2, 3, 4)$ satisfy the conditions

$$\begin{gathered} 2c_{2tx} - 3c_{1tt} - c_{3xx} - 3c_{1t}c_3 + 3c_{1x}c_4 + 2c_{2t}c_2 \\ -3c_{3t}c_1 - c_{3x}c_2 + 6c_{4x}c_1 = 0, \end{gathered} \tag{3.20}$$

$$\begin{gathered} 2c_{3tx} - c_{2tt} - 3c_{4xx} - 6c_{1t}c_4 + c_{2t}c_3 + 3c_{2x}c_4 \\ -2c_{3x}c_3 - 3c_{4t}c_1 + 3c_{4x}c_2 = 0. \end{gathered} \tag{3.21}$$

Assuming that a solution $x(t)$ of the equation (3.19) is given, the function f of equation (3.18) becomes

$$\ddot{y} = f(t, x(t), y, \dot{x}(t), \dot{y}). \tag{3.22}$$

According to the Lie criterion, equation (3.22) is linearizable if and only if

$$\begin{gathered} f(t, x(t), y, \dot{x}(t), \dot{y}) = h_1(t, x(t), y, \dot{x}(t))\dot{y}^3 + h_2(t, x(t), y, \dot{x}(t))\dot{y}^2 \\ +h_3(t, x(t), y, \dot{x}(t))\dot{y} + h_4(t, x(t), y, \dot{x}(t)), \end{gathered} \tag{3.23}$$

where the coefficients $h_i(t, x(t), y, \dot{x}(t))$, $(i = 1, 2, 3, 4)$ satisfy the condition (1.3), with $a = h_1$, $b = h_2$, $c = h_3$, $d = h_4$. These conditions become

$$\begin{gathered} 2D_t h_{2y} - 3D_t^2 h_1 - h_{3yy} - 3(D_t h_1)h_3 + 3h_{1y}h_4 + 2(D_t h_2)h_2 \\ -3(D_t h_3)h_1 - h_{3y}h_2 + 6h_{4y}h_1 = 0, \end{gathered} \tag{3.24}$$

$$2D_t h_{3y} - D_t^2 h_2 - 3h_{4yy} - 6(D_t h_1)h_4 + (D_t h_2)h_3 + 3h_{2y}h_4$$
$$-2h_{3y}h_3 - 3(D_t h_4)h_1 + 3h_{4y}h_2 = 0. \tag{3.25}$$

Here the operator D_t is the operator of the total derivative with respect to t

$$D_t = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \dddot{x}\frac{\partial}{\partial \ddot{x}}\,.$$

Substituting $\ddot{x} = g(t, x(t), \dot{x}(t))$, $\dddot{x} = D_t(g)$ where g is defined as (3.19) into equations (3.24) and (3.25) one obtains the conditions

$$\sum_{i=1}^{6} \gamma_i(\dot{x})^{7-i} + \gamma_7 = 0,\ \sum_{i=1}^{6} \gamma_{i+7}(\dot{x})^{7-i} + \gamma_{14} = 0, \tag{3.26}$$

with the coefficients

$$\gamma_1 = -3h_{1\dot{x}\dot{x}}{c_1}^2,\quad \gamma_2 = 3c_1(-2h_{1\dot{x}\dot{x}}c_2 - 3h_{1\dot{x}}c_1),$$

$$\gamma_3 = 3(-c_{1x}h_{1\dot{x}} - 2h_{1\dot{x}x}c_1 - 2h_{1\dot{x}\dot{x}}c_1c_3 - h_{1\dot{x}\dot{x}}{c_2}^2 - 5h_{1\dot{x}}c_1c_2),$$

$$\gamma_4 = -3c_{1t}h_{1\dot{x}} - 3c_{2x}h_{1\dot{x}} - 6h_{1t\dot{x}}c_1 - 6h_{1\dot{x}x}c_2 - 6h_{1\dot{x}\dot{x}}c_1c_4 - 6h_{1\dot{x}\dot{x}}c_2c_3$$
$$-12h_{1\dot{x}}c_1c_3 - 3h_{1\dot{x}}c_1h_3 - 6h_{1\dot{x}}{c_2}^2 - 3h_{1x}c_1 + 2h_{2\dot{x}y}c_1 + 2h_{2\dot{x}}c_1h_2$$
$$-3h_{3\dot{x}}c_1h_1,$$

$$\gamma_5 = -3c_{2t}h_{1\dot{x}} - 3c_{3x}h_{1\dot{x}} - 6h_{1t\dot{x}}c_2 - 6h_{1\dot{x}x}c_3 - 6h_{1\dot{x}\dot{x}}c_2c_4 - 3h_{1\dot{x}\dot{x}}{c_3}^2$$
$$-9h_{1\dot{x}}c_1c_4 - 9h_{1\dot{x}}c_2c_3 - 3h_{1\dot{x}}c_2h_3 - 3h_{1xx} - 3h_{1x}c_2 + 2h_{2\dot{x}y}c_2$$
$$+2h_{2\dot{x}}c_2h_2 - 3h_{3\dot{x}}c_2h_1,$$

$$\gamma_6 = -3c_{3t}h_{1\dot{x}} - 3c_{4x}h_{1\dot{x}} - 6h_{1t\dot{x}}c_3 - 6h_{1tx} - 6h_{1\dot{x}x}c_4 - 6h_{1\dot{x}\dot{x}}c_3c_4 - 6h_{1\dot{x}}c_2c_4$$
$$-3h_{1\dot{x}}{c_3}^2 - 3h_{1\dot{x}}c_3h_3 - 3h_{1x}c_3 - 3h_{1x}h_3 + 2h_{2\dot{x}y}c_3 + 2h_{2\dot{x}}c_3h_2 + 2h_{2xy}$$
$$+2h_{2x}h_2 - 3h_{3\dot{x}}c_3h_1 - 3h_{3x}h_1,$$

$$\gamma_7 = -3c_{4t}h_{1\dot{x}} - 6h_{1t\dot{x}}c_4 - 3h_{1tt} - 3h_{1t}h_3 - 3h_{1\dot{x}\dot{x}}{c_4}^2 - 3h_{1\dot{x}}c_3c_4 - 3h_{1\dot{x}}c_4h_3$$
$$-3h_{1x}c_4 + 3h_{1y}h_4 + 2h_{2ty} + 2h_{2t}h_2 + 2h_{2\dot{x}y}c_4 + 2h_{2\dot{x}}c_4h_2 - 3h_{3t}h_1$$
$$-3h_{3\dot{x}}c_4h_1 - h_{3yy} - h_{3y}h_2 + 6h_{4y}h_1,$$

$$\gamma_8 = -h_{2\dot{x}\dot{x}}{c_1}^2,\quad \gamma_9 = c_1(-2h_{2\dot{x}\dot{x}}c_2 - 3h_{2\dot{x}}c_1),$$

$$\gamma_{10} = -c_{1x}h_{2\dot{x}} - 2h_{2\dot{x}x}c_1 - 2h_{2\dot{x}\dot{x}}c_1c_3 - h_{2\dot{x}\dot{x}}{c_2}^2 - 5h_{2\dot{x}}c_1c_2,$$

$$
\begin{aligned}
\gamma_{11} = & -c_{1t}h_{2\dot{x}} - c_{2x}h_{2\dot{x}} - 6h_{1\dot{x}}c_1h_4 - 2h_{2t\dot{x}}c_1 - 2h_{2\dot{x}x}c_2 - 2h_{2\dot{x}\dot{x}}c_1c_4 \\
& -2h_{2\dot{x}\dot{x}}c_2c_3 - 4h_{2\dot{x}}c_1c_3 + h_{2\dot{x}}c_1h_3 - 2h_{2\dot{x}}{c_2}^2 - h_{2x}c_1 + 2h_{3\dot{x}y}c_1 \\
& -3h_{4\dot{x}}c_1h_1, \\
\gamma_{12} = & -c_{2t}h_{2\dot{x}} - c_{3x}h_{2\dot{x}} - 6h_{1\dot{x}}c_2h_4 - 2h_{2t\dot{x}}c_2 - 2h_{2\dot{x}x}c_3 - 2h_{2\dot{x}\dot{x}}c_2c_4 \\
& -h_{2\dot{x}\dot{x}}{c_3}^2 - 3h_{2\dot{x}}c_1c_4 - 3h_{2\dot{x}}c_2c_3 + h_{2\dot{x}}c_2h_3 - h_{2xx} - h_{2x}c_2 + 2h_{3\dot{x}y}c_2 \\
& -3h_{4\dot{x}}c_2h_1, \\
\gamma_{13} = & -c_{3t}h_{2\dot{x}} - c_{4x}h_{2\dot{x}} - 6h_{1\dot{x}}c_3h_4 - 6h_{1x}h_4 - 2h_{2t\dot{x}}c_3 - 2h_{2tx} - 2h_{2\dot{x}x}c_4 \\
& -2h_{2\dot{x}\dot{x}}c_3c_4 - 2h_{2\dot{x}}c_2c_4 - h_{2\dot{x}}{c_3}^2 + h_{2\dot{x}}c_3h_3 - h_{2x}c_3 + h_{2x}h_3 + 2h_{3\dot{x}y}c_3 \\
& +2h_{3xy} - 3h_{4\dot{x}}c_3h_1 - 3h_{4x}h_1, \\
\gamma_{14} = & -c_{4t}h_{2\dot{x}} - 6h_{1t}h_4 - 6h_{1\dot{x}}c_4h_4 - 2h_{2t\dot{x}}c_4 - h_{2tt} + h_{2t}h_3 - h_{2\dot{x}\dot{x}}{c_4}^2 \\
& -h_{2\dot{x}}c_3c_4 + h_{2\dot{x}}c_4h_3 - h_{2x}c_4 + 3h_{2y}h_4 + 2h_{3ty} + 2h_{3\dot{x}y}c_4 - 2h_{3y}h_3 \\
& -3h_{4t}h_1 - 3h_{4\dot{x}}c_4h_1 - 3h_{4yy} + 3h_{4y}h_2.
\end{aligned}
$$

Theorem 3.2. System (3.18) is sequentially linearizable if and only if it satisfies the conditions (3.19)-(3.21), (3.23) and (3.26).

In the next section we will demonstrate systems of two second-order ordinary differential equations which are linearizable in this way, but are not linearizable by point transformations.

3.4 Application to a System of Second-order Quadratically Semi-linear Ordinary Differential Equations

In this section we show that a system of two second-order quadratically semi-linear ordinary differential equations

$$
\begin{aligned}
\ddot{x} &= a(x,y)\dot{x}^2 + 2b(x,y)\dot{x}\dot{y} + c(x,y)\dot{y}^2, \\
\ddot{y} &= d(x,y)\dot{x}^2 + 2e(x,y)\dot{x}\dot{y} + f(x,y)\dot{y}^2,
\end{aligned}
\tag{3.27}
$$

which is linearizable via point transformations, is also sequentially linearizable. Notice that some types of Newtonian systems are of the form (3.27).

A linearization criterion for system (3.27) to be equivalent to the simplest equations via point transformations was obtained in (Mahomed and Qadir, 2007). These criteria are

$$S_i = 0, \quad (i = 1, 2, 3, 4), \tag{3.28}$$

where

$$S_1 = a_y - b_x + be - cd,\ S_2 = b_y - c_x + (ac - b^2) + (bf - ce),$$

$$S_3 = d_y - e_x - (ae - bd) - (df - e^2),\ S_4 = b_x + f_x - a_y - e_y.$$

Notice that system (3.27) is a projectable system with

$$\lambda(x, y, y') = -cy'^3 + (f - 2b)y'^2 + (2e - a)y' + d. \tag{3.29}$$

Applying Theorem 3.1 proven above, one obtains the conditions for system (3.27) to be sequentially linearizable:

$$\begin{aligned} &3S_{1y} - 3S_{2x} + 2S_{4y} + 3(f - b)S_1 - 3eS_2 - 3cS_3 + (2f - b)S_4 = 0, \\ &3S_{1x} + 3S_{3y} + S_{4x} - 3(e - a)S_1 + 3dS_2 + 3bS_3 - (2e - a)S_4 = 0. \end{aligned} \tag{3.30}$$

Relations (3.28) make (3.30) vanish. Thus in general there are quadratically semi-linear system (3.27) which are not linearizable via point transformations, but are sequentially linearizable. Furthermore, equations (3.30) show that the set of systems (3.27) which are linearizable via point transformations is a subset of the set of equations which are sequentially linearizable.

3.5 Illustration of the Linearization Theorem

In this section we demonstrate examples of systems of two second-order ordinary differential equations which are sequentially linearizable, but not linearizable via point transformations. Consider a system

$$\ddot{x} = y,\ \ddot{y} = \tfrac{\dot{y}}{\dot{x}} y. \tag{3.31}$$

Applying the linearization criteria obtained in either Aminova and Aminov (2006) or Neut, Petitot and Dridi (2009) to system (3.31), one obtains that system (3.31) is not equivalent to the simplest equations under point transformations. Let us show that system (3.31) is sequentially linearizable.

For system (3.31), $\lambda = 0$ which implies that $y'' = 0$. The first equation of (3.31) becomes $\ddot{x} = c_1 x + c_2$, which is a linear second-order equation. Therefore, system (3.31) is sequentially linearizable.

The presented example shows that a system of two second-order ordinary differential equations which is not linearizable by point transformations might be sequentially linearizable.

Let us make another observation. System (3.31) is equivalent to the fourth-order ordinary differential equation

$$x^{(4)} = \tfrac{x^{(3)}}{\dot{x}} \ddot{x}. \tag{3.32}$$

Applying the linearization criteria obtained in Ibragimov, Meleshko and Suksern (2008) to equation (3.32), one notes that equation (3.32) is not linearizable by point transformations either.

Remark 3.2. System (3.31) is a particular case of the system

$$\begin{aligned} \ddot{x} &= f(y - \tfrac{\dot{y}}{\dot{x}} x, \tfrac{\dot{y}}{\dot{x}}, t) + x g(y - \tfrac{\dot{y}}{\dot{x}} x, \tfrac{\dot{y}}{\dot{x}}, t), \\ \ddot{y} &= \tfrac{\dot{y}}{\dot{x}} (f(y - \tfrac{\dot{y}}{\dot{x}} x, \tfrac{\dot{y}}{\dot{x}}, t) + x g(y - \tfrac{\dot{y}}{\dot{x}} x, \tfrac{\dot{y}}{\dot{x}}, t)), \end{aligned} \tag{3.33}$$

which is also sequentially linearizable. Here the functions f and g are arbitrary. For system (3.33), $\lambda = 0$ which implies $y'' = 0$. Thus the first equation of (3.33) becomes $\ddot{x} = f(c_2, c_1, t) + x g(c_2, c_1, t)$ which is linear equation. Therefore, this system is sequentially linearizable.

3.6 Summary

In this chapter, a new method for linearizing a system of ordinary differential equations was introduced. This method consists of consecutively reducing the number of the dependent variables and using the linearization criterion for the reduced equations. The method was applied to a system of two second-order ordinary differential equations. Moreover, it was shown that for systems of two second-order quadratically semi-linear ordinary differential equations the class of equations linearizable by the new method is lager than the class of equations linearizable via point transformations. Finally, examples of applications of the method are given.

CHAPTER IV

LINEARIZATION OF TWO SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS VIA FIBER PRESERVING POINT TRANSFORMATIONS

4.1 Establishment of the Second Main Problem

The linearization problem of a system of second-order ordinary differential equations

$$y_1'' = F_1(x, y_1, y_2, y_1', y_2'), \quad y_2'' = F_2(x, y_1, y_2, y_1', y_2'). \tag{4.1}$$

via a point transformation is to find an invertible transformation

$$t = \varphi(x, y_1, y_2), \; u_1 = \psi_1(x, y_1, y_2), \; u_2 = \psi_2(x, y_1, y_2), \tag{4.2}$$

which transforms the system of equations (4.1) into a linear system of equations

$$\ddot{u} + K(t)u = 0. \tag{4.3}$$

Note that system (4.1) is the same as system (3.1), however we have changed the variables x, y and t to y_1, y_2 and x, for ease of notation.

In the next section the form of a linearizable system (4.1) is obtained. This form coincides with the form obtained in Aminova and Aminov (2006) for a system (4.1) to be equivalent to the simplest equations. Some invariants of this form with respect to the general set of point transformations related with a linearizable systems (4.1) were obtained in Sookmee (2005). The necessary and sufficient conditions for system (4.1) to be equivalent with respect to a fiber preserving point transformation

$$t = \varphi(x),\ u_1 = \psi_1(x, y_1, y_2),\ u_2 = \psi_2(x, y_1, y_2) \tag{4.4}$$

to system

$$\ddot{u}_1 + k_1 u_1 + k_3 u_2 = 0, \quad \ddot{u}_2 + k_4 u_1 + k_2 u_2 = 0, \tag{4.5}$$

where k_i, $(i = 1, 2, 3, 4)$ are constant, are discussed in this chapter.

