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ON SOME FRACTIONAL STOCHASTIC MODELS IN FINANCE

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วิทยานิพนธ์ฉบับนี้พิจารณาภาคเศษส่วน ของแบบจำลองแบลค-โชล เนื่องจาก ได้มีการพิสูจน์ แล้วว่า แบบจำลองตลาดเชิงคณิตศาสตร์ ที่ขับเคลื่อนด้วยการเคลื่อนที่แบบบราวเนียนเศษส่วนมีอาบิทราจเราจึงเสนอแบบจำลองตลาดโดยประมาณแบบแบลค-โชลเศษส่วน เราพิสูจน์ว่า แบบจำลองแบลค-โชลเศษส่วนโดยประมาณของเราไม่มี อาบิทราจ ขณะที่ผลเฉลยของแบบจำลองเศษส่วน สามารถประมาณให้ใกล้เคียงแค่ไหนก็ได้ และนี่ แสดงถึงข้อได้เปรียบ อย่างมากของวิธีการประมาณของเรา

ยิ่งไปกว่านั้นเราได้พิจารณาภาคเศษส่วนของแบบจำลองอัตราดอกเบี้ยที่สำคัญที่สุด ได้แก่แบบจำลอง Vasicek, Ho-Lee, และ Hull-White เราได้หาผลเฉลยโดยประมาณของแบบจำลองเหล่านี้ และพิสูจน์ว่า ผลเฉลยโดยประมาณนี้ ลู่เข้าสู่ผลเฉลยเดิมของแต่ละแบบจำลอง

สุดท้าย ได้แสดงวิถีตัวอย่าง ของแบบจำลองราคาแบลค-โชล และของแบบจำลองแบลค-โชลเศษส่วนประมาณค่าสำหรับราคาหุ้น IBM เทียบกับข้อมูลเชิงประสบการณ์

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RATTIKAN SAELIM : ON SOME FRACTIONAL STOCHASTIC MODELS IN FINANCE. THESIS ADVISOR: ASSOC. PROF. PAIROTE SATTAYATHAM, Ph.D. 118 PP. ISBN 974-533-315-8

FRACTIONAL BROWNIAN MOTION/ ARBITRAGE/ APPROXIMATE MODEL/ FRACTIONAL BLACK-SCHOLES MODEL/ EQUIVALENT MARTINGALE MEASURE

In this thesis, a fractional version of Black-Scholes model is considered. Since it was proved that the market mathematical models driven by fractional Brownian motion could have arbitrage, we introduce the approximate model of the fractional Black-Scholes market. We prove that our approximate fractional Black-Scholes model has no more arbitrage while the solution of the fractional model can be approximated at any exactitude and this shows a considerable advantage of our approximate approach.

Moreover, fractional versions of most important models of interest rate as Vasicek, Ho-Lee, and Hull-White models, are also considered. Their approximate solutions are derived and proved to converge to each of its original solution.

Finally, sample paths of Black-Scholes pricing model and approximate fractional Black-Scholes model for the IBM prices are illustrated against the empirical data.

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Rattikan Saelim

Contents

Page

Contents (Continued)

Page

Contents (Continued)

List of Figures

Chapter I

Preliminaries on Fractional Stochastics

In this chapter, besides the literature review, we discuss the mathematical background and tools ranging from stochastic calculus, properties of fractional Brownian motion to a new approach to fractional stochastic calculus.

1.1 Introduction

Since the celebrated papers by Black and Scholes (1973) and Merton (1973) the idea of using Brownian motion for modelling prices of risky assets (share prices of stock, stock indices such as the Dow Jones, Nikkei or DAX, foreign exchange rates, interest rates, etc.) has been generally accepted. In the classical Black-Scholes pricing model the randomness of the log-returns of financial indices is modelled by Brownian motion. In fact, in the early studies the systems were usually modelled by Brownian motion which implies that the events are independent and identically distributed. Physically, this means that the events must not influence one another and they must all be equally likely to occur.

However, in many problems related to mathematical finance and other fields the processes under study seem empirically to exhibit self-similarity property and the long-range dependence property (the latter property is absent in Brownian motion). In fact, this history is dated back to the year 1951 when H. E. Hurst studied the long term water flow characteristics of the Nile River (based on more than 800 years of data). He noted that the water level of the Nile obeyed a self similar pattern with a long range dependence. Moreover, in recent studies, Leland et al. (1994) showed that, in communication networks, real inputs exhibit long range dependence. It is also reported, for example by Decreusefond et al. (1999) and Beran (1994), that the random processes arising from hydrological and economic time series exhibit long range dependence and self similarity. It is also shown by Alvarez-Ramirez et al. (2002) that rescaled range Hurst analysis provides evidence that the crude oil market is a persistent process with long-range memory effects (see also Shiryaev, 1999).

Definition 1.1. Let $H \in (0,1)$ be a constant. The fractional Brownian motion with Hurst parameter H is a Gaussian process $(W_t^H)_{t\geq 0} = (W_t^H(\omega)), t \geq 0, \omega \in \Omega$, satisfying

$$EW_t^H = 0 \text{ for all } t \geq 0$$

and the covariance function

$$R(s,t) := E[W_s^H W_t^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right); s,t \geq 0.$$

Here E denotes the expectation with respect to the probability law P for $(W_t^H)_{t\geq 0}$, where $(\Omega, \mathcal{F})$ is a measurable space.

If $H = \frac{1}{2}$ then W_t^H coincide with the classical Brownian motion, denoted by W_t. If $H > \frac{1}{2}$ then W_t^H is *persistent,* in the sense that

$$\rho(n) := E[W_1^H(W_{n+1}^H - W_n^H)] > 0 \text{ for all } n = 1, 2, 3, ...$$

and

$$\sum_{k=0}^{\infty} \rho(k) = \infty$$

If $H < \frac{1}{2}$ then W_t^H is *anti-persistent*, in the sense that

$$\rho(n) < 0 \text{ for all } n = 1, 2, 3, ...$$

In this case

$$\sum_{k=0}^{\infty} |\rho(k)| < \infty.$$

Another important property of fractional Brownian motion is self-similarity, i.e., for $H \in (0,1)$ and $\alpha > 0$ the law of $(W_{\alpha t}^H)_{t\geq 0}$ is the same as the law of $(\alpha^H W_t^H)_{t\geq 0}$.

In order to be able to apply fractional Brownian motion to study the market situations we need a stochastic calculus for fractional Brownian motion. Since for $H \neq \frac{1}{2}$ the fBm B_t^H is neither a semimartingale (Theorem 1.2) nor a Markov process (Theorem 1.3) then the well developed stochastic calculus is not applicable. In particular, for $H > \frac{1}{2}$, it is a long memory process (see page 14). In other words, the behavior of a real process after a given time t does not only depend on the situation at t but also of the whole history of the process up to time t. This significant property makes fractional Brownian motion a natural candidate as a model of noise in mathematical finance (see, e.g., Rogers, 1997) and in communication networks (Leland et al., 1994).

Many authors tried to understand what a stochastic integral of the form

$$\int_0^T f(t,\omega) dW_t^H$$

should mean. The most common constructions of such a stochastic integral are the following.

I **The pathwise or forward integral**

The integral is denoted by

$$\int_0^T \phi(t,\omega) d^- W_t^H.$$

If the integrand $\phi(t,\omega)$ is *caglad* (left-continuous with right sided limits) then this integral can be defined by Riemann sums, as follows:

Let $0 = t_0 < t_1 < ... < t_N = T$ be a partition of $[0, T]$. Put $\triangle t_k = t_{k+1} - t_k$ and define

$$\int_0^T \phi(t,\omega)d^- W_t^H := \lim_{\triangle t_k \to 0} \sum_{k=0}^{N-1} \phi(t_k) \left(W_{t_{k+1}}^H - W_{t_k}^H \right), \tag{1}$$

if the limit exists in probability.

Note that with this definition the integration takes place with respect to t for each fixed "path" $\omega \in \Omega$. Therefore, this integral is often called *pathwise integral*. Using a classical integration theory due to Young (i.e., the Riemann-Stieltjes integral $\int f dg$ exists if $f(t)$ is a function of bounded p-variation (for its definition see Section A.2 in Appendices) and $g(t)$ is a function of bounded q-variation for $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} > 1$) one can prove that the pathwise integral (1) exists if the p-variation of $t \mapsto \phi(t, \omega)$ is finite for all $p > \frac{1}{1-H}$. Since $t \mapsto W_t^H$ has finite q-variation iff $q \geq \frac{1}{H}$, we see that if $H < \frac{1}{2}$ then this theory does not even include integrals like

$$\int_0^T W_s^H d^- W_s^H.$$

For this reason one often assumes that $H > \frac{1}{2}$ when dealing with forward integrals with respect to W_t^H. In general

$$E \int_0^T W_s^H d^- W_s^H \neq 0,$$

even if the forward integral belongs to $L^1(\Omega, \mathcal{F}, P)$.

For $H > \frac{1}{2}$ the forward integral obeys *Stratonovich type* of integration rules. For example, if $f \in C^1(\mathbb{R})$ and

$$X_t := \int_0^t \phi(s,\omega) d^- W_s^H \text{ exists for all } t \geq 0$$

then

$$f(X_t) = f(0) + \int_0^t f'(X_s) d^- X_s, \tag{2}$$

where

$$d^- X_s = \phi(s,\omega) d^- W_s^H.$$

For this reason the forward integral is also sometimes called integral of Stratonovich type with respect to fractional Brownian motion. In fact, this is the Newton-Leibnitz's rule of integration.

As special case of (2) we note that

$$\int_0^T W_s^H d^- W_s^H = \frac{1}{2} \left(W_T^H\right)^2 \quad \text{for } H > \frac{1}{2}.$$

Moreover, a slight extension of (2) gives that the unique solution X_t of the fractional forward stochastic differential equation

$$d^- X_t = \alpha(t,\omega) X_t dt + \beta(t,\omega) X_t d^- W_t^H; \quad X_0 = x > 0 \tag{3}$$

is

$$X_t = x \exp\left(\int_0^t \alpha(s,\omega) ds + \int_0^t \beta(s,\omega) d^- W_s^H \right)$$

for $H > \frac{1}{2}$, provided that the integrals on the right hand side exist.

II **The Skorohod (Wick-Ito) integral**

This integral is denoted by

$$\int_0^T \phi(t,\omega) \delta W_t^H.$$

It is defined in terms of Riemann sums, as follows:

$$\int_0^T \phi(t,\omega) \delta W_t^H = \lim_{\triangle t_k \to 0} \sum_{k=0}^{N-1} \phi(t_k) \diamond \left(W_{t_{k+1}}^H - W_{t_k}^H \right), \tag{4}$$

where $\diamond$ denotes the *Wick product* (see Definition A.6 in Appendices). The difference between this integral and the forward integral is the use of the Wick product instead of the ordinary product in the Riemann sums (4) and (1), respectively.

The Skorohod integral behaves in many ways like the Ito integral of classical Brownian motion. For example, we have

$$E \int_0^T \phi(t,\omega)\delta W_t^H = 0$$

if the integral belongs to $L^2(\Omega, \mathcal{F}, P)$. Moreover, if $f \in C^2(\mathbb{R})$ then we have the following Ito type formula

$$f(W_t^H) = f(0) + \int_0^t f'(W_s^H)\delta W_t^H + H \int_0^t \int_0^t f''(W_s^H) s^{2H} ds, \qquad (5)$$

valid for all $H \in (0,1)$, provided that the left hand side and the last term on the right hand side both belong to $L^2(\Omega, \mathcal{F}, P)$.

Note that as special case of (5) we get

$$\int_0^T W_s^H \delta W_s^H = \frac{1}{2}\left(W_T^H\right)^2 - \frac{1}{2} T^{2H} \quad \text{for } H \in (0,1). \qquad (6)$$

The Wick-Skorohod-Ito analogue of (3) is the equation

$$\delta X_t = \alpha(t,\omega) X_t dt + \beta(t,\omega) X_t \delta W_t^H; \quad X_0 = x > 0. \qquad (7)$$

Assume that $\alpha(t,\omega) = a$ and $\beta(t,\omega) = b$ are constant. Then by a slight extension of the Ito formula (5) one obtains that the unique solution of (7) is

$$X_t = x \exp\left(\beta W_t^H + \alpha t - \frac{1}{2}\beta^2 t^{2H}\right); \qquad H \in (0,1). \qquad (8)$$

Note that if $H = \frac{1}{2}$ then the formulas (6) and (8) reduce to the formulas obtained by the Ito formula for the classical Brownian motion.

After the pathwise theory for fractional Brownian motion was developed (see, e.g., Lin, 1995 and Decreusefond et al., 1998&1999) it was proved that the market mathematical model driven by fractional Brownian motion could have arbitrage (Cheridito, 2003, Rogers, 1997, Sottinen, 2001 and Sottinen&Valkeila,

2001&2003). However after the development of the Skorohod integral based on the Wick product (e.g., Duncan et al., 2000 and Hu&Oksendal, 2003) it was proved (Hu&Oksendal, 2003) that the corresponding Ito type fractional Black-Scholes market has no arbitrage. Unfortunately, this integral does not allow economical interpretation. Worse still, these two types of definition (the pathwise and Skorohod integrals) are difficult for numerics.

In 2002, Thao tried to solve this problem. He proposed another definition of *fractional stochastic integral* motivated by a formulae of integration by parts and an approximate approach to fractional Brownian motion.

- Suppose first that $f(t,\omega)$ is a stochastic process of finite variation on $[0,t]$. The *fractional stochastic integral* of $f(t,\omega)$ is defined as

$$\int_0^t f(s,\omega)dB_s = f(t,\omega)B_t(\omega) - \int_0^t B_s(\omega)df(s,\omega) - [f,B]_t \tag{9}$$

provided that the integral on the right hand side exists in the Riemann-Stieltjes sense for almost all ω. In (9), $[f,B]_t$ is the *quadratic variation* of f_t and B_t where the fractional process B_t is defined as

$$B_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$$

and $W = (W_t, \mathcal{F}_t^W)_{t\geq 0}$ is standard Brownian motion on $(\Omega, \mathcal{F}, P)$ with its natural filtration $(\mathcal{F}_t^W)_{t\geq 0}$. See Section 1.7.2 for the reason why the process B_t is used instead of the fractional Brownian motion W_t^H.

But in this preliminary definition the integrand $f(t,\omega)$ must be of finite variation, which is a restrictive requirement. Thao (2002) has proposed a definition of the integral for any stochastic as the L^2-limit of $I_t^\varepsilon = \int_0^t f(s,\omega)dB_s^\varepsilon$ provided it exists, where B_t^ε is an L^2-approximation of B_t and is a semimartingale, when $\varepsilon \to 0$. This idea is motivated by the fact that for the integral of a process having finite variation defined above, it is proved that $\int_0^t f(s,\omega)dB_s = L^2 - \lim \int_0^t f(s,\omega)dB_s^{\varepsilon}$.

The purpose of this thesis is using the definition proposed by Thao (2002) to study some fractional models in finance: a fractional Vasicek model, a fractional Ho-Lee model, and a fractional Hull-White model, using a new approximate approach (Chapters III, IV, V). Moreover, the arbitrage problem for fractional Black-Scholes pricing model is also investigated. It is found that the approximate model is arbitrage-free so that one can give arbitrage-free prices with long memory (Chapter II). Finally, the sample paths of IBM stock prices simulated by classical Black-Scholes model and by fractional Black-Scholes model are illustrated against the empirical data (Chapter VI). All of basic terms and definitions can be found in Appendices.

1.2 Long Memory and Short Memory

A stochastic process, in general, is characterized by two quantities, namely, the probability density and the correlation function. The probability density describes the random nature of the fluctuations while the correlation function describes how a fluctuation at a given time influences subsequent fluctuations. If the correlation between two observations that are far apart decreases fairly slowly and is summed up to infinity then this is interpreted as a *long memory*. In fact, if $X = (X_t)_{t\geq 0}$ is a stochastic process on $(\Omega, \mathcal{F}, P)$ and $\rho(k) = E[X_1(X_{k+1} - X_k)]$ and if

$$\sum_{k=0}^{\infty} \rho(k) = \infty$$

then the process X is said to have *long memory* or *long-range dependence* or *strong aftereffect*. This means that the process today may influence the process at some time in the future. In other words, the process at long time before may influence the process today.

On the other hand, if the correlation between two observations that are far apart decreases fast enough so that they are summed up to a finite number then it is interpreted as a *short memory* or *short-range dependence*. For example, since the Brownian motion $W = (W_t)_{t\leq 0}$ has independent increments (see for definition in Section A.2) so that $E[W_1(W_{k+1} - W_k)] = 0$, for all $k \geq 1$ and, hence,

$$\sum_{k=0}^{\infty} E[W_1(W_{k+1} - W_k)] < \infty.$$

Therefore, Brownian motion, as well as the processes of martingale property and Markov type, have short memory.

In Sections 1.3 - 1.6, we review all definitions and tools for the further uses.

1.3 Fractional Brownian Motion (fBm)

Recall here that a stochastic process $X = (X_t)_{t\geq 0}$ is H-*self-similar* with parameter $H > 0$ if

$$(X_{at})_{t\geq 0} \stackrel{d}{=} \left(a^H X_t\right)_{t\geq 0}$$

for all $a > 0$, where $\stackrel{d}{=}$ means equality in distributions.

Suppose that $Y = (Y_t)_{t\geq 0}$ is a self-similar process with parameter H. Then

$$Y_t \stackrel{d}{=} t^H Y_1 \text{ for } t > 0$$

and hence

$$Var(Y_t) = Var(t^H Y_1) = t^{2H} Var(Y_1).$$

In the following, we consider the values of $H \in (0, 1)$, and in particular $Y_0 = 0$ with probability 1. Assume further that Y_t has zero mean, is normalized so that $Var(Y_1) = 1$, and stationary increments, i.e., the random n-vectors $(Y_{t_1}, Y_{t_2}, ..., Y_{t_n})$ and $(Y_{t_1+h}, Y_{t_2+h}, ..., Y_{t_n+h}), h > 0$ are identically distributed. Hence

Corollary 1.1. *Let $(Y_t)_{t\geq 0}$ be real-valued H-self-similar with stationary increments and suppose that $Var(Y_1) = 1$. Then, for $s < t$,*

$$R(t, s) := Cov(Y_t, Y_s) = \frac{1}{2}\left(t^{2H} + s^{2H} - (t-s)^{2H}\right). \tag{10}$$

Proof. By self similarity and stationary of the increments,

$$E(Y_t - Y_s)^2 = E(Y_{t-s} - Y_0)^2 = EY_{t-s}^2 = (t-s)^{2H}.$$

On the other hand,

$$\begin{aligned} E(Y_t - Y_s)^2 &= EY_t^2 - 2EY_tY_s + EY_s^2 \\ (t-s)^{2H} &= t^{2H} - 2Cov(Y_t, Y_s) + s^{2H}, \end{aligned}$$

hence,

$$R(t, s) = \frac{1}{2}\left(t^{2H} + s^{2H} - (t-s)^{2H}\right). \qquad \square$$

Theorem 1.1. *A fractional Brownian motion $(W_t^H)_{t\geq 0}$ is H-self-similar with stationary increments. When $H \in (0, 1)$, it has a stochastic integral representation:*

$$\frac{1}{C_H}\int_{\mathbb{R}}\left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}}\right] dW_s, t \geq 0 \tag{11}$$

where $H \in (0, 1)$, $f^+ = \max\{f, 0\}$ and

$$C_H = \left(\int_0^\infty \left[(1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right]^2 ds + \frac{1}{2H}\right)^{\frac{1}{2}}.$$

If $H = 1$, $W_t^1 = tW_1^1$ almost surely. Fractional Brownian motion is unique in the sense that the class of all fractional Brownian motions coincides with that of all Gaussian selfsimilar processes with stationary increments.

Proof.[Embrechts et al., 2002] Assume that $Var(W_1^H) = 1$.

(i) **Self-similarity**: We have that, for $a > 0$,

$$\begin{aligned} E\left(W_{at}^H W_{as}^H\right) &= \frac{1}{2}\left[(at)^{2H} + (as)^{2H} - a^{2H}\left|t-s\right|^{2H}\right] \\ &= a^{2H}\frac{1}{2}\left[t^{2H} + s^{2H} - \left|t-s\right|^{2H}\right] \\ &= a^{2H}E(W_t^H W_s^H) \\ &= E\left[\left(a^H W_t^H\right)\left(a^H W_s^H\right)\right]. \end{aligned}$$

Since all processes here are Gaussian, this equality in covariance implies that $\left(W_{at}^H\right) \overset{d}{=} (a^H W_t^H)$.