4.2 Necessary Form of a Linearizable System (4.1)

For obtaining necessary conditions for system (4.1) to be linearizable via point transformations (4.2) one assumes that system (4.1) is obtained from the linear system of differential equations (4.3) by an invertible transformation (4.2). The derivatives are changed by the formulae*,

$$u_1' = g_1(x, y_1, y_2, y_1', y_2') = \tfrac{D_x\psi_1}{D_x\varphi}, \quad u_1'' = \tfrac{D_x g_1}{D_x\varphi},$$
$$u_2' = g_2(x, y_1, y_2, y_1', y_2') = \tfrac{D_x\psi_2}{D_x\varphi}, \quad u_2'' = \tfrac{D_x g_2}{D_x\varphi},$$

where

$$D_x = \frac{\partial}{\partial x} + y_1'\frac{\partial}{\partial y_1} + y_2'\frac{\partial}{\partial y_2} + y_1''\frac{\partial}{\partial y_1'} + y_2''\frac{\partial}{\partial y_2'}.$$

Replacing u_1', u_1'', u_2' and u_2'' in system (4.3), it becomes

$$\begin{aligned} y_1'' &= y_1'(a_{11}{y_1'}^2 + a_{12}y_1'y_2' + a_{13}{y_2'}^2) + a_{14}{y_1'}^2 + a_{15}y_1'y_2' \\ &\qquad + a_{16}{y_2'}^2 + a_{17}y_1' + a_{18}y_2' + a_{19}, \\ y_2'' &= y_2'(a_{11}{y_1'}^2 + a_{12}y_1'y_2' + a_{13}{y_2'}^2) + a_{24}{y_1'}^2 + a_{25}y_1'y_2' \\ &\qquad + a_{26}{y_2'}^2 + a_{27}y_1' + a_{28}y_2' + a_{29}, \end{aligned} \tag{4.6}$$

*See more details in Appendix D.

where the coefficients a_{ij} are expressed through the functions φ, ψ_1, ψ_2, their partial derivatives and the entries of the matrix $K = (k_{ij}(t))$ as follows:

$$a_{11} = (h_1\psi_{1y_1y_1} + h_2\psi_{2y_1y_1} - v\varphi_{y_1y_1} + f_1{\varphi_{y_1}}^3 - f_2{\varphi_{y_1}}^2\varphi_{y_2})/\Delta, \tag{4.7}$$

$$a_{12} = 2(h_1\psi_{1y_1y_2} + h_2\psi_{2y_1y_2} - v\varphi_{y_1y_2} + f_1{\varphi_{y_1}}^2\varphi_{y_2} - f_2\varphi_{y_1}{\varphi_{y_2}}^2)/\Delta, \tag{4.8}$$

$$a_{13} = (h_1\psi_{1y_2y_2} + h_2\psi_{2y_2y_2} - v\varphi_{y_2y_2} + f_1\varphi_{y_1}{\varphi_{y_2}}^2 - f_2{\varphi_{y_2}}^3)/\Delta, \tag{4.9}$$

$$\begin{aligned} a_{14} = {} & (2h_1\psi_{1xy_1} + 2h_2\psi_{2xy_1} + h_3\psi_{1y_1y_1} + h_4\psi_{2y_1y_1} + h_5\varphi_{y_1y_1} \\ & -2v\varphi_{xy_1} + 3f_1\varphi_x{\varphi_{y_1}}^2 - 2f_2\varphi_x\varphi_{y_1}\varphi_{y_2} - f_3{\varphi_{y_1}}^2\varphi_{y_2})/\Delta, \end{aligned} \tag{4.10}$$

$$\begin{aligned} a_{15} = {} & 2(h_1\psi_{1xy_2} + h_2\psi_{2xy_2} + h_3\psi_{1y_1y_2} + h_4\psi_{2y_1y_2} + h_5\varphi_{y_1y_2} \\ & -2v\varphi_{xy_2} + 2f_1\varphi_x\varphi_{y_1}\varphi_{y_2} - f_2\varphi_x{\varphi_{y_2}}^2 - f_3\varphi_{y_1}{\varphi_{y_2}}^2)/\Delta, \end{aligned} \tag{4.11}$$

$$a_{16} = (h_3\psi_{1y_2y_2} + h_4\psi_{2y_2y_2} + h_5\varphi_{y_2y_2} + f_1\varphi_x{\varphi_{y_2}}^2 - f_3{\varphi_{y_2}}^3)/\Delta, \tag{4.12}$$

$$\begin{aligned} a_{17} = {} & (h_1\psi_{1xx} + h_2\psi_{2xx} + 2h_3\psi_{1xy_1} + 2h_4\psi_{2xy_1} + 2h_5\varphi_{xy_1} - v\varphi_{xx} \\ & +3f_1{\varphi_x}^2\varphi_{y_1} - f_2{\varphi_x}^2\varphi_{y_2} - 2f_3\varphi_x\varphi_{y_1})/\Delta, \end{aligned} \tag{4.13}$$

$$a_{18} = 2(h_3\psi_{1xy_2} + h_4\psi_{2xy_2} + 2h_5\varphi_{xy_2} + f_1{\varphi_x}^2\varphi_{y_2} - f_3\varphi_x{\varphi_{y_2}}^2)/\Delta, \tag{4.14}$$

$$a_{19} = (h_3\psi_{1xx} + h_4\psi_{2xx} + h_5\varphi_{xx} + f_1{\varphi_x}^3 - f_3{\varphi_x}^2\varphi_{y_2})/\Delta, \tag{4.15}$$

$$a_{24} = (h_6\psi_{1y_1y_1} + h_7\psi_{2y_1y_1} + h_8\varphi_{y_1y_1} - f_2\varphi_x{\varphi_{y_1}}^2 + f_3{\varphi_{y_1}}^3)/\Delta, \tag{4.16}$$

$$\begin{aligned} a_{25} = {} & 2(h_1\psi_{1xy_1} + h_2\psi_{2xy_1} - v\varphi_{xy_1} - h_6\psi_{1y_1y_2} + h_7\psi_{2y_1y_2} \\ & +f_1\varphi_x\varphi_x{\varphi_{y_1}}^2 - 2f_2\varphi_x\varphi_{y_1}\varphi_{y_2} - f_3{\varphi_{y_1}}^2\varphi_{y_2})/\Delta, \end{aligned} \tag{4.17}$$

$$\begin{aligned} a_{26} = {} & (2h_1\psi_{1xy_2} + 2h_2\psi_{2xy_2} - 2v\varphi_{xy_2} + h_6\psi_{1y_2y_2} + h_7\psi_{2y_2y_2} \\ & +h_8\varphi_{y_2y_2} + 2f_1\varphi_x\varphi_{y_1}\varphi_{y_2} - 3f_2\varphi_x{\varphi_{y_2}}^2 - f_3\varphi_{y_1}{\varphi_{y_2}}^2)/\Delta, \end{aligned} \tag{4.18}$$

$$a_{27} = 2(h_6\psi_{1xy_1} + h_7\psi_{2xy_1} + h_8\varphi_{xy_1} - f_2{\varphi_x}^2\varphi_{y_1} + f_3\varphi_x{\varphi_{y_1}}^2)/\Delta, \tag{4.19}$$

$$\begin{aligned} a_{28} = {} & (h_1\psi_{1xx} + h_2\psi_{2xx} - v\varphi_{xx} + 2h_6\psi_{1xy_2} + 2h_7\psi_{2xy_2} + 2h_8\varphi_{xy_2} \\ & +f_1{\varphi_x}^2\varphi_{y_1} - 3f_2{\varphi_x}^2\varphi_{y_2} + 2f_3\varphi_x\varphi_{y_1}\varphi_{y_2})/\Delta, \end{aligned} \tag{4.20}$$

$$a_{29} = (h_6\psi_{1xx} + h_7\psi_{2xx} + h_8\varphi_{xx} - f_2{\varphi_x}^3 + f_3{\varphi_x}^2\varphi_{y_1})/\Delta, \tag{4.21}$$

where $\Delta \neq 0$ is the Jacobian of the change of variables (4.2),

$$\Delta = (\varphi_x\psi_{1y_1}\psi_{2y_2} - \varphi_x\psi_{1y_2}\psi_{2y_1} - \varphi_{y_1}\psi_{1x}\psi_{2y_2} + \varphi_{y_1}\psi_{1y_2}\psi_{2x} + \varphi_{y_2}\psi_{1x}\psi_{2y_1} - \varphi_{y_2}\psi_{1y_1}\psi_{2x}),$$

$$f_1 = \psi_{1y_2}(k_{22}\psi_2 + k_{21}\psi_1) - \psi_{2y_2}(k_{11}\psi_1 + k_{12}\psi_2),$$

$$f_2 = \psi_{1y_1}(k_{22}\psi_2 + k_{21}\psi_1) - \psi_{2y_1}(k_{11}\psi_1 + k_{12}\psi_2),$$

$$f_3 = \psi_{1x}(k_{22}\psi_2 + k_{21}\psi_1) - \psi_{2x}(k_{11}\psi_1 + k_{12}\psi_2),$$

$$v = \psi_{1y_2}\psi_{2y_1} - \psi_{1y_1}\psi_{2y_2}, \quad h_1 = \varphi_{y_2}\psi_{2y_1} - \varphi_{y_1}\psi_{2y_2}, \quad h_2 = \varphi_{y_1}\psi_{1y_2} - \varphi_{y_2}\psi_{1y_1},$$

$$h_3 = \varphi_{y_2}\psi_{2x} - \varphi_x\psi_{2y_2}, \quad h_4 = \varphi_x\psi_{1y_2} - \varphi_{y_2}\psi_{1x}, \quad h_5 = \psi_{1x}\psi_{2y_2} - \psi_{1y_2}\psi_{2x},$$

$$h_6 = \varphi_x\psi_{2y_1} - \varphi_{y_1}\psi_{2x}, \quad h_7 = \varphi_{y_1}\psi_{1x} - \varphi_x\psi_{1y_1}, \quad h_8 = \psi_{1y_1}\psi_{2x} - \psi_{1x}\psi_{2y_1}.$$

Equation (4.6) presents the necessary form of a system of two second-order ordinary differential equations which can be mapped via point transformations into a system of linear equations (4.3) .

4.3 Sufficient Conditions for Equivalency to (4.5) via Fiber Preserving Transformations

For obtaining sufficient conditions of linearizability of system (4.6), one has to solve the compatibility problem of the system of equations (4.7)-(4.21), considering it as an overdetermined system of partial differential equations for the functions ψ_1, ψ_2 and φ with given coefficients a_{ij} of system (4.6).

The next part of the present thesis deals with a fiber preserving set of point transformations (4.4):

$$t = \varphi(x), \ u_1 = \psi_1(x, y_1, y_2), \ u_2 = \psi_2(x, y_1, y_2),$$

and constant matrix

$$K = \begin{pmatrix} k_1 & k_3 \\ k_4 & k_2 \end{pmatrix}.$$

The compatibility analysis depends on the value of ψ_{1y_1}.

4.3.1 Case $\psi_{1y_1} \neq 0$

Substitution of $\varphi_{y_1} = 0$ and $\varphi_{y_2} = 0$ into (4.7)-(4.21) gives

$$a_{11} = 0,\ a_{12} = 0,\ a_{13} = 0, \tag{4.22}$$

$$\begin{aligned}
v_x &= (2\varphi_{xx} v - \varphi_x v(a_{17} + a_{28}))/(2\varphi_x),\\
v_{y_1} &= -v(2a_{14} + a_{25})/2,\\
v_{y_2} &= -v(a_{15} + 2a_{26})/2,\\
\psi_{1xx} &= (\varphi_{xx}\psi_{1x} - \varphi_x{}^3(k_1\psi_1 + k_3\psi_2) - \varphi_x\psi_{1y_1}a_{19} - \varphi_x\psi_{1y_2}a_{29})/\varphi_x, && (4.23)\\
\psi_{1xy_1} &= (\varphi_{xx}\psi_{1y_1} - \varphi_x\psi_{1y_1}a_{17} - \varphi_x\psi_{1y_2}a_{27})/(2\varphi_x), && (4.24)\\
\psi_{1xy_2} &= (\varphi_{xx}\psi_{1y_2} - \varphi_x\psi_{1y_1}a_{18} - \varphi_x\psi_{1y_2}a_{28})/(2\varphi_x), && (4.25)\\
\psi_{1y_1y_1} &= -(\psi_{1y_1}a_{14} + \psi_{1y_2}a_{24}), && (4.26)\\
\psi_{1y_1y_2} &= -(\psi_{1y_1}a_{15} + \psi_{1y_2}a_{25})/2, && (4.27)\\
\psi_{1y_2y_2} &= -(\psi_{1y_1}a_{16} + \psi_{1y_2}a_{26}), && (4.28)\\
\psi_{2y_2} &= (\psi_{1y_2}\psi_{2y_1} - v)/\psi_{1y_1}, && (4.29)\\
\psi_{2y_1y_1} &= (a_{24}v - \psi_{1y_1}\psi_{2y_1}a_{14} - \psi_{1y_2}\psi_{2y_1}a_{24})/\psi_{1y_1}, && (4.30)\\
\psi_{2xy_1} &= (\varphi_{xx}\psi_{1y_1}\psi_{2y_1} - \varphi_x\psi_{1y_1}\psi_{2y_1}a_{17} - \varphi_x\psi_{1y_2}\psi_{2y_1}a_{27} && (4.31)\\
&\quad +\varphi_x a_{27} v)/(2\varphi_x\psi_{1y_1}),\\
\psi_{2xx} &= (\varphi_{xx}\psi_{1y_1}\psi_{2x} - \varphi_x{}^3\psi_{1y_1}(k_2\psi_2 + k_4\psi_1) - \varphi_x\psi_{1y_1}\psi_{2y_1}a_{19} && (4.32)\\
&\quad -\varphi_x\psi_{1y_2}\psi_{2y_1}a_{29} + \varphi_x a_{29} v)/(\varphi_x\psi_{1y_1}),
\end{aligned}$$

where $\Delta = -\varphi_x v \neq 0$. Comparing the mixed derivatives $(v_x)_{y_1} = (v_{y_1})_x, (v_x)_{y_2} = (v_{y_2})_x$ and $(v_{y_2})_{y_1} = (v_{y_1})_{y_2}$, one obtains the equations

$$\begin{gathered}
2a_{14x} - a_{17y_1} + a_{25x} - a_{28y_1} = 0,\ \ a_{15x} - a_{17y_2} + 2a_{26x} - a_{28y_2} = 0,\\
a_{15y_1} - 2a_{14y_2} - a_{25y_2} + 2a_{26y_1} = 0.
\end{gathered} \tag{4.33}$$

Considering the conditions $(\psi_{1xx})_{y_1} = (\psi_{1xy_1})_x$, $(\psi_{2xx})_{y_1} = (\psi_{2xy_1})_x$, $(\psi_{1xx})_{y_2} = (\psi_{1xy_2})_x$ and $(\psi_{2xx})_{y_2} = (\psi_{2y_2})_{xx}$, one has

$$\begin{aligned}\varphi_{xxx} &= (3{\varphi_{xx}}^2\psi_{1_{y_1}} - 4{\varphi_x}^4\psi_{1_{y_1}}k_1 - 4{\varphi_x}^4\psi_{2_{y_1}}k_3 + {\varphi_x}^2\psi_{1_{y_1}}(\lambda_{20} - \lambda_{16}) \\ &\quad + {\varphi_x}^2\psi_{1_{y_2}}\lambda_{12})/(2\varphi_x\psi_{1_{y_1}}), \end{aligned} \tag{4.34}$$

$$k_4 = (4{\varphi_x}^2\psi_{1_{y_1}}\psi_{2_{y_1}}(k_1 - k_2) + 4{\varphi_x}^2{\psi_{2_{y_1}}}^2k_3 - v\lambda_{12})/(4{\varphi_x}^2{\psi_{1_{y_1}}}^2), \tag{4.35}$$

$$k_3 = ({\psi_{1_{y_2}}}^2\lambda_{12} - {\psi_{1_{y_1}}}^2\lambda_{15} - \psi_{1_{y_1}}\psi_{1_{y_2}}\lambda_{16})/(4{\varphi_x}^2v), \tag{4.36}$$

$$\begin{aligned}k_2 &= (4{\varphi_x}^2\psi_{1_{y_1}}k_1v - 2{\psi_{1_{y_1}}}^2\psi_{2_{y_1}}\lambda_{15} - 2\psi_{1_{y_1}}\psi_{1_{y_2}}\psi_{2_{y_1}}\lambda_{16} \\ &\quad + \psi_{1_{y_1}}v\lambda_{16} + 2{\psi_{1_{y_2}}}^2\psi_{2_{y_1}}\lambda_{12} - 2\psi_{1_{y_2}}v\lambda_{12})/(4{\varphi_x}^2\psi_{1_{y_1}}v).\end{aligned} \tag{4.37}$$

Here the functions $\lambda_n(x, y_1, y_2)$ are defined through $a_{ij}(x, y_1, y_2)$ and their derivatives (presented in Appendix G).

Equating the mixed derivatives $(\psi_{1xy_1})_{y_2} = (\psi_{1xy_2})_{y_1}$, $(\psi_{2xy_1})_{y_2} = (\psi_{2y_2})_{xy_1}$, $(\psi_{1xy_1})_{y_1} = (\psi_{1_{y_1y_1}})_x$, $(\psi_{2xy_1})_{y_1} = (\psi_{2_{y_1y_1}})_x$, $(\psi_{1xy_2})_{y_1} = (\psi_{1_{y_1y_2}})_x$, $(\psi_{1xy_2})_{y_2} = (\psi_{1_{y_2y_2}})_x$, $(\psi_{1_{y_1y_1}})_{y_2} = (\psi_{1_{y_1y_2}})_{y_1}$, $(\psi_{2_{y_1y_1}})_{y_2} = (\psi_{2_{y_2}})_{y_1y_1}$, $(\psi_{1_{y_1y_2}})_{y_2} = (\psi_{1_{y_2y_2}})_{y_1}$ and using the conditions $(k_3)_{y_2} = 0$, $(k_2)_{y_2} = 0$, one obtains

$$\lambda_n = 0, \quad (n = 1, 2, ..., 11). \tag{4.38}$$

Note that the equation $(\psi_{1xy_2})_{y_1} - (\psi_{1_{y_1y_2}})_x = 0$ is also satisfied. Differentiating equation (4.34) and (4.35) with respect to y_1 and y_2, one has

$$2\lambda_{15_{y_1}} - 2a_{14}\lambda_{15} - a_{15}\lambda_{16} + a_{25}\lambda_{15} = 0, \quad \lambda_{27+j} = 0, \ (j = 0, 1, 2). \tag{4.39}$$

The equation $(k_4)_x = 0$ is

$$4\varphi_{xx}\lambda_{12}v + \varphi_x\lambda_{14}v = 0. \tag{4.40}$$

Considering the equation $(k_3)_x = 0$, one gets

$$
\begin{aligned}
4\varphi_{xx}{\psi_{1y_1}}^2\lambda_{15} + 4\varphi_{xx}\psi_{1y_1}\psi_{1y_2}\lambda_{16} - 4\varphi_{xx}{\psi_{1y_2}}^2\lambda_{12} \\
+\varphi_x{\psi_{1y_1}}^2(-2\lambda_{15x} + a_{17}\lambda_{15} + a_{18}\lambda_{16} - a_{28}\lambda_{15}) \\
+2\varphi_x\psi_{1y_1}\psi_{1y_2}(-\lambda_{16x} - a_{18}\lambda_{12} + a_{27}\lambda_{15}) - \varphi_x{\psi_{1y_2}}^2\lambda_{14} = 0,
\end{aligned}
\tag{4.41}
$$

and the condition $(k_2)_x = 0$ gives

$$
\begin{aligned}
4\varphi_{xx}{\psi_{1y_1}}^2\psi_{2y_1}\lambda_{15} + 4\varphi_{xx}\psi_{1y_1}\psi_{1y_2}\psi_{2y_1}\lambda_{16} - 2\varphi_{xx}\psi_{1y_1}\lambda_{16}v \\
-4\varphi_{xx}{\psi_{1y_2}}^2\psi_{2y_1}\lambda_{12} + 4\varphi_{xx}\psi_{1y_2}\lambda_{12}v + \varphi_x{\psi_{1y_1}}^2\psi_{2y_1}(-2\lambda_{15x} + a_{17}\lambda_{15} \\
+a_{18}\lambda_{16} - a_{28}\lambda_{15}) + 2\varphi_x\psi_{1y_1}\psi_{1y_2}\psi_{2y_1}(-\lambda_{16x} - a_{18}\lambda_{12} + a_{27}\lambda_{15}) \\
+\varphi_x\psi_{1y_1}v(\lambda_{16x} + a_{18}\lambda_{12} - a_{27}\lambda_{15}) - \varphi_x{\psi_{1y_2}}^2\psi_{2y_1}\lambda_{14} + \varphi_x\psi_{1y_2}\lambda_{14}v = 0.
\end{aligned}
\tag{4.42}
$$

Adding j_1 times (4.40) to (4.41), where $j_1 = {\psi_{1y_2}}^2/v$, one has

$$
\begin{aligned}
4\varphi_{xx}{\psi_{1y_1}}^2\lambda_{15} + 4\varphi_{xx}\psi_{1y_1}\psi_{1y_2}\lambda_{16} + \varphi_x{\psi_{1y_1}}^2(-2\lambda_{15x} + a_{17}\lambda_{15} \\
+a_{18}\lambda_{16} - a_{28}\lambda_{15}) + 2\varphi_x\psi_{1y_1}\psi_{1y_2}(-\lambda_{16x} - a_{18}\lambda_{12} + a_{27}\lambda_{15}) = 0.
\end{aligned}
\tag{4.43}
$$

Subtracting equation j_2(4.40) + j_3(4.43) from (4.42), where $j_2 = (\psi_{1y_2}v - {\psi_{1y_2}}^2\psi_{2y_1})/v$ and $j_3 = \psi_{2y_1}$, one has

$$
\varphi_x\psi_{1y_1}v(\lambda_{16x} + a_{18}\lambda_{12} - a_{27}\lambda_{15}) - 2\varphi_{xx}\psi_{1y_1}\lambda_{16}v = 0. \tag{4.44}
$$

Subtracting equation j_4(4.44) from (4.43), where $j_4 = (-2\psi_{1y_2})/v$, one has

$$
4\varphi_{xx}{\psi_{1y_1}}^2\lambda_{15} + \varphi_x{\psi_{1y_1}}^2(a_{17}\lambda_{15} - 2\lambda_{15x} + a_{18}\lambda_{16} - a_{28}\lambda_{15}) = 0. \tag{4.45}
$$

Next consider the equations (4.40)$/v$, (4.45)$/j_5$ and (4.44)$/j_6$, where $j_5 = {\psi_{1y_1}}^2$ and $j_6 = \psi_{1y_1}v$, one achieves

$$
4\varphi_{xx}\lambda_{12} + \varphi_x\lambda_{14} = 0, \tag{4.46}
$$

$$
2\varphi_{xx}\lambda_{16} - \varphi_x(\lambda_{16x} + a_{18}\lambda_{12} - a_{27}\lambda_{15}) = 0, \tag{4.47}
$$

$$
4\varphi_{xx}\lambda_{15} + \varphi_x(a_{17}\lambda_{15} - 2\lambda_{15x} + a_{18}\lambda_{16} - a_{28}\lambda_{15}) = 0. \tag{4.48}
$$

Further analysis of the compatibility depends on the values of the coefficients λ_{12}, λ_{15} and λ_{16} of the last three equations (4.46)-(4.48).

Case $\lambda_{12} \neq 0$.

Substituting φ_{xx}, found from equation (4.46), into (4.48) and (4.47), one obtains

$$\begin{gathered} a_{17}\lambda_{12}\lambda_{15} - 2\lambda_{15x}\lambda_{12} + a_{18}\lambda_{12}\lambda_{16} - a_{28}\lambda_{12}\lambda_{15} - \lambda_{14}\lambda_{15} = 0, \\ 2\lambda_{16x}\lambda_{12} + 2a_{18}{\lambda_{12}}^2 - 2a_{27}\lambda_{12}\lambda_{15} + \lambda_{14}\lambda_{16} = 0. \end{gathered} \tag{4.49}$$

Differentiating equation (4.46) with respect to y_1 and y_2, one has

$$\lambda_{12y_1}\lambda_{14} - \lambda_{14y_1}\lambda_{12} = 0, \ \lambda_{12y_2}\lambda_{14} - \lambda_{14y_2}\lambda_{12} = 0. \tag{4.50}$$

Equation (4.34) becomes

$$\begin{aligned} k_1 \ = \ & (16{\psi_{1y_1}}^2\psi_{2y_1}{\lambda_{12}}^2\lambda_{15} + 16\psi_{1y_1}\psi_{1y_2}\psi_{2y_1}{\lambda_{12}}^2\lambda_{16} \\ & +16\psi_{1y_1}{\lambda_{12}}^2 v(\lambda_{20} - \lambda_{16}) + 4\psi_{1y_1}\lambda_{12}v(2\lambda_{14x} + a_{17}\lambda_{14} \\ & -a_{28}\lambda_{14}) + \psi_{1y_1}\lambda_{14}v(4a_{27}\lambda_{16} + 5\lambda_{14}) - 16{\psi_{1y_2}}^2\psi_{2y_1}{\lambda_{12}}^3 \\ & +16\psi_{1y_2}{\lambda_{12}}^3 v)/(64{\varphi_x}^2\psi_{1y_1}{\lambda_{12}}^2 v). \end{aligned} \tag{4.51}$$

Differentiating equation (4.51) with respect to x, one gets the condition

$$32{\lambda_{12}}^3\lambda_{17} + 8{\lambda_{12}}^2\lambda_{18} + 2\lambda_{12}\lambda_{19} + \lambda_{14}(8{a_{27}}^2{\lambda_{16}}^2 + 18a_{27}\lambda_{14}\lambda_{16} + 15{\lambda_{14}}^2) = 0. \tag{4.52}$$

Notice that the equations $(k_1)_{y_1} = 0$ and $(k_1)_{y_2} = 0$ are satisfied. Hence, there are no new conditions for the functions $\varphi(x)$, $\psi_1(x, y_1, y_2)$ and $\psi_2(x, y_1, y_2)$. In summary, the criteria for linearization are conditions (4.22), (4.33), (4.38), (4.39), (4.49), (4.50) and (4.52). Note also that updating the k_i, $(i = 2, 3, 4)$, these become as follows:

$$k_2 = (\lambda_{30}/{\varphi_x}^2) - k_1, \tag{4.53}$$

$$k_3 = ({\psi_{1y_2}}^2\lambda_{12} - {\psi_{1y_1}}^2\lambda_{15} - \psi_{1y_1}\psi_{1y_2}\lambda_{16})/(4{\varphi_x}^2 v), \tag{4.54}$$

$$\begin{aligned} k_4 = {} & ({\psi_{1y_1}}^2{\psi_{2y_1}}^2\lambda_{15} + \psi_{1y_1}\psi_{1y_2}{\psi_{2y_1}}^2\lambda_{16} - \psi_{1y_1}\psi_{2y_1}\lambda_{16}v \\ & -{\psi_{1y_2}}^2{\psi_{2y_1}}^2\lambda_{12} + 2\psi_{1y_2}\psi_{2y_1}\lambda_{12}v - \lambda_{12}v^2)/(4{\varphi_x}^2{\psi_{1y_1}}^2 v). \end{aligned} \tag{4.55}$$

Case $\lambda_{12} = 0$ and $\lambda_{16} \neq 0$.

Since $\lambda_{12} = 0$ and $\varphi_x \neq 0$, equation (4.46) leads to the condition $\lambda_{14} = 0$, and equation (4.47) becomes

$$\varphi_{xx} = \varphi_x\lambda_{16x}/(2\lambda_{16}).$$

Substituting φ_{xx} into (4.48) and (4.34), one gets

$$2(\lambda_{16x}\lambda_{15} - \lambda_{15x}\lambda_{16}) + \lambda_{15}\lambda_{16}(a_{17} - a_{28}) + a_{18}{\lambda_{16}}^2 = 0, \tag{4.56}$$

$$\begin{aligned} k_1 = {} & (4\psi_{1y_1}\psi_{2y_1}{\lambda_{16}}^2\lambda_{15} + 4\psi_{1y_2}\psi_{2y_1}{\lambda_{16}}^3 - 4{\lambda_{16}}^3 v + 4{\lambda_{16}}^2\lambda_{23}v \\ & -4\lambda_{16}\lambda_{16xx}v + 5{\lambda_{16x}}^2 v)/(16{\varphi_x}^2{\lambda_{16}}^2 v). \end{aligned}$$

The equation $(k_1)_x = 0$, leads to the condition

$$8{\lambda_{16}}^3\lambda_{21} + 4{\lambda_{16}}^2\lambda_{22} + 18\lambda_{16}\lambda_{16xx}\lambda_{16x} - 15{\lambda_{16x}}^3 = 0. \tag{4.57}$$

Note that the equations $(\varphi_{xx})_{y_i} = 0$ and $(k_1)_{y_i} = 0$, $(i = 1, 2)$ are satisfied. Hence, there are no other conditions for the functions $\varphi(x)$, $\psi_1(x, y_1, y_2)$ and $\psi_2(x, y_1, y_2)$. Summarizing, the linearization criteria in the case $\lambda_{12} = 0$ and $\lambda_{16} \neq 0$ are conditions (4.22), (4.33), (4.38), (4.39), (4.56) and (4.57). Note also that updating k_i, $(i = 2, 3, 4)$, these become as follows:

$$k_2 = (\lambda_{31}/{\varphi_x}^2) - k_1, \tag{4.58}$$

$$k_3 = (-{\psi_{1y_1}}^2\lambda_{15} - \psi_{1y_1}\psi_{1y_2}\lambda_{16})/(4{\varphi_x}^2 v), \tag{4.59}$$

$$k_4 = (\psi_{1y_1}{\psi_{2y_1}}^2\lambda_{15} + \psi_{1y_2}{\psi_{2y_1}}^2\lambda_{16} - \psi_{2y_1}\lambda_{16}v)/(4{\varphi_x}^2\psi_{1y_1}v). \tag{4.60}$$

Case $\lambda_{12} = 0,\ \lambda_{16} = 0$ **and** $\lambda_{15} \neq 0$

Substituting φ_{xx}, found from (4.48), into (4.34), one has

$$k_1 = (16\psi_{1y_1}\psi_{2y_1}{\lambda_{15}}^3 + \lambda_{26}v)/(64{\varphi_x}^2{\lambda_{15}}^2v). \quad (4.61)$$

Differentiating equation (4.61) with respect to x, one gets

$$\begin{aligned}{\lambda_{15}}^3\lambda_{24} + 2{\lambda_{15}}^2\lambda_{25} - 120{\lambda_{15x}}^3 + 36\lambda_{15}\lambda_{15x}(4\lambda_{15xx} \\ +\lambda_{15x}a_{17} - \lambda_{15x}a_{28}) = 0.\end{aligned} \quad (4.62)$$

Note that the equations $(\varphi_{xx})_{y_i} = 0$ and $(k_1)_{y_i} = 0$, $(i = 1, 2)$ are satisfied. Hence, there are no more conditions for the compatibility, and the linearization criteria in the studied case are (4.22), (4.33), (4.38), (4.39) and (4.62). Note also that updating the k_i, $(i = 2, 3, 4)$, these become as follows:

$$k_2 = (\lambda_{32}/{\varphi_x}^2) - k_1, \quad (4.63)$$

$$k_3 = (-{\psi_{1y_1}}^2\lambda_{15})/(4{\varphi_x}^2v), \quad (4.64)$$

$$k_4 = ({\psi_{2y_1}}^2\lambda_{15})/(4{\varphi_x}^2v). \quad (4.65)$$

Remark 4.1. In the case $\lambda_{12} = 0,\ \lambda_{16} = 0$ and $\lambda_{15} = 0$, one has

$$k_1 = k_2 = (3{\varphi_{xx}}^2 - 2\varphi_{xxx}\varphi_x + {\varphi_x}^2\lambda_{33})/(4{\varphi_x}^4), \quad k_3 = k_4 = 0.$$

This case corresponds to (2.34).

Combining all derived results in the case $\psi_{1y_1} \neq 0$, the following theorem is proven.

Theorem 4.1. Necessary and sufficient conditions for system (4.6) to be equivalent to a linear system (4.3) with constant matrix K via fiber preserving transformations are

(I.) The conditions are equations (4.22), (4.33), (4.38) and (4.39), together with the additional conditions:

(I.1.) If $\lambda_{12} \neq 0$, then the additional conditions are equations (4.49), (4.50) and (4.52).

(I.2.) If $\lambda_{12} = 0$ and $\lambda_{16} \neq 0$, then the additional conditions are equations (4.56) and (4.57).

(I.3.) If $\lambda_{12} = 0$, $\lambda_{16} = 0$ and $\lambda_{15} \neq 0$, then the additional condition is equation (4.62).

(I.4.) If $\lambda_{12} = 0$, $\lambda_{16} = 0$ and $\lambda_{15} = 0$, then there are no additional conditions.

4.3.2 Case $\psi_{1y_1} = 0$

Without loss of generality, we can assume that $\psi_{2y_2} = 0$ as well. Otherwise the change of variables (which is indeed an equivalence transformation)

$$\bar{x} = x,\ \bar{y_1} = y_2,\ \bar{y_2} = y_1,$$

will bring us back to the case $\psi_{1y_1} \neq 0$. Thus, substituting $\varphi_{y_1} = 0$, $\varphi_{y_2} = 0$ and $\psi_{2y_2} = 0$ into (4.7)-(4.21), one obtains the equations as follows:

$$\begin{gathered} a_{11} = 0, a_{12} = 0, a_{13} = 0, a_{15} = 0, a_{16} = 0, \\ a_{18} = 0, a_{24} = 0, a_{25} = 0, a_{27} = 0, \end{gathered} \tag{4.66}$$

$$\psi_{2y_1y_1} = -\psi_{2y_1} a_{14}, \tag{4.67}$$

$$\psi_{2xy_1} = (\varphi_{xx}\psi_{2y_1} - \varphi_x\psi_{2y_1} a_{17})/(2\varphi_x), \tag{4.68}$$

$$\psi_{1y_2y_2} = -\psi_{1y_2} a_{26}, \tag{4.69}$$

$$\psi_{1xy_2} = (\varphi_{xx}\psi_{1y_2} - \varphi_x\psi_{1y_2} a_{28})/(2\varphi_x), \tag{4.70}$$

$$\psi_{1xx} = (\varphi_{xx}\psi_{1x} - {\varphi_x}^3(k_1\psi_1 + k_3\psi_2) - \varphi_x\psi_{1y_2} a_{29})/\varphi_x, \tag{4.71}$$

$$\psi_{2xx} = (\varphi_{xx}\psi_{2x} - {\varphi_x}^3(k_2\psi_2 + k_4\psi_1) - \varphi_x\psi_{2y_1} a_{19})/\varphi_x, \tag{4.72}$$

where $\Delta = -\varphi_x\psi_{1y_2}\psi_{2y_1} \neq 0$.

From equations (4.67)-(4.72), one can compare the mixed derivatives $(\psi_{1y_1})_{xy_2} = (\psi_{1xy_2})_{y_1}$, $(\psi_{1xy_2})_{y_2} = (\psi_{1y_2y_2})_x$, $(\psi_{1y_1})_{y_2y_2} = (\psi_{1y_2y_2})_{y_1}$, $(\psi_{2y_2})_{xy_1} = (\psi_{2xy_1})_{y_2}$, $(\psi_{2xy_1})_{y_1} = (\psi_{2y_1y_1})_x$, $(\psi_{2y_2})_{y_1y_1} = (\psi_{2y_1y_1})_{y_2}$, $(\psi_{1xx})_{y_1} = (\psi_{1y_1})_{xx}$, $(\psi_{2xx})_{y_2} = (\psi_{2y_2})_{xx}$, $(\psi_{1xx})_{y_2} = (\psi_{1xy_2})_x$, $(\psi_{2xx})_{y_1} = (\psi_{2xy_1})_x$, one obtains

$$\begin{gathered} a_{28y_1} = 0,\ a_{28y_2} = 2a_{26x},\ a_{26y_1} = 0,\ a_{17y_2} = 0, \\ a_{17y_1} = 2a_{14x},\ a_{14y_2} = 0, \end{gathered} \tag{4.73}$$

$$k_3 = (-a_{29y_1}\psi_{1y_2})/({\varphi_x}^2\psi_{2y_1}), \tag{4.74}$$

$$k_4 = (-a_{19y_2}\psi_{2y_1})/({\varphi_x}^2\psi_{1y_2}), \tag{4.75}$$

$$k_1 = (3{\varphi_{xx}}^2 - 2\varphi_{xxx}\varphi_x + {\varphi_x}^2\mu_1)/(4{\varphi_x}^4), \tag{4.76}$$

$$k_2 = (3{\varphi_{xx}}^2 - 2\varphi_{xxx}\varphi_x - {\varphi_x}^2\mu_2)/(4{\varphi_x}^4), \tag{4.77}$$

where the coefficients μ_n are defined through a_{ij} and their derivatives, shown in Appendix G.

Since k_i, $(i = 1, ..., 4)$ are constant, the equations $(k_2)_x = 0$, $(k_2)_{y_2} = 0$, $(k_1)_{y_1} = 0$, $(k_3)_{y_1} = 0$, $(k_4)_{y_2} = 0$, $(k_1)_{y_2} = 0$ and $(k_2)_{y_1} = 0$ give

$$\varphi_{xxxx} = (12\varphi_{xxx}\varphi_{xx}\varphi_x - 12{\varphi_{xx}}^3 + 2\varphi_{xx}{\varphi_x}^2\mu_2 - {\varphi_x}^3\mu_{2x})/(2{\varphi_x}^2), \tag{4.78}$$

$$\begin{gathered} a_{19y_2}a_{14} - a_{19y_1y_2} = 0,\ a_{29y_1}a_{26} - a_{29y_1y_2} = 0, \\ a_{29y_1y_1} + a_{29y_1}a_{14} = 0,\ a_{19y_2y_2} + a_{19y_2}a_{26} = 0, \\ a_{26xx} - a_{26x}a_{28} + a_{26y_2}a_{29} - a_{29y_2y_2} + a_{29y_2}a_{26} = 0, \\ a_{14xx} - a_{14x}a_{17} + a_{14y_1}a_{19} - a_{19y_1y_1} + a_{19y_1}a_{14} = 0. \end{gathered} \tag{4.79}$$

Notice that the equations $(\varphi_{xxxx})_{y_1} = 0$, $(\varphi_{xxxx})_{y_2} = 0$, $(k_3)_{y_2} = 0$ and $(k_4)_{y_1} = 0$ are satisfied. Considering the derivatives $(k_1)_x = 0$, $(k_3)_x = 0$ and $(k_4)_x = 0$, one achieves

$$2\varphi_{xx}\mu_5 - \varphi_x\mu_{5x} = 0, \tag{4.80}$$

$$(\varphi_{xx} - \varphi_x\mu_3)a_{29y_1} = 0, \tag{4.81}$$

$$(\varphi_{xx} - \varphi_x\mu_4)a_{19y_2} = 0. \tag{4.82}$$

Further analysis of the compatibility depends on a_{29y_1}, a_{19y_2} and μ_5.

Case $a_{29y_1} \neq 0$.

From equation (4.81), one obtains that

$$\varphi_{xx} = \varphi_x\mu_3.$$

Substituting φ_{xx} into (4.80) and (4.82), one has

$$\begin{aligned} 2\mu_5\mu_3 - \mu_{5x} &= 0, \\ a_{19y_2}(\mu_3 - \mu_4) &= 0. \end{aligned} \tag{4.83}$$

Substitution of φ_{xx} into (4.78) gives

$$2\mu_{3xx} + \mu_{2x} - 6\mu_{3x}\mu_3 - 2\mu_2\mu_3 + 2{\mu_3}^3 = 0. \tag{4.84}$$

Note that the equations $(\varphi_{xx})_{y_1} = 0$ and $(\varphi_{xx})_{y_2} = 0$ are satisfied. Hence, there are no new conditions. In summary, the linearization criteria are equations (4.66), (4.73), (4.79), (4.83) and (4.84). Note also that updating the k_i, $(i = 1, 2, 3, 4)$, these become as follows:

$$\begin{aligned} k_1 &= ({\mu_3}^2 - 2\mu_{3x} - \mu_5 - \mu_2)/(4{\varphi_x}^2), \\ k_2 &= k_1 + \mu_5/(4{\varphi_x}^2), \\ k_3 &= (-a_{29y_1}\psi_{1y_2})/({\varphi_x}^2\psi_{2y_1}), \\ k_4 &= (-a_{19y_2}\psi_{2y_1})/({\varphi_x}^2\psi_{1y_2}). \end{aligned}$$

Case $a_{29y_1} = 0$ and $a_{19y_2} \neq 0$.