(ii) For the case $H = 1$, first note that because of (10), $E\left(W_t^1 W_s^1\right) = st$. Then,

$$\begin{aligned} E\left(W_t^1 - tW_1^1\right)^2 &= E(W_t^1)^2 - 2tE\left(W_t^1 W_1^1\right) + t^2 E(W_1^1)^2 \\ &= t^2 - 2t\frac{1}{2}\left(t^2 + 1 - (t-1)^2\right) + t^2 \\ &= 0, \end{aligned}$$

so that $W_t^1 = tW_1^1$ almost surely.

(iii) **Fractional Brownian motion**: Let the integral (11) be denoted by Y_t. The expression (11) is a zero mean Gaussian process with covariance function (10). To see this, let us first consider the process

$$X_t = \int_{\mathbb{R}} \left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}}\right] dW_s, t \geq 0.$$

Clearly, by Ito isometry property we have, $EX_t = 0$ and

$$
\begin{aligned}
Var(X_t) &= E(X_t)^2 \\
&= E\left(\int_{\mathbb{R}} \left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}}\right] dW_s\right)^2 \\
&= \int_{\mathbb{R}} \left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}}\right]^2 ds \\
&= \int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right]^2 ds + \int_0^t (t-s)^{2H-1}\, ds, t \geq 0 \\
&= \int_0^{\infty} \left[(t+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right]^2 du + \frac{t^{2H}}{2H}, t \geq 0 \\
&= t^{2H-1} \int_0^{\infty} \left[\frac{(t+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}}{t^{H-\frac{1}{2}}}\right]^2 du + \frac{t^{2H}}{2H}, t \geq 0 \\
&= t^{2H} \int_0^{\infty} \left[\left(1+\frac{u}{t}\right)^{H-\frac{1}{2}} - \left(\frac{u}{t}\right)^{H-\frac{1}{2}}\right]^2 d\left(\frac{u}{t}\right) + \frac{t^{2H}}{2H}, t \geq 0 \\
&= C_H^2 t^{2H}, t \geq 0.
\end{aligned}
$$

Similarly, for any $s < t$

$$
E\left(X_t - X_s\right)^2 = E\left[\int_{\mathbb{R}} \left[\left((t-u)^+\right)^{H-\frac{1}{2}} - \left((s-u)^+\right)^{H-\frac{1}{2}}\right]^2 dW_u\right]
$$

and by change of variables ($v = u - s$), one obtains

$$
\begin{aligned}
E\left(X_t - X_s\right)^2 &= \int_{\mathbb{R}} \left[\left((t-s-v)^+\right)^{H-\frac{1}{2}} - \left((-v)^+\right)^{H-\frac{1}{2}}\right]^2 dv \\
&= C_H^2 (t-s)^{2H}.
\end{aligned}
$$

It follows that for all $t \geq s \in \mathbb{R}$,

$$
\begin{aligned}
Cov(X_t, X_s) &= \frac{1}{2}\left[Var(X_t) + Var(X_s) - Var(X_t - X_s)\right] \\
&= \frac{1}{2}\left[Var(X_t) + Var(X_s) - Var(X_{t-s})\right] \\
&= \frac{1}{2}\left[C_H^2 t^{2H} + C_H^2 s^{2H} - C_H^2 (t-s)^{2H}\right] \\
&= \frac{C_H^2}{2}\left[t^{2H} + s^{2H} - (t-s)^{2H}\right].
\end{aligned}
$$

Moreover,

$$\begin{aligned} Cov(Y_t, Y_s) &= Cov(C_H^{-1}X_t, C_H^{-1}X_s) \\ &= C_H^{-2}Cov(X_t, X_s) \\ &= \frac{1}{2}\left[t^{2H} + s^{2H} - (t-s)^{2H}\right]. \end{aligned}$$

Therefore, W_t^H for $0 < H < 1$ is a fractional Brownian motion.

(iv) **Stationary increments**: It is enough to consider only covariances. We have

$$\begin{aligned} E\left(W_{t+h}^H - W_h^H\right)\left(W_{s+h}^H - W_h^H\right) &= E\left[W_{t+h}^H W_{s+h}^H - W_{t+h}^H W_h^H - W_h^H W_{s+h}^H + \left(W_h^H\right)^2\right] \\ &= \frac{1}{2}\left[(t+h)^{2H} + (s+h)^{2H} - |t-s|^{2H}\right] \\ &\quad -\frac{1}{2}\left[(t+h)^{2H} + h^{2H} - t^{2H}\right] \\ &\quad -\frac{1}{2}\left[h^{2H} + (s+h)^{2H} - s^{2H}\right] + h^{2H} \\ &= \frac{1}{2}\left[t^{2H} + s^{2H} - (t-s)^{2H}\right] \\ &= E(W_t^H W_s^H) \end{aligned}$$

concluding that $\left(W_{t+h}^H - W_h^H\right) \stackrel{d}{=} (W_t^H)$.

(v) For the uniqueness, first note that once a process $(Y_t)_{t\geq 0}$ is H-self-similar and has stationary increments, then by Corollary 1.1, it has the same covariance function as in (10). Since $(Y_t)_{t\geq 0}$ is zero mean Gaussian, it is the same as (W_t^H) in law. $\square$

Another basic property of fBm, W_t^H on $(\Omega, \mathcal{F}, P)$, is *long-range dependence.* In fact, for $n \geq 1$,

$$\begin{aligned}
\rho(n) &= E[W_1^H(W_{n+1}^H - W_n^H)] \\
&= EW_1^H W_{n+1}^H - EW_1^H W_n^H \\
&= \frac{1}{2}\left(1^{2H} + (n+1)^{2H} - n^{2H}\right) - \frac{1}{2}\left(1^{2H} + n^{2H} - (n-1)^{2H}\right) \\
&= \frac{1}{2}\left((n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right) \\
&= \frac{1}{2} n^{2H} g(n^{-1}),
\end{aligned}$$

where $g(x) = (1+x)^{2H} - 2 + (1-x)^{2H}$. If $0 < H < 1$ and $H \neq \frac{1}{2}$, then the Taylor expansion of $g(x)$ about the origin gives

$$g(x) = 2H(2H-1)x^2 + o(x^4).$$

Therefore,

$$\rho(n) = \frac{1}{2} n^{2H} g(n^{-1}) = \frac{1}{2} n^{2H}\left[2H(2H-1)n^{-2} + o(n^{-4})\right]$$

and as n tends to infinity,

$$\rho(n) = H(2H-1)n^{2H-2}. \tag{12}$$

Moreover, for $\frac{1}{2} < H < 1$ the correlation decay to zero so slowly that

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

Hence, for $\frac{1}{2} < H < 1$, fBm W_t^H has *long-range dependence*. For $H = \frac{1}{2}$ it can easily be seen, by (10), that the observations are uncorrelated. In fact, the fBm with Hurst index H is a semimartingale if and only if $H = \frac{1}{2}$ (Roger, 1997). Finally, for $0 < H < \frac{1}{2}$, we have $2 - 2H > 1$ and hence

$$\sum_{n=0}^{\infty} \rho(n) = H(2H-1) \sum_{n=0}^{\infty} \frac{1}{n^{2-2H}} < \infty.$$

Therefore in this case, the process exhibits short-range dependence.

1.4 The Need to Study Fractional Brownian Motions

Almost all statistical analysis of economic and financial systems begins by assuming that the dynamics are primarily random. Models considered earlier in Mathematical finance assume that the price of an asset should follow a martingale property in which each price change is unaffected by its predecessor.

Stochastic differential equations driven by Brownian motion are traditionally used to model the dynamic of stock prices. It is well known that Brownian motion is a typical semimartingale with short-range dependence: when $H = \frac{1}{2}$, the autocorrelation $\rho(n)$ of (12) is zero for all n, hence $\sum_{n=1}^{\infty} \rho(n) < \infty$. However, in recent years it has become increasingly obvious that long-range dependence phenomena are widespread in financial data. The dependence structure of the financial data have been studied using the so-called *Hurst index (Hurst parameter)* H. In the uncorrelated case one should have $H = \frac{1}{2}$. If $H < \frac{1}{2}$ the time series is *antipersistent*. This means that whenever the price has been up, it is more likely that it will be down in the close future. Conversely, if $H > \frac{1}{2}$ one has *persistence* with positive correlations. This means that all price flunctuations are correlated with all future price flunctuations. Persistence implies that if the price has been up or down then the chances are that it will continue to up or down in the future, respectively.

Many studies indicated Hurst indices $H > \frac{1}{2}$. For example, for the monthly S&P500 index (from January, 1963 through December 1989) the estimated Hurst index is $H = 0.78$ (see Shiryaev, 1999, p. 377 and the reference therein). In 2002, Alvarez-Ramirez et al. studied the daily records of international crude oil prices and found that the rescaled range Hurst analysis provideds evidence that the crude oil market is a persistent process with long memory effect. In fact, they found that the Hurst indices are all above $\frac{1}{2}$ with different time scales.

1.5 A Fractional Brownian Motion Is Not A Semimartingale

Definition 1.2 (Martingale). Let a family $\mathcal{M} = (\mathcal{M}_t)_{t\geq 0}$, on $(\Omega, \mathcal{F})$, of σ-algebras $\mathcal{M}_t \subset \mathcal{F}$ such that

$$0 \leq s < t \Longrightarrow \mathcal{M}_s \subset \mathcal{M}_t$$

(i.e. $\mathcal{M}$ is a filtration) be given. A stochastic process $(M_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ *adapted* (for its definition see Section A.1) to the filtration $\mathcal{M} = (\mathcal{M}_t)_{t\geq 0}$ is called a martingale (with respect to filtration $(\mathcal{M}_t)_{t\geq 0}$ and the measure P) if for any t, M_t is integrable, i.e., $E\,|M_t| < \infty$ and for any t and s with $0 \leq s \leq t \leq T$,

$$E(M_t|\mathcal{M}_s) = M_s \quad a.s.$$

One can see that Brownian motion $W = (W_t)_{t\geq 0}$ is a martingale with respect to the $\sigma-$algebras $\mathcal{F}_t^W$ generated by $W_s, s \leq t$. Since $W_t - W_s$ is independent of $\mathcal{F}_t^W$, $s < t$, we have

$$E\left(W_t - W_s|\mathcal{F}_s^W\right) = E\left(W_t - W_s\right) = 0$$

and since W_t is adapted to $\mathcal{F}_t^W$ then $E(W_t|\mathcal{F}_t^W) = W_t$. Therefore,

$$\begin{aligned} E\left(W_t|\mathcal{F}_s^W\right) &= E\left(W_t - W_s + W_s|\mathcal{F}_s^W\right) \\ &= E\left(W_t - W_s|\mathcal{F}_s^W\right) + E\left(W_s|\mathcal{F}_s^W\right) \\ &= W_s. \end{aligned}$$

A stochastic process $S = (S_t)_{t\geq 0}$ is called a *semimartingale* if it can be represented as sums

$$S_t = S_0 + M_t + A_t, \tag{13}$$

where $A = (A_t)_{t\geq 0}$ is a process of bounded variation and $M = (M_t)_{t\geq 0}$ is a (local) martingale both defined on some filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$$

satisfying the usual conditions, i.e., the $\sigma-$algebras $\mathcal{F}$ is P-complete and $\mathcal{F}_t, t \geq 0$ must contain all the sets in $\mathcal{F}$ of P-probability zero, and be right continuous ($\mathcal{F}_t = \mathcal{F}_{t^+}, t \geq 0$). Since a standard Brownian motion, W, is a martingale one can see from (13) that with $A_t = 0$ and $M_t = W_t$, for $t \geq 0$, the standard Brownian motion is a semimartingale.

We will see in the following theorems that the fractional Brownian motion $X = (X_t)_{t\geq 0}$ with Hurst parameter $H \in (0,1)$ is neither a semimartingale (Theorem 1.2) nor a Markov process (Theorem 1.3).

Theorem 1.2. *(Rogers, 1997) The fBm, is a semimartingale only if* $H = \frac{1}{2}$.

Proof. Let $X = (X_t)_{t\geq 0}$ be a fractional Brownian motion with self-similar parameter $H \in (0,1)$. We know that when $H = \frac{1}{2}$ fractional Brownian motion is in fact a standard Brownian motion and hence a semimartingale.

Now fix the parameter H and consider for $p > 0$ fixed

$$Y_{n,p} := \sum_{j=1}^{2^n} \left| X_{j2^{-n}} - X_{(j-1)2^{-n}} \right|^p (2^n)^{pH-1}. \tag{14}$$

From self-similarity property we obtain that (14) has (for each n) the same law as

$$\begin{aligned}\sum_{j=1}^{2^n} \left| 2^{-nH} X_j - 2^{-nH} X_{j-1} \right|^p (2^n)^{pH-1} &= \sum_{j=1}^{2^n} \left| X_j - X_{j-1} \right|^p 2^{-npH} (2^n)^{pH-1} \\ &= 2^{-n} \sum_{j=1}^{2^n} \left| X_j - X_{j-1} \right|^p.\end{aligned}$$

Noticing that the sequence $(X_k - X_{k-1})_{k\in\mathbb{Z}}$ is stationary and ergodic, the ergodic theorem tells us that

$$\widetilde{Y}_{n,p} := 2^{-n}\sum_{j=1}^{2^n}|X_j - X_{j-1}|^p \to E\,|X_1 - X_0|^p =: \gamma_p \quad (n\to\infty)$$

almost surely and in L^1. Hence

$$Y_{n,p} \xrightarrow{d} \gamma_p \quad (n\to\infty)$$

and therefore (Theorem A.1b) $Y_{n,p} \xrightarrow{P} \gamma_p$. Hence,

$$V_{n,p} := \sum_{j=1}^{2^n}\left|X_{j2^{-n}} - X_{(j-1)2^{-n}}\right|^p \xrightarrow{P} \begin{cases} 0 & if \quad pH > 1, \\ \infty & if \quad pH < 1. \end{cases} \tag{15}$$

If $H > \frac{1}{2}$, we can choose $p \in (H^{-1}, 2)$ such that $V_{n,p} \to 0$ in probability, and therefore almost surely down a fast subsequence. This implies that the quadratic variation of X is zero, and so (if X were to be a semimartingale) X must be a finite-variation process. But since for $p \in (1, H^{-1}), V_p := \lim_{n\to\infty} V_{n,p}$ is almost surely infinite, and (by scaling) the p-variation on any interval is infinite almost surely, X can not be finite variation. If $H < \frac{1}{2}$, we can choose $p > 2$ such that $pH < 1$, and the p-variation of X on $[0,1]$ (and hence on any fixed interval) must be infinite. This contradicts the almost-sure finiteness of the quadratic variation of X, assuming X is a semimartingale. In either way, if $H \neq \frac{1}{2}$, X is not a semimartingale. $\square$

1.6 A Fractional Brownian Motion Is Not A Markov Process

1.6.1 Markov Processes

The Markov property states that if we know the present state of the process, then the future behavior of the process is independent of its past. The process

X_t has the Markov property if the conditional distribution of X_{t+s} given $X_t = x$, does not depend on the past values (but it may depend on the present value x). That means the process *does not remember* how it got to the present state x. Let $\mathcal{F}_t$ denote the $\sigma-$algebra generated by the process up to time t.

Definition 1.3 (Markov Process). Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration under $\mathcal{F}$. Let $(X_t)_{t\geq 0}$ be a stochastic process on $(\Omega, \mathcal{F}, P)$. This process is said to be *Markovian* or *Markov process* if:

- the stochastic process $(X_t)_{t\geq 0}$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, and
- The Markov property: for any $t, s > 0$, the distribution of X_{t+s} conditional on $\mathcal{F}_t$ is the same as the distribution of X_{t+s} conditional on X_t, that is,

$$P(X_{t+s} \leq y|\mathcal{F}_t) = P(X_{t+s} \leq y|X_t), \text{ a.s.}$$

A Markov process is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variable and the way in which the present has emerged from the past are irrelevant.

Lemma 1.1. *Let $R(t, s)$ be covariance function of a centered Gaussian process, Y_t is Markovian then for all t, s, t_0 such that $t > s > t_0$ we have*

$$R(t, t_0) = \frac{R(t, s)R(s, t_0)}{R(s, s)} \tag{16}$$

(see for example, E.Wong and B. Hajek, 1985)

Theorem 1.3. *Every fractional Brownian motion with Hurst index $H \neq \frac{1}{2}$ is not a Markov process*

Proof. (Huy,2003) Let $(W_t^H)_{t\geq 0}$ be a fractional Brownian motion with $H \neq \frac{1}{2}$. Suppose that it is a Markov process. Put

$$f_s(t) = \frac{R(t,s)}{s^\alpha} = \frac{1}{2}\left[(\frac{t}{s})^\alpha + 1 - (\frac{t}{s} - 1)^\alpha\right], \quad t > s \tag{17}$$

where $\alpha = 2H$. Consider the derivative of $f_s(t)$ *w.r.t.* t:

$$f_s^{'}(t) = \frac{1}{2}\frac{\alpha}{s}\left[(\frac{t}{s})^{\alpha-1} - (\frac{t}{s} - 1)^{\alpha-1}\right], \quad t > s \tag{18}$$

We see for $s < t$ that

$$\begin{aligned} f_s^{'}(t) &< 0 \quad \text{if} \quad \alpha < 1 \\ f_s^{'}(t) &> 0 \quad \text{if} \quad \alpha > 1 \\ f_s^{'}(t) &= 0 \quad \text{if} \quad \alpha = 1 \end{aligned}$$

So, if $\alpha \neq 1$, $f_s(t)$ is either decreasing or increasing. On the other hand, for $\alpha < 1$ we have

$$\begin{aligned} \lim_{t\to\infty} f_s(t) &= \frac{1}{2}\lim_{t\to\infty}\frac{\frac{1}{t^\alpha} + \frac{1}{s^\alpha} - (\frac{1}{s} - \frac{1}{t})^\alpha}{\frac{1}{t^\alpha}} \\ &= \frac{1}{2}\lim_{t\to\infty}\left(1 + (\frac{t}{s} - 1)^{\alpha-1}\right) = \frac{1}{2}; \quad t > s \end{aligned} \tag{19}$$

Hence for $\alpha < 1$, $f_s(t)$ is decreasing from 1 to $\frac{1}{2}$ when t varies from 0 to infinity. Now for $0 < r < s < t$ it follows from the above Lemma that

$$\frac{R(t,r)}{r^\alpha} = \frac{R(t,s)}{s^\alpha}\frac{R(s,r)}{r^\alpha}$$

or

$$f_r(t) = f_s(t).f_r(s)$$

Taking the limit of both sides of (4) when $t \to \infty$ get

$$\frac{1}{2} = \frac{1}{2}f_r(s), \quad r < s$$

or

$$f_r(s) \equiv 1, \quad r < s.$$

This is contrary to the property of the function $f_r(s)$ given by (18);

$$f_r(s) = \frac{1}{2}\left[(\frac{s}{r})^\alpha + 1 - (\frac{s}{r} - 1)^\alpha\right]$$

Then W_t^H should not be a Markov process. $\square$

1.7 A New Approach to Fractional Stochastic Calculus

In this section, we prepare mathematical tools for defining stochastic integral with respect to fractional Brownian motion via integration by parts. Moreover, in this thesis we choose to use the approximate approach, namely, using the L^2-convergence of a semimartingale to a fractional process.

1.7.1 A New Approximation of Fractional Brownian Motion

The following theorem was proved by Thao (2002). The result of this theorem will be frequently referred to throughout the text.

For every $\varepsilon > 0$ we define

$$B_t^\varepsilon = \int_0^t (t - s + \varepsilon)^{H-\frac{1}{2}} dW_s, H \neq \frac{1}{2}, 0 < H < 1.$$

Theorem 1.4. *The process $(B_t^\varepsilon, t \geq 0)$ is a semimartingale.*

Proof. (Thao, 2002) Consider the stochastic process φ_t^ε defined as

$$\varphi_t^\varepsilon = \int_0^t (t - u + \varepsilon)^{\alpha - 1} dW_u$$

where $\alpha = H - \frac{1}{2}$ (then $-\frac{1}{2} < \alpha < \frac{1}{2}$, since $0 < H < 1$).