From equation (4.82), one obtains that

$$\varphi_{xx} = \varphi_x \mu_4.$$

Substituting φ_{xx} into (4.80), one gets

$$\mu_{5x} - 2\mu_5\mu_4 = 0. \tag{4.85}$$

Substitution of φ_{xx} into (4.78) gives

$$6\mu_{4x}\mu_4 - \mu_{2x} - 2\mu_{4xx} + 2\mu_2\mu_4 - 2\mu_4{}^3 = 0. \tag{4.86}$$

Note that the equations $(\varphi_{xx})_{y_1} = 0$ and $(\varphi_{xx})_{y_2} = 0$ are satisfied. Hence, there are no other conditions. Thus, the linearization criteria in this case are (4.66), (4.73), (4.79), (4.85) and (4.86). Note also that updating the k_i, $(i = 1, 2, 3, 4)$, these become as follows:

$$k_1 = (\mu_4{}^2 - 2\mu_{4x} - \mu_5 - \mu_2)/(4\varphi_x{}^2),$$
$$k_2 = k_1 + \mu_5/(4\varphi_x{}^2),$$
$$k_3 = 0,$$
$$k_4 = (-a_{19y_2}\psi_{2y_1})/(\varphi_x{}^2\psi_{1y_2}).$$

Case $a_{29y_1} = 0,\ a_{19y_2} = 0$ and $\mu_5 \neq 0$.

From equation (4.80), one obtains that

$$\varphi_{xx} = (\varphi_x\mu_{5x})/(2\mu_5).$$

Substitution of φ_{xx} into (4.78) leads to the condition

$$\mu_5{}^2(4\mu_{5x}\mu_2 - 4\mu_{5xxx}) + 18\mu_{5xx}\mu_{5x}\mu_5 - 15\mu_{5x}{}^3 - 4\mu_{2x}\mu_5{}^3 = 0. \tag{4.87}$$

Note that the equations $(\varphi_{xx})_{y_1} = 0$ and $(\varphi_{xx})_{y_2} = 0$ are satisfied. Hence, there are no more conditions. In brief, the linearization criteria are conditions (4.66), (4.73), (4.79) and (4.87). Notice also that

$$k_2 = k_1 + \mu_5/(4{\varphi_x}^2), \quad k_3 = 0, \quad k_4 = 0.$$

Remark 4.2. In the case $a_{29y_1} = 0,\ a_{19y_2} = 0,\ \mu_5 = 0$, one has

$$k_1 = k_2 = (3{\varphi_{xx}}^2 - 2\varphi_{xxx}\varphi_x + {\varphi_x}^2\mu_2)/(4{\varphi_x}^4), \quad k_3 = k_4 = 0.$$

This case corresponds to (2.34).

Combining all obtained results in the case $\psi_{1y_1} = 0$ and $\psi_{2y_2} = 0$, the following theorem is proven.

Theorem 4.2. Necessary and sufficient conditions for system (4.6) to be equivalent to a linear system (4.3) with constant matrix K by fiber preserving transformations are

(II.) The conditions are equations (4.66), (4.73) and (4.79),

and the additional conditions:

(II.1.) If $a_{29y_1} \neq 0$, then the additional conditions are equations (4.83), (4.84).

(II.2.) If $a_{29y_1} = 0$ and $a_{19y_2} \neq 0$, then the additional conditions are equations (4.85), (4.86).

(II.3.) If $a_{29y_1} = 0,\ a_{19y_2} = 0$ and $\mu_5 \neq 0$, then the additional condition is equation (4.87).

(II.4.) If $a_{29y_1} = 0,\ a_{19y_2} = 0$ and $\mu_5 = 0$, then there are no additional conditions.

Remark 4.3. If one assumes that the conditions (II.) of Theorem 4.2 are valid, then the conditions (I.) of Theorem 4.1 vanish. Moreover, these conditions also imply that $\lambda_{12} = -4a_{29y_1}$, $\lambda_{15} = -4a_{19y_2}$, $\lambda_{16} = -\mu_5$, and the following is valid: (a) the conditions (II.1.) become a particular case of the conditions (I.1.); (b) the conditions (II.3.) are a particular case of the conditions (I.2.); (c) the conditions

(II.2.) with $\mu_5 \neq 0$ and $\mu_5 = 0$ form particular cases of the conditions (I.2.) and (I.3.), respectively. This allows to propose the conjecture that Theorem 4.1 is valid independently of the values of ψ_{1y_1} and ψ_{2y_2}.

Notice that this conjecture is to be expected. For example, for a linearizable single second-order ordinary differential equation via a point transformation the criteria of linearization are combined to only two conditions, whereas during compatibility analysis one has to study two separable cases, see in Meleshko (2005).

4.4 Necessary Conditions of Linearization under Point Transformations

During the study presented in the previous section several relations for linearizability for the general case of point transformations (4.2) and for the general case of the matrix $K(t)$ were noted. These relations are the necessary conditions for linearization and they were obtained as follows. For example, assuming that $\psi_{1y_1} \neq 0$, from equations (4.7)-(4.21) one obtains the derivatives v_x, v_{y_j}, φ_{xx}, φ_{xy_j}, $\varphi_{y_jy_k}$, ψ_{1xx}, $\psi_{l_{xy_j}}$, $\psi_{l_{y_jy_k}}$, $(j, k, l = 1, 2)$. Comparing the mixed derivatives of the functions v, φ, ψ_1 and ψ_2, one can find the expressions of the quantities ω_n, $(n = 1, 2, ..., 15)$, where ω_n are expressed through a_{ij} and their derivatives (shown in Appendix G). Excluding the functions v, φ, ψ_1 and ψ_2 from these expressions, one obtains the conditions

$$J_i = 0, \quad (i = 1, 2, ..., 15), \tag{4.88}$$

where

$$J_1 = \omega_1\omega_{11} - 2\omega_1\omega_9 + 2\omega_{10}\omega_2 - \omega_3\omega_6,\ J_2 = \omega_1\omega_5 + 2\omega_2\omega_6,$$

$$J_3 = 6\omega_1\omega_8 - 2\omega_1\omega_{12} + 10\omega_{11}\omega_2 - 20\omega_2\omega_9 - 5\omega_3^2,$$

$$J_4 = 2\omega_{10}\omega_2 - \omega_1\omega_9,\ J_5 = 10\omega_1\omega_7 + \omega_{12}\omega_2 - 3\omega_2\omega_8,$$

$$J_6 = \omega_1\omega_8 + \omega_{11}\omega_2 - 3\omega_2\omega_9,\ J_7 = 4\omega_1\omega_{12} - 2\omega_1\omega_8 + 10\omega_3^2 + 5\omega_3\omega_5,$$

$$J_8 = 10\omega_1\omega_{13} + \omega_{12}\omega_3 - 3\omega_3\omega_8,\ J_9 = \omega_1\omega_{15} - \omega_{10}\omega_3,$$

$$J_{10} = \omega_1\omega_{14} + \omega_{11}\omega_3 - 3\omega_3\omega_9,\ J_{11} = 2\omega_{12}\omega_2 - \omega_2\omega_8 + 5\omega_3\omega_4,$$

$$J_{12} = \omega_{13}\omega_2 - \omega_3\omega_7,\ J_{13} = 2\omega_{15}\omega_2 - \omega_3\omega_9,$$

$$J_{14} = \omega_{14}\omega_2 - \omega_3\omega_8,\ J_{15} = 2\omega_1\omega_4 - 2\omega_2\omega_3 - \omega_2\omega_5.$$

After obtaining these relations, one can directly check by substituting (4.7)-(4.21) into (4.88), that they are satisfied for the general case of point transformations (4.2) and for the general case of the matrix $K(t)$.

Thus, the following theorem can be stated.

Theorem 4.3. The conditions (4.88) are necessary for system (4.6) to be linearizable under point transformations.

Remark 4.4. Notice also that considering the conditions obtained in Aminova and Aminov (2006); Neut, Petitot and Dridi (2009), one notes that they are not satisfied unless the matrix $K = 0$.

4.5 Illustration of the Linearization Theorem

In this section, examples demonstrating the procedure of using the linearization theorems are presented.

Example 4.4.1. Some types of Newtonian systems are of the form of a system of two second-order quadratically semi-linear ordinary differential equations

$$y_1'' = a(y_1,y_2){y_1'}^2 + 2b(y_1,y_2)y_1'y_2' + c(y_1,y_2){y_2'}^2,$$
$$y_2'' = d(y_1,y_2){y_1'}^2 + 2e(y_1,y_2)y_1'y_2' + f(y_1,y_2){y_2'}^2. \tag{4.89}$$

In Aminova and Aminov (2006); Mahomed and Qadir (2007) showed that system (4.89) is equivalent via point transformations to the simplest equations $\ddot{u}_1 = 0, \quad \ddot{u}_2 = 0$ if and only if

$$S_i = 0, \quad (i = 1,2,3,4), \tag{4.90}$$

where

$$S_1 = a_{y_2} - b_{y_1} + be - cd, \; S_2 = b_{y_2} - c_{y_1} + (ac - b^2) + (bf - ce),$$
$$S_3 = d_{y_2} - e_{y_1} - (ae - bd) - (df - e^2), \; S_4 = b_{y_1} + f_{y_1} - a_{y_2} - e_{y_2}.$$

Application of fiber preserving transformation to system (4.89) also leads to the same conditions (4.90).

Example 4.4.2. Consider a nonlinear system

$$y_1'' = -{y_1'}^2 - {y_2'}^2 - q_1, \quad y_2'' = q_2 - 2y_1'y_2', \tag{4.91}$$

where q_1, q_2 are constant. Applying the linearization criteria obtained in Aminova and Aminov (2006); Neut, Petitot and Dridi (2009) to system (4.91), one obtains that system (4.91) is equivalent to the free particle equations via point transformations if and only if $q_2 = 0$. Let us consider the case $q_2 \neq 0$. Note that for system (4.91):

$$\lambda_{12} = -4q_2, \; \lambda_{14} = 0, \; \lambda_{15} = -4q_2, \; \lambda_{16} = 0.$$

Since $q_2 \neq 0$, then $\lambda_{12} \neq 0$ and equation (4.46) becomes $\varphi_{xx} = 0$. Taking the simplest solution $\varphi = x$ of this equation and solving the compatible system of equations (4.23)-(4.28) and (4.29)-(4.32) for the functions ψ_1 and ψ_2, one gets the solution $\psi_1 = \frac{1}{2}e^{(y_1 - y_2)}$ and $\psi_2 = \frac{1}{2}e^{(y_1+y_2)}$. Notice that we assume $k_3 = k_4 = 0$

for convenience, during solving the compatibility system of equations (4.23)-(4.32). Substituting φ, ψ_1 and ψ_2 into equations (4.37) and (4.51), one obtains $k_1 = q_1+q_2$ and $k_2 = q_1 - q_2$. Thus, Theorem 4.1 guarantees that system (4.91) can be transformed to the system of linear equations

$$\ddot{u}_1 + k_1 u_1 = 0, \ \ddot{u}_2 + k_2 u_2 = 0,$$

and the linearizing transformation is

$$t = x, \quad u_1 = \frac{1}{2} e^{(y_1 - y_2)}, \quad u_2 = \frac{1}{2} e^{(y_1 + y_2)}.$$

Example 4.4.3. A variety of applications in science and engineering such as the well-known oscillator system, the vibration of springs and some types of the conservative systems with two degrees of freedom, are of the form:

$$y_1'' = g_1(x)y_1 + g_2(x)y_2, \quad y_2'' = g_3(x)y_2 + g_4(x)y_1. \tag{4.92}$$

For system (4.92):

$$\lambda_{12} = -4g_4, \ \lambda_{16} = 4(g_1 - g_3), \ \lambda_{15} = -4g_2, \ \lambda_{21} = -2g_{3x}, \ \lambda_{22} = 4g_3\lambda_{16x} - \lambda_{16xxx}.$$

Then by virtue of Theorem 4.1, system (4.92) can be reduced via a fiber preserving transformation to a linear system with constant coefficients if the functions $g_i(x)$, $(i = 1, 2, 3, 4)$ are as follows. If $g_4 \neq 0$, then the conditions are

$$g_{4x}g_2 - g_{2x}g_4 = 0, \quad g_4(g_{3x} - g_{1x}) + g_{4x}(g_1 - g_3) = 0,$$
$$16g_{3x}{g_4}^3 + 4g_{4xxx}{g_4}^2 - 18g_{4xx}g_{4x}g_4 + 15{g_{4x}}^3 - 16g_{4x}g_3{g_4}^2 = 0,$$

if $g_4 = 0$ and $g_1 \neq g_3$, then the conditions are

$$g_2(g_{3x} - g_{1x}) + g_{2x}(g_1 - g_3) = 0,$$
$$8{\lambda_{16}}^3\lambda_{21} + 4{\lambda_{16}}^2\lambda_{22} + 18\lambda_{16}\lambda_{16xx}\lambda_{16x} - 15{\lambda_{16x}}^3 = 0, \tag{4.93}$$

if $g_4 = 0$, $g_1 = g_3$ and $g_2 \neq 0$, then the conditions are

$$15{g_{2x}}^3 - 18g_{2xx}g_{2x}g_2 + 16{g_2}^3g_{3x} + 4{g_2}^2(g_{2xxx} - 4g_{2x}g_3) = 0,$$

if $g_4 = 0$, $g_1 = g_3$ and $g_2 = 0$, then this case corresponds to (2.34).

For instance, considering the oscillator system ($g_2 = g_4 = 0$)

$$y_1'' = g_1(x)y_1, \quad y_2'' = g_3(x)y_2,$$

the criteria of (Aminova and Aminov, 2006) and (Neut, Petitot and Dridi, 2009) are only satisfied when $g_1 = g_3$. If $g_1 \neq g_3$, then there is only the single condition (4.93) for two functions g_1 and g_3 which guarantees that a fiber preserving transformation can transform this system to the case

$$\ddot{u}_1 + k_1u_1 + k_3u_2 = 0, \quad \ddot{u}_2 + k_2u_2 + k_4u_1 = 0.$$

Example 4.4.4. We consider a predator-prey population model, the non-linear Lotka-Volterra system:

$$\begin{aligned} y_1' &= g(y_1, y_2) = l_1y_1 - l_2y_1y_2, \\ y_2' &= h(y_1, y_2) = l_3y_1y_2 - l_4y_2, \end{aligned} \tag{4.94}$$

where $l_i > 0$, $(i = 1, 2, 3, 4)$ are constant. System (4.94) is a particular case of the system

$$\begin{aligned} y_1'' &= f_1(y_2)y_1' - f_2(y_1)y_2', \\ y_2'' &= f_3(y_1)y_2' + f_4(y_2)y_1'. \end{aligned} \tag{4.95}$$

with

$$f_1 = l_1 - l_2y_2,\ f_2 = l_2y_1,\ f_3 = l_3y_1 - l_4,\ f_4 = l_3y_2. \tag{4.96}$$

System (4.95) is not only the differential form of (4.94), but also a type of system (4.6). Thus applying conditions (I.) of Theorem 4.1 to system (4.95), one obtains the needed linearizing conditions:

$$f_i' = 0, (i = 1, 2, 3, 4). \tag{4.97}$$

The linearizing conditions (4.97) force (4.96) to give $l_2 = l_3 = 0$, these make λ_{12} and λ_{15} of Theorem 4.1 vanish. Therefore, the differential form of system (4.94) is linearizable by a fiber preserving point transformation. Note that the transformed linear system depends on the value of $\lambda_{16} = {l_1}^2 - {l_4}^2$.

Example 4.4.5. Consider the fourth-order ordinary differential equation

$$y_1'''' - a(x, y_1)(y_1''')^2 - b(x, y_1) = 0. \tag{4.98}$$

Applying the linearization criteria obtained in Ibragimov, Meleshko and Suksern (2008) to equation (4.98), one obtains that equation (4.98) does not satisfy even the necessary condition for linearization. On the other hand, equation (4.98) is equivalent to the system

$$y_1'' = y_2,\ y_2'' = a(x, y_1){y_2'}^2 + b(x, y_1). \tag{4.99}$$

Applying the linearization criteria obtained in Aminova and Aminov (2006); Neut, Petitot and Dridi (2009) to system (4.99), one obtains that this system is not linearizable. On the other hand, system (4.99) is a type of equation (4.6). Applying Theorem 4.1, one obtains that linearization criteria are

$$b_{xy_1} = 0,\ b_{y_1y_1} = 0,\ a = 0.$$

These conditions require that equation (4.98) is already linear. Note that for system (4.99), $\lambda_{12} = -4b_{y_1}$, $\lambda_{16} = 0$ and $\lambda_{15} = -4$.

4.6 Summary

In this chapter, the necessary form of a linearizable system of two second-order ordinary differential equations $y_1'' = f_1(x, y_1, y_2, y_1', y_2')$, $y_2'' = f_2(x, y_1, y_2, y_1', y_2')$ via point transformations was presented. Some other necessary conditions were also found. Necessary and sufficient conditions for a system of two second-order ordinary differential equations to be transformed to the general form of linear system with constant coefficients via fiber preserving transformations were obtained. On the way of establishing of main theorems, we also gave an explicit procedure for constructing this linearizing transformation. Illustrative examples of linearization theorems were given.

CHAPTER V

CONCLUSIONS

This thesis was devoted to the study of the linearization problem of a system of two second-order ordinary differential equations

$$\ddot{x} = G(t, x, y, \dot{x}, \dot{y}),\ \ddot{y} = F(t, x, y, \dot{x}, \dot{y}). \tag{5.1}$$

The method of the study was separated into two parts as follows.

5.1 Linearization of a Projectable System (5.1)

A new method for linearizing a system of ordinary differential equations was introduced. This method consists of sequentially reducing number of the dependent variables and using the Lie criteria for the reduced equations. The method was applied to a system of two second-order ordinary differential equations. Moreover, it was shown that for systems of two second-order quadratically semi-linear ordinary differential equations the new method gives a more general set of linearizable systems than is possible via point transformations. An example of equations which are not linearizable by point transformations, but are sequentially linearizable by the new method, was given.

5.2 Linearization of System (5.1) via Fiber Preserving Point Transformations

The necessary form of a linearizable system of two second-order ordinary differential equations (5.1) via point transformations was obtained. Some other necessary conditions were also found. Necessary and sufficient conditions for a system of two second-order ordinary differential equations to be transformed to the general form of linear system with constant coefficients via fiber preserving transformations were obtained. A linear system with constant coefficients was chosen because of its simplicity of finding the general solution. Along the way of establishing of main theorems, we also gave an explicit procedure for constructing this linearizing transformation.

REFERENCES

REFERENCES

Aminova, A. V. and Aminov, N. A.-M. (2006). Projective geometry of systems of second-order differential equations. **Matem. Sbornik.** 197(7): 951-975.

Bocharov, A. V., Sokolov, V. V. and Svinolupov, S. I. (1993). On some equivalence problems for differential equations. **Preprint ESI.** 54.

Cartan, E. (1924). Sur les variétés à connexion projective. **Bull. Soc. Math. France.** 52: 205-241.

Cartan, E. (1946). **Les systems differentiels exterieurs et leurs applications scientifiques**, Hermann.

Chern, S.-S. (1940). The geometry of the differential equation $y''' = F(x, y, y', y'')$. **Rep. Nat. Tsing Hua Univ.** 4: 97-111.

Davenport, J. H. (1994). Computer algebra - past, present and future. **Euromath Bulletin.** 1(2): 25-45.

Doubrov, B., Komrakov, B. and Morimoto, T. (1999). Equivalence of holonomic differential equations. **Lobachevskii Journal of Mathematics.** 3: 39-71.

Doubrov, B. (2001). Contact trivialization of ordinary differential equations. In **O. Kowalski, D. Krupka, and J. Slovak (eds.), Proceedings of the 8th International Conference on Differential Geometry and Its Applications** (pp 73-84). Opava, Czech Republic: Silesian University in Opava.

Duarter, L. G. S., Moreira, I. C. and Santos, F. C. (1994). Linearisation under nonpoint transformations. **J. Phys. A: Math. Gen.** 27: L739-L743.

Euler, N., Wolf, T., Leach, P. G. L. and Euler, M. (2003). Linearizable third-order ordinary differential equations and generalized Sundman transformations: The Case $X''' = 0$. **Acta Applicandae Mathematicae.** 76: 89-115.

Euler, N. and Euler, M. (2004). Sundman symmetries of nonlinear second-order and third-order ordinary differential equations. **Journal of Nonlinear Mathematical Physics.** 11(3): 399-421.

Finikov, S.P. (1948). **Method of Cartan exterior forms**. Moscow-Leningrad: Gostechizdat.

Grebot, G. (1997). The characterization of third order ordinary differential equations admitting a transitive fiber-preserving point symmetry group. **Journal of Mathematical Analysis and Applications.** 206: 364-388.

Gusyatnikova, V. N. and Yumaguzhin, V. A. (1999). Contact transformations and local reducibility of ODE to the form $y''' = 0$. **Acta Applicandae Mathematicae.** 56: 155-179.

Hearn, A. C. (1987). **Reduce users manual, ver. 3.3.** Santa Monica: The Rand Corporation CP 78.

Ibragimov, N.H. and Meleshko, S.V. (2004). Linearization of third-order ordinary differential equations by point transformations. **Archives of ALGA.** 1: 71-92.

Ibragimov, N. H. and Meleshko, S. V. (2005). Linearization of third-order ordinary differential equations by point and contact transformations. **Journal of Mathematical Analysis and Applications.** 308: 266-289.

Ibragimov, N. H., Meleshko, S. V. and Suksern, S. (2008). Linearization of fourth-order ordinary differential equations by point transformations. **J. Phys. A: Math. Theor.** 41(235206): 19pp.

Kuranishi, M. (1967). **Lectures on involutive systems of partial differential equations.** Säo Paulo: Publ.Soc. Math. Säo Paulo.

Lie, S. (1883). Klassifikation und integration von gewöhnlichen differentialgleichungen zwischen x, y, die eine gruppe von transformationen gestatten. III. **Archiv for Matematik og Naturvidenskab.** 8(4): 371-427.

Liouville, R. (1889). Sur les invariantes de certaines équations. **J. de l'École Polytechnique.** 59: 7-88.

Mahomed, F. M. and Qadir, A. (2007). Linearization criteria for a system of second-order quadratically semi-linear ordinary differential equations. **Nonlinear Dynamics.** 48: 417-422.

Meleshko, S. V. (2005). **Methods for constructing exact solutions of partial differential equations.** New York: Springer Science+Business Media, Inc.

Meleshko, S. V. (2006). On linearization of third-order ordinary differential equations. **Journal of Physics A: Mathematical and Theoretical.** 39: 15135-15145.

Nakpim, W. and Meleshko, S. V. (2010). Linearization of Second-Order Ordinary Differential Equations by Generalized Sundman Transformations. **Sym-**

metry, Integrability and Geometry:Methods and Applications (SIGMA). 6(051).

Nakpim, W. and Meleshko, S. V. (2010). Linearization of third-order ordinary differential equations by generalized Sundman transformations: The case $X''' + \alpha X = 0$. **Communications in Nonlinear Science and Numerical Simulation.** 15: 1717-1723.

Neut, S. and Petitot, M. (2002). La géométrie de l'équation $y''' = f(x, y, y', y'')$. **C. R. Acad. Sci. Paris.** 335: 515-518.

Neut, S., Petitot, M. and Dridi, R. (2009). Élie Cartan's geometrical vision or how to avoid expression swell. **Journal of Symbolic Computation.** 44: 261-270.

Pommaret, J. E. (1978). **Systems of partial differential equations and Lie pseudogroups.** New York: Gordon & Breach.

Sookmee, S. (2005). **Invariants of the equivalence group of a system of second-order ordinary differential equations.** M.Sc. thesis, School of Mathematics, Suranaree University of Technology.

Suksern, S., Ibragimov, N. H. and Meleshko, S. V. (2009). Criteria for fourth-order ordinary differential equations to be linearizable by contact transformations. **Communications in Nonlinear Science and Numerical Simulation.** 14: 2619-2628.

Thailert, E. (2008). **Equivalence of linear second-order parabolic partial differential equations to one of the canonical forms.** Ph.D. thesis, School of Mathematics, Suranaree University of Technology.

Tresse, A. M. (1896). **Détermination des invariants ponctuels de l'équation différentielle ordinaire du second ordre** $y'' = f(x, y, y')$**.** Preiss-chriften der fürstlichen Jablonowskischen Geselschaft XXXII, Leipzig, S.Herzel.

Wafo Soh, C. and Mahomed, F. M. (2001). Linearization criteria for a system of second-order ordinary differential equations. **Int. J. Nonlinear Mech.** 36: 671-677.

APPENDICES

APPENDIX A

CANONICAL FORM OF A LINEAR SECOND-ORDER ODE

This part shows that a linear second-order ordinary differential equation:

$$y''(x) + a(x)\, y'(x) + b(x)\, y(x) = c(x)\,. \tag{A.1}$$

can be reduced to the simplest equation

$$u'' = 0,$$

under point transformations.

If $c(x) \neq 0$, then equation (A.1) is nonhomogeneous. If $c(x) = 0$, then equation (A.1) is homogeneous. Note that the general solution of equation (A.1) is as follows

$$y = y_h + y_p,$$

where y_h is the general solution of the homogeneous equation

$$y'' + a(x)y' + b(x)y = 0$$

and y_p is a particular solution of the equation (A.1).

Next let us construct the point transformation

$$t = x, \;\; z = y - y_p(x). \tag{A.2}$$

Then the derivatives are changed by the formulae

$$y' = z' + y_p' \,,$$

$$y'' = z'' + y_p''.$$

Substitution of these y' and y'' into equation (A.1), gives

$$(z'' + a(x)z' + b(x)z) + \left(y_p'' + a(x)y_p' + b(x)y_p\right) \;=\; c(x). \tag{A.3}$$

Since $y_p'' + a(x)y_p' + b(x)y_p = c(x)$. Thus equation (E.3) becomes

$$z'' + a(x)z' + b(x)z \;=\; 0. \tag{A.4}$$

That is, we can eliminate the coefficient $c(x)$ from equation (A.1).

Next will show the elimination of the coefficients $a(x)$ and $b(x)$ from the equation (A.4). Let us construct the point transformation

$$t = x, \;\; w = \frac{z}{\alpha(x)}, \tag{A.5}$$

where $\alpha(x) \neq 0$. Then the derivatives are changed by the formulae

$$z' = w'\alpha(x) + w\alpha'(x),$$

$$z'' = w''\alpha(x) + 2w'\alpha'(x) + w\alpha''(x).$$

Substitution of these z' and z'' into equation (A.4), gives

$$w''\alpha(x) + w'(2\alpha'(x) + a(x)\alpha(x)) + w(\alpha''(x) + a(x)\alpha'(x) + b(x)\alpha(x)) = 0. \tag{A.6}$$

Next construct the Cauchy problem

$$\alpha''(x) + a(x)\alpha'(x) + b(x)\alpha(x) = 0, \tag{A.7}$$

$$\alpha(x_0) = 1, \;\alpha'(x_0) = 1. \tag{A.8}$$

Then there exists the unique solution of (A.7) satisfying the initial condition (A.8). Therefore, the equation (A.6) is reduced to

$$w''\alpha(x) + w'(2\alpha'(x) + a(x)\alpha(x)) = 0. \tag{A.9}$$

Since by equation (A.8), $\alpha(x) \neq 0$ in some neighborhood of x_0, equation (A.9) can be rewritten as follows

$$w'' + w'\tilde{a}(x) = 0, \tag{A.10}$$

where $\tilde{a}(x) = \frac{2\alpha'(x)}{\alpha(x)} + a(x)$.

Next multiply the equation (A.10) by the nonzero term $e^{\int_{x_0}^{x} \tilde{a}(s)ds}$, one gets

$$e^{\int_{x_0}^{x} \tilde{a}(s)ds} (w'' + w'\tilde{a}(x)) = 0. \tag{A.11}$$

Defining $u' = w' e^{\int_{x_0}^{x} \tilde{a}(s)ds}$, one obtains that the equation (A.11) is reducible to simplest equation

$$u'' = 0.$$

Remark A.1. The values of Jacobians of the point transformations (A.2) and (A.5) are equal to 1 and $\frac{1}{\alpha(x)}$, respectively.

Remark A.2. The composition of two point transformations is still a point transformation.

APPENDIX B

THE INVERSE FUNCTION THEOREM

Theorem. *(The Inverse Function Theorem)*

Let V be open in R^n and $f : V \to R^n$ be C^1 on V. If $\Delta_f(a) \neq 0$ for some $a \in V$, then there exits an open set $W \subset V$ containing a, such that

i) f is 1-1 on W,

ii) f^{-1} is C^1 on $f(W)$, and

iii) for each $z := f(s) \in f(W)$,

$$D(f^{-1})(z) := [Df(s)]^{-1},$$

where D is the Jacobian matrix and $[\quad]^{-1}$ represents matrix inversion.

Example. Let $f(x, y) = (3x - y, \frac{x}{y})$. Prove that f^{-1} exists.

Defining

$$t := \varphi(x, y) = 3x - y, \quad u := \psi(x, y) = \frac{x}{y}.$$

Then $\varphi_x = 3, \quad \varphi_y = -1, \quad \psi_x = \frac{1}{y}, \ \psi_y = \frac{-x}{y^2}$, these imply that $f \in C^1(E)$, where $E = \{\ (x, y) \mid x \in R,\ y \in R \setminus \{0\}\ \}$.

Next consider

$$\Delta_f(a) = det \begin{pmatrix} 3 & -1 \\ \frac{1}{y_0} & \frac{-x_0}{{y_0}^2} \end{pmatrix} = \frac{y_0 - 3x_0}{{y_0}^2} \neq 0,$$

for all $a = (x_0, y_0) \in E$ with $y_0 \neq 3x_0$.

Thus by *the Inverse Function Theorem*, the function f^{-1} exists at least locally and direct computation shows that

$$x = \tilde{\varphi}(t, u) := \frac{ut}{3u - 1}, \quad y = \tilde{\psi}(t, u) := \frac{t}{3u - 1},$$

for any $(t, u) \in f(W)$, where W is some open set containing a.

APPENDIX C

ANOTHER EXAMPLE TO EXPRESS THE POINT TRANFORMATION

Consider the problem* of finding all partial differential equations of the form

$$F(t, x, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0, \tag{C.1}$$

which are equivalent to a linear second-order parabolic partial equation

$$v_\tau + b_1(\tau, y)v_{yy} + b_2(\tau, y)v_y + b_3(\tau, y)v = 0. \tag{C.2}$$

The essential part of this linearization problem under point transformations, is to find an invertible change of the independent and dependent variables

$$\tau = H(t, x, u),\ y = Y(t, x, u),\ v = V(t, x, u), \tag{C.3}$$

which transforms the nonlinear equation (C.1) into a linear second-order parabolic partial differential equation (C.2). Here the independent variables are t, x and the dependent variable is u. Note that $\Delta \neq 0$ is the Jacobian of the change of variables (C.3)

$$\Delta = (D_tH)(D_xY) - (D_xH)(D_tY),$$

where

$$D_t = \partial_t + u_t\partial_u + u_{tx}\partial_{u_x} + u_{tt}\partial_{u_t},\ \ D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{tx}\partial_{u_t}.$$

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*See more details in Thailert (2008).

APPENDIX D

DEFINING DERIVATIVES IN POINT TRANSFORMATIONS IN CASE OF A SYSTEM OF TWO SECOND-ORDER ODES

Consider the linearization problem of a system of second-order ordinary differential equations

$$y_1'' = F_1(x, y_1, y_2, y_1', y_2'), \quad y_2'' = F_2(x, y_1, y_2, y_1', y_2'). \tag{D.1}$$

via a point transformation. The problem consists of finding an invertible transformation

$$t = \varphi(x, y_1, y_2),\ u_1 = \psi_1(x, y_1, y_2),\ u_2 = \psi_2(x, y_1, y_2), \tag{D.2}$$

which transforms the system of equations (D.1) into a linear system of equations

$$\ddot{u}_1 + k_{11}(t)u_1 + k_{12}(t)u_2 = 0, \quad \ddot{u}_2 + k_{21}(t)u_1 + k_{22}(t)u_2 = 0. \tag{D.3}$$

Notice that in the equation (D.2), x is the independent variable and $y_1,\ y_2$ are dependent variables, and $\Delta \neq 0$ is the Jacobian of the change of variables (D.2),

$$\Delta = (\varphi_x\psi_{1y_1}\psi_{2y_2} - \varphi_x\psi_{1y_2}\psi_{2y_1} - \varphi_{y_1}\psi_{1x}\psi_{2y_2} + \varphi_{y_1}\psi_{1y_2}\psi_{2x} + \varphi_{y_2}\psi_{1x}\psi_{2y_1} - \varphi_{y_2}\psi_{1y_1}\psi_{2x}).$$

In order to find the necessary condition for system (D.1) to be linearizable to the form (D.3), one need to find the formulae of the derivatives $\ddot{u}_1$ and $\ddot{u}_2$.

Let us assume that $y(x)$ is a given function*. The first equation of (D.2) becomes

$$t = \varphi\left(x, y_1\left(x\right), y_2\left(x\right)\right) =: \tilde{\varphi}(x).$$

*This function need not to be the solution of equation (D.1).

Suppose that $\tilde{\varphi}^{'}(x) = 0$, then $t = \varphi = constant$. This contradicts the nonzero value of the Jacobian of the change of variables (D.2). Thus, $\tilde{\varphi}^{'}(x) = \varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \neq 0$, then by virtue of the Inverse Function Theorem, one finds

$$x = \xi(t). \tag{D.4}$$

Substitution of x into the second and third equation of (D.2), one obtains

$$u_1(t) = \psi_1(\xi(t), y_1(\xi(t)), y_2(\xi(t))), \tag{D.5}$$

$$u_2(t) = \psi_2(\xi(t), y_1(\xi(t)), y_2(\xi(t))). \tag{D.6}$$

Thus the first-order derivatives of u_1, u_2 with respect to t, are defined by the formula

$$\dot{u_1} = \frac{du_1}{dt} = \frac{\partial\psi_1}{\partial x}\frac{d\xi}{dt} + \frac{\partial\psi_1}{\partial y_1}\frac{dy_1}{dx}\frac{d\xi}{dt} + \frac{\partial\psi_1}{\partial y_2}\frac{dy_2}{dx}\frac{d\xi}{dt} = (\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2})\frac{d\xi}{dt}, \tag{D.7}$$

$$\dot{u_2} = \frac{du_1}{dt} = \frac{\partial\psi_2}{\partial x}\frac{d\xi}{dt} + \frac{\partial\psi_2}{\partial y_1}\frac{dy_1}{dx}\frac{d\xi}{dt} + \frac{\partial\psi_2}{\partial y_2}\frac{dy_2}{dx}\frac{d\xi}{dt} = (\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2})\frac{d\xi}{dt}. \tag{D.8}$$

To find $\frac{d\xi}{dt}$, let us consider the identity

$$t = \varphi(\xi(t), y_1(\xi(t)), y_2(\xi(t))). \tag{D.9}$$

Differentiating the equation (D.9) with respect to t, one obtains

$$\begin{aligned} \frac{dt}{dt} &= \frac{\partial\varphi}{\partial x}\frac{d\xi}{dt} + \frac{\partial\varphi}{\partial y_1}\frac{dy_1}{dx}\frac{d\xi}{dt} + \frac{\partial\varphi}{\partial y_2}\frac{dy_2}{dx}\frac{d\xi}{dt} \\ 1 &= (\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2})\frac{d\xi}{dt} \\ \frac{d\xi}{dt} &= \frac{1}{(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2})}. \end{aligned} \tag{D.10}$$

Substitution of $\frac{d\xi}{dt}$ into equations (D.7) and (D.8), one obtains

$$\dot{u_1} = \frac{\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2}}{\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2}} = \frac{D_x\psi_1}{D_x\varphi} = h_{11}(x, y_1, y_2, y_1', y_2'),$$

$$\dot{u_2} = \frac{\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2}}{\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2}} = \frac{D_x\psi_2}{D_x\varphi} = h_{21}(x, y_1, y_2, y_1', y_2'),$$

where

$$D_x = \frac{\partial}{\partial x} + y_1' \frac{\partial}{\partial y_1} + y_2' \frac{\partial}{\partial y_2} + y_1'' \frac{\partial}{\partial y_1'} + y_2'' \frac{\partial}{\partial y_2'}$$

is the total derivative with respect to x.