An application of the stochastic theorem of Fubini gives us:

$$\begin{aligned}
\int_0^t \varphi_s^\varepsilon ds &= \int_0^t \int_0^s (s-u+\varepsilon)^{\alpha-1} dW_u ds \\
&= \int_0^t \left(\int_u^t (s-u+\varepsilon)^{\alpha-1} ds \right) dW_u \\
&= \int_0^t \left(\frac{(t-u+\varepsilon)^\alpha}{\alpha} - \frac{\varepsilon^\alpha}{\alpha} \right) dW_u \\
&= \frac{1}{\alpha} \left[\int_0^t (t-u+\varepsilon)^\alpha dW_u - \int_0^t \varepsilon^\alpha dW_u \right] \\
&= \frac{1}{\alpha} \left(B_t^\varepsilon - \varepsilon^\alpha W_t \right).
\end{aligned}$$

Hence

$$B_t^\varepsilon = \alpha \int_0^t \varphi_s^\varepsilon ds + \varepsilon^\alpha W_t.$$

Since $\alpha \int_0^t \varphi_s^\varepsilon ds$ is of bounded variation and W_t is a martingale so B_t^ε is a semi-martingale. $\square$

Theorem 1.5. *B_t^ε converges to B_t in $L^2(\Omega)$ when ε tends to 0. This convergence is uniform with respect to $t \in [0,T]$.*

Proof. The Mean Value Theorem applied to the function $f(u) = u^\alpha$ yields:

$$\begin{aligned}
\left|(t-s+\varepsilon)^\alpha - (t-s)^\alpha\right| &\leq |\alpha|\, \varepsilon \sup_{0\leq\theta\leq 1} \left|(t-s+\theta\varepsilon)^{\alpha-1}\right| \\
&= |\alpha|\, \varepsilon (t-s)^{\alpha-1}, \alpha = H - \frac{1}{2} \qquad (20) \\
& \quad\; (0 < s < t).
\end{aligned}$$

By virtue of Ito integration isometry we see that

$$\begin{aligned}
E\left|B_t^\varepsilon - B_t\right|^2 &= E\left| \int_0^t \left[(t-s+\varepsilon)^\alpha - (t-s)^\alpha\right] dW_s \right|^2 \\
&= \int_0^t \left|(t-s+\varepsilon)^\alpha - (t-s)^\alpha\right|^2 ds. \qquad (21)
\end{aligned}$$

(i) (Thao, 2003) If $\frac{1}{2} < H < 1$, that is, $0 < \alpha < \frac{1}{2}$ we have from (20)

$$\begin{aligned}
\int_0^t |(t-s+\varepsilon)^\alpha - (t-s)^\alpha|^2\, ds &\le \alpha^2\varepsilon^2 \int_0^t |t-s|^{2\alpha-2}\, ds \\
&= \alpha^2\varepsilon^2 \left(\int_0^{t-\varepsilon} |t-s|^{2\alpha-2}\, ds + \int_{t-\varepsilon}^t |t-s|^{2\alpha-2}\, ds \right) \\
&\le \alpha^2\varepsilon^2 \frac{\varepsilon^{2\alpha-1}}{1-2\alpha} + \alpha^2\varepsilon^2 \frac{\varepsilon^{2\alpha-1}}{1-2\alpha} \\
&= C_1(\alpha)\varepsilon^{2\alpha+1} \to 0 \qquad (22)
\end{aligned}$$

as $\varepsilon \to 0$, where $C_1(\alpha) = \frac{2\alpha^2}{1-2\alpha} > 0$.

(ii) (Thao et al., 2002) If $0 < H < \frac{1}{2}$, that is, $-\frac{1}{2} < \alpha < 0$, we put $\alpha = -\beta$, so $0 < \beta < \frac{1}{2}$ and we have

$$\begin{aligned}
\left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right| &\le \beta\varepsilon \sup_{0\le\theta\le1} \left|(t-s+\theta\varepsilon)^{-\beta-1}\right| \\
&= \beta\varepsilon(t-s)^{-\beta-1}, \qquad (23)
\end{aligned}$$

$$\begin{aligned}
E\left|B_t^\varepsilon - B_t\right|^2 &= E\left|\int_0^t \left[(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right] dW_s\right|^2 \\
&= \int_0^t \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 ds \\
&= \int_0^{t-\varepsilon} \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 ds \\
&\quad + \int_{t-\varepsilon}^t \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 ds, \qquad (24)
\end{aligned}$$

The evaluation of (23) applied to the first term of (24) gives us

$$\int_0^{t-\varepsilon} \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 ds \le \beta^2\varepsilon^2 \int_0^{t-\varepsilon} (t-s)^{-2\beta-2} ds. \qquad (25)$$

For the second term of the right hand side of (24) we have

$$\int_{t-\varepsilon}^t \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 ds \le \int_{t-\varepsilon}^t (t-s)^{-2\beta} ds. \qquad (26)$$

It follows from (24), (25) and (26) that

$$E\left|B_t^\varepsilon - B_t\right|^2 \le \beta^2\varepsilon^2 \int_0^{t-\varepsilon} (t-s)^{-2\beta-2} ds + \int_{t-\varepsilon}^t (t-s)^{-2\beta} ds.$$

After some calculation we get

$$E\left|B_t^\varepsilon - B_t\right|^2 \leq C_2(\beta)\varepsilon^{1-2\beta} \to 0, \text{ as } \varepsilon \to 0, \tag{27}$$

where $C_2(\beta)$ is a positive constant depending only on β.

From (22) and (27) we see that in both cases ($H > \frac{1}{2}$ and $H < \frac{1}{2}$) there is an estimation for $\|B_t^\varepsilon - B_t\|^2 = E\left[|B_t^\varepsilon - B_t|^2\right]$ as follows:

$$\|B_t^\varepsilon - B_t\|^2 \leq C_3(\alpha)\varepsilon^{1+2\alpha}, \tag{28}$$

where $0 < \alpha < \frac{1}{2}$ for $\frac{1}{2} < H < 1$ and $-\frac{1}{2} < \alpha < 0$, for $0 < H < \frac{1}{2}$, and $C_3(\alpha) = \max\{C_1(\alpha), C_2(\beta)\}$depending only on α $(= -\beta)$.

The relation (28) is valid for every $t \geq 0$, so

$$\sup_{0\leq t\leq T} \|B_t^\varepsilon - B_t\| \leq C(\alpha)\varepsilon^{\frac{1}{2}+\alpha} \to 0, \text{ as } \varepsilon \to 0, \tag{29}$$

where $C(\alpha) = \sqrt{C_3(\alpha)}$ which proves that $B_t^\varepsilon \to B_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$. $\square$

1.7.2 A New Approach to Fractional Stochastic Calculus

The main theoretical problem raised by the fractional Brownian motion is that it is not a semimartingale nor a Markov process, hence the usual stochastic calculus cannot be applied. Even though some authors have developed the calculus for which the stochastic integral with respect to fractional Brownian motion is defined, it is difficult for numerics. Since the Ito calculus is well-defined for an integral with respect to semimartingale, it is more convenient to use the well-developed one.

Recent advances in the branch of mathematics known as dynamical systems promise to revolutionize the way scientists view many different kinds of evolutionary processes. These processes occur in all branches of science, ranging from the fluctuations of temperature, pressure, wind speed and so on in meteorology to the ups and downs of stock market in economics. Indeed, any process which involves time is an example of a dynamical system.

Let a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t\geq 0}, P)$ be given and $W = (W_t)_{t\geq 0}$ be the standard Brownian motion with its natural filtration $(\mathcal{F}_t^W)_{t\geq 0}$. Recall that $f^+ = \max\{f, 0\}$. Then for $t \geq 0$, we have

$$\begin{aligned}\int_{\mathbb{R}} \left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}}\right] dW_s &= \int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right] dW_s \\ &\quad + \int_0^t (t-s)^{H-\frac{1}{2}} dW_s.\end{aligned}$$

Therefore, as proved in Theorem (1.1), our fractional Brownian motion is of the form

$$\begin{aligned}W_t^H &= \frac{1}{C_H}\int_{\mathbb{R}} \left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}}\right] dW_s, t \geq 0 \\ &= \frac{1}{C_H}\left(Z_t + \int_0^t (t-s)^{H-\frac{1}{2}} dW_s\right) \end{aligned} \tag{30}$$

where

$$C_H = \left(\int_0^{\infty} \left[(1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right]^2 ds + \frac{1}{2H}\right)^{\frac{1}{2}},$$

$(W_s)_{s\geq 0}$ is a standard Brownian motion, $H \in (0,1)$, $f^+ = \max\{f, 0\}$ and

$$Z_t = \int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right] dW_s.$$

Let us consider the fractional stochastic dynamical system of $(X_t)_{0\leq t\leq T}$ expressed by the following fractional stochastic differential equation

$$\begin{aligned} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t^H \\ X_t|_{t=0} &= X_0, t \in [0, T], \end{aligned} \tag{31}$$

where X_0 is a given random variable.

In fact, to make (31) having a sense, we have to define the fractional stochastic integral

$$\int_0^t f(s,\omega)dW_s^H.$$

However, in 2000, Alos et al. have proposed to use

$$B_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s, \tag{32}$$

instead of W_t^H in fractional stochastic calculus, since Z_t has absolutely continuous trajectories and it is the term B_t that has long memory. Therefore, instead of (31), we consider the fractional stochastic differential equation

$$\begin{aligned} dX_t &= b(t,X_t)dt + \sigma(t,X_t)dB_t \\ X_t|_{t=0} &= X_0, t \in [0,T]. \end{aligned}$$

To define the fractional stochastic integral

$$\int_0^t f(s,\omega)dB_s$$

where B_t is given by (32) and $H \in (0,1)$, we follow the work by Thao (2002). Even though some authors (e.g., Alos et al., 2000, Alos&Nualart, 2001, Decreusefond et al., 1999 and Duncan et al., 2000) have discussed many contents for a fractional stochastic calculus with respect to this process B_t, they are complicated to apply in practice and also difficult for numerics. The definition given by Thao is quite a simple approach to fractional stochastic integration with an orientation to possible applications in physics and finance. Of course a simple approach usually leads to a less general result, but his aim is an essay of doing something for feasible applications.

1.7.3 Fractional Stochastic Integration

Let a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t\geq 0}, P)$ be given where $\mathcal{F}_t^W$ is a σ-algebra generated by standard Brownian motion $W = (W_t)_{t\geq 0}$. Suppose that

$f(t)$ is a deterministic function of bounded variation on $[0,T]$ and the fractional process B_t is given as in (32):

$$B_t = \int_0^t (t-s)^\alpha dW(s), \quad \alpha = H - \frac{1}{2}, 0 < H < 1.$$

Then the integral $\int_0^t B_s df(s)$ is well defined in the sense of Riemann-Stieltjes for almost all w.

Definition 1.4. The fractional stochastic integral of $f(t)$ is a stochastic process I_t defined as

$$I_t := \int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df(s)$$

Now suppose $(f_t(\omega))_{t\geq 0}$ is a stochastic process on $(\Omega, \mathcal{F}, P)$ whose sample paths are of bounded variation on $[0,T]$ for almost every $\omega \in \Omega$.

Definition 1.5. The fractional stochastic integral of $f(t,\omega)$ is a stochastic process I_t defined as

$$I_t = \int_0^t f(s,\omega)dB_s = f(t,\omega)B_t - \int_0^t B_s df(s,\omega) - [f,B]_t. \tag{33}$$

Remark 1.1. (i) The pathwise integral in the right hand side of (33) exists in the sense of Riemann-Stieltjes for almost all ω.

(ii) If the function $f(t,\omega)$ has absolutely continuous sample paths (for instance, if it is Lipschitzian with respect to t) then it is of bounded variation and so its integral $I_t = \int_0^t f(s,\omega)dB_s$ exists.

Theorem 1.6. *Suppose that the process $f(t,\omega)$ has continuous sample paths and is of bounded variation on $[0,T]$ such that $E\int_0^t f^2(s,\omega)ds < \infty$. Then the stochastic integral*

$$I_t^\varepsilon = \int_0^t f(s,\omega)dB_s^\varepsilon, \tag{34}$$

where $B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s, \alpha = H - \frac{1}{2}, 0 < H < 1,$ *converges in* $L^2(\Omega)$ *as* $\varepsilon \to 0$ *to* $I_t = \int_0^t f(s,\omega) dB_s$ *defined as in (33). This convergence is uniform with respect to* $t \in [0,T]$.

Proof. (Thao, 2003) Since

$$\begin{aligned} E\left(B_t^\varepsilon\right)^2 &= E\left(\int_0^t (t-s+\varepsilon)^\alpha dW_s\right)^2 = E\int_0^t (t-s+\varepsilon)^{2\alpha} ds \\ &= -\left.\frac{(t-s+\varepsilon)^{2\alpha+1}}{2\alpha+1}\right|_{s=0}^{s=t} = -\left(\frac{\varepsilon^{2\alpha+1}}{2\alpha+1} - \frac{(t+\varepsilon)^{2\alpha+1}}{2\alpha+1}\right) < \infty \end{aligned}$$

it follows from Theorem 1.5 that B_t^ε is a square integrable martingale. Therefore the stochastic integral $I_t^\varepsilon = \int_0^t f(s,\omega) dB_s^\varepsilon$ exists. An application of the formula of integration by parts to I_t^ε gives us

$$I_t^\varepsilon = \int_0^t f(s,\omega) dB_s^\varepsilon = f(t,\omega) B_t^\varepsilon - \int_0^t B_s^\varepsilon df(s,\omega) - [f, B^\varepsilon]_t.$$

Denote by $\|\cdot\|$ the norm in the space $L^2(\Omega)$ and taking account of properties of quadratic variations we have

$$\begin{aligned} \|I_t - I_t^\varepsilon\| &= \left\|\int_0^t f(s,\omega) dB_s - \int_0^t f(s,\omega) dB_s^\varepsilon\right\| \\ &= \left\|\int_0^t f(s,\omega) d\left(B_s - B_s^\varepsilon\right)\right\| \\ &= \left\|f(t,\omega)\left(B_t - B_t^\varepsilon\right) - \int_0^t \left(B_s - B_s^\varepsilon\right) df(s,\omega) - [f, B - B^\varepsilon]_t\right\| \\ &\leq \|f(t,\omega)\| \|B_t - B_t^\varepsilon\| + \left\|\int_0^t \left(B_s - B_s^\varepsilon\right) df(s,\omega)\right\| + \|[f, B - B^\varepsilon]_t\|. \end{aligned}$$

An analogous argument as in the proof of Theorem 1.5 yields

$$\sup_{0 \leq t \leq T} \|B_t - B_t^\varepsilon\| \leq C\varepsilon^{\frac{1}{2}+\alpha},$$

where $\alpha = H - \frac{1}{2}, H \in (0,1)$ and $C > 0$ is some constant. Then

$$\|f(t,\omega)\| \|B_t - B_t^\varepsilon\| \leq MC\varepsilon^{\frac{1}{2}+\alpha} \tag{35}$$

where $M = \max_{0 \le t \le T} \|f(t,\omega)\|$ (the maximum exists since $E\,|f(t,\omega)|^2$ is continuous with respect to $t \in [0,T]$). Moreover, we have

$$\|[f, B - B^{\varepsilon}]_t\| \le \|f(t,\omega)\| \, \|B_t - B_t^{\varepsilon}\| \le MC\varepsilon^{\frac{1}{2}+\alpha}. \tag{36}$$

On the other hand we see that

$$\begin{aligned}
\left\| \int_0^t (B_s - B_s^{\varepsilon})\, df(s,\omega) \right\| &\le \left\| \int_0^t \|B_s - B_s^{\varepsilon}\|\, df(s,\omega) \right\| \\
&\le C\varepsilon^{\frac{1}{2}+\alpha} \left\| \int_0^t df(s,\omega) \right\| \\
&\le C\varepsilon^{\frac{1}{2}+\alpha} \left(\|f(t,\omega)\| + \|f(0,\omega)\| \right) \\
&\le 2CM\varepsilon^{\frac{1}{2}+\alpha}.
\end{aligned} \tag{37}$$

It follows from (35), (36) and (37) that

$$\|I_t - I_t^{\varepsilon}\| \le 4CM\varepsilon^{\frac{1}{2}+\alpha}.$$

Hence

$$\sup_{0 \le t \le T} \|I_t - I_t^{\varepsilon}\| \le 4CM\varepsilon^{\frac{1}{2}+\alpha} \to 0, \text{ as } \varepsilon \to 0.$$

Therefore, $I_t \to I_t^{\varepsilon}$ in $L^2(\Omega)$ as $\varepsilon \to 0$ uniformly with respect to $t \in [0,T]$. □

Remark 1.2. Theorem 1.6 is proved for the L^2-convergence of $I_t \to I_t^{\varepsilon}$ in the case that f is of bounded variation. This motivates us to define the fractional stochastic integral for any stochastic process $f(t,\omega)$ as follows.

Definition 1.6. Let $f(t,\omega)$ be a stochastic process with continuous path. Then the fractional stochastic integral of $f(t,\omega)$ is defined by

$$\int_0^t f(s,\omega) dB_s := L^2 - \lim_{\varepsilon \to 0} \int_0^t f(s,\omega) dB_s^{\varepsilon},$$

whenever the limit exists in $L^2(\Omega, \mathcal{F}, P)$, where $B_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ and $B_t^{\varepsilon} = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dW_s$ for $0 < H < 1$.

1.7.4 Fractional Stochastic Differential Equations

Consider an equation formally written as

$$\begin{aligned} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_t|_{t=0} &= X_0, t \in [0, T], \end{aligned} \tag{38}$$

where $B_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$.

Definition 1.7. A solution of (38) is a stochastic process $(X_t, t \geq 0)$ adapted to the $\sigma-$algebra $\mathcal{F}_t \equiv \sigma(X_0, B_s, s \leq t), t \in [0, T]$ and satisfying the following relation:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, 0 \leq t \leq T,$$

where the stochastic integral in the right hand side exists in the sense of Definition 1.6 if $\sigma(t, X_t)$ has sample paths of bounded variation in $[0, T]$.

Theorem 1.7. *Suppose that coefficients $b(t, x)$ and $\sigma(t, x)$ of the equation (38) satisfy the Lipschitz conditions with respect to x:*

1. $|b(t, x) - b(t, y)| \leq k_1 |x - y|$

2. $|\sigma(t, x) - \sigma(t, y)| \leq k_2 |x - y|$

and $\sigma(t, x)$ is of bounded variation with respect to $t \in [0, T]$ for every fixed x, and $\sigma(t, x)$ has bounded derivative with respect to x. Moreover, X_0 is supposed to be a square-integrable random variable: $E|X_0|^2 < \infty$. Then there exists a unique solution to equation (38).

Proof. (see Thao, 2002)

Chapter II

On The Absence of Arbitrage Opportunity for The Fractional Black-Scholes Model

In this Chapter, after introducing an approximate approach to a fractional Black-Scholes Model and to a fractional Langevin equation, we discuss the principle of Absence of Arbitrage Opportunity (principle of AAO) for this model: We prove that in spite of the fact that, in general, there exists an arbitrage opportunity for a fractional model, there will be no arbitrage for our approximate model while the approximation can be made with any exactitude. This is an advantage of our approximate approach.

2.1 Introduction to A Fractional Black-Scholes Model

Following Black-Scholes (1973), we first assume that the price $S = (S_t)_{t\geq 0}$ of a risky asset at time t is given by *geometric Brownian motion* of the form

$$S_t = S_0 \exp\left(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t\right), \forall t \in [0, T], \tag{1}$$

where $W = (W_t)_{t\geq 0}$ is Brownian motion, S_0 is a given random variable such that $ES_0^2 < \infty$, μ and σ are constants. The motivation for this assumption on S_t comes from the fact that S_t is the unique strong solution of the linear stochastic differential equation

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dW_u, \forall t \in [0, T],$$

which can be formally written as

$$dS_t = \mu S_t dt + \sigma S_t dW_t, S_{t=0} = S_0. \tag{2}$$

where μ is also known as the *drift rate* or *rate of return* of the price S_t and σ as the *volatility* (which measures the standard deviation of the return $\frac{dS_t}{S_t}$). Let us note that the Brownian motion W_t is called the *driving process* of the stochastic differential equation (2) or, in other words, the stochastic differential equation (2) *is driven by* Brownian motion W_t.

If $\sigma = 0$, the equation (2) becomes an ordinary differential equation which, in fact, describes an investment on a non-risky asset (e.g., a bank account). The initial capital S_0 grows, from $t = 0$, continuously compounded with the interest rate μ to be

$$S_t = S_0 e^{\mu t}$$

at time t. If μ is a function of t then S_0 grows from the initial time $t = 0$ to be

$$S_t = S_0 \exp\left(\int_0^t \mu(s) ds\right)$$

at time t. On the other hand, if one knows the amount, say, S_t that one would get at the future time t, one can also find its present value by discounting it at the same rate of growth. That is, if it grows to become S_t continuously compounded with the rate μ then its present value at $t = 0$ is

$$e^{-\mu t} S_t.$$

The value $e^{-\mu t} S_t$ is called the *discounted value* or the *present value* of S_t at the rate μ.

Let us observe that the drift rate μ and the volatility σ could be some adapted stochastic processes satisfying some conditions. So that the equation (2) could be in the form

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t, S_{t=0} = S_0. \tag{3}$$

Since the strong solution of (3) possesses the Markov property, that means to say roughly that its future behavior depends only on its immediately previous values and not on its values long time ago, then, the strong solutions of (3) have no memory. However, in practice, the stock price S_t at t may leave long-range consequences: S_t must thus be a long memory process.