Next consider the second-order derivative of u_1 with respect to t, is defined by the formula

$$\begin{aligned}
\ddot{u}_1 &= \frac{d^2 u_1}{dt^2} = \frac{d\dot{u}_1}{dt} = \frac{dh_{11}\left(\xi\left(t\right), y_1\left(\xi\left(t\right)\right), y_2\left(\xi\left(t\right)\right), y_1'\left(\xi\left(t\right)\right), y_2'\left(\xi\left(t\right)\right)\right)}{dt} \\
&= \frac{\partial h_{11}}{\partial x}\frac{d\xi}{dt} + \frac{\partial h_{11}}{\partial y_1}\frac{dy_1}{dx}\frac{d\xi}{dt} + \frac{\partial h_{11}}{\partial y_2}\frac{dy_2}{dx}\frac{d\xi}{dt} + \frac{\partial h_{11}}{\partial y_1'}\frac{dy_1'}{dx}\frac{d\xi}{dt} + \frac{\partial h_{11}}{\partial y_2'}\frac{dy_2'}{dx}\frac{d\xi}{dt} \\
&= \left(h_{11x} + y_1' h_{1y_1} + y_2' h_{2y_2} + y_1'' h_{11y_1'} + y_2'' h_{11y_2'}\right)\frac{d\xi}{dt} \\
&= \frac{h_{11x} + y_1' h_{1y_1} + y_2' h_{2y_2} + y_1'' h_{11y_1'} + y_2'' h_{11y_2'}}{\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2}} \\
&= \frac{D_x h_{11}}{D_x \varphi} \\
&= h_{12}(x, y_1, y_2, y_1', y_2', y_1'', y_2'').
\end{aligned}$$

Note that the formula for the second-order derivative of u_2 with respect to t, is similar to $\ddot{u}_1$. That is, one obtains

$$\ddot{u}_2 = h_{22}(x, y_1, y_2, y_1', y_2', y_1'', y_2'').$$

Repeating the same process, one obtains the higher order derivatives by the formulae

$$\begin{aligned}
{u_1}^{(3)} &= \frac{d\ddot{u}_1}{dt} = \frac{Dh_{12}}{D\varphi} = h_{13}(x, y_1, y_2, y_1', y_2', y_1'', y_2'', y_1''', y_2'''), \\
{u_2}^{(3)} &= \frac{d\ddot{u}_2}{dt} = \frac{Dh_{22}}{D\varphi} = h_{23}(x, y_1, y_2, y_1', y_2', y_1'', y_2'', y_1''', y_2'''), \\
{u_1}^{(4)} &= \frac{d{u_1}^{(3)}}{dt} = \frac{Dh_{13}}{D\varphi} = h_{14}(x, y_1, y_2, y_1', y_2', y_1'', y_2'', y_1''', y_2''', y_1^{(4)}, y_2^{(4)}), \\
{u_2}^{(4)} &= \frac{d{u_2}^{(3)}}{dt} = \frac{Dh_{23}}{D\varphi} = h_{24}(x, y_1, y_2, y_1', y_2', y_1'', y_2'', y_1''', y_2''', y_1^{(4)}, y_2^{(4)}).
\end{aligned}$$

The recurrent formula is as follows

$$u_j^{(n)} = \frac{du_j^{(n-1)}}{dt} = \frac{Dh_{(n-1)j}}{D\varphi} = h_{nj}(x, y, y', y'', ..., y^{(n)}), \ \ (n = 1, 2, ..., k), \ (j = 1, 2).$$

Note that $u_j^{(0)} = u_j$ and $h_{(0)j} = \psi_j$. Observe that the order of the given ordinary differential equation is preserved under the invertible point transformation (D.2).

For the relation of the solutions $\tilde{y}_1(x), \tilde{y}_2(x)$ and $\tilde{u}_1(t), \tilde{u}_2(t)$, according to (D.4), (D.5) and (D.6), one can convert the solution $\tilde{y}_1(x), \tilde{y}_2(x)$ to the solution $\tilde{u}_1(t), \tilde{u}_2(t)$. Conversely, since the point transformation (D.2) is invertible, x, y_1 and y_2 can be written as follows

$$x = \hat{\varphi}\left(t, u_1, u_2\right), \quad y_1 = \hat{\psi}_1\left(t, u_1, u_2\right), \quad y_2 = \hat{\psi}_2\left(t, u_1, u_2\right). \tag{D.11}$$

If we have the solution $\tilde{u}_1(t), \tilde{u}_2(t)$, by applying the Inverse Function Theorem to the first equation of (D.11), one obtains

$$t = \sigma(x).$$

Substitution of this t into the second and third equation of (D.11), one obtains the solution

$$\tilde{y}_1(x) := \hat{\psi}_1(\sigma(x), \tilde{u}_1(\sigma(x)), \tilde{u}_2(\sigma(x))) \ ,$$

$$\tilde{y}_2(x) := \hat{\psi}_2(\sigma(x), \tilde{u}_1(\sigma(x)), \tilde{u}_2(\sigma(x))) \ .$$

APPENDIX E

PROOF FOR CANONICAL FORMS OF A SYSTEM OF *N* LINEAR SECOND-ORDER ODES

Let us consider a linear system of n second-order ordinary differential equations

$$\ddot{v} + C\dot{v} + Dv + E = 0, \tag{E.1}$$

where $v = v(t)$ and $E = E(t)$ are vectors, $C = C(t)$ and $D = D(t)$ are $n \times n$ square matrices.

If $E(t) \neq 0$, then system (E.1) is called nonhomogeneous. If $E(t) = 0$, then system (E.1) is called homogeneous. Note that the general solution of system (E.1) is of the form

$$v = v_h + v_p,$$

where v_h is the general solution of the homogeneous system

$$\ddot{v} + C(t)\dot{v} + D(t)v = 0,$$

and v_p is any particular solution of the system (E.1).

Next defining the point transformation

$$x = t, \;\; z = v - v_p(t). \tag{E.2}$$

Then the derivatives are changed by the formulae

$$\dot{v} = \dot{z} + \dot{v}_p \; ,$$

$$\ddot{v} = \ddot{z} + \ddot{v}_p.$$

Substitution of these $\dot{v}$ and $\ddot{v}$ into system (E.1), gives

$$(\ddot{z} + C(t)\dot{z} + D(t)z) + (\ddot{v}_p + C(t)\dot{v}_p + D(t)v_p) \quad = \quad E(t). \tag{E.3}$$

Since $\ddot{v}_p + C(t)\dot{v}_p + D(t)v_p = E(t)$. Thus system (E.3) becomes

$$\ddot{z} + C(t)\dot{z} + D(t)z \quad = \quad 0. \tag{E.4}$$

That is, we can eliminate the vector $E(t)$ from system (E.1).

E.1 First Candidate for the Canonical Form.

Since there exists $G_1(t)$, where $G_1(t)$ is nonsingular $n \times n$ square matrix satisfying the Cauchy problem

$$2\dot{G}_1(t) + C(t)G_1(t) = 0, \tag{E.5}$$

$$det\ G_1(t_0) \neq 0, \tag{E.6}$$

we can define $u = (G_1(t))^{-1}z$. Hence the derivatives are changed by the formulae

$$\dot{z} = \dot{u}G_1(t) + u\dot{G}_1(t), \quad \ddot{z} = \ddot{u}G_1(t) + 2\dot{u}\dot{G}_1(t) + u\ddot{G}_1(t).$$

Substitution of these $\dot{z}$ and $\ddot{z}$ into system (E.4), gives

$$\ddot{u} + (G_1(t))^{-1}(2\dot{G}_1(t) + C(t)G_1(t))\dot{u} \tag{E.7}$$

$$+(G_1(t))^{-1}(\ddot{G}_1(t) + C(t)\dot{G}_1(t) + D(t)G_1(t))u = 0.$$

The property (E.5) implies that the system (E.11) reduces to

$$\ddot{u} + Ku = 0, \tag{E.8}$$

where $K = (G_1(t))^{-1}(\ddot{G}_1(t) + C(t)\dot{G}_1(t) + D(t)G_1(t))$. Notice that system (E.8) is called the first candidate of canonical forms.

E.2 Second Candidate for the Canonical Form.

Since there exists $G_2(t)$, where $G_2(t)$ is nonsingular $n \times n$ square matrix satisfying the Cauchy problem

$$\ddot{G}_2(t) + C(t)\dot{G}_2(t) + D(t)G_2(t) = 0, \tag{E.9}$$

$$det\ G_2(t_0) \neq 0, \tag{E.10}$$

we can define $u = (G_2(t))^{-1}z$. Hence the derivatives are changed by the formulae

$$\dot{z} = \dot{u}G_2(t) + u\dot{G}_2(t), \quad \ddot{z} = \ddot{u}G_2(t) + 2\dot{u}\dot{G}_2(t) + u\ddot{G}_2(t).$$

Substitution of these $\dot{z}$ and $\ddot{z}$ into system (E.4), gives

$$\ddot{u} + (G_2(t))^{-1}(2\dot{G}_2(t) + C(t)G_2(t))\dot{u} \tag{E.11}$$

$$+(G_2(t))^{-1}(\ddot{G}_2(t) + C(t)\dot{G}_2(t) + D(t)G_2(t))u = 0.$$

The property (E.9) implies that the system (E.11) reduces to

$$\ddot{u} + K_1\dot{u} = 0, \tag{E.12}$$

where $K_1 = (G_2(t))^{-1}(2\dot{G}_2(t) + C(t)G_2(t))$. Notice that system (E.12) is called the second candidate of canonical forms.

Remark C.1. The value of the Jacobian of the point transformation (E.2) is equal to 1.

Remark C.2. The initial conditions (E.6) and (E.10) imply that the solutions $G_1(t)$ and $G_2(t)$ are nonsingular in some neighborhoods of t_0, respectively.

Remark C.3. The system (E.5) can be written as a linear system of n^2 first-order ordinary differential equations. Meanwhile, the system (E.9) can be written as a linear system of $2n^2$ first-order ordinary differential equations. One may thus apply the theorems* of existence and uniqueness to those linear systems to guarantee existence of solutions.

*The details are presented in Appendix F.

APPENDIX F

THE THEORY OF EXISTENCE AND UNIQUENESS OF LINEAR SYSTEMS

Definition. Any system of m first-order ordinary differential equations of the form

$$\dot{X} = F(t, X), \quad X \in R^m, \tag{F.1}$$

is called a normal system of first order ordinary differential equations.

Definition. A normal system of p^{th} order ordinary differential equations for the unknown functions $\xi_1(t), \xi_1(t), ..., \xi_n(t)$ is any system of the form

$$\frac{d^p\xi_k}{dt^p} = F_k(\xi_1, \frac{d\xi_1}{dt}, ..., \frac{d^{p-1}\xi_1}{dt^{p-1}}; \xi_2, \frac{d\xi_2}{dt}, ..., \frac{d^{p-1}\xi_2}{dt^{p-1}}; ...; \xi_n, \frac{d\xi_n}{dt}, ..., \frac{d^{p-1}\xi_n}{dt^{p-1}}; t) \tag{F.2}$$

where $k = 1, 2, ..., n$. In other words, the highest derivatives of each function ξ_k, $(k = 1, 2, ..., n)$ can be found only in the left side.

Theorem. A normal system (F.2) of ordinary differential equations is equivalent to a normal system of the type (F.1).

Proof.

We introduce new unknown functions:

$$X_1 = \xi_1, \; X_2 = \frac{d\xi_1}{dt}, \; X_3 = \frac{d^2\xi_1}{dt^2}, ..., \; X_p = \frac{d^{p-1}\xi_1}{dt^{p-1}},$$

$$X_{p+1} = \xi_2, \; X_{p+2} = \frac{d\xi_2}{dt}, \; X_{p+3} = \frac{d^2\xi_2}{dt^2}, ..., \; X_{2p} = \frac{d^{p-1}\xi_2}{dt^{p-1}},$$

.... ,

$$X_{\tilde{p}+1} = \xi_n, \; X_{\tilde{p}+2} = \frac{d\xi_n}{dt}, \; X_{\tilde{p}+3} = \frac{d^2\xi_n}{dt^2}, ..., \; X_{np} = \frac{d^{p-1}\xi_n}{dt^{p-1}},$$

where $\tilde{p} = (n-1)p$. Hence the normal systems of the type (F.2) is transformed to a normal system of first order ordinary differential equations:

$$\frac{dX_i}{dt} = H_i(t, X_1, X_2, X_3, ..., X_m), \ (i = 1, 2, ..., m), \tag{F.3}$$

where $m = np$. Note that if $i = jp, (j = 1, 2, ..., n)$, then $H_i = F_i$.

Thus, the study of a normal system of first-order ordinary differential equations provides insight into any kind normal systems (F.2).

Therefore, in the scope of this thesis, it is enough to study the existence and uniqueness theorems for normal systems of first-order ordinary differential equations as follows.

Theorem. *(Local theorem)*

Consider a Cauchy problem consisting of a normal system of n first-order ordinary differential equations,

$$\dot{X} = F(t, X), \ \ X(t_0) = X_0,$$

which satisfy the properties:

(a) $F(t, X) \in C(D)$, where D is an open set in R^{n+1},

(b) for the cylinder $G = \{ \ (t, X) \in D \mid |t - t_0| \leq a, \ \| \ X - X_0 \ \| \leq b \ \}$, there are constants $m = \max_{(t,X)\in G} \| \ F(t, X) \ \|$ and $h = min(a, \frac{b}{m})$,

(c) $F(t, X)$ satisfies a Lipschitz condition in G.

Then there exists one and only one solution of the Cauchy problem in the interval $J = [t_0 - h, t_0 + h]$.

Theorem. *(Global theorem)*

Let $F(t, X) \in C(D)$ satisfy a Lipschitz condition in D with the Lipschitz constant $L(t)$, which can depend on t: there is a function $L(t) \in C(J)$, $J = (a, b)$ that

$$\| \ F(t, X_1) - F(t, X_2) \ \| \leq L(t) \ \| \ X_1 - X_2 \ \|,$$

where $D = \{ (t, X) \in R^{n+1} | \ t \in J \}$. Then there exists one and only one solution of the Cauchy problem:

$$\dot{X} = F(t, X), \ \ X(t_0) = X_0, \ t_0 \in J,$$

on the interval J, $\forall (X_0, t_0) \in D$.

F.1 Systems of linear equations

Let us consider a linear system of n first-order ordinary differential equations. In matrix form it can be written as

$$\dot{X}(t) = A(t)X(t) + B(t),$$

where $X(t)$ and $B(t)$ are column vectors of the length n, $A(t)$ is $n \times n$ square matrix.

Theorem. If $A(t)$ and $B(t) \in C(J)$, then there exists one and only one solution of the Cauchy problem:

$$\dot{X}(t) = A(t)X(t) + B(t),$$

$$X(t_0) = X_0, \ t_0 \in J,$$

defined on the whole interval J.

Proof.

For proving the theorem one needs to check conditions of the *global theorem.* Here $F(t, X) = A(t)X + B(t)$. Thus, $F(t, X) \in C(D)$, where $D = \{ (t, X) \mid t \in J, \ X \in R^n \}$. For checking the Lipschitz condition in D, one has to study

$$F(t, X_1) - F(t, X_2) = A(t)(X_1 - X_2).$$

Therefore F(t,X) satisfies a Lipschitz condition in D with the Lipschitz constant $L(t) = \| A(t) \|_2$.