A stochastic process $X = (X_t)_{t\geq 0}$ is said to have *long memory* (or *long-range dependence*) if $\sum_{n=1}^{\infty} r(n) = \infty$ where $r(n) = E[X_1(X_n - X_{n-1})]$. Instead of (2), we consider *a fractional Black-Scholes model* defined by the following fractional stochastic differential equation

$$\begin{aligned} dS_t &= S_t(\mu dt + \sigma dB_t), 0 \leq t \leq T, \qquad (4) \\ S_{t=0} &= S_0, \end{aligned}$$

where $B_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ and H is the Hurst index, $0 < H < 1$.

Now we consider an approximate model defined for each $\varepsilon > 0$ by

$$\begin{aligned} dS_t^\varepsilon &= S_t^\varepsilon(\mu dt + \sigma dB_t^\varepsilon), 0 \leq t \leq T \qquad (5) \\ S_{t=0}^\varepsilon &= S_0 \text{ (same initial condition as in (4))}, \end{aligned}$$

where $B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dW_s$. We can prove that:

(i) B_t^ε is a semimartingale and $B_t^\varepsilon \to B_t$, in $L^2(\Omega)$ $t \in [0,T]$, as $\varepsilon \to 0$ (this assertion is mentioned already in Chapter I)

(ii) The solution S_t^ε of (5) converges in $L^2(\Omega)$ to the exact solution S_t of (4) as $\varepsilon \to 0$.

Furthermore, the convergence mentioned in (i) and (ii) is uniform with respect to t. That is we have the following theorem.

Theorem 2.1. *The solution of (5), for* $\|S_0\|^2 = E\,|S_0|^2 < \infty$, *is given by*

$$S_t^{\varepsilon} = S_0 \exp(-\frac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma\varepsilon^{\alpha}W_t + \int_0^t H_s^{\varepsilon}ds)$$

where $\alpha = H - \frac{1}{2}$,

$$H_t^{\varepsilon} = \mu + \alpha\sigma\int_0^t (t-s+\varepsilon)^{\alpha-1}dW_s.$$

Furthermore, for $H > \frac{1}{2}$ *the stochastic process* S_t^* *defined by*

$$S_t^* = S_0 \exp(\mu t + \sigma B_t)$$

is the limit in $L^2(\Omega)$ *of* S_t^{ε} *as* $\varepsilon \to 0$. *This limit is uniform with respect to* $t \in [0,T]$. *Moreover,* S_t^* *is the unique solution of the fractional stock pricing model (4).*

Proof. [Thao&Thomas-Agnan, 2003] Replacing $dB_t^{\varepsilon} = \alpha\varphi_t^{\varepsilon}dt + \varepsilon^{\alpha}dW_t$, where $\varphi_t^{\varepsilon} = \int_0^t (t-s+\varepsilon)^{\alpha-1}dW_s$, in the equation (5) we obtain

$$dS_t^{\varepsilon} = S_t^{\varepsilon}\left[\mu + \alpha\sigma\varphi_t^{\varepsilon}\right]dt + \sigma\varepsilon^{\alpha}S_t^{\varepsilon}dW_t \tag{6}$$

or

$$\frac{dS_t^{\varepsilon}}{S_t^{\varepsilon}} = H_t^{\varepsilon}dt + \sigma\varepsilon^{\alpha}dW_t \tag{7}$$

where $H_t^{\varepsilon} = \mu + \alpha\sigma\varphi_t^{\varepsilon}$.

An application of Ito formula to the function $f(u) = \log u$ for $u = S_t^{\varepsilon} > 0$ yields

$$\log S_t^{\varepsilon} = \log S_0 + \int_0^t \frac{dS_s^{\varepsilon}}{S_s^{\varepsilon}} + \frac{1}{2}\int_0^t -\frac{1}{(S_s^{\varepsilon})^2}\left(\sigma\varepsilon^{\alpha}S_s^{\varepsilon}\right)^2 ds$$

or

$$\int_0^t \frac{dS_s^{\varepsilon}}{S_s^{\varepsilon}} = \log\frac{S_t^{\varepsilon}}{S_0} + \frac{1}{2}\left(\sigma\varepsilon^{\alpha}\right)^2 t. \tag{8}$$

In combining (7) and (8) we get

$$\log \frac{S_t^\varepsilon}{S_0} + \frac{1}{2}\left(\sigma\varepsilon^\alpha\right)^2 t = \int_0^t H_s^\varepsilon ds + \sigma\varepsilon^\alpha W_t \tag{9}$$

or

$$S_t^\varepsilon = S_0 \exp\left(-\frac{1}{2}\left(\sigma\varepsilon^\alpha\right)^2 t + \sigma\varepsilon^\alpha W_t + \int_0^t H_s^\varepsilon ds\right). \tag{10}$$

On the other hand we have

$$\int_0^t H_s^\varepsilon ds = \mu t + \alpha\sigma \int_0^t \varphi_s^\varepsilon ds,$$

and it follows from the semimartingale expression of B_t^ε (see the proof of Theorem 1.4) that

$$\int_0^t \varphi_s^\varepsilon ds = \frac{1}{\alpha}\left(B_t^\varepsilon - \varepsilon^\alpha W_t\right).$$

Therefore

$$\int_0^t H_s^\varepsilon ds = \mu t + \sigma B_t^\varepsilon - \sigma\varepsilon^\alpha W_t.$$

And we have at last:

$$S_t^\varepsilon = S_0 \exp\left(\mu t - \frac{1}{2}\left(\sigma\varepsilon^\alpha\right)^2 t + \sigma B_t^\varepsilon\right).$$

One can see that if $\varepsilon \to 0, \alpha = H - \frac{1}{2} > 0$ then $\frac{1}{2}\left(\sigma\varepsilon^\alpha\right)^2 t \to 0$ and we have shown (Theorem 1.5) that $B_t^\varepsilon \to B_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$. In fact, we consider the process S_t^* defined as

$$S_t^* = S_0 \exp\left(\mu t + \sigma B_t\right).$$

We are now to show that S_t^* is the limit of S_t^ε in $L^2(\Omega)$ as $\varepsilon \to 0$. We have

$$\begin{aligned} S_t^\varepsilon - S_t^* &= S_0 \exp\left(\mu t - \frac{1}{2}\left(\sigma\varepsilon^\alpha\right)^2 t + \sigma B_t^\varepsilon\right) - S_0 \exp\left(\mu t + \sigma B_t\right) \\ &= S_0 \exp\left(\mu t + \sigma B_t\right)\left(\exp\left[-\frac{1}{2}\left(\sigma\varepsilon^\alpha\right)^2 t + \sigma\left(B_t^\varepsilon - B_t\right)\right] - 1\right). \end{aligned} \tag{11}$$

Denoting the norm in $L^2(\Omega)$ by $\|\cdot\|$, we see that

$$\|S_0\|^2 = ES_0^2 < \infty \text{ by hypothesis of the theorem,}$$

and

$$\|\exp(\mu t + \sigma B_t)\| \le e^{\mu t}\exp(\sigma\|B_t\|) \le e^{\mu T}\exp\left(\sigma\frac{T^{\alpha-\frac{1}{2}}}{2\alpha-1}\right)$$

since, by virtue of Ito integration isometry,

$$\|B_t\|^2 = E\left[\int_0^t (t-s)^\alpha dW_s\right]^2 = E\int_0^t (t-s)^{2\alpha} ds = \frac{t^{2\alpha-1}}{2\alpha-1}.$$

Moreover, it follows form the relation $e^{\|A\|} - 1 = \|A\| + o(\|A\|)$ that

$$\begin{aligned} &\exp\left(\left\|-\tfrac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma\left(B_t^\varepsilon - B_t\right)\right\|\right) - 1 \\ &\qquad = \left\|-\tfrac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma\left(B_t^\varepsilon - B_t\right)\right\| + o\left(\left\|-\tfrac{1}{2}\sigma^2\varepsilon^{2\alpha} + \sigma\left(B_t^\varepsilon - B_t\right)\right\|\right) \\ &\qquad \le \tfrac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma\left\|B_t^\varepsilon - B_t\right\| + o\left(\tfrac{1}{2}\sigma^2\varepsilon^{2\alpha} + \sigma\left\|B_t^\varepsilon - B_t\right\|\right). \end{aligned}$$

It follows from Theorem 1.5 that $\|B_t^\varepsilon - B_t\| \le C(\alpha)\varepsilon^{\frac{1}{2}+\alpha}$, $\alpha = H - \frac{1}{2} > 0$ since $H > \frac{1}{2}$. Hence

$$\begin{aligned} &\left\|\exp\left(\left\|-\tfrac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma\left(B_t^\varepsilon - B_t\right)\right\|\right) - 1\right\| \\ &\qquad \le \frac{1}{2}\sigma^2\varepsilon^{2\alpha}T + \sigma C(\alpha)\varepsilon^{\frac{1}{2}+\alpha} + o\left(\sigma C(\alpha)\varepsilon^{\frac{1}{2}+\alpha} + \frac{1}{2}\sigma^2\varepsilon^{2\alpha}\right). \end{aligned} \tag{12}$$

Since the right hand side of (12) does not depend on t and approaches zero when $\varepsilon \to 0$. Therefore, one can see from (11) and (12) that $S_t^\varepsilon \to S_t^*$ in $L^2(\Omega)$ as $\varepsilon \to 0$ and the convergence is uniform with respect to t. $\square$

2.2 Fractional Langevin Equations

One knows from mathematical finance that an interest model can be expressed by a Langevin equation. So for the fractional model case, we try to introduce a fractional Langevin equation. The classical Langevin dynamics describes the motion of a linear dynamical system of particles perturbed by a Wiener-white noise:

$$\begin{aligned} dX_t &= -bX_t dt + \sigma dW_t \\ X_t|_{t=0} &= X_0, t \in [0,T] \end{aligned} \tag{13}$$

where b and σ are some positive constants. The solution X_t of (13) expressing the state of the system at time t, is known as an Ornstein-Uhlenbeck process:

$$X_t = X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)} dW_s, 0 \leq t \leq T. \tag{14}$$

This process is of crucial importance in the study of theory of stochastic processes. It is a typical process having properties of a stationary Gauss-Markov and self-similar process. As a Markov process, it presents a loss-memory evolution of the system (13).

So, in order to study long memory dynamics for this kind linear dynamical system, Thao & Nguyen (2002) consider the following equation

$$\begin{aligned} dX_t &= -bX_t dt + \sigma dB_t \\ X_t|_{t=0} &= X_0, t \in [0,T], \end{aligned} \tag{15}$$

The equation (15) is called a *fractional Langevin equation.* This is a simplest particular case of fractional stochastic differential equations where the volatility σ is a constant and the *drift* is a linear function of X_t. We can see (from (15)) that X_t is a stochastic process, adapted to $\sigma-$algebra $\mathcal{F}_t = \sigma(X_s, s \leq t; X_0)$ and satisfying the following relation

$$X_t = X_0 - b \int_0^t X_s ds + \sigma B_t.$$

And the approximately fractional equation corresponding to (15) is given for every $\varepsilon > 0$ by:

$$dX_t^\varepsilon = -bX_t^\varepsilon dt + \sigma dB_t^\varepsilon, X_0^\varepsilon = X_0, \tag{16}$$

where $B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s, \alpha = H - \frac{1}{2}$, $H \in (0, \frac{1}{2})$. The solution of (16) is explicitly expressed by:

$$X_t^\varepsilon = X_0 e^{-bt} + \sigma\varepsilon^\alpha \int_0^t e^{-b(t-s)} dW_s + \sigma \int_0^t e^{-b(t-s)} \varphi(s) ds, \tag{17}$$

where $\varphi(s) = \alpha \int_0^t (t-s+\varepsilon)^{\alpha-1} dW_s$.

To see this let us consider the equations

$$dZ_t = -bZ_t dt + \sigma\varepsilon^\alpha dW_t. \tag{18}$$

and

$$dY_t = -bY_t dt - \sigma\varphi(t) dt \tag{19}$$

We see that (18) is a classical Langevin equation whose solution is an Ornstein-Uhlenbeck process:

$$Z_t = Z_0 e^{-bt} + \sigma\varepsilon^\alpha \int_0^t e^{-b(t-s)} dW_s$$

where Z_0 is an initial value of Z_t, that is supposed to be a random variable independent of $(W_t)_{t\in[0,T]}$. The equation (19) is an ordinary differential equation for every fixed ω and its solution is

$$Y_t = Y_0 e^{-bt} - \sigma \int_0^t e^{-b(t-s)} \varphi(s) ds$$

where Y_0, an initial value of Y_t, is independent of $(Wt)_{t\in[0,T]}$. It is easy to see that for $X_t := Y_t + Z_t$ with $X_0 := Y_0 + Z_0$, we have

$$\begin{aligned} dX_t \quad : & \quad = dY_t + dZ_t = -bX_t dt + \sigma\varepsilon^\alpha dW_t - \sigma\varphi(t) dt \\ = & \quad -bX_t dt + \sigma\left(\varepsilon^\alpha dW_t - \varphi(t) dt\right) \\ = & \quad -bX_t dt + \sigma dB_t^\varepsilon \end{aligned}$$

which is in fact (16). Moreover the solution $Y_t + Z_t$ is exactly the right hand side of (17) and hence solves (16).

Now, for the convergence of solution, suppose that X_t and X_t^ε are solutions of (15) and (16), respectively:

$$dX_t = -bX_t dt + \sigma dB_t, 0 \le t \le T,$$

and

$$dX_t^\varepsilon = -bX_t^\varepsilon dt + \sigma dB_t^\varepsilon, 0 \le t \le T.$$

Then

$$X_t - X_t^\varepsilon = -b \int_0^t (X_s - X_s^\varepsilon)\, ds + \sigma (B_t - B_t^\varepsilon)$$

and hence

$$\|X_t - X_t^\varepsilon\| = \left\| b \int_0^t (X_s - X_s^\varepsilon)\, ds \right\| + \sigma \|B_t - B_t^\varepsilon\| .$$

Moreover, it follows from Theorem A.6 that

$$\|X_t - X_t^\varepsilon\| = b \int_0^t \|X_s - X_s^\varepsilon\|\, ds + \sigma C(\alpha) \varepsilon^{\frac{1}{2}+\alpha}, 0 \le t \le T \tag{20}$$

where $\|\cdot\|$ denotes the norm in $L^2(\Omega)$.

A standard application of Gronwall's lemma starting from (20) gives us:

$$\|X_t - X_t^\varepsilon\| \le \sigma C(\alpha) \varepsilon^{\frac{1}{2}+\alpha} e^{bt}.$$

Therefore,

$$\sup_{0 \le t \le T} \|X_t - X_t^\varepsilon\| \le \sigma C(\alpha) \varepsilon^{\frac{1}{2}+\alpha} e^{bT} \to 0$$

as $\varepsilon \to 0$. So $X_t^\varepsilon \to X_t$ in $L^2(\Omega)$ as $\varepsilon \to 0$ uniformly with respect to t.

2.3 Arbitrage and Martingale Measure

In this section, we discuss several terms in the financial market such as derivative security, options, riskless/risky assets, portfolio, arbitrage opportunity, etc. The concept of (absence of) arbitrage opportunity is discussed via the derivation of

Black-Scholes partial differential equation (Black-Scholes PDE). Furthermore, we explain what it is meant by a martingale measure and how it relates to the absence of arbitrage opportunity. Finally, at the end of this section, a criterion of free arbitrage is given.

In financial markets, every item that is *traded* is of one of two types. The first type is a *basic asset* (*basic security*) such as a stock, a bond, or a unit of currency. The second type is a *derivative security* (also called *contingent claim*) whose value depends on the value of another basic security such as a stock or a bond. In this case the more basic security is called an *underlying asset*.

In this thesis, the derivative security of interest is European (call/put) option. A *European call (put) option* is a contract that gives the purchaser a *right*, but not obligation, to *purchase (sell)* the underlying asset, say stock, at a fixed price K, called *exercise price* or *strike price*, at a specified time T, called time of *maturity* or *expiration*. The person or firm who formulates the contract and offers it for sale is termed *the writer*. The person or firm who purchases the contract is termed *the holder*. If at the maturity time T the stock price S_T is greater than the exercise price K it makes sense to a holder of a call option to exercise it (claim his right). He would purchase a share of stock (worth S_T) at price K and making a profit of $S_T - K$. On the other hand, if $S_T < K$ at maturity a sensible holder would not exercise the option (to purchase a share of stock price K and sell it at a lower price S_T) since otherwise he would make a loss of $K - S_T$. In this case the option expires worthless. Thus the *value (payoff)* of a call option at maturity time T can be written as

$$\max(S_T - K, 0) = (S_T - K)^+ . \tag{21}$$

Similarly, if at the maturity time T the stock price $S_T < K$ it would make financial sense to a holder of put option to exercise it so that he can sell a share of stock worth S_T at a higher price K and make a profit of $K - S_T$. Again, if $K < S_T$ at maturity, a sensible holder of put option would not exercise it since otherwise he would make a loss of $S_T - K$. Thus the payoff of a put option at maturity T can be written as

$$\max\left(K - S_T, 0\right) = \left(K - S_T\right)^+ . \tag{22}$$

The payoffs (21) and (22) are also called the *option values*. Since nobody knows what is going to happen in the market situation, a trader usually invests in several securities at the same time in order to distribute his risk. These securities constitute his *portfolio*.

- **Arbitrage Opportunity**

An arbitrage opportunity is the opportunity in which one can obtain a profit without risk (gain from zero). However, in the financial market it is assumed that the market is fair i.e., there is no arbitrage opportunity in the market. This can be loosely stated as "there is no such thing as a free lunch". Moreover, if one wants a high return one should be willing to face a high risk. Hence the contents of the absence of arbitrage opportunity is of interest in this content.

Let us consider Black-Scholes analysis. Suppose that trader A has an option (no matter put or call) whose value $X(t, S)$ depends only on time t and stock price S satisfied the stochastic differential equation (2). It is assumed that there are no transaction costs and during the life of the option the underlying asset pays no dividend. Moreover, trading of the underlying asset can take place continuously, and this asset is divisible. The function X is required to have at least one t derivative and two S derivatives. Using Ito's lemma, we can write

$$dX = \sigma S \frac{\partial X}{\partial S} dW + \left(\mu S \frac{\partial X}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 X}{\partial S^2} + \frac{\partial X}{\partial t} \right) dt. \tag{23}$$

Now suppose that his portfolio consists of one option and $-\Delta$ shares of stock (the underlying asset). The value (wealth) of the portfolio is

$$\Pi = X - \Delta S. \tag{24}$$

In Black-Scholes analysis, it is assumed that the portfolio is *self-financing* i.e., the value of the portfolio depends only on the movement of the stock price. Hence, the jump in the value of this portfolio in one time-step (written in infinitesimal notation) is

$$d\Pi = dX - \Delta dS.$$

Here Δ is held fixed during the time-step. Putting (2), (23) and (24) together, we find that

$$d\Pi = \sigma S \left(\frac{\partial X}{\partial S} - \Delta \right) dW + \left(\mu S \frac{\partial X}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 X}{\partial S^2} + \frac{\partial X}{\partial t} - \mu \Delta S \right) dt. \tag{25}$$

In the equation above we can eliminate the random component by choosing

$$\Delta = \frac{\partial X}{\partial S}. \tag{26}$$

This results in the equation (25):

$$d\Pi = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 X}{\partial S^2} + \frac{\partial X}{\partial t} \right) dt. \tag{27}$$

On the other hand, with the assumption of no transaction costs, if trader B invests an amount Π in riskless assets, he would see a growth (in infinitesimal notation) of $r\Pi dt$ in a time-step dt where r is the rate of return of riskless assets investment. The risk free interest rate r and the asset volatility σ are known function of time during the life of the option. By the assumption that the market is fair (no arbitrage), with the same initial capital, the two traders should gain an equal amount during the same period of time. Thus we have

$$r\Pi dt = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 X}{\partial S^2} + \frac{\partial X}{\partial t} \right) dt. \tag{28}$$

Substituting (24) and (26) into (28) and dividing throughout by dt we arrive at

$$\frac{\partial X}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 X}{\partial S^2} + rS\frac{\partial X}{\partial S} - rX = 0. \tag{29}$$

This equation is known as the *Black-Scholes partial differential equation.*

- **Martingale Measure**

Recall that a stochastic process $M = (M_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ adapted to the filtration $\mathcal{M} = (\mathcal{M}_t)_{t\geq 0}$ is called a martingale if $E_P\,|M| < \infty$ and for any s, t such $0 \leq s \leq t \leq T$,

$$E_P(M_t|\mathcal{M}_s) = M_s \text{ a.s.} \tag{30}$$

Sometimes, in order to be specified which probability we are working under, the martingale M is also written P-martingale.

Let a probability space $(\Omega, \mathcal{F}, P)$ be given, $W = (W_t, \mathcal{F}_t^W)_{t\geq 0}$ be a standard Brownian motion where $(\mathcal{F}_t^W)_{t\geq 0}$ satisfies the usual condition (see section A.1) and, as usual, $\mathcal{F}_s^W \subset \mathcal{F}_t^W \subset \mathcal{F}, s \leq t$.

Definition 2.1. (a) A market is an $\mathcal{F}_t^W-$adapted $(n+1)$-dimensional Ito process $X(t) = (X_0(t), X_1(t), ..., X_n(t))$; $t \in [0, T]$ which we will assume has the form

$$dX_0(t) = \rho(t,\omega)X_0(t)dt; X_0(0) = 1 \tag{31}$$

and

$$dX_i(t) = \mu_i(t,\omega)dt + \sum_{i=1}^{m} \sigma_{ij}(t,\omega)dW_j(t); X_i(0) = x_i \tag{32}$$

where σ_i is row number i of the $n \times m$ matrix $[\sigma_{ij}]; 1 \leq i \leq n \in \mathbb{N}$, set of positive integer numbers.

(b) The market $(X_t)_{t\in[0,T]}$ is called normalized if $X_0(t) \equiv 1$.

(c) A portfolio in the market $(X_t)_{t\in[0,T]}$ is an $(n+1)$-dimensional (t,ω)-measurable and $\mathcal{F}_t^W$-adapted stochastic process

$$\theta(t,\omega)=(\theta_0(t,\omega),\theta_1(t,\omega),...,\theta_n(t,\omega))\,;t\in[0,T]. \tag{33}$$

(d) The value at time t of a portfolio $\theta(t)$ is defined by

$$V(t,\omega)=V^{\theta}(t,\omega)=\theta(t)\cdot X(t)=\sum_{i=0}^{n}\theta_i(t)X_i(t) \tag{34}$$

where $\cdot$ denotes inner product in $\mathbb{R}^{n+1}$.

(e) The portfolio $\theta(t)$ is called *self-financing* if

$$dV(t)=\theta(t)\cdot dX(t) \tag{35}$$

i.e.

$$V(t)=V(0)+\int_0^t\theta(s)dX(s)\text{ for }t\in[0,T].$$

(f) A portfolio $\theta(t)$ which is self-financing is called admissible if the corresponding value process $V^{\theta}(t)$ is (t,ω) a.s. lower bounded, i.e. there exists $K=K(\theta)<\infty$ such that

$$V^{\theta}(t,\omega)\geq -K\text{ for a.s. }(t,\omega)\in[0,T]\times\Omega$$

(There must be a limit to how much debt the creditors can tolerate).

Remark 2.1. (a) The market can always be nomalized by defining

$$\overline{X}_i(t)=X_0^{-1}(t)X_i(t);i=1,2,...,n. \tag{36}$$

The market

$$\overline{X}(t)=\left(1,\overline{X}_1(t),\overline{X}_2(t),...,\overline{X}_n(t)\right)$$

is called the *normalization* of $X(t)$.

(b) Let us consider the normalized market

$$\overline{X}(t)=\left(1,\frac{1}{X_0(t)}X_1(t),\frac{1}{X_0(t)}X_2(t),...,\frac{1}{X_0(t)}X_n(t)\right).$$

The term $\frac{1}{X_0(t)}$ is called the *discounted coefficient* and $\left(\frac{1}{X_0(t)}X_i(t)\right)_{t\geq 0}, i = 1, 2, ..., n$ are called *discounted price* process.

The following notion of probability measure leads us to the (mathematical) concept of arbitrage free market. If P and Q are two probability measures on $(\Omega, \mathcal{F})$, we say that Q is *absolutely continuous with respect to* P (written $Q \ll P$) if and only if $P(A) = 0$ implies $Q(A) = 0$ for $A \in \mathcal{F}$. Moreover, if Q is an *indefinite integral* with respect to P, i.e.,

$$Q(A) = \int_A hdP, \forall A \in \mathcal{F}$$

where h is a Borel measurable function provided that $\int_A hdP < \infty$, then $Q \ll P$. The converse assertion is guaranteed by the *Radon Nikodym Theorem* (see, e.g., Dudley, 2002) that if P and Q are two probability measures on $(\Omega, \mathcal{F})$ and $Q \ll P$ then there exists a Borel measurable function $g : \Omega \to \mathbb{R}$ such that $Q(A) = \int_A hdP, \forall A \in \mathcal{F}$. Furthermore, the two measures are said to be *equivalent* if $Q \ll P$ and $P \ll Q$, written $Q \sim P$.

Let us consider a market $X(t), t \in [0, T]$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, P)$. A probability Q on $(\Omega, \mathcal{F})$ is called a *martingale measure* (or *risk-neutral probability*) if and only if

(i) $Q \sim P$, i.e. the measure Q and P have the same null sets, and

(ii) almost surely,

$$E_Q\left[\frac{1}{X_0(t)}X_i(t)\middle|\mathcal{F}_s\right] = \frac{1}{X_0(s)}X_i(s), 0 \leq s < t \leq T.$$

Now we are in position to state a criterion of free arbitrage given by Oksendal (1998) of a financial market. It is very useful in many applications. And we will use it to prove that there is no more arbitrage opportunity for our approximate model.

In the theorem following, let $W = (W_t, \mathcal{F}_t^W)_{t\geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}_s^W \subset \mathcal{F}_t^W \subset \mathcal{F}, s \leq t$ and $X = (X_t, \mathcal{F}_t^W)_{t\geq 0}$ be a market consisting of two assets given by

$$dX_t^0 = \rho(t,\omega)X_t^0 dt$$

and

$$dX_t^1 = \mu(t,\omega)dt + \sigma(t,\omega)dW_t$$

where $\mu(t,\omega)$, $\sigma(t,\omega)$, $\rho(t,\omega) \in \mathbb{R}$.

Theorem 2.2. *Suppose there exists a square integrable process $u(t,\omega)$ (t,ω)-measurable adapted to the Brownian filtration $(\mathcal{F}_t^W)_{t\geq 0}$ such that*

$$\sigma(t,\omega)u(t,\omega) = \mu(t,\omega) - \rho(t,\omega)X_t^1 \tag{37}$$

for almost all (t,ω) and such that

$$E_P \exp\left(\frac{1}{2}\int_0^T u^2(t,\omega)dt\right) < \infty. \tag{38}$$

Then the market $(X_t)_{t\in[0,T]}$ has no arbitrage. Conversely, if the market $(X_t)_{t\in[0,T]}$ has no arbitrage, then there exists an $\mathcal{F}_t^W$-adapted, (t,ω)-measurable process $u(t,\omega)$ such that

$$\sigma(t,\omega)u(t,\omega) = \mu(t,\omega) - \rho(t,\omega)X_t^1$$

for almost all (t,ω).

Proof. (see Oksendal, 1998, p.256)

2.4 Principle of AAO for Approximate Models

In this section, we give a very important result concerning AAO Principle by proving that there will be no more arbitrage for our approximate model. This is one of our main results in the Thesis.

The solution of the classical Black-Scholes model driven by standard Brownian motion is a Markov process. However, in some empirical studies of financial time series (see, e.g. Shiryaev, 1999. p. 365) it has been demonstrated that the log-returns have strong aftereffects (long-range dependence), i.e. the prices remember their past. The self-similar and long-range dependence properties of the fractional Brownian motion make this process a suitable candidate for replacing the Brownian motion. The fractional Brownian motion, however, is not a semimartingale when $H \neq \frac{1}{2}$. Therefore it may be suspected that a stock price model driven by it would admit arbitrage opportunities. Indeed, Rogers (1997) constructed the arbitrage by using the path properties of the fractional Brownian motion. He showed that arbitrage is possible when the risky asset has a log-normal price driven by a fractional Brownian motion. In 1999, Shiryaev gave an explicit construction of an arbitrage strategy based on continuous time trading (see also Cheridito, 2001, for more information on arbitrage in this kind of models and Sottinen, 2001, where arbitrage appears in a natural discrete time binomial approximation to the continuous time model).

Let us consider the fractional Black-Scholes model of the form

$$dS_t = \mu S_t dt + \sigma S_t dB_t, S_{t=0} = S_0 \tag{39}$$

where $B_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$, a fractional process and S_t is the option price at time $t \in [0, T]$. One can see that by the result of Rogers (1997), there exist arbitrage opportunities for the fractional Black-Scholes model (39). Hence our interest is of an approximate model for (39) which has no more arbitrage. In stead of (39), the approximate model driven by $B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dW_s$ of the form

$$dS_t^\varepsilon = \mu S_t^\varepsilon dt + \sigma S_t^\varepsilon dB_t^\varepsilon, S_{t=0}^\varepsilon = S_0 \tag{40}$$

is considered. It is seen in section 2.1 that B_t^ε and S_t^ε converge to B_t and S_t in

L^2, uniformly in t as $\varepsilon \to 0$, respectively.

Based on the criterion of free arbitrage for a financial market given in the previous section we will prove that for this approximate model there is no more arbitrage opportunity, since the option price S_t can be approximated at any exactitude by a suitable S_t^ε. In (40), for every $\varepsilon > 0$,

$$dB_t^\varepsilon = \alpha\varphi_t^\varepsilon dt + \varepsilon^\alpha dW_t$$

with $\alpha = H - \frac{1}{2}, H \in (0,1)$ and

$$\varphi_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{\alpha-1} dW_s.$$

By substituting the expression of dB_t^ε above into (40) one gets

$$\begin{aligned} dS_t^\varepsilon &= \mu S_t^\varepsilon dt + \sigma S_t^\varepsilon \left(\alpha\varphi_t^\varepsilon dt + \varepsilon^\alpha dW_t\right) \\ &= \left(\mu S_t^\varepsilon + \alpha\sigma\varphi_t^\varepsilon S_t^\varepsilon\right) dt + \varepsilon^\alpha \sigma S_t^\varepsilon dW_t \\ &= \left(\mu + \alpha\sigma\varphi_t^\varepsilon\right) S_t^\varepsilon dt + \varepsilon^\alpha \sigma S_t^\varepsilon dW_t \end{aligned} \tag{41}$$

Our main result here is to prove that the approximate model (40) has no-arbitrage. We introduce an additional asset X_t^0 satisfying

$$dX_t^0 = \rho(t,\omega) X_t^0 dt$$

where $\rho(t,\omega) = (\alpha\sigma - \varepsilon^{\alpha+1})\varphi_t^\varepsilon$. Then according to (37) we have

$$\begin{aligned} u(t,\omega) &= \frac{\mu(t,\omega) - \rho(t,\omega)}{\sigma(t,\omega)} \\ &= \frac{\mu + \alpha\sigma\varphi_t^\varepsilon - (\alpha\sigma - \varepsilon^{\alpha+1})\varphi_t^\varepsilon}{\varepsilon^\alpha \sigma} \\ &= \frac{\mu + \varepsilon^{\alpha+1}\varphi_t^\varepsilon}{\varepsilon^\alpha \sigma} \\ &= \frac{\mu}{\varepsilon^\alpha \sigma} + \frac{\varepsilon\varphi_t^\varepsilon}{\sigma}. \end{aligned} \tag{42}$$

Since $(a+b)^2 \le 2(a^2+b^2)$ then from (42) we have

$$u^2 \le 2\left[\left(\frac{\mu}{\varepsilon^\alpha\sigma}\right)^2 + \left(\frac{\varepsilon\varphi_t^\varepsilon}{\sigma}\right)^2\right]. \tag{43}$$

Let a probability space $(\Omega, \mathcal{F}, P)$ be given, $W = (W_t, \mathcal{F}_t^W)_{t\geq 0}$ be a standard Brownian motion where $(\mathcal{F}_t^W)_{t\geq 0}$ satisfies the usual condition and, as usual, $\mathcal{F}_s^W \subset \mathcal{F}_t^W \subset \mathcal{F}, s \leq t$. In order to prove that the approximate model (40) has no-arbitrage, the following lemmas will be needed.

Lemma 2.1. *Suppose that X is a $\mathcal{N}(0,\nu^2)$-random variable and a is a constant such that $(1-2\nu^2a^2) > 0$. Then $Ee^{a^2X^2}$ is finite.*

Proof. Since X is $\mathcal{N}(0,\nu^2)$-distributed, by definition

$$Ee^{a^2X^2} = \int_{-\infty}^{\infty} e^{a^2x^2} f(x)dx$$

where $f(x)$ is the probability density function of X, i.e. $f(x) = \frac{1}{\nu\sqrt{2\pi}}\exp(-\frac{x^2}{2\nu^2})$. Hence

$$\begin{aligned} Ee^{a^2X^2} &= \frac{1}{\nu\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{a^2x^2}e^{-\frac{x^2}{2\nu^2}}dx \\ &= \frac{1}{\nu\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}(\frac{1}{\nu^2}-2a^2)}dx \\ &= \frac{1}{\nu\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{2\left(\frac{\nu^2}{1-2\nu^2a^2}\right)}\right]dx. \end{aligned} \tag{44}$$

Since $1-2\nu^2a^2 > 0$ and by putting $\sigma = \frac{|\nu|}{\sqrt{1-2\nu^2a^2}}$, we see that

$$Ee^{a^2X^2} = \sigma < \infty. \qquad \square$$

Lemma 2.2. *$E\exp\left(\frac{\varepsilon}{\sigma}\varphi_t^\varepsilon\right)^2$ is finite for every $t \geq 0$ and sufficiently small ε.*

*Proof***.** We notice that for every t, φ_t^ε is a Gaussian random variable. Indeed, we see that

$$\varphi_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{\alpha-1}dW_s = P - \lim_{|\Delta|\to 0}\sum_{k=1}^{n}(t-s_{k-1}+\varepsilon)\left[W_{s_k} - W_{s_{k-1}}\right],$$

where Δ is any partition of $[0,t]$ into n subintervals $[s_{k-1}, s_k)$, $k = 1, ..., n$. Since each increment $W_{s_k} - W_{s_{k-1}}$ is a Gaussian random variable, so is φ_t^ε. We now find the expectation μ and the variance ν^2 of φ_t^ε.

In account of a well-known property of Itô integral saying that the expectation of an Itô integral is equal to zero, we can write:

$$\mu = E\varphi_t^\varepsilon = E\int_0^t (t-s+\varepsilon)^{\alpha-1}\, dW_s = 0.$$

On the other hand,

$$\nu^2 = E\left[\varphi_t^\varepsilon - E\varphi_t^\varepsilon\right]^2 = E\left[\varphi_t^\varepsilon\right]^2 = E\left[\int_0^t (t-s+\varepsilon)^{\alpha-1}\, dW_s\right]^2.$$

By the isometry from the Itô integration we see that

$$E\left[\int_0^t (t-s+\varepsilon)^{\alpha-1}\, dW_s\right]^2 = \int_0^t (t-s+\varepsilon)^{2\alpha-2}\, ds.$$

Now we have

$$\int_0^t (t-s+\varepsilon)^{2\alpha-2}\, ds = \frac{1}{2\alpha-1}\left[(t+\varepsilon)^{2\alpha-1} - \varepsilon^{2\alpha-1}\right].$$

So, for every t, φ_t^ε is a centered normal random variable $\mathcal{N}(0,\nu^2)$ with the variance ν^2 defined by

$$\nu^2 = \frac{1}{1-2\alpha}\left[(t+\varepsilon)^{2\alpha-1} - \varepsilon^{2\alpha-1}\right], \quad 1-2\alpha = 2(1-H) > 0.$$

In order to apply Lemma (2.1) we calculate $1-2\nu^2a^2$, where $a = \frac{\varepsilon}{\sigma}$. We see that

$$2\nu^2\frac{\varepsilon^2}{\sigma^2} = \frac{2}{\sigma^2(1-2\alpha)}\left[\varepsilon^2\left((t+\varepsilon)^{2\alpha-1} - \varepsilon^{2\alpha-1}\right)\right]$$

which can be made smaller than 1 for small enough ε. Then we have $Ee^{\left(\frac{\varepsilon}{\sigma}\varphi_t^\varepsilon\right)^2} < \infty$ as required. $\square$

The theorem below gives the most important result showing that our approximate model (40) is free of arbitrage. It makes use of the preceding two lemmas and the Theorem 2.2 above in Section 2.3. One can see that the market in Theorem 2.2, (32), is driven by Brownian motion W_t while our approximate model (40) is driven by semimartingale B_t^ε. However, we still make use of Theorem 2.2 *since the semimartingale B_t^ε can be expressed as a sum of a process of bounded variation and a Brownian motion. Therefore, the approximate model (40) driven by the semimartingale B_t^ε is in fact the model (41) driven by Brownian motion to which Theorem 2.2 can be applied.*

Theorem 2.3. *For sufficiently small $\varepsilon > 0$, there is no-arbitrage for the approximate model (40).*

Proof. By Theorem 2.2, one needs to show that the Novikov condition:

$$E\exp\left(\frac{1}{2}\int_0^T u^2(t,\omega)dt\right) < \infty \tag{45}$$

holds where u is defined by (42). Since μ, σ, ε and α are constants and by (43) one gets

$$\begin{aligned} E\exp\left(\frac{1}{2}\int_0^T u^2(t,\omega)dt\right) &\leq E\exp\left[\frac{1}{2}\int_0^T \left[2\left(\frac{\mu}{\varepsilon^\alpha\sigma}\right)^2 + 2\left(\frac{\varepsilon\varphi_t^\varepsilon}{\sigma}\right)^2\right]dt\right] \\ &= E\left[\exp\left(\int_0^T \left(\frac{\mu}{\varepsilon^\alpha\sigma}\right)^2 dt\right)\exp\left(\int_0^T \left(\frac{\varepsilon\varphi_t^\varepsilon}{\sigma}\right)^2 dt\right)\right] \\ &= \exp\left(\left(\frac{\mu}{\varepsilon^\alpha\sigma}\right)^2 T\right) E\exp\left(\int_0^T \left(\frac{\varepsilon\varphi_t^\varepsilon}{\sigma}\right)^2 dt\right). \end{aligned} \tag{46}$$

It follows from (46) one only needs to show that $E\exp\left(\int_0^T \left(\frac{\varepsilon\varphi_t^\varepsilon}{\sigma}\right)^2 dt\right)$ is finite. By Jensen's inequality,

$$\exp\left(\int_0^T \left(\frac{\varepsilon\varphi_t^\varepsilon}{\sigma}\right)^2 dt\right) \leq \int_0^T \exp\left(\frac{\varepsilon\varphi_t^\varepsilon}{\sigma}\right)^2 dt$$

and by Fubini's theorem and by lemma (2.2), $E\exp\left(\frac{\varepsilon\varphi_t^\varepsilon}{\sigma}\right)^2 < \infty$, one gets

$$E\exp\left(\int_0^T \left(\frac{\varepsilon\varphi_t^\varepsilon}{\sigma}\right)^2 dt\right) \leq \int_0^T \left(E\exp\left(\frac{\varepsilon\varphi_t^\varepsilon}{\sigma}\right)^2\right) dt < \infty. \tag{47}$$

Hence the Novikov's condition (45) holds by (46) and (47). The proof of Theorem 2.3 is thus complete. $\square$

2.5 Conclusion

As said before, the presence of arbitrage opportunities in fractional models for finance, based on path-wise constructions of fractional stochastic calculus, is very often met and this is a considerable gap in applying these models to financial practice. Our results presented in this Chapter overcomes this difficulty: In spite of the fact that there exists arbitrage opportunity for the fractional model, this model can be approximated with any level of exactitude by another model where there is no more arbitrage! So in practice one can use a suitable approximate model which expresses good enough the long-range dependence of financial prices.

Chapter III

A Fractional Vasiček Model

As we have seen in Section 2.3, in the derivation of Black-Scholes PDE, we assumed the interest rate to be a constant (or at least a known function of time) throughout a period of time, e.g., [0,T]. In reality this is far from the case. In fact, it has been studied that the movement of interest rate also exhibits long-range dependence (see, e.g., Duan et al., 2001 and Gil-Alana, 2004). Namely, they exhibit significant autocorrelation between observations widely separated in time.

In this chapter, the fractional Vasiček model is considered and the approximate model for the fractional Vasiček model is also given. The solution to the approximate model is found and we proved that it converges, in $L^2(\Omega)$, to the solution of the original model.