APPENDIX G

DEFINITION OF λ_n, μ_n AND ω_n

$$
\begin{aligned}
\lambda_1 &= 2a_{28y_1} - 2a_{27y_2} + a_{17}a_{25} - 2a_{18}a_{24} - a_{25}a_{28} + 2a_{26}a_{27},\\
\lambda_2 &= 2a_{18y_1} - 2a_{17y_2} - 2a_{14}a_{18} + a_{15}a_{17} - a_{15}a_{28} + 2a_{16}a_{27},\\
\lambda_3 &= 4a_{24x} - 2a_{27y_1} - 2a_{14}a_{27} + 2a_{17}a_{24} - 2a_{24}a_{28} + a_{25}a_{27},\\
\lambda_4 &= 2a_{28y_1} - 2a_{25x} + a_{15}a_{27} - 2a_{18}a_{24},\\
\lambda_5 &= 2a_{28y_2} - 4a_{26x} + 2a_{16}a_{27} - a_{18}a_{25},\\
\lambda_6 &= 2a_{25y_1} - 4a_{24y_2} + 2a_{14}a_{25} - 2a_{15}a_{24} + 4a_{24}a_{26} - {a_{25}}^2,\\
\lambda_7 &= 4a_{16x} - 2a_{18y_2} + a_{15}a_{18} - 2a_{16}a_{17} + 2a_{16}a_{28} - 2a_{18}a_{26},\\
\lambda_8 &= 4a_{16y_1} - 2a_{15y_2} - 4a_{14}a_{16} + {a_{15}}^2 - 2a_{15}a_{26} + 2a_{16}a_{25},\\
\lambda_9 &= 2a_{25y_2} - 4a_{26y_1} - a_{15}a_{25} + 4a_{16}a_{24},\\
\lambda_{10} &= 2a_{18x}a_{15} - 8a_{16y_1}a_{19} - 8a_{16y_2}a_{29} - 4a_{17x}a_{16} - 4a_{18xy_2}\\
&\quad -4a_{18x}a_{26} + 2a_{18y_1}a_{18} + 2a_{18y_2}a_{17} + 2a_{18y_2}a_{28} + 8a_{19y_1}a_{16}\\
&\quad +8a_{19y_2y_2} - 8a_{19y_2}a_{15} + 8a_{19y_2}a_{26} + 4a_{26x}a_{18} + 4a_{28x}a_{16}\\
&\quad -16a_{29y_2}a_{16} - 2a_{14}{a_{18}}^2 - 2a_{15}a_{18}a_{28} + 2a_{16}{a_{17}}^2 - 2a_{16}{a_{28}}^2\\
&\quad +2a_{17}a_{18}a_{26} + {a_{18}}^2a_{25} + 2a_{18}a_{26}a_{28},\\
\lambda_{11} &= 2a_{18y_2}a_{27} - 4a_{19y_2}a_{25} - 4a_{26xx} + 4a_{26x}a_{28} - 4a_{26y_1}a_{19} - 4a_{26y_2}a_{29}\\
&\quad +4a_{29y_1}a_{16} + 4a_{29y_2y_2} - 4a_{29y_2}a_{26} - a_{15}a_{18}a_{27} + 2a_{16}a_{17}a_{27}\\
&\quad -2a_{16}a_{27}a_{28} + 2a_{18}a_{26}a_{27},\\
\lambda_{12} &= 2a_{27x} - 4a_{29y_1} - a_{17}a_{27} + 4a_{19}a_{24} + 2a_{25}a_{29} - a_{27}a_{28},
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{14} &= \lambda_{12}(a_{28} - a_{17}) - a_{27}\lambda_{16} - 2\lambda_{12x}, \\
\lambda_{15} &= 2a_{18x} - 4a_{19y_2} + 2a_{15}a_{19} + 4a_{16}a_{29} - a_{17}a_{18} - a_{18}a_{28}, \\
\lambda_{16} &= 4a_{19y_1} - 2a_{17x} + 2a_{28x} - 4a_{29y_2} - 4a_{14}a_{19} - 2a_{15}a_{29} + a_{17}{}^2 \\
&\quad +2a_{19}a_{25} + 4a_{26}a_{29} - a_{28}{}^2, \\
\lambda_{17} &= 2a_{19x}a_{25} - 2a_{19y_2}a_{27} + 2a_{25x}a_{19} + 4a_{26x}a_{29} + 2a_{28xx} - 2a_{28x}a_{28} \\
&\quad -4a_{29xy_2} + 4a_{29x}a_{26} - 2a_{29y_1}a_{18} + a_{15}a_{19}a_{27} + 2a_{16}a_{27}a_{29} - a_{17}a_{18}a_{27} \\
&\quad +2a_{18}a_{19}a_{24} + a_{18}a_{25}a_{29} - a_{18}a_{27}a_{28} - a_{27}\lambda_{15}, \\
\lambda_{18} &= 2a_{19y_1}\lambda_{14} + 4a_{28x}\lambda_{14} - 10a_{29y_2}\lambda_1 4 + 2\lambda_{14xx} + 2\lambda_{14x}a_{17} - 2\lambda_{14x}a_{28} \\
&\quad -2a_{14}a_{19}\lambda_{14} - a_{15}a_{29}\lambda_{14} + a_{17}{}^2\lambda_{14} - a_{17}a_{28}\lambda_{14} - 3a_{18}a_{27}\lambda_{14} \\
&\quad +5a_{19}a_{25}\lambda_{14} + 10a_{26}a_{29}\lambda_{14} - 2a_{28}{}^2\lambda_{14}, \\
\lambda_{19} &= 8a_{29y_1}\lambda_{14}\lambda_{16} + 8\lambda_{14x}a_{27}\lambda_{16} + 18\lambda_{14x}\lambda_{14} + 8a_{17}a_{27}\lambda_{14}\lambda_{16} + 9a_{17}\lambda_{14}{}^2 \\
&\quad -8a_{19}a_{24}\lambda_{14}\lambda_{16} - 4a_{25}a_{29}\lambda_{14}\lambda_{16} + 4a_{27}{}^2\lambda_{14}\lambda_{15} - 4a_{27}a_{28}\lambda_{14}\lambda_{16} \\
&\quad -9a_{28}\lambda_{14}{}^2, \\
\lambda_{20} &= 2a_{28x} - 4a_{29y_2} - a_{18}a_{27} + 2a_{19}a_{25} + 4a_{26}a_{29} - a_{28}{}^2, \\
\lambda_{21} &= 2a_{26x}a_{29} + a_{28xx} - a_{28x}a_{28} - 2a_{29xy_2} + 2a_{29x}a_{26}, \\
\lambda_{22} &= 4a_{29y_2}\lambda_{16x} - 2a_{28x}\lambda_{16x} - \lambda_{16xxx} - 4\lambda_{16x}a_{26}a_{29} + \lambda_{16x}a_{28}{}^2, \\
\lambda_{23} &= 2a_{28x} - 4a_{29y_2} + 4a_{26}a_{29} - a_{28}{}^2, \\
\lambda_{24} &= 32(a_{19xy_1} - a_{19x}a_{14}) - 16(a_{17y_1}a_{19} + a_{18y_1}a_{29} + a_{29x}a_{15} - a_{14}a_{18}a_{29}) \\
&\quad +56(a_{19y_1}a_{17} - a_{14}a_{17}a_{19}) - 24(a_{19y_1}a_{28} + a_{14}a_{19}a_{28}) + 21a_{28}{}^3 \\
&\quad +160(a_{26x}a_{29} - a_{29xy_2} + a_{29x}a_{26}) + 120(a_{17}a_{26}a_{29} - a_{29y_2}a_{17}) \\
&\quad +64a_{28xx} + 48a_{28x}a_{17} - 112a_{28x}a_{28} + 88(a_{29y_2}a_{28} - a_{26}a_{28}a_{29}) \\
&\quad -36a_{15}a_{17}a_{29} + 20a_{15}a_{28}a_{29} + 15a_{17}{}^3 - 9a_{17}{}^2a_{28} - 27a_{17}a_{28}{}^2, \\
\lambda_{25} &= 12(\lambda_{15xx}a_{28} - \lambda_{15xx}a_{17} + \lambda_{15x}a_{15}a_{29}) + 24(\lambda_{15x}a_{14}a_{19} - a_{19y_1}\lambda_{15x}) \\
&\quad -16\lambda_{15xxx} - 32a_{28x}\lambda_{15x} - 9\lambda_{15x}a_{17}{}^2 + 6\lambda_{15x}a_{17}a_{28} + 19\lambda_{15x}a_{28}{}^2 \\
&\quad +88(a_{29y_2}\lambda_{15x} - \lambda_{15x}a_{26}a_{29}),
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{26} &= ({\lambda_{15}}^2(16a_{19y_1} + 32a_{28x} - 80a_{29y_2} - 16a_{14}a_{19} - 8a_{15}a_{29} + 5{a_{17}}^2 \\
&\quad -2a_{17}a_{28} + 80a_{26}a_{29} - 19{a_{28}}^2) + 4\lambda_{15}(\lambda_{15x}a_{28} - 4\lambda_{15xx} - \lambda_{15x}a_{17}) \\
&\quad +20{\lambda_{15x}}^2)/(32{\lambda_{15}}^2), \\
\lambda_{27} &= 8a_{14y_1}a_{19} + 4a_{15y_1}a_{29} + 4a_{17xy_1} - 4a_{17y_1}a_{17} + 4a_{18x}a_{24} - 2a_{18y_1}a_{27} \\
&\quad -8a_{19y_1y_1} + 8a_{19y_1}a_{14} - 8a_{19y_2}a_{24} - 4a_{24x}a_{18} - 2a_{27x}a_{15} + 8a_{29y_1}a_{15} \\
&\quad +2a_{14}a_{18}a_{27} + a_{15}a_{17}a_{27} - 2a_{15}a_{25}a_{29} + a_{15}a_{27}a_{28} + 8a_{16}a_{24}a_{29} \\
&\quad -4a_{17}a_{18}a_{24} - a_{18}a_{25}a_{27}, \\
\lambda_{28} &= 2a_{17y_1}a_{27} - 8a_{19y_1}a_{24} - 4a_{24xx} + 4a_{24x}a_{28} - 4a_{24y_1}a_{19} \\
&\quad -4a_{24y_2}a_{29} + 4a_{29y_1y_1} + 4a_{29y_1}a_{14} - 4a_{29y_1}a_{25} + 4a_{29y_2}a_{24} \\
&\quad -a_{15}{a_{27}}^2 + 2a_{18}a_{24}a_{27}, \\
\lambda_{29} &= 2a_{18y_1}a_{27} - 2a_{19y_1}a_{25} - 4a_{19y_2}a_{24} - 4a_{24y_2}a_{19} - 2a_{25xx} + 2a_{25x}a_{28} \\
&\quad -4a_{26y_1}a_{29} + 4a_{29y_1y_2} + 2a_{29y_1}a_{15} - 4a_{29y_1}a_{26} - 2a_{14}a_{18}a_{27} \\
&\quad +2a_{14}a_{19}a_{25} + a_{15}a_{17}a_{27} - 2a_{15}a_{19}a_{24} - a_{15}a_{25}a_{29} - a_{15}a_{27}a_{28} \\
&\quad +4a_{16}a_{24}a_{29} + a_{18}a_{25}a_{27} + 4a_{19}a_{24}a_{26} - a_{19}{a_{25}}^2, \\
\lambda_{30} &= (8{\lambda_{12}}^2(-\lambda_{16} + 2\lambda_{20}) + 4\lambda_{12}(2\lambda_{14x} + a_{17}\lambda_{14} - a_{28}\lambda_{14}) + \lambda_{14}(4a_{27}\lambda_{16} \\
&\quad +5\lambda_{14}))/(32{\lambda_{12}}^2), \\
\lambda_{31} &= (4{\lambda_{16}}^2\lambda_{20} - 2{\lambda_{16}}^3 - 4\lambda_{16}\lambda_{16xx} + 5{\lambda_{16x}}^2)/(8{\lambda_{16}}^2), \\
\lambda_{32} &= ({\lambda_{15}}^2(16a_{19y_1} + 32a_{28x} - 80a_{29y_2} - 16a_{14}a_{19} - 8a_{15}a_{29} + 5{a_{17}}^2 \\
&\quad -2a_{17}a_{28} + 80a_{26}a_{29} - 19{a_{28}}^2) + 4\lambda_{15}(\lambda_{15x}a_{28} - 4\lambda_{15xx} - \lambda_{15x}a_{17}) \\
&\quad +20{\lambda_{15x}}^2)/(32{\lambda_{15}}^2), \\
\lambda_{33} &= 2a_{28x} - 4a_{29y_2} - a_{18}a_{27} + 2a_{19}a_{25} + 4a_{26}a_{29} - {a_{28}}^2, \\
\mu_1 &= 2a_{28x} - 4a_{29y_2} + 4a_{26}a_{29} - {a_{28}}^2, \\
\mu_2 &= 4a_{19y_1} - 2a_{17x} - 4a_{14}a_{19} + {a_{17}}^2, \\
a_{29y_1}\mu_3 &= (2a_{29xy_1} + a_{29y_1}a_{17} - a_{29y_1}a_{28})/4,
\end{aligned}
$$

$$
\begin{aligned}
a_{19y_2}\mu_4 &= (2a_{19xy_2} - a_{19y_2}a_{17} + a_{19y_2}a_{28})/4, \\
\mu_5 &= -(\mu_1 + \mu_2), \\
\omega_1 &= 2a_{12y_1} - 4a_{11y_2} - 2a_{11}a_{15} + 2a_{12}a_{14} - a_{12}a_{25} + 4a_{13}a_{24}, \\
\omega_2 &= 2a_{12y_2} - 4a_{13y_1} + 4a_{11}a_{16} - a_{12}a_{15} + 2a_{12}a_{26} - 2a_{13}a_{25}, \\
\omega_3 &= a_{15y_1} - 2a_{14y_2} - a_{25y_2} + 2a_{26y_1} - 4a_{11}a_{18} + 2a_{12}a_{17} \\
&\quad -2a_{12}a_{28} + 4a_{13}a_{27}, \\
\omega_4 &= 2a_{15y_2} - 4a_{13x} - 4a_{16y_1} - 2a_{13}a_{17} - 2a_{13}a_{28} + 4a_{14}a_{16} \\
&\quad -{a_{15}}^2 + 2a_{15}a_{26} - 2a_{16}a_{25}, \\
\omega_5 &= 4a_{12x} + 4a_{25y_2} - 8a_{26y_1} + 8a_{11}a_{18} - 2a_{12}a_{17} + 6a_{12}a_{28} - 8a_{13}a_{27} \\
&\quad -2a_{15}a_{25} + 8a_{16}a_{24} + \omega_3, \\
\omega_6 &= 4a_{11x} + 4a_{24y_2} - 2a_{25y_1} + 2a_{11}a_{17} + 2a_{11}a_{28} - 2a_{14}a_{25} + 2a_{15}a_{24} \\
&\quad -4a_{24}a_{26} + {a_{25}}^2, \\
\omega_7 &= 2a_{18y_2} - 4a_{16x} - 4a_{13}a_{19} - a_{15}a_{18} + 2a_{16}a_{17} - 2a_{16}a_{28} + 2a_{18}a_{26}, \\
\omega_8 &= 5a_{15x} - a_{17y_2} - 4a_{18y_1} - 6a_{26x} + 3a_{28y_2} + 4a_{12}a_{19} - 8a_{13}a_{29} \\
&\quad +4a_{14}a_{18} - 2a_{15}a_{17} + 2a_{15}a_{28} + 4a_{16}a_{27} - 4a_{18}a_{25}, \\
\omega_9 &= a_{17y_1} - 2a_{14x} + 3a_{25x} - 3a_{28y_1} + 4a_{12}a_{29} - 2a_{15}a_{27} + 4a_{18}a_{24}, \\
\omega_{10} &= 2a_{27y_1} - 4a_{24x} - 4a_{11}a_{29} + 2a_{14}a_{27} - 2a_{17}a_{24} + 2a_{24}a_{28} - a_{25}a_{27}, \\
\omega_{11} &= 4a_{25x} + 4a_{27y_2} - 8a_{28y_1} + 8a_{11}a_{19} + 8a_{12}a_{29} - 2a_{15}a_{27} - 2a_{17}a_{25} \\
&\quad +8a_{18}a_{24} + 2a_{25}a_{28} - 4a_{26}a_{27}, \\
\omega_{12} &= 12a_{17y_2} - 12a_{18y_1} - 8a_{26x} + 4a_{28y_2} - 8a_{12}a_{19} - 24a_{13}a_{29} + 12a_{14}a_{18} \\
&\quad -6a_{15}a_{17} + 6a_{15}a_{28} - 8a_{16}a_{27} - 2a_{18}a_{25}, \\
\omega_{13} &= 2\lambda_{15},\ \omega_{14} = 4\lambda_{16},\ \omega_{15} = -2\lambda_{12}.
\end{aligned}
$$

APPENDIX H

THE COEFFICIENTS a_{ij} OF SYSTEM (4.6), EXPRESSED THROUGH THE FUNCTIONS φ, ψ_1, ψ_2, AND k_{ij}

$$
\begin{aligned}
a_{11} = {} & (\varphi_{y_1y_1}\psi_{1_{y_1}}\psi_{2_{y_2}} - \varphi_{y_1y_1}\psi_{1_{y_2}}\psi_{2_{y_1}} + {\varphi_{y_1}}^3\psi_{1_{y_2}}k_{22}\psi_2 \\
& + {\varphi_{y_1}}^3\psi_{1_{y_2}}k_{21}\psi_1 - {\varphi_{y_1}}^3\psi_{2_{y_2}}k_{11}\psi_1 - {\varphi_{y_1}}^3\psi_{2_{y_2}}k_{12}\psi_2 \\
& - {\varphi_{y_1}}^2\varphi_{y_2}\psi_{1_{y_1}}k_{22}\psi_2 - {\varphi_{y_1}}^2\varphi_{y_2}\psi_{1_{y_1}}k_{21}\psi_1 + {\varphi_{y_1}}^2\varphi_{y_2}\psi_{2_{y_1}}k_{11}\psi_1 \\
& + {\varphi_{y_1}}^2\varphi_{y_2}\psi_{2_{y_1}}k_{12}\psi_2 - \varphi_{y_1}\psi_{1_{y_1y_1}}\psi_{2_{y_2}} + \varphi_{y_1}\psi_{1_{y_2}}\psi_{2_{y_1y_1}} \\
& + \varphi_{y_2}\psi_{1_{y_1y_1}}\psi_{2_{y_1}} - \varphi_{y_2}\psi_{1_{y_1}}\psi_{2_{y_1y_1}})/\Delta,
\end{aligned}
\tag{H.1}
$$

$$
\begin{aligned}
a_{12} = {} & 2(\varphi_{y_1y_2}\psi_{1_{y_1}}\psi_{2_{y_2}} - \varphi_{y_1y_2}\psi_{1_{y_2}}\psi_{2_{y_1}} + {\varphi_{y_1}}^2\varphi_{y_2}\psi_{1_{y_2}}k_{22}\psi_2 \\
& + {\varphi_{y_1}}^2\varphi_{y_2}\psi_{1_{y_2}}k_{21}\psi_1 - {\varphi_{y_1}}^2\varphi_{y_2}\psi_{2_{y_2}}k_{11}\psi_1 - {\varphi_{y_1}}^2\varphi_{y_2}\psi_{2_{y_2}}k_{12}\psi_2 \\
& - \varphi_{y_1}{\varphi_{y_2}}^2\psi_{1_{y_1}}k_{22}\psi_2 - \varphi_{y_1}{\varphi_{y_2}}^2\psi_{1_{y_1}}k_{21}\psi_1 + \varphi_{y_1}{\varphi_{y_2}}^2\psi_{2_{y_1}}k_{11}\psi_1 \\
& + \varphi_{y_1}{\varphi_{y_2}}^2\psi_{2_{y_1}}k_{12}\psi_2 - \varphi_{y_1}\psi_{1_{y_1y_2}}\psi_{2_{y_2}} + \varphi_{y_1}\psi_{1_{y_2}}\psi_{2_{y_1y_2}} \\
& + \varphi_{y_2}\psi_{1_{y_1y_2}}\psi_{2_{y_1}} - \varphi_{y_2}\psi_{1_{y_1}}\psi_{2_{y_1y_2}})/\Delta,
\end{aligned}
\tag{H.2}
$$

$$
\begin{aligned}
a_{13} = {} & (\varphi_{y_1}{\varphi_{y_2}}^2\psi_{1_{y_2}}k_{22}\psi_2 + \varphi_{y_1}{\varphi_{y_2}}^2\psi_{1_{y_2}}k_{21}\psi_1 - \varphi_{y_1}{\varphi_{y_2}}^2\psi_{2_{y_2}}k_{11}\psi_1 \\
& - \varphi_{y_1}{\varphi_{y_2}}^2\psi_{2_{y_2}}k_{12}\psi_2 - \varphi_{y_1}\psi_{1_{y_2y_2}}\psi_{2_{y_2}} + \varphi_{y_1}\psi_{1_{y_2}}\psi_{2_{y_2y_2}} \\
& + \varphi_{y_2y_2}\psi_{1_{y_1}}\psi_{2_{y_2}} - \varphi_{y_2y_2}\psi_{1_{y_2}}\psi_{2_{y_1}} - {\varphi_{y_2}}^3\psi_{1_{y_1}}k_{22}\psi_2 \\
& - {\varphi_{y_2}}^3\psi_{1_{y_1}}k_{21}\psi_1 + {\varphi_{y_2}}^3\psi_{2_{y_1}}k_{11}\psi_1 + {\varphi_{y_2}}^3\psi_{2_{y_1}}k_{12}\psi_2 \\
& - \varphi_{y_2}\psi_{1_{y_1}}\psi_{2_{y_2y_2}} + \varphi_{y_2}\psi_{1_{y_2y_2}}\psi_{2_{y_1}})/\Delta,
\end{aligned}
\tag{H.3}
$$

$$a_{14} = (2\varphi_{xy_1}\psi_{1_{y_1}}\psi_{2_{y_2}} - 2\varphi_{xy_1}\psi_{1_{y_2}}\psi_{2_{y_1}} + 3\varphi_x{\varphi_{y_1}}^2\psi_{1_{y_2}}k_{22}\psi_2 \quad \text{(H.4)}$$
$$+3\varphi_x{\varphi_{y_1}}^2\psi_{1_{y_2}}k_{21}\psi_1 - 3\varphi_x{\varphi_{y_1}}^2\psi_{2_{y_2}}k_{11}\psi_1 - 3\varphi_x{\varphi_{y_1}}^2\psi_{2_{y_2}}k_{12}\psi_2$$
$$-2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{1_{y_1}}k_{22}\psi_2 - 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{1_{y_1}}k_{21}\psi_1$$
$$+2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{2_{y_1}}k_{11}\psi_1 + 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{2_{y_1}}k_{12}\psi_2 - \varphi_x\psi_{1_{y_1y_1}}\psi_{2_{y_2}}$$
$$+\varphi_x\psi_{1_{y_2}}\psi_{2_{y_1y_1}} + \varphi_{y_1y_1}\psi_{1_x}\psi_{2_{y_2}} - \varphi_{y_1y_1}\psi_{1_{y_2}}\psi_{2_x}$$
$$-{\varphi_{y_1}}^2\varphi_{y_2}\psi_{1_x}k_{22}\psi_2 - {\varphi_{y_1}}^2\varphi_{y_2}\psi_{1_x}k_{21}\psi_1 + {\varphi_{y_1}}^2\varphi_{y_2}\psi_{2_x}k_{11}\psi_1$$
$$+{\varphi_{y_1}}^2\varphi_{y_2}\psi_{2_x}k_{12}\psi_2 - 2\varphi_{y_1}\psi_{1_{xy_1}}\psi_{2_{y_2}} + 2\varphi_{y_1}\psi_{1_{y_2}}\psi_{2_{xy_1}}$$
$$+2\varphi_{y_2}\psi_{1_{xy_1}}\psi_{2_{y_1}} - \varphi_{y_2}\psi_{1_x}\psi_{2_{y_1y_1}} + \varphi_{y_2}\psi_{1_{y_1y_1}}\psi_{2_x}$$
$$-2\varphi_{y_2}\psi_{1_{y_1}}\psi_{2_{xy_1}})/\Delta,$$