3.1 Introduction

In the continuous time $t \geq 0$ the standard definition of the bank interest rate $r = (r_t)_{t\geq 0}$ is based on the relation

$$dB_t = r_t B_t dt$$

where $B = (B_t)_{t\geq 0}$ is a bank account. Clearly,

$$r_t = \frac{d}{dt}(\ln B_t)$$

and

$$B_t = B_0 \exp\left(\int_0^t r_s ds\right).$$

The *interest rate* (also called *short rate* of interest, *spot rate* or *instantaneous rate* of interest), in fact, reflects the price of *borrowing/investing* money from/in the bank. The concept of interest rate plays even more important role in the "indirect data" of the evolution of share prices. This explains the existence of a variety of models with interest rate $r = (r_t)_{t\geq 0}$ described by diffusion equations

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t \tag{1}$$

where $(\mu(t,x))_{t\geq 0}, (\sigma(t,x))_{t\geq 0}$ are given stochastic processes and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion. It is known that the solution to (1) is always a Markov process that has no memory. So the model (1) is not suitable since, in the financial markets, each value of r_t can influence upon its behavior in some time range. Correspondingly, the prices of bonds at a time t can have some consequences on their price some time later.

Let the filtered probability space $(\Omega, \mathcal{F}_t, (\mathcal{F}_t^W)_{t\geq 0}, P)$ satisfying the usual conditions be given and $r = (r_t)_{t\geq 0}$ be an $\mathcal{F}_t^W$-measurable stochastic process. As usual, $W = (W_t)_{t\geq 0}$ is a standard Brownian motion. The classical Vasicek model is the model of the form

$$dr_t = (b - ar_t)dt + \sigma dW_t \tag{2}$$

where a, σ are positive constants and b is any real number.

However, with $B_t = \int_0^t (t-s)^\alpha dW_s$, $\alpha = H - \frac{1}{2}$, $H(0,1)$ we consider the model of the form

$$\begin{aligned} dr_t &= (b - ar_t)dt + \sigma dB_t, \quad a > 0 \qquad (3) \\ r_t|_{t=0} &= r_0. \end{aligned}$$

where r_0 is a given square integrable random variable. The SDE (3) is also called the *fractional Vasicek model.* According to our definition of fractional stochastic integral (Definition (1.6)) the exact solution of (3) is

$$r_t = \int_0^t (b - ar_s)ds + \sigma B_t.$$

3.2 Approximate Fractional Vasiček Model

We have seen that the driving process B_t of (3) is not a semimartingale unless $H = \frac{1}{2}$ (Theorem 1.2). Hence, in order to solve (3) we consider the approximate equation

$$dr_t^\varepsilon = (b - ar_t^\varepsilon)dt + \sigma dB_t^\varepsilon \tag{4}$$

where $B_t^\varepsilon = \int_0^t (t - s + \varepsilon)^\alpha dW_s$ is a semimartingale (Theorem 1.4). In fact B_t^ε can be expressed (see the proof of Theorem 1.4) as

$$B_t^\varepsilon = \alpha \int_0^t \varphi_s^\varepsilon ds + \varepsilon^\alpha W_t \tag{5}$$

where $\varphi_t^\varepsilon = \int_0^t (t - s + \varepsilon)^{\alpha-1} dW_s$. Writing (5) in differential form:

$$dB_t^\varepsilon = \alpha \varphi_t^\varepsilon dt + \varepsilon^\alpha dW_t \tag{6}$$

and substituting it into (4) to obtain

$$dr_t^\varepsilon = (b - ar_t^\varepsilon)dt + \sigma \left(\alpha \varphi_t^\varepsilon dt + \varepsilon^\alpha dW_t\right), \quad r_t^\varepsilon|_{t=0} = r_0, \tag{7}$$

where r_0 is given at time $t = 0$. Rewrite (7) to obtain

$$dr_t^\varepsilon = \left[(b - ar_t^\varepsilon) + \sigma\varphi(t)\right] dt + \sigma\varepsilon^\alpha dW_t \quad r_t^\varepsilon|_{t=0} = r_0, \tag{8}$$

where $\varphi(t) = \alpha \varphi_t^\varepsilon$. We will solve the approximate model (8) (by Theorem 3.1) and then we prove that its solution converges in L^2 to the solution of the original model (by Theorem 3.2).

Remark 3.1. According to Theorem 1.7 in Chapter I, the problem (3) with $b(t,r) = b - ar$ and $\sigma(t,r) = \sigma$ satisfy the Lipschitz conditions with respect to r and since σ is constant, it is of bounded variation and has bounded derivative. Then there exists a unique solution to the problem (3).

The following theorem yields the solution r_t^ε to the problem (4). In fact:

Theorem 3.1. *The solution of the approximate model (8) is given by:*

$$r_t^\varepsilon = \frac{b}{a} + \left(r_0 - \frac{b}{a}\right) e^{-at} + \sigma\varepsilon^\alpha \int_0^t e^{-a(t-s)} dW_s + \sigma \int_0^t \varphi(s) e^{-a(t-s)} ds.$$

Proof. Let us consider first the SDE

$$dx(t) = (b - ax(t))\, dt + \sigma\varepsilon^\alpha dW_t, \quad x(0) = x_0. \tag{9}$$

Set $u(t) = b - ax(t)$. Hence $u_0 = b - ax_0$, $\quad dx(t) = -\frac{du(t)}{a}$ and (9) becomes

$$-\frac{du(t)}{a} = u(t)dt + \sigma\varepsilon^\alpha dW_t$$

or

$$du(t) = -au(t)dt - a\sigma\varepsilon^\alpha dW_t. \tag{10}$$

The equation (10) is in fact the classical stochastic Langevin equation whose solution is given by

$$u(t) = u_0 e^{-at} - a\sigma\varepsilon^\alpha \int_0^t e^{-a(t-s)} dW_s. \tag{11}$$

Substituting back $u(t)$, as a function of $x(t)$, (11) becomes

$$b - ax(t) = (b - ax(t))\, e^{-at} - a\sigma\varepsilon^\alpha \int_0^t e^{-a(t-s)} dW_s$$

or

$$x(t) = \frac{b}{a} - \left(\frac{b}{a} - x_0\right) e^{-at} + \sigma\varepsilon^\alpha \int_0^t e^{-a(t-s)} dW_s. \tag{12}$$

Let us consider further an ordinary differential equations:

$$dy(t) = -ay(t)dt + \sigma\varphi(t)dt, \quad y(0) = y_0. \tag{13}$$

Solving (13) we get

$$y(t) = y_0 e^{-at} + \sigma \int_0^t \varphi(s) e^{-a(t-s)} ds. \tag{14}$$

Now, let $z(t) = x(t) + y(t)$ and $x_0 = y_0 = \frac{r_0}{2}$. Then

$$\begin{aligned} z(t) &= x(t) + y(t) \\ &= \frac{b}{a} - \left(\frac{b}{a} - x_0\right) e^{-at} + \sigma\varepsilon^\alpha \int_0^t e^{-a(t-s)} dW_s + y_0 e^{-at} + \sigma \int_0^t \varphi(s) e^{-a(t-s)} ds \\ &= \frac{b}{a} - \frac{b}{a} e^{-at} + (x_0 + y_0) e^{-at} + \sigma\varepsilon^\alpha \int_0^t e^{-a(t-s)} dW_s + \sigma \int_0^t \varphi(s) e^{-a(t-s)} ds. \\ &= \frac{b}{a} + \left(r_0 - \frac{b}{a}\right) e^{-at} + \sigma\varepsilon^\alpha \int_0^t e^{-a(t-s)} dW_s + \sigma \int_0^t \varphi(s) e^{-a(t-s)} ds. \end{aligned} \tag{15}$$

The process $\varphi(t)$ appearing in (15) can be simulated as illustrated in the proof of lemma 2.2. Moreover, we also have

$$\begin{aligned} dz(t) &= dx(t) + dy(t) \\ &= (b - ax(t))\, dt + \sigma\varepsilon^\alpha dW_t - ay(t)dt + \sigma\varphi(t)dt \\ &= (b - az(t) + \sigma\varphi(t))\, dt + \sigma\varepsilon^\alpha dW_t \end{aligned}$$

with $z(0) = x_0 + y_0 = r_0$, which is indeed the problem (8). Therefore, by uniqueness of the solution of (8) we get (15) is the solution to (8). $\square$

A natural question arises whether the solution r_t^ε of (4) would converge to the solution r_t of (3).

3.3 Convergence

Suppose that r_t and r_t^ε are solutions of (3) and (4), respectively:

$$dr_t = (b - ar_t)dt + \sigma dB_t, \quad 0 \le t \le T,$$

and

$$dr_t^\varepsilon = (b - ar_t^\varepsilon)dt + \sigma dB_t^\varepsilon, \quad 0 \le t \le T.$$

Now the convergence of r_t^ε to r_t as $\varepsilon \to 0$ can be shown below.

Theorem 3.2. *r_t^ε converges to r_t uniformly in $L^2(\Omega)$ as $\varepsilon \to 0$.*

Proof. We have

$$r_t - r_t^\varepsilon = -a \int_0^t (r_s - r_s^\varepsilon)ds + \sigma(B_t - B_t^\varepsilon),$$

then

$$\|r_t - r_t^\varepsilon\| \le \left\| a \int_0^t (r_s - r_s^\varepsilon)ds \right\| + \sigma \left\| B_t - B_t^\varepsilon \right\|, \tag{16}$$

where $\|\cdot\|$ denotes the norm in $L^2(\Omega)$. Since B_t^ε converges to B_t in $L^2(\Omega)$ when ε tends to zero and this convergence is uniform with respect to $t \in [0, T]$ (Theorem 1.5). We have,

$$\sup_{0 \le t \le T} \|B_t - B_t^\varepsilon\| \le C(\alpha)\varepsilon^{\alpha + \frac{1}{2}},$$

where $0 < \alpha < \frac{1}{2}$ and $C(\alpha)$ depends only on α (see the proof of Theorem 1.5). Therefore (16) becomes

$$\|r_t - r_t^\varepsilon\| \le a \int_0^t \|r_s - r_s^\varepsilon\| \, ds + \sigma C(\alpha)\varepsilon^{\alpha + \frac{1}{2}}. \tag{17}$$

A standard application of Gronwall's lemma to equation (17) gives:

$$\|r_t - r_t^\varepsilon\| \le e^{at}\sigma C(\alpha)\varepsilon^{\alpha + \frac{1}{2}}.$$

It follows that

$$\sup_{0 \le t \le T} \|r_t - r_t^\varepsilon\| \le e^{aT} C(\alpha)\varepsilon^{\alpha + \frac{1}{2}} \to 0,$$

as $\varepsilon \to 0$. The proof is completed. $\square$

It is known that a fractional Brownian motion differs from a standard Brownian motion and one of differences is that its increments are dependent and exhibit some long memory. Namely, the process r_t^ε at long time ago may influence upon its behavior today. Hence, with the fractional Vasiček model (3), the situation in real market is reflected more precisely than that by the classical one.

3.4 Conclusion

We have seen that in this Chapter, after introducing a fractional version for the well-known interest model of Vasicek, we have perfectly solved this model from our approximate approach by establishing and solving approximately fractional model and by showing that its solution converges to the exact solution. These results answer increasing demands from practice in considering long range consequence of interest values.

Chapter IV

A Fractional Ho-Lee Model

In this chapter, we consider the fractional version of the classical Ho-Lee model. We will investigate the solution of this model by studying its corresponding approximate model. It is also found that the solution to the approximate model converges, in $L^2(\Omega)$, to the solution of the original model.

4.1 Introduction

In the classical Vasicek model:

$$dr_t = (b - ar_t)dt + \sigma dW_t,$$

when r_t is large, the negative coefficient in front of dt mean that r_t will move down on average, if r_t is small, similarly the positive coefficient will raise r_t on average again. This phenomenon is called *mean reversion*. Mean-reverting models are used for modeling a process that "does not go anywhere". That is why they are used for interest rates. However, in the Vasiček model, interest rates can easily become negative which is not a very good property.

Let the filtered probability space $(\Omega, \mathcal{F}_t, (\mathcal{F}_t^W)_{t\geq 0}, P)$ satisfying the usual conditions be given and $r = (r_t)_{t\geq 0}$ be an $\mathcal{F}_t^W$-measurable stochastic process. As usual, $W = (W_t)_{t\geq 0}$ is a standard Brownian motion. The classical Ho-Lee Model is the model of the form

$$dr_t = b(t)r_tdt + \sigma dW_t.$$

where $b(t)$ is a deterministic function of t and finite on $[0, T]$ and σ is a positive constant.

4.2 Approximate Fractional Ho-Lee Model

Definition 4.1. The model for interest rate r_t expressed by

$$dr_t = b(t)r_t dt + \sigma dB_t \tag{1}$$

$$r_t|_{t=0} = r_0$$

where

$$B_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s, \quad H \in (0,1), \tag{2}$$

$b(t)$ is a deterministic function and r_0 is a square integrable random variable, is called the *fractional Ho-Lee model.*

Solving Method: In order to solve (1) we consider the approximate equation

$$dr_t^\varepsilon = b(t)r_t^\varepsilon dt + \sigma dB_t^\varepsilon, \tag{3}$$

where

$$B_t^\varepsilon = \alpha \int_0^t \varphi_s^\varepsilon ds + \varepsilon^\alpha W_t \tag{4}$$

with $\varphi_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{\alpha-1} dW_s$. Writing (4) in differential form

$$dB_t^\varepsilon = \alpha \varphi_t^\varepsilon dt + \varepsilon^\alpha dW_t. \tag{5}$$

Substituting (5) into (3) to obtain

$$dr_t^\varepsilon = b(t)r_t^\varepsilon dt + \sigma\left(\alpha\varphi_t^\varepsilon dt + \varepsilon^\alpha dW_t\right), \quad r^\varepsilon|_{t=0} = r_0, \tag{6}$$

where r_0 is given at time $t = 0$. Rewrite (6) to obtain

$$dr_t^\varepsilon = \left[b(t)r_t^\varepsilon + \sigma\alpha\varphi_t^\varepsilon\right]dt + \sigma\varepsilon^\alpha dW_t \quad r^\varepsilon|_{t=0} = r_0, \tag{7}$$

where $\varphi(t) = \alpha\varphi_t^\varepsilon$. We will solve the approximate model (7) (by Theorem 4.1) and then we prove that its solution converges in L^2 to the solution of the original model (by Theorem 4.2).

Remark 4.1. The problem (1) with $b(t,r) = b(t)r$ and $\sigma(t,r) = \sigma$ satisfies the Lipschitz conditions with respect to r_t. In fact, for any $s, t \in R$,

$$\begin{aligned} \|b(t)r_t - b(t)r_s\| &\leq |b(t)| \, \|r_t - r_s\| \\ &\leq L \, \|r_t - r_s\| \, , \end{aligned}$$

where $\|\cdot\|$ denotes the L^2-norm and $L = \max_{0\leq t\leq T} |b(t)|$. Moreover, since σ is constant, it is of bounded variation and has bounded derivative. Then by Theorem 1.7 there exists a unique solution to the problem (1).

We are now in the position to find the solution of (7) which is later shown to be an approximate of the solution of (1).

Theorem 4.1. *The solution of the approximate model (7) is the following:*

$$r_t^\varepsilon = e^{\int_0^t b(s)ds} \left(r_0 + \sigma\varepsilon^\alpha \int_0^t e^{-\int_0^s b(\tau)d\tau} dW_s + \sigma \int_0^t \varphi(s) e^{-\int_0^s b(\tau)d\tau} ds \right).$$

Proof. Recall that the solution of the general linear stochastic differential equation

$$dX(t) = (\alpha(t) + \beta(t)X(t))\,dt + (\gamma(t) + \delta(t)X(t))\,dW(t)$$

where α, β, γ, and δ are given adapted processes and continuous functions of t is given by

$$X(t) = U(t) \left(X(0) + \int_0^t \frac{\alpha(s) - \delta(s)\gamma(s)}{U(s)} ds + \int_0^t \frac{\gamma(s)}{U(s)} dW(s) \right)$$

where

$$U(t) = U(0) \exp \left(\int_0^t \left((\beta(s) - \frac{1}{2}\delta^2(s))ds \right) + \int_0^t \delta(s) dW(s) \right).$$

Consider first the SDE:

$$dx(t) = b(t)x(t)dt + \sigma\varepsilon^{\alpha}dW_t \tag{8}$$

with $x(0) = x_0$. Hence the solution of (8) is given by

$$x(t) = e^{\int_0^t b(s)ds}\left(x_0 + \int_0^t \sigma\varepsilon^{\alpha}e^{-\int_0^s b(u)du}dW_s\right).$$

Consider next an ordinary differential equation

$$dy(t) = b(t)y(t)dt + \sigma\varphi(t)dt \tag{9}$$

with $y(0) = y_0$. It is an ordinary differential equation and its solution is given by

$$y(t) = e^{\int_0^t b(s)ds}\left(y_0 + \sigma\int_0^t \varphi(s)e^{-\int_0^s b(\tau)d\tau}ds\right)$$

Now with $x_0 = y_0 = \frac{r_0}{2}$ and $z(t) = x(t) + y(t)$ we get

$$\begin{aligned}
dz(t) &= dx(t) + dy(t) \\
&= \left[b(t)\left(x(t) + y(t)\right) + \sigma\varphi(t)\right]dt + \sigma\varepsilon^{\alpha}dW_t \\
&= \left[b(t)z(t) + \sigma\varphi(t)\right]dt + \sigma\varepsilon^{\alpha}dW_t
\end{aligned}$$

and $z(0) = r_0$ which is, in fact, the problem (7). Therefore, by existence and uniqueness of solution (Theorem 1.9)

$$\begin{aligned}
z(t) &= x(t) + y(t) \\
&= e^{\int_0^t b(s)ds}\Bigg[(x_0 + y_0) + \sigma\varepsilon^{\alpha}\int_0^t e^{-\int_0^s b(\tau)d\tau}dW_s \\
&\quad + \sigma\int_0^t \varphi(s)e^{-\int_0^s b(\tau)d\tau}ds\Bigg] \\
&= e^{\int_0^t b(s)ds}\left(r_0 + \sigma\varepsilon^{\alpha}\int_0^t e^{-\int_0^s b(\tau)d\tau}dW_s + \sigma\int_0^t \varphi(s)e^{-\int_0^s b(\tau)d\tau}ds\right)
\end{aligned}$$

is the solution of (7). $\square$

4.3 Convergence

Suppose that r_t and r_t^ε are solutions of (1) and (3), respectively:

$$dr_t = b(t)r_t dt + \sigma dB_t, \quad 0 \le t \le T,$$

and

$$dr_t^\varepsilon = b(t)r_t^\varepsilon dt + \sigma dB_t^\varepsilon, \quad 0 \le t \le T.$$

Now the convergence of r_t^ε to r_t as $\varepsilon \to 0$ can be shown below.

Theorem 4.2. *r_t^ε converges to r_t uniformly in $L^2(\Omega)$ as $\varepsilon \to 0$.*

Proof. We have

$$r_t - r_t^\varepsilon = \int_0^t b(s)(r_s - r_s^\varepsilon)ds + \sigma(B_t - B_t^\varepsilon),$$

then with $\|\cdot\|$ represents the L^2-norm,

$$\|r_t - r_t^\varepsilon\| \le \int_0^t \|b(s)\| \, \|r_s - r_s^\varepsilon\| \, ds + \sigma \|B_t - B_t^\varepsilon\| \, . \tag{10}$$

Since B_t^ε converges to B_t in $L^2(\Omega)$ when ε tends to zero (Theorem 1.7) and this convergence is uniform with respect to $t \in [0, T]$. Hence,

$$\sup_{0 \le t \le T} \|B_t - B_t^\varepsilon\| \le C(\alpha)\varepsilon^{\alpha + \frac{1}{2}},$$

where $0 < \alpha < \frac{1}{2}$ and $L(\alpha)$ depends only on α as in the previous chapter. Set

$$b = \sup_{0 \le t \le T} |b(t)| \, .$$

Therefore (10) becomes

$$\|r_t - r_t^\varepsilon\| \le b \int_0^t \|r_s - r_s^\varepsilon\| \, ds + \sigma C(\alpha)\varepsilon^{\alpha + \frac{1}{2}}. \tag{11}$$

A standard application of Gronwall's lemma to (11) will give us:

$$\|r_t - r_t^\varepsilon\| \le e^{bt}\sigma C(\alpha)\varepsilon^{\alpha + \frac{1}{2}}.$$

It follows that

$$\sup_{0\le t\le T} \|r_t - r_t^{\varepsilon}\| \le e^{bT} C(\alpha)\varepsilon^{\alpha+\frac{1}{2}} \to 0,$$

as $\varepsilon \to 0$. The proof is completed. $\square$

Since the random source of the fractional Ho-Lee model has long memory, the model therefore reflects more precise the situation in real market where each state of interest can influence many reactions later in the market.

4.4 Conclusion

In this chapter, the same approximate approach has been used to completely solve the fractional case for Ho-Lee model that is frequently met in studying verious problems in finance.

Chapter V

A Fractional Hull-White Model

In this chapter, the fractional version of the classical Hull-White model is studied. Its approximate model is also given. The solution to the approximate model is found and its convergence is shown. In fact, the approximation can be made with any exactitude. Our main results presented in this Chapter are to appear in the Vietnam Journal of Mathematics.

5.1 Introduction

Let the filtered probability space $(\Omega, \mathcal{F}_t, (\mathcal{F}_t^W)_{t\geq 0}, P)$ satisfying the usual conditions be given and $r = (r_t)_{t\geq 0}$ be an $\mathcal{F}_t^W$-measurable stochastic process. As usual, $W = (W_t)_{t\geq 0}$ is a standard Brownian motion. It is well-known in mathematical finance that the Hull-White model for interest r_t has the following form

$$dr_t = (b(t) - a(t)r_t)\, dt + \sigma(t)dW_t,$$

where $a(t)$, $b(t)$ and $\sigma(t)$ are deterministic continuous functions of t and $a(t) > 0$, $\sigma(t) > 0$, and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion. This model is very useful in practice of financial markets, it gives also the price of zero-coupon bonds corresponding to each value of the rate r_t.

But each value of r_t can influence upon its behavior in some time range. Correspondingly, the prices of bonds at a time t can have some consequences on their price some time later. Similar to the Vasiček and Ho-Lee models, the classical Hull-White model is not suitable since its solution is always a Markov process that has no memory.

The purpose of this chapter is to introduce a fractional Hull-White model for the interest rate r_t for which the driving process is replaced by a fractional Brownian motion, $B_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s, H \in (0,1)$, a process of long memory.

5.2 Approximate Fractional Hull-White Model

Definition 5.1. The model for interest rate r_t expressed by

$$dr_t = (b(t) - a(t)r_t)\, dt + \sigma(t) dB_t, \quad a(t) > 0 \tag{1}$$

$$r_t|_{t=0} = r_0$$

where B_t is defined by (2), $a(t), b(t), \sigma(t)$ are deterministic functions with $a(t) > 0$, $\sigma(t)$ is of finite variation and r_0 a square integrable random variable, is called the *fractional Hull-White model.*

Solving Method: In order to solve (1) we consider and solve for the approximate equation

$$dr_t^\varepsilon = (b(t) - a(t)r_t^\varepsilon)\, dt + \sigma(t) dB_t^\varepsilon \tag{2}$$

where

$$B_t^\varepsilon = \alpha \int_0^t \varphi_s^\varepsilon ds + \varepsilon^\alpha W_t \tag{3}$$

with $\varphi_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{\alpha-1} dW_s$. Writing (3) in differential form:

$$dB_t^\varepsilon = \alpha \varphi_t^\varepsilon dt + \varepsilon^\alpha dW_t. \tag{4}$$

Substituting (4) into (2) to obtain

$$dr_t^\varepsilon = (b(t) - a(t)r_t^\varepsilon)\, dt + \sigma(t)\left(\alpha \varphi_t^\varepsilon dt + \varepsilon^\alpha dW_t\right), \quad r^\varepsilon|_{t=0} = r_0, \tag{5}$$

where r_0 is given at time $t = 0$. Rewrite (5) with $\varphi(t) = \alpha\varphi_t^\varepsilon$ to obtain

$$dr_t^\varepsilon = [b(t) - a(t)r_t^\varepsilon + \sigma(t)\varphi(t)]\,dt + \sigma(t)\varepsilon^\alpha dW_t, \quad r^\varepsilon|_{t=0} = r_0. \tag{6}$$

We will solve the approximate model (6) (by Theorem 5.1) and then we prove that its solution converges in L^2 to the solution of the original model (by Theorem 5.2).

Theorem 5.1. *The solution of the approximate model (6) is given as follows:*

$$r_t^\varepsilon = e^{-\int_0^t a(s)ds}\left[r_0 + \varepsilon^\alpha \int_0^t \sigma(s)e^{\int_0^s a(u)du}dW_s + \int_0^t (b(u) + \sigma(u)\varphi(u))\,e^{\int_0^u a(s)ds}du\right].$$

Proof. Consider a stochastic differential equation

$$dx(t) = -a(t)x(t)dt + \sigma(t)\varepsilon^\alpha dW_t \tag{7}$$

with $x(0) = x_0$. The SDE (7) is a linear stochastic differential equation whose solution is given by

$$x(t) = e^{-\int_0^t a(s)ds}\left[x_0 + \varepsilon^\alpha \int_0^t \sigma(s)e^{\int_0^s a(r)dr}dW_s\right].$$

Let us consider next an ordinary differential equation

$$dy(t) = (b(t) - a(t)y(t))\,dt + \sigma(t)\varphi(t)dt \tag{8}$$

with $y(0) = y_0$. Rewriting the equation (8) as:

$$\frac{dy(t)}{dt} + a(t)y(t) = b(t) + \sigma(t)\varphi(t).$$

This is a linear ordinary differential equation whose solution is given by:

$$y(t) = e^{-\int_0^t a(s)ds}\left[y_0 + \int_0^t (b(u) + \sigma(u)\varphi(u)e^{\int_0^u a(s)ds}du\right]. \tag{9}$$

Similar to the previous two chapters, let $z(t) = x(t) + y(t)$ and with $x_0 = y_0 = \frac{r_0}{2}$ we get $z(0) = r_0$ and

$$\begin{aligned} dz(t) &= (b(t) - a(t)\,(x(t) + y(t)) + \sigma(t)\varphi(t))\,dt + \sigma(t)\varepsilon^\alpha dW_t \\ &= (b(t) - a(t)z(t) + \sigma(t)\varphi(t))\,dt + \sigma(t)\varepsilon^\alpha dW_t \end{aligned}$$

which, in fact, is the problem (6). By existence and uniqueness of the solution for (6) we get

$$\begin{aligned} z(t) &= x(t) + y(t) \\ &= e^{-\int_0^t a(s)ds}\left[r_0 + \varepsilon^\alpha \int_0^t \sigma(s) e^{\int_0^s a(r)dr} dW_s + \int_0^t (b(u) + \sigma(u)\varphi(u) e^{\int_0^u a(s)ds} du\right] \end{aligned}$$

is the solution to (6). $\square$

5.3 Convergence

We note that the equation (1) is a fractional linear stochastic differential equation whose solution is defined by

$$r_t = r_0 + \int_0^t (b(s) - a(s) r_s)\, ds + \int_0^t \sigma(s) dB_s.$$

Under the regularity assumptions on $a(t)$ and $b(t)$, it is easy to verify that there exists such a unique solution for (1). Denote this solution by r_t and suppose that r_t^ε is the solution of the corresponding approximate model (6). Thus r_t and r_t^ε satisfy the following equations:

$$dr_t = (b(t) - a(t) r_t) dt + \sigma(t) dB_t, \quad 0 \le t \le T$$

$$dr_t^\varepsilon = (b(t) - a(t) r_t^\varepsilon) dt + \sigma(t) dB_t^\varepsilon, \quad 0 \le t \le T.$$

Theorem 5.2. *r_t^ε converges to r_t in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$ as $\varepsilon \to 0$.*

Proof. We have

$$r_t - r_t^\varepsilon = -\int_0^t a(s)\,(r_s - r_s^\varepsilon) ds + \sigma(t)(B_t - B_t^\varepsilon) - \int_0^t (B_s - B_s^\varepsilon) d\sigma(s). \tag{10}$$

Denote by $\|\cdot\|$ the norm in $L^2(\Omega)$. Then

$$\begin{aligned} \|r_t - r_t^\varepsilon\| \le & \int_0^t |a(s)|\, \|r_s - r_s^\varepsilon\| ds + |\sigma(t)| \|B_t - B_t^\varepsilon\| \\ & + \|B_t - B_t^\varepsilon\|\, |\sigma(t) - \sigma(0)|\,. \end{aligned} \tag{11}$$

Since $a(t)$ and $\sigma(t)$ are continuous, hence bounded, on $[0,T]$ then there exist positive constants M_1, M_2 such that

$$|a(t)| \leq M_1 = \max_{0\leq t\leq T} |a(t)| \text{ and } |\sigma(t)| \leq M_2 = \max_{0\leq t\leq T} |\sigma(t)|.$$

Hence,

$$\|r_t - r_t^\varepsilon\| \leq M_1 \int_0^t \|r_s - r_s^\varepsilon\| ds + 2M_2\|B_t - B_t^\varepsilon\|. \tag{12}$$

We know that $B_t^\varepsilon \to B_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0,T]$ and we have also the following estimate (see the proof of Theorem 1.7):

$$\|B_t - B_t^\varepsilon\|^2 = E\,|B_t - B_t^\varepsilon|^2 \leq C_3(\alpha)\,\varepsilon^{1+2\alpha}, \tag{13}$$

where $C_3(\alpha)$ is a constant depending only on α. Then

$$\|B_t - B_t^\varepsilon\| \leq K(\alpha)\,\varepsilon^{\frac{1}{2}+\alpha}, \tag{14}$$

where $K(\alpha) = \sqrt{C_3(\alpha)}$. It follows from (11), (12) and (13) that

$$\|r_t - r_t^\varepsilon\| \leq M_1 \int_0^t \|r_s - r_s^\varepsilon\| ds + M\,\varepsilon^{\frac{1}{2}+\alpha}, \tag{15}$$

where $M = 2M_2K(\alpha)$. A standard application of Gronwall Lemma will give us:

$$\|r_t - r_t^\varepsilon\| \leq e^{M_1 t} M\,\varepsilon^{\frac{1}{2}+\alpha}. \tag{16}$$

Hence

$$\sup_{0\leq t\leq T} \|r_t - r_t^\varepsilon\| \leq e^{M_1 T} M\,\varepsilon^{\frac{1}{2}+\alpha} \to 0 \tag{17}$$

as $\varepsilon \to 0$. The proof of Theorem 5.2 is thus complete. $\square$

5.4 Conclusion

Once again, by the long-range dependence property of the fractional Brownian motion, the fractional Hull-White model (1) reflects more precisely the movement of the interest rate in the market. Namely, each state of interest at time

t can influence upon the state of interest at long time after t. Our approximate method giving an answer to requirements of study on financial interest rates where the long-range dependence is mentioned.

Chapter VI

Applications

In this chapter, a sample path of standard Brownian motion is given. As examples, the empirical historical IBM-prices are simulated by classical Black-Scholes model and by approximate fractional Black-Scholes model. Both paths are illustrated against the empirical data. As we expected, the result of simulation shows that the latter pricing model give a better fit with the empirical data.

6.1 Sample Paths

For the simulation of Gaussian process in general, and Brownian motion in particular, there are numerous procedures. See also, Beran (1994, pp. 215-217), Embrechts et al. (2002, p. 71), Lamperton et al. (1996, p. 165) and Mikosch (1998, p. 51) for various methods of simulation.

Let us recall here that a process $X = (X_t)_{t \geq 0}$ has a self-similarity property with Hurst parameter H provided that for any $c > 0$,

$$X_{ct} \stackrel{d}{=} c^H X_t.$$

Since Brownian motion $W = (W_t)_{t \geq 0}$ is self-similar wiht Hurst parameter $H = \frac{1}{2}$, then

$$W_t \stackrel{d}{=} t^{\frac{1}{2}} W_1,$$

$W_1 \sim \mathcal{N}(0, 1)$. Therefore a sample path of Brownian motion can be easily simulated by an independent and identically $\mathcal{N}(0, 1)$ distributed random variable.

Figure 6.1a illustrates a sample path of standard Brownian motion (sample path of Brownian motion with $W_0 = 0$). Self-similarity is a distributional, not a path-wise property. In fact, sample paths of self-similar process look quantitatively similar, but they are not simply scaled copies of each other. Figure 6.1, illustrate this on different scales.

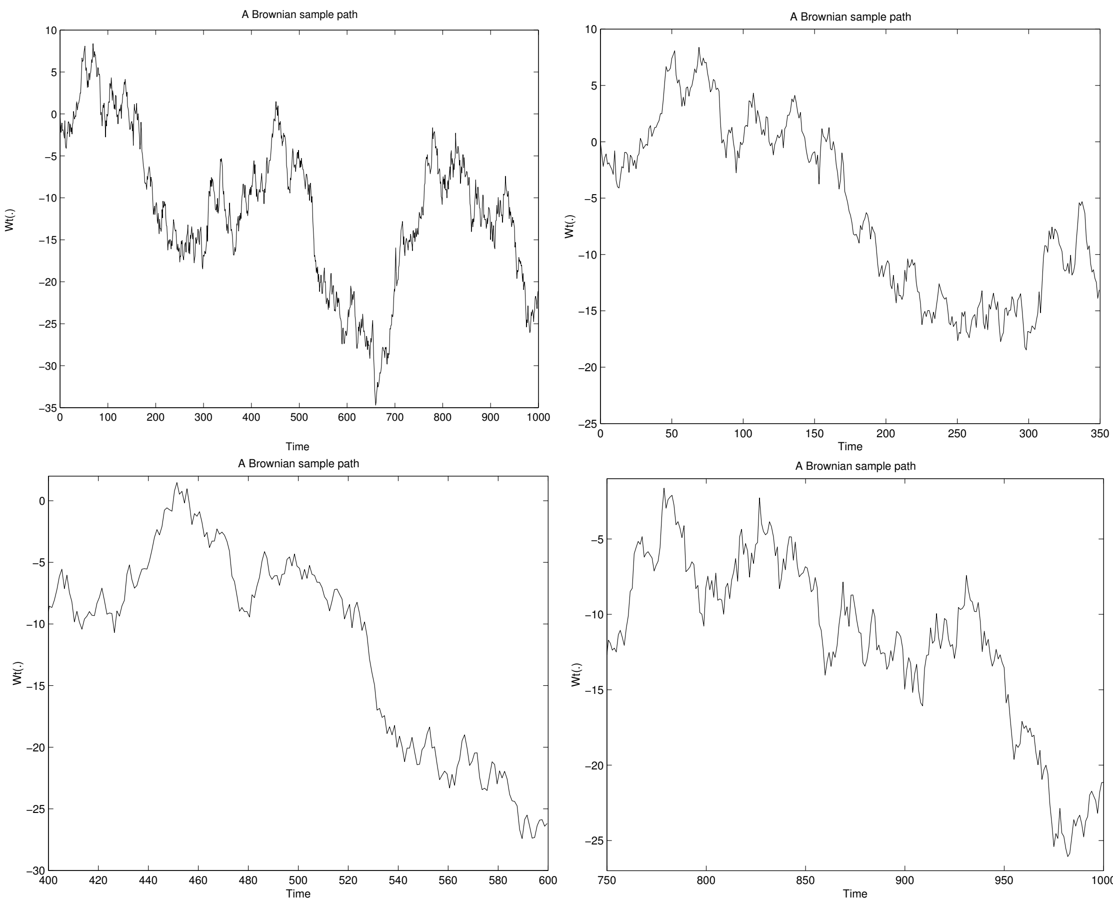

Figure 6.1: Self-similarity: the same Brownian sample paths on different scales. Sample paths of self-similar process look quantitatively similar but they are not scaled copied of each other.

For a sample path of fractional process, let us recall that the fractional process used in this thesis is

$$B_t = \int_0^t (t-s)^{\alpha} dW_s, \tag{1}$$

where $\alpha = H - \frac{1}{2}$, and the Hurst parameter $H \in (0,1)$. Using the same idea of the simulation of standard Brownian motion, a sample path of the fractional process (1) can be simulated, for fixed $t > 0$, as

$$\begin{aligned} B_t &\simeq \sum_{k-1}^{N} (t - \frac{k}{N} t)^{\alpha} \left[W_{(k+1)\frac{t}{N}} - W_{k\frac{t}{N}} \right] \\ &= \sum_{k-1}^{N} (t - \frac{k}{N} t)^{\alpha} \sqrt{\frac{t}{N}} \left[W_{(k+1)} - W_k \right] \\ &= \sqrt{\frac{t}{N}} \sum_{k-1}^{N} (t - \frac{k}{N} t)^{\alpha} g_k \end{aligned}$$

where $g_k \sim \mathcal{N}(0,1)$.

6.2 IBM-Simulated Prices

In this section we give the IBM[1]-simulated prices produced by geometric Brownian motion:

$$S_t = S_0 \exp\left(\sigma W_t + (\mu - \frac{1}{2}\sigma^2) t \right), \forall t \in [0,T], \tag{2}$$

whose random source is standard Brownian motion W_t and the prices simulated by our approximate solution to the fractional Black-Scholes model (4):

$$S_t^{\varepsilon} = S_0 \exp\left(\mu t - \frac{1}{2} \left(\sigma \varepsilon^{\alpha} \right)^2 t + \sigma B_t^{\varepsilon} \right) \tag{3}$$

plotted against empirical data illustrated in Figure 6.2 and Figure 6.3 , respectively. For comparative purposes, we compute the average relative percentage error (ARPE):

$$ARPE = \frac{1}{N} \sum_{k=1}^{N} \frac{|X_k - Y_k|}{X_k} \cdot 100$$

where N is the number of prices, $X = (X_k)_{k \geq 1}$ is the market price and $Y = (Y_k)_{k \geq 1}$ is the model prices.

[1]International Business Machines Corp (Paris)

The historical stock prices of IBM was obtained from *http://finance.yahoo.com.* The dataset consists of 264 open-prices of IBM starting from Aug 12, 2003 to August 20, 2004. In both pricing models, (2) and (3), the drift μ and volatility σ are kept fixed the same. The random source of (2) is the standard Brownian motion W_t while the random source of (3) is the process $B_t^{\varepsilon} = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dW_s$ in which the simulation of its paths is similar to that of our fractional process (1).

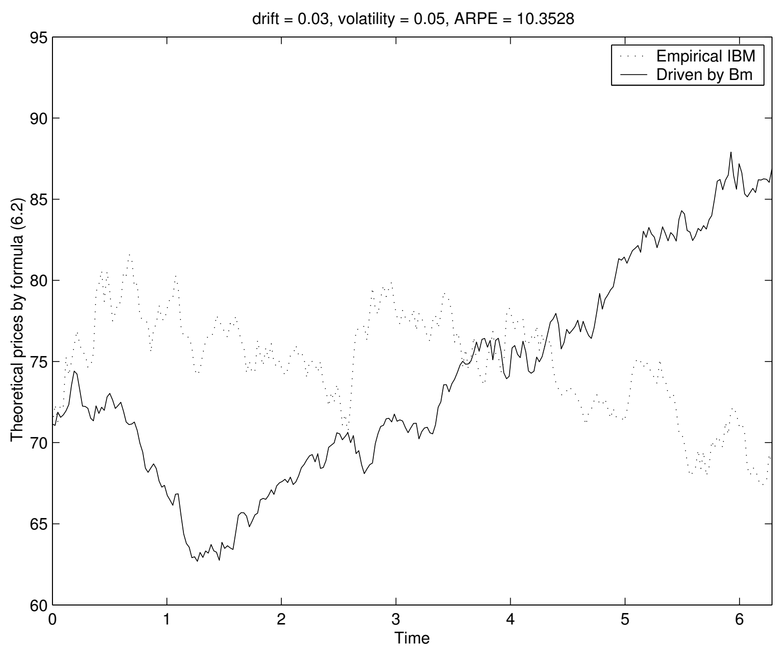

Figure 6.2: IBM-prices is simulated by the pricing models with the initial price = 71.15, drift = 0.03 and volatility = 0.05.

Now, let the ARPE by the model (2) and (3) be denoted by ARPE(B) and ARPE(FB), respectively. With $\mu = 0.03, \sigma = 0.05, H = 0.62$ and $\varepsilon = 0.00001$ are fixed. We worked out for 3,000 trails and stored the ARPE(B) and ARPE(FB) for each sampling. It is found that the averages of ARPE(B) and ARPE(FB) are 10.45% and 7.67%, respectively. While the variances are 23.28% and 8.07%, respectively. One can see (Figures 6.2 and 6.3) that the pricing model (3) gives a better fit the with real data than the pricing model (2). Moreover, in this case,

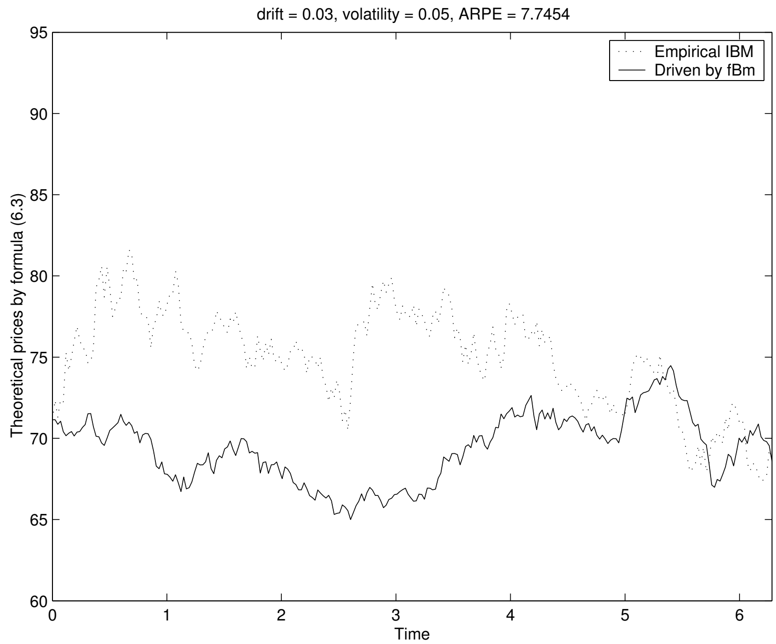

Figure 6.3: IBM-prices is simulated by the pricing models with the initial price = 71.15, drift = 0.03, volatility = 0.05, the Hurst parameter = 0.62 and $\varepsilon = 0.00001$.

the pricing model (3) fits with real data 2.78% better than the pricing model (2).