$$a_{15} = 2(\varphi_{xy_2}\psi_{1_{y_1}}\psi_{2_{y_2}} - \varphi_{xy_2}\psi_{1_{y_2}}\psi_{2_{y_1}} + 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{1_{y_2}}k_{22}\psi_2 \quad \text{(H.5)}$$
$$+2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{1_{y_2}}k_{21}\psi_1 - 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{2_{y_2}}k_{11}\psi_1 - 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{2_{y_2}}k_{12}\psi_2$$
$$-\varphi_x{\varphi_{y_2}}^2\psi_{1_{y_1}}k_{22}\psi_2 - \varphi_x{\varphi_{y_2}}^2\psi_{1_{y_1}}k_{21}\psi_1 + \varphi_x{\varphi_{y_2}}^2\psi_{2_{y_1}}k_{11}\psi_1$$
$$+\varphi_x{\varphi_{y_2}}^2\psi_{2_{y_1}}k_{12}\psi_2 - \varphi_x\psi_{1_{y_1y_2}}\psi_{2_{y_2}} + \varphi_x\psi_{1_{y_2}}\psi_{2_{y_1y_2}} + \varphi_{y_1y_2}\psi_{1_x}\psi_{2_{y_2}}$$
$$-\varphi_{y_1y_2}\psi_{1_{y_2}}\psi_{2_x} - \varphi_{y_1}{\varphi_{y_2}}^2\psi_{1_x}k_{22}\psi_2 - \varphi_{y_1}{\varphi_{y_2}}^2\psi_{1_x}k_{21}\psi_1$$
$$+\varphi_{y_1}{\varphi_{y_2}}^2\psi_{2_x}k_{11}\psi_1 + \varphi_{y_1}{\varphi_{y_2}}^2\psi_{2_x}k_{12}\psi_2 - \varphi_{y_1}\psi_{1_{xy_2}}\psi_{2_{y_2}}$$
$$+\varphi_{y_1}\psi_{1_{y_2}}\psi_{2_{xy_2}} + \varphi_{y_2}\psi_{1_{xy_2}}\psi_{2_{y_1}} - \varphi_{y_2}\psi_{1_x}\psi_{2_{y_1y_2}} + \varphi_{y_2}\psi_{1_{y_1y_2}}\psi_{2_x}$$
$$-\varphi_{y_2}\psi_{1_{y_1}}\psi_{2_{xy_2}})/\Delta,$$

$$a_{16} = (\varphi_x{\varphi_{y_2}}^2\psi_{1_{y_2}}k_{22}\psi_2 + \varphi_x{\varphi_{y_2}}^2\psi_{1_{y_2}}k_{21}\psi_1 - \varphi_x{\varphi_{y_2}}^2\psi_{2_{y_2}}k_{11}\psi_1 \quad \text{(H.6)}$$
$$-\varphi_x{\varphi_{y_2}}^2\psi_{2_{y_2}}k_{12}\psi_2 - \varphi_x\psi_{1_{y_2y_2}}\psi_{2_{y_2}} + \varphi_x\psi_{1_{y_2}}\psi_{2_{y_2y_2}} + \varphi_{y_2y_2}\psi_{1_x}\psi_{2_{y_2}}$$
$$-\varphi_{y_2y_2}\psi_{1_{y_2}}\psi_{2_x} - {\varphi_{y_2}}^3\psi_{1_x}k_{22}\psi_2 - {\varphi_{y_2}}^3\psi_{1_x}k_{21}\psi_1 + {\varphi_{y_2}}^3\psi_{2_x}k_{11}\psi_1$$
$$+{\varphi_{y_2}}^3\psi_{2_x}k_{12}\psi_2 - \varphi_{y_2}\psi_{1_x}\psi_{2_{y_2y_2}} + \varphi_{y_2}\psi_{1_{y_2y_2}}\psi_{2_x})/\Delta,$$

$$a_{17} = (2\varphi_{xy_1}\psi_{1_x}\psi_{2_{y_2}} - 2\varphi_{xy_1}\psi_{1_{y_2}}\psi_{2_x} + \varphi_{xx}\psi_{1_{y_1}}\psi_{2_{y_2}} - \varphi_{xx}\psi_{1_{y_2}}\psi_{2_{y_1}} \quad \text{(H.7)}$$
$$+3{\varphi_x}^2\varphi_{y_1}\psi_{1_{y_2}}k_{22}\psi_2 + 3{\varphi_x}^2\varphi_{y_1}\psi_{1_{y_2}}k_{21}\psi_1 - 3{\varphi_x}^2\varphi_{y_1}\psi_{2_{y_2}}k_{11}\psi_1$$
$$-3{\varphi_x}^2\varphi_{y_1}\psi_{2_{y_2}}k_{12}\psi_2 - {\varphi_x}^2\varphi_{y_2}\psi_{1_{y_1}}k_{22}\psi_2 - {\varphi_x}^2\varphi_{y_2}\psi_{1_{y_1}}k_{21}\psi_1$$
$$+{\varphi_x}^2\varphi_{y_2}\psi_{2_{y_1}}k_{11}\psi_1 + {\varphi_x}^2\varphi_{y_2}\psi_{2_{y_1}}k_{12}\psi_2 - 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{1_x}k_{22}\psi_2$$

$$-2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{1_x}k_{21}\psi_1 + 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{2_x}k_{11}\psi_1 + 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{2_x}k_{12}\psi_2$$
$$-2\varphi_x\psi_{1_{xy_1}}\psi_{2_{y_2}} + 2\varphi_x\psi_{1_{y_2}}\psi_{2_{xy_1}} - \varphi_{y_1}\psi_{1_{xx}}\psi_{2_{y_2}} + \varphi_{y_1}\psi_{1_{y_2}}\psi_{2_{xx}}$$
$$+2\varphi_{y_2}\psi_{1_{xy_1}}\psi_{2_x} + \varphi_{y_2}\psi_{1_{xx}}\psi_{2_{y_1}} - 2\varphi_{y_2}\psi_{1_x}\psi_{2_{xy_1}} - \varphi_{y_2}\psi_{1_{y_1}}\psi_{2_{xx}})/\Delta,$$

$$a_{18} = 2(\varphi_{xy_2}\psi_{1_x}\psi_{2_{y_2}} - \varphi_{xy_2}\psi_{1_{y_2}}\psi_{2_x} + \varphi_x{}^2\varphi_{y_2}\psi_{1_{y_2}}k_{22}\psi_2 \tag{H.8}$$
$$+\varphi_x{}^2\varphi_{y_2}\psi_{1_{y_2}}k_{21}\psi_1 - \varphi_x{}^2\varphi_{y_2}\psi_{2_{y_2}}k_{11}\psi_1 - \varphi_x{}^2\varphi_{y_2}\psi_{2_{y_2}}k_{12}\psi_2$$
$$-\varphi_x\varphi_{y_2}{}^2\psi_{1_x}k_{22}\psi_2 - \varphi_x\varphi_{y_2}{}^2\psi_{1_x}k_{21}\psi_1 + \varphi_x\varphi_{y_2}{}^2\psi_{2_x}k_{11}\psi_1$$
$$+\varphi_x\varphi_{y_2}{}^2\psi_{2_x}k_{12}\psi_2 - \varphi_x\psi_{1_{xy_2}}\psi_{2_{y_2}} + \varphi_x\psi_{1_{y_2}}\psi_{2_{xy_2}} + \varphi_{y_2}\psi_{1_{xy_2}}\psi_{2_x}$$
$$-\varphi_{y_2}\psi_{1_x}\psi_{2_{xy_2}})/\Delta,$$

$$a_{19} = (\varphi_{xx}\psi_{1_x}\psi_{2_{y_2}} - \varphi_{xx}\psi_{1_{y_2}}\psi_{2_x} + \varphi_x{}^3\psi_{1_{y_2}}k_{22}\psi_2 + \varphi_x{}^3\psi_{1_{y_2}}k_{21}\psi_1 \tag{H.9}$$
$$-\varphi_x{}^3\psi_{2_{y_2}}k_{11}\psi_1 - \varphi_x{}^3\psi_{2_{y_2}}k_{12}\psi_2 - \varphi_x{}^2\varphi_{y_2}\psi_{1_x}k_{22}\psi_2 - \varphi_x{}^2\varphi_{y_2}\psi_{1_x}k_{21}\psi_1$$
$$+\varphi_x{}^2\varphi_{y_2}\psi_{2_x}k_{11}\psi_1 + \varphi_x{}^2\varphi_{y_2}\psi_{2_x}k_{12}\psi_2 - \varphi_x\psi_{1_{xx}}\psi_{2_{y_2}} + \varphi_x\psi_{1_{y_2}}\psi_{2_{xx}}$$
$$+\varphi_{y_2}\psi_{1_{xx}}\psi_{2_x} - \varphi_{y_2}\psi_{1_x}\psi_{2_{xx}})/\Delta,$$

$$a_{24} = (\varphi_x\varphi_{y_1}{}^2\psi_{2_{y_1}}k_{11}\psi_1 - \varphi_x\varphi_{y_1}{}^2\psi_{1_{y_1}}k_{22}\psi_2 - \varphi_x\varphi_{y_1}{}^2\psi_{1_{y_1}}k_{21}\psi_1 \tag{H.10}$$
$$+\varphi_x\varphi_{y_1}{}^2\psi_{2_{y_1}}k_{12}\psi_2 + \varphi_x\psi_{1_{y_1y_1}}\psi_{2_{y_1}} - \varphi_x\psi_{1_{y_1}}\psi_{2_{y_1y_1}}$$
$$-\varphi_{y_1y_1}\psi_{1_x}\psi_{2_{y_1}} + \varphi_{y_1y_1}\psi_{1_{y_1}}\psi_{2_x} + \varphi_{y_1}{}^3\psi_{1_x}k_{22}\psi_2 + \varphi_{y_1}{}^3\psi_{1_x}k_{21}\psi_1$$
$$-\varphi_{y_1}{}^3\psi_{2_x}k_{11}\psi_1 - \varphi_{y_1}{}^3\psi_{2_x}k_{12}\psi_2 + \varphi_{y_1}\psi_{1_x}\psi_{2_{y_1y_1}} - \varphi_{y_1}\psi_{1_{y_1y_1}}\psi_{2_x})/\Delta,$$

$$a_{25} = 2(\varphi_{xy_1}\psi_{1_{y_1}}\psi_{2_{y_2}} - \varphi_{xy_1}\psi_{1_{y_2}}\psi_{2_{y_1}} + \varphi_x\varphi_{y_1}{}^2\psi_{1_{y_2}}k_{22}\psi_2 \tag{H.11}$$
$$+\varphi_x\varphi_{y_1}{}^2\psi_{1_{y_2}}k_{21}\psi_1 - \varphi_x\varphi_{y_1}{}^2\psi_{2_{y_2}}k_{11}\psi_1 - \varphi_x\varphi_{y_1}{}^2\psi_{2_{y_2}}k_{12}\psi_2$$
$$-2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{1_{y_1}}k_{22}\psi_2 - 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{1_{y_1}}k_{21}\psi_1 + 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{2_{y_1}}k_{11}\psi_1$$
$$+2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{2_{y_1}}k_{12}\psi_2 + \varphi_x\psi_{1_{y_1y_2}}\psi_{2_{y_1}} - \varphi_x\psi_{1_{y_1}}\psi_{2_{y_1y_2}}$$
$$-\varphi_{y_1y_2}\psi_{1_x}\psi_{2_{y_1}} + \varphi_{y_1y_2}\psi_{1_{y_1}}\psi_{2_x} + \varphi_{y_1}{}^2\varphi_{y_2}\psi_{1_x}k_{22}\psi_2 + \varphi_{y_1}{}^2\varphi_{y_2}\psi_{1_x}k_{21}\psi_1$$
$$-\varphi_{y_1}{}^2\varphi_{y_2}\psi_{2_x}k_{11}\psi_1 - \varphi_{y_1}{}^2\varphi_{y_2}\psi_{2_x}k_{12}\psi_2 - \varphi_{y_1}\psi_{1_{xy_1}}\psi_{2_{y_2}} + \varphi_{y_1}\psi_{1_x}\psi_{2_{y_1y_2}}$$
$$-\varphi_{y_1}\psi_{1_{y_1y_2}}\psi_{2_x} + \varphi_{y_1}\psi_{1_{y_2}}\psi_{2_{xy_1}} + \varphi_{y_2}\psi_{1_{xy_1}}\psi_{2_{y_1}} - \varphi_{y_2}\psi_{1_{y_1}}\psi_{2_{xy_1}})/\Delta,$$

$$a_{26} = (2\varphi_{xy_2}\psi_{1_{y_1}}\psi_{2_{y_2}} - 2\varphi_{xy_2}\psi_{1_{y_2}}\psi_{2_{y_1}} + 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{1_{y_2}}k_{22}\psi_2 \tag{H.12}$$
$$+2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{1_{y_2}}k_{21}\psi_1 - 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{2_{y_2}}k_{11}\psi_1 - 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{2_{y_2}}k_{12}\psi_2$$

$$-3\varphi_x{\varphi_{y_2}}^2\psi_{1_{y_1}}k_{22}\psi_2 - 3\varphi_x{\varphi_{y_2}}^2\psi_{1_{y_1}}k_{21}\psi_1 + 3\varphi_x{\varphi_{y_2}}^2\psi_{2_{y_1}}k_{11}\psi_1$$
$$+3\varphi_x{\varphi_{y_2}}^2\psi_{2_{y_1}}k_{12}\psi_2 - \varphi_x\psi_{1_{y_1}}\psi_{2_{y_2y_2}} + \varphi_x\psi_{1_{y_2y_2}}\psi_{2_{y_1}}$$
$$+\varphi_{y_1}{\varphi_{y_2}}^2\psi_{1_x}k_{22}\psi_2 + \varphi_{y_1}{\varphi_{y_2}}^2\psi_{1_x}k_{21}\psi_1 - \varphi_{y_1}{\varphi_{y_2}}^2\psi_{2_x}k_{11}\psi_1$$
$$-\varphi_{y_1}{\varphi_{y_2}}^2\psi_{2_x}k_{12}\psi_2 - 2\varphi_{y_1}\psi_{1_{xy_2}}\psi_{2_{y_2}} + \varphi_{y_1}\psi_{1_x}\psi_{2_{y_2y_2}} - \varphi_{y_1}\psi_{1_{y_2y_2}}\psi_{2_x}$$
$$+2\varphi_{y_1}\psi_{1_{y_2}}\psi_{2_{xy_2}} - \varphi_{y_2y_2}\psi_{1_x}\psi_{2_{y_1}} + \varphi_{y_2y_2}\psi_{1_{y_1}}\psi_{2_x} + 2\varphi_{y_2}\psi_{1_{xy_2}}\psi_{2_{y_1}}$$
$$-2\varphi_{y_2}\psi_{1_{y_1}}\psi_{2_{xy_2}})/\Delta,$$

$$a_{27} = 2(\varphi_{xy_1}\psi_{1_{y_1}}\psi_{2_x} - \varphi_{xy_1}\psi_{1_x}\psi_{2_{y_1}} - {\varphi_x}^2\varphi_{y_1}\psi_{1_{y_1}}k_{22}\psi_2 \tag{H.13}$$
$$-{\varphi_x}^2\varphi_{y_1}\psi_{1_{y_1}}k_{21}\psi_1 + {\varphi_x}^2\varphi_{y_1}\psi_{2_{y_1}}k_{11}\psi_1 + {\varphi_x}^2\varphi_{y_1}\psi_{2_{y_1}}k_{12}\psi_2$$
$$+\varphi_x{\varphi_{y_1}}^2\psi_{1_x}k_{22}\psi_2 + \varphi_x{\varphi_{y_1}}^2\psi_{1_x}k_{21}\psi_1 - \varphi_x{\varphi_{y_1}}^2\psi_{2_x}k_{11}\psi_1$$
$$-\varphi_x{\varphi_{y_1}}^2\psi_{2_x}k_{12}\psi_2 + \varphi_x\psi_{1_{xy_1}}\psi_{2_{y_1}} - \varphi_x\psi_{1_{y_1}}\psi_{2_{xy_1}} - \varphi_{y_1}\psi_{1_{xy_1}}\psi_{2_x}$$
$$+\varphi_{y_1}\psi_{1_x}\psi_{2_{xy_1}})/\Delta,$$

$$a_{28} = (2\varphi_{xy_2}\psi_{1_{y_1}}\psi_{2_x} - 2\varphi_{xy_2}\psi_{1_x}\psi_{2_{y_1}} + \varphi_{xx}\psi_{1_{y_1}}\psi_{2_{y_2}} \tag{H.14}$$
$$-\varphi_{xx}\psi_{1_{y_2}}\psi_{2_{y_1}} + {\varphi_x}^2\varphi_{y_1}\psi_{1_{y_2}}k_{22}\psi_2 + {\varphi_x}^2\varphi_{y_1}\psi_{1_{y_2}}k_{21}\psi_1$$
$$-{\varphi_x}^2\varphi_{y_1}\psi_{2_{y_2}}k_{11}\psi_1 - {\varphi_x}^2\varphi_{y_1}\psi_{2_{y_2}}k_{12}\psi_2 - 3{\varphi_x}^2\varphi_{y_2}\psi_{1_{y_1}}k_{22}\psi_2$$
$$-3{\varphi_x}^2\varphi_{y_2}\psi_{1_{y_1}}k_{21}\psi_1 + 3{\varphi_x}^2\varphi_{y_2}\psi_{2_{y_1}}k_{11}\psi_1 + 3{\varphi_x}^2\varphi_{y_2}\psi_{2_{y_1}}k_{12}\psi_2$$
$$+2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{1_x}k_{22}\psi_2 + 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{1_x}k_{21}\psi_1 - 2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{2_x}k_{11}\psi_1$$
$$-2\varphi_x\varphi_{y_1}\varphi_{y_2}\psi_{2_x}k_{12}\psi_2 + 2\varphi_x\psi_{1_{xy_2}}\psi_{2_{y_1}} - 2\varphi_x\psi_{1_{y_1}}\psi_{2_{xy_2}}$$
$$-2\varphi_{y_1}\psi_{1_{xy_2}}\psi_{2_x} - \varphi_{y_1}\psi_{1_{xx}}\psi_{2_{y_2}} + 2\varphi_{y_1}\psi_{1_x}\psi_{2_{xy_2}} + \varphi_{y_1}\psi_{1_{y_2}}\psi_{2_{xx}}$$
$$+\varphi_{y_2}\psi_{1_{xx}}\psi_{2_{y_1}} - \varphi_{y_2}\psi_{1_{y_1}}\psi_{2_{xx}})/\Delta,$$

$$a_{29} = (\varphi_{xx}\psi_{1_{y_1}}\psi_{2_x} - \varphi_{xx}\psi_{1_x}\psi_{2_{y_1}} - {\varphi_x}^3\psi_{1_{y_1}}k_{22}\psi_2 \tag{H.15}$$
$$-{\varphi_x}^3\psi_{1_{y_1}}k_{21}\psi_1 + {\varphi_x}^3\psi_{2_{y_1}}k_{11}\psi_1 + {\varphi_x}^3\psi_{2_{y_1}}k_{12}\psi_2$$
$$+{\varphi_x}^2\varphi_{y_1}\psi_{1_x}k_{22}\psi_2 + {\varphi_x}^2\varphi_{y_1}\psi_{1_x}k_{21}\psi_1 - {\varphi_x}^2\varphi_{y_1}\psi_{2_x}k_{11}\psi_1$$
$$-{\varphi_x}^2\varphi_{y_1}\psi_{2_x}k_{12}\psi_2 + \varphi_x\psi_{1_{xx}}\psi_{2_{y_1}} - \varphi_x\psi_{1_{y_1}}\psi_{2_{xx}} - \varphi_{y_1}\psi_{1_{xx}}\psi_{2_x}$$
$$+\varphi_{y_1}\psi_{1_x}\psi_{2_{xx}})/\Delta.$$

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