Remark 6.1. 1. In fact, by observation, during the experiments we have done many of 3000-sampling trails and found that even though, each time of the experiment, the averages of ARPE(B) and ARPE(FB) are not certain, the difference between them is quite certain. For example, in 18 3,000-sampling trails, the averages of ARPE(B) vary from 10.20 to 10.54 and ARPE(FB) from 7.53 to 7.77. Moreover, we found that, each time, the minimum and maximum of ARPE(B) are mostly higher than those of the ARPE(FB).

2. In some rare case the ARPE(FB) is down to 2.82% (see Figure 6.4).

3. However, in any case, the results depend on what data one uses. For some sets of data the theoretical price of classical Black-Scholes model is better and for some data the fractional Black-Scholes is better.

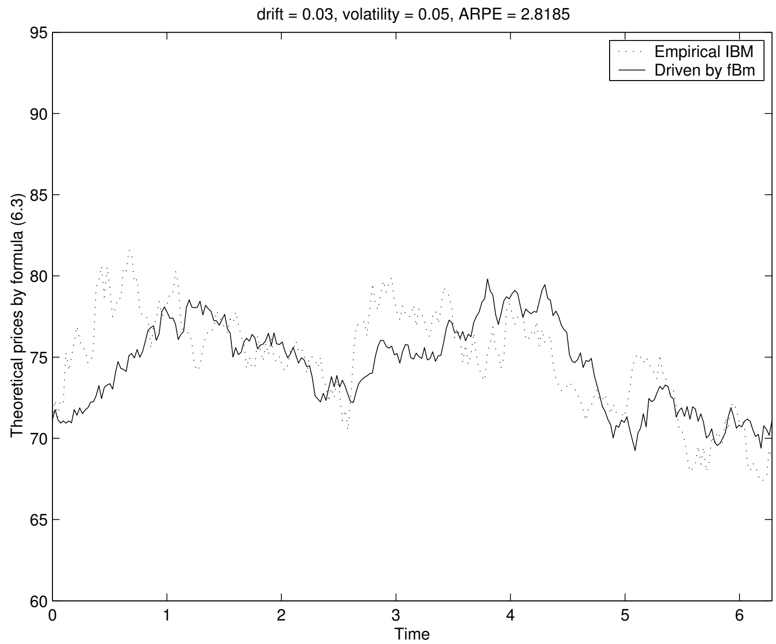

Figure 6.4: IBM-prices simulated by the pricing model with the same set up as Figure 6.3.

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Appendices

Appendix I

Tools

A.1 Definitions

We shall assume that all our considerations are Ω, the space of elementary events ω (*market situations* in this context); $\mathcal{F}$, some $\sigma-$algebras of subsets of Ω (the set of *observable market events*) and P, a probability (or probability measure) on $\mathcal{F}$. A probability P on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ such that

(a) $P(\phi) = 0, P(\Omega) = 1$ and

(b) if $A_1, A_2, ... \in \mathcal{F}$ with $A_i \cap A_j = \phi$ for $i \neq j$ then

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space*. It is called a *complete* probability space if $\mathcal{F}$ contains all subsets of sets of (probability) measure zero.

To define our probability space $(\Omega, \mathcal{F}, P)$ more specifically assume that we have a flow $(\mathcal{F}_t)_{t\geq 0}$ of $\sigma-$algebras such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \quad \forall s \leq t, s \geq 0, t \in [0, T].$$

This flow of nondecreasing $\sigma-$algebras $\mathcal{F}_t$ is also called a *filtration* and $(\Omega, \mathcal{F}_t, (\mathcal{F}_t)_{t\geq 0}, P)$ is called a *filtered probability space* or a *stochastic basis*. If each $\sigma-$algebras $\mathcal{F}_t, t \geq 0$ is completed with all sets of probability zero and the whole family is *right continuous*, i.e.,

$$\mathcal{F}_t = \cap_{t<s}\mathcal{F}_s$$

one say that the corresponding basis $(\Omega, \mathcal{F}_t, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfies *usual conditions*. Throughout the thesis, any filtered probability space is assumed to satisfy the usual conditions. A *random variable* X on a probability space $(\Omega, \mathcal{F}, P)$ is a Borel measurable function from Ω to $\mathbb{R}$, i.e.,

$$X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})).$$

A family

$$X_t(\omega) = X(t, \omega), \quad \omega \in \Omega, t \in [0, T]$$

of random variables, also written $X = (X_t)_{t\geq 0}$, is called a *stochastic process* with index or parameter set $\mathbb{R}_+$ and state space $\mathbb{R}$. Stochastic processes are functions of two variables; the usual notation suppresses the probability space variable ω. For a fixed instant of time $t \in [0, T]$,

$$X_t = X(t) = X(t, \cdot)$$

denote a random variable on the probability space $(\Omega, \mathcal{F}, P)$; for a fixed random outcome $\omega \in \Omega$,

$$X(\cdot, \omega)$$

corresponds to a real-valued function defined on $[0, T]$. The latter is called a *sample path*, *trajectory* or *realization* of the process X. For the sake of convenience we may sometimes use the notation

$$X = (X_t)_{t\geq 0} \text{ or } X = \{X(t), t \geq 0\}$$

for a stochastic process.

The $\sigma-$algebras $\mathcal{F}_t$ represents the information available up to time t. We say that the process $(X_t)_{t\geq 0}$ is (*nonanticipating* or) *adapted* to the filtration $(\mathcal{F}_t)_{t\geq 0}$

if for any $t \in [0,T]$, X_t is a random variable on $\mathcal{F}_t$, that is, X_t is $\mathcal{F}_t-$measurable (measurable relative to $\mathcal{F}_t$). The intuitive significance is that X is adapted if and only if for each t and ω, $X_t(\omega)$ is known (to the 'observer') at time t. Often, we have

$$\mathcal{F}_t = \sigma\left(X_s, s \leq t\right)$$

so that the information about ω available at time t is $(X_s)_{s \leq t}$.

Since $\sigma-$algebras generated by $X_s, s \leq t$, $\sigma(X_s, s \leq t)$, is the smallest $\sigma-$algebras making $X_s, s \leq t$ measurable (see for example, Ash 2000, theorem 5.2(b)), we have, in other words, $\sigma(X_s, s \leq t) \subset \mathcal{F}$. Such a smallest $\sigma-$algebras is also called a *natural filtration* denoted by $(\mathcal{F}_t^X)_{t \geq 0}$. Obviously, a process is adapted to its natural filtration. Moreover, if a stochastic process $X = (X_t, \mathcal{F}_t)_{t \geq 0}$ is given, this means the process X_t is adapted to the given $\sigma-$algebras $\mathcal{F}_t$ for all $t \geq 0$.

Definition A.1. If X is a random variable on the probability space $(\Omega, \mathcal{F}, P)$, the expectation of X, with respect to the probability measure P, is defined by

$$E_P(X) = \int_\Omega X dP$$

provided the integral exists.

Remark A.1. (a) Since $E(X)$ is the integral of the Borel measurable function X with respect to the probability measure P, so that the results of integration theory are applicable.

(b) The subscript P indicates in which measure the expectation is taken. However, if the (underlying) measure is clear we simply write E (instead of E_P).

In the following we let $(\Omega, \mathcal{F}, P)$ denote a given complete probability space. If X is a random variable on $(\Omega, \mathcal{F}, P)$ the *probability measure induced* by X is the probability measure F_X on $\mathcal{B}(\mathbb{R})$ given by

$$F_X(B) = P\{\omega \in \Omega : X(\omega) \in B\}, B \in \mathcal{B}(\mathbb{R}).$$

The numbers $F_X(B), B \in \mathcal{B}(\mathbb{R})$, completely characterize the random variable X in the sense that they provide the probability of all events involving X. In fact, the function $F = F_X$ from $\mathbb{R}$ to $[0, 1]$ given by

$$F(x) = P\{\omega \in \Omega : X(\omega) \leq x\}, x \text{ real}$$

is called the *distribution function* of the random variable X and the probability measure P is also referred to as the P-law of X. Moreover, for $p > 0$ we define the space $L^p = L^p(\Omega, \mathcal{F}, P)$ as the collection of all complex-valued Borel measurable function X such that

$$E\left(|X|^p\right) < \infty.$$

In particular, if $p = 2$ then X is said to be *square-integrable.* Moreover, we set

$$\|X\|_p = \left(E\,|X|^p\right)^{\frac{1}{p}}, X \in L^p.$$

- **Modes of Convergence**

Now, let $A, A_n, n = 1, 2, 3, ...$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$.

Definition A.2 (Convergence in Distribution). The sequence (A_n) *converges in distribution* or *converges weakly* to the random variable A, written $A_n \xrightarrow{d} A$, if for all bounded continuous functions f the relation

$$Ef(A_n) \to Ef(A), \quad n \to \infty,$$

holds.

Definition A.3 (Convergence in Probability). We say that (A_n) converges in *probability* to the random variable A ($A_n \xrightarrow{P} A$) if for all positive ε the relation

$$P\left(|A_n - A| > \varepsilon\right) \to 0, n \to \infty$$

holds.

Theorem A.1. *(a) Convergence in probability implies convergence in distribution (Ash, 2000, Theorem 7.1.7 a).*

(b) The converse is true if and only if $A = a$ a.s. for some constant a (Ash, 2000, Theorem 7.1.7 c).

Definition A.4 (Almost Sure Convergence). We say the (A_n) converges *almost surely (a.s.)* to the random variable A ($A_n \overset{a.s.}{\to} A$) if for P-almost all $\omega \in \Omega$ the relation

$$A_n(\omega) \to A(\omega), n \to \infty$$

holds.

Remark A.2. 1. This means that

$$P(A_n \to A) = P(\{w : A_n(\omega) \to A(\omega)\}) = 1.$$

In such a case the property is said to hold *with probability 1* or *almost surely*. In nonprobabilistic contexts, a property that holds for ω outside a set of (probability) measure zero is said to hold *almost everywhere* or for *almost all* ω.

2. Convergence with probability 1 implies convergence in probability and hence convergence in distribution. In fact, for every $\varepsilon > 0$,

$$P(A_n \to A) = 1 \Longrightarrow P(|A_n - A| > \varepsilon) \to 0.$$

Definition A.5 (L^p-Convergence). Let $p > 0$. We say that (A_n) converges in L^p or in the pth mean to the random variable A ($A_n \overset{L^p}{\to} A$) if $E|A|^p < \infty$ and $E|A_n|^p < \infty$ and

$$E|A_n - A|^p \to 0, n \to \infty$$

Remark A.3. 1. By Markov's inequality, $P(|A_n - A| > \varepsilon) \leq \varepsilon^{-p} E\,|A_n - A|^p$ for positive p and ε. Thus $A_n \xrightarrow{L^p} A$ implies $A_n \xrightarrow{P} A$. The converse is in general not true.

2. For $p = 2$, the L^2-convergence is also referred to as *convergence in mean square*.

- **Wick Product**

Let

$$h_n(x) := (-1)^n \exp(\frac{x^2}{2}) \frac{d^n}{dx^n} \left(\exp(-\frac{x^2}{2}) \right); \quad n = 0, 1, 2, ...$$

and

$$\xi_n(x) := \pi^{-1/4} \left((n-1)!\right)^{-1/2} h_{n-1} \left(\sqrt{2}x \right) \exp(-\frac{x^2}{2}) \quad n = 1, 2, ...$$

Further let J be the set of all multi-indices $\alpha = (\alpha_1, ...)$ of finite length, with $\alpha_i \in \mathbb{N} \cup \{0\}$ for all i. For $\alpha = (\alpha_1, ..., \alpha_m) \in J$ define

$$\mathcal{H}_\alpha(\omega) = h_{\alpha_1}(<\omega, \xi_1>) h_{\alpha_2}(<\omega, \xi_2>) ... h_{\alpha_m}(<\omega, \xi_m>).$$

Moreover, the space $(S)^*$ of Hida distributions is the set of all formal expansions

$$G(\omega) = \sum_{\alpha \in J} b_\alpha \mathcal{H}_\alpha(\omega)$$

such that

$$\sum_{\alpha \in J} b_\alpha^2 \alpha! \, (2\mathbb{N})^{-q\alpha} < \infty \text{ for some } q \in \mathbb{N}.$$

Definition A.6 (The Wick Product). Let

$$F(\omega) = \sum_{\alpha \in J} a_\alpha \mathcal{H}_\alpha(\omega) \in (S)^*$$

and

$$G(\omega) = \sum_{\beta \in J} b_\beta \mathcal{H}_\beta(\omega) \in (S)^*.$$

Then the *Wick Product* of F and G, $F \diamond G$, is defined by

$$\begin{aligned} (F \diamond G)(\omega) &= \sum_{\alpha,\beta \in J} a_\alpha b_\beta \mathcal{H}_{\alpha+\beta}(\omega) \\ &= \sum_{\gamma \in J} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) \mathcal{H}_\gamma(\omega). \end{aligned}$$

A.2 Brownian Motion

In the continuous time case, many models of 'complex' structure one played by Brownian motion. The concept of Brownian motion goes back to 1828 when the botanist R. Brownian described the random movement of particles of pollen in water. It was observed that a particle moved in an irregular, random fashion. Brownian motion as a mathematical concept was introduced for the first time by L. Bachelier (1900) and A. Einstein (1905). A rigorous mathematical theory was given by N. Wiener (1923). For this reason the Brownian motion is often called the *Wiener process* and denoted by W_t.

Since Brownian motion and fractional Brownian motion are both Gaussian processes, in the following, we define all terms for a Gaussian process.

If V is any d-dimensional random vector, then we define the mean vector of

$$V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_d \end{bmatrix} \text{ to be the vector } \mu = E[V] = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_d \end{bmatrix},$$

and we define the covariance matrix of V to be

$$\sum = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2d} \\ \vdots & \vdots & & \vdots \\ \sigma_{d1} & \sigma_{d2} & \dots & \sigma_{dd} \end{bmatrix}, \text{ where } \sigma_{ij} = E\left[(V_i - \mu_i)(V_j - \mu_j)\right].$$

A d-dimensional random vector V is said to have the *multivariate Gaussian distribution* with mean μ and covariance $\sum$ if the density of V is given by

$$(2\pi \det \textstyle\sum)^{-\frac{d}{2}} \exp\left(-\frac{1}{2}(x-\mu)^T \textstyle\sum^{-1}(x-\mu)\right) \quad \forall x \in \mathbb{R}^d.$$

A stochastic process $(X_t)_{t\geq 0}$ is called a *Gaussian process* if it has the property that the vector $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ has the multivariate Gaussian distribution for any finite sequence $0 \leq t_1 < t_2 < ... < t_n$. A Gaussian process with zero mean is called a *centered Gaussian process.* The distribution of a process is determined by all joint distributions and the density of a multivariate Gaussian distribution is explicitly given through its mean and covariance matrix. Indeed, if $(X_t)_{t\geq 0}$ is a Gaussian process, then its distribution is determined by its mean function $\mu_t = EX_t$ and its covariance function

$$R(t,s) = E\left[\left(X_t - \mu_t\right)\left(X_s - \mu_s\right)\right].$$

Definition A.7 (Brownian motion). Brownian motion $(W_t)_{t\geq 0}$ is a stochastic process defined on some probability space $(\Omega, \mathcal{F}, P)$ with

- (*Normal increments*) $W_t - W_s$ has normal distribution with mean 0 and covariance $t-s, s<t$. This implies with $s=0$ that $W_t - W_0$ has $\mathcal{N}(0,t)$ distribution.

- (*Independent increments*) $W_t - W_s$ is independent of the past i.e., of $W_u, 0 \leq u \leq s$.

- (*Continuous path*) P-a.s. $t \mapsto W_t(\omega)$ is a continuous function of t.

(For the existence of Brownian motion see, for example, Billingsley, 1995 p. 503) The Brownian motion $(W_t)_{t\geq 0}$ is *standard* if

$$W_0 = 0 \ P\text{-a.s.} \ EW_t = 0, EW_t^2 = t$$

where P-a.s. means almost surely with respect to the measure P. Moreover, one can easily see that the covariance function of standard Brownian motion $(W_t)_{t\geq 0}$ is given by

$$cov(W_t, W_s) = \min(s,t). \tag{A.1}$$

In fact, suppose, without loss of generality, that $s < t$. Then

$$\begin{aligned} E(W_t W_s) &= E\left[(W_t - W_s + W_s)W_s\right] \\ &= E\left[(W_t - W_s)W_s + W_s^2\right] \\ &= E\left[(W_t - W_s)W_s\right] + EW_s^2 \\ &= EW_s^2 = s \\ &= \min(s,t). \end{aligned}$$

One can see from the covariance function of fractional Brownian motion (10) on page 10 that when $H = \frac{1}{2}$ the covariance function becomes

$$R(t,s) = \frac{1}{2}\left(t + s - |t-s|\right).$$

The right hand side is, in fact, $\min(s,t)$. This means that when $H = \frac{1}{2}$ the fractional Brownian motion becomes Brownian motion. Notice that some authors define a standard Brownian motion to be a centered Gaussian process having the covariance function (A.1).

Another important notion of Brownian motion is the variation of the paths. Suppose $X = (X_t)_{t\geq 0}$ is a continuous stochastic process on $(\Omega, \mathcal{F}, P)$, then for

$p > 0$ the $p-$*variation* of the process of $(X_t)_{t\geq 0}$ is defined as

$$[X,X]_t := P - \lim_{n\to\infty} \sum_{k=1}^{n} \left|X_{t_k} - X_{t_{k-1}}\right|^p \tag{A.2}$$

where $P - lim$ means limit in probability, for each n, $\{t_k\}_{k=0}^n$ is a partition of $[0,t] \subset [0,T]$ with $0 = t_0 < t_1 < t_2 < ... < t_n = t$, and the limit is taken over all partition with, for each n,

$$\delta_n = \max_{1\leq k\leq n} \{t_k - t_{k-1}\} \to 0$$

as $n \to \infty$. When $p = 1$ and $p = 2$, (A.2) is called the *(first) variation* and *quadratic variation* of the process $(X_t)_{t\geq 0}$, respectively. The process $(X_t)_{t\geq 0}$ is said to have *bounded (finite) p-variation* provided that the limit (A.2) exists on any fixed finite interval. Suppose further that $Y = (Y_t)_{t\geq 0}$ is another stochastic process defined on the same space. Then the *quadratic covariation* of X_t and Y_t on $[0,t]$ is defined by the following limit

$$[X,Y]_t = P - \lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(X_{t_k} - X_{t_{k-1}}\right)\left(Y_{t_k} - Y_{t_{k-1}}\right)\right].$$

Remark A.4. The mathematical analysis gives the following results:

1. If X is continuous and of bounded variation (i.e., $p = 1$) then its quadratic variation is zero (see, e.g., Klebaner, 1998 Theorem 1.10).

2. If X and Y are continuous and either X or Y is of bounded variation. Then $[X,Y]_t = 0$ (Klebaner, 1998 Theorem 1.11)

The following theorem shows that almost every Brownian path is of unbounded variation on every interval while its quadratic variation over $[0,T]$ *is* t.

Theorem A.2. *Quadratic variation of Brownian motion over* $[0,t]$ *is* t.

Corollary A.1. *Almost every Brownian path is of unbounded variation on* $[0, t]$.

However, as in Mikosch (1998) and the references therein, it is well known that the sample paths of Brownian motion have bounded $p-$variation on any fixed finite interval, provided that $p > 2$ and unbounded $p-$variation for $p \leq 2$. According to the proof of Theorem A.2 we have

$$E\left[\left(W_{t_k} - W_{t_{k-1}}\right)^2 - (t_k - t_{k-1})\right]^2 = 2\left(t_k - t_{k-1}\right)^2$$

when $t_k - t_{k-1}$ is very small, $\left(t_k - t_{k-1}\right)^2$ is very small and we have the approximate equation

$$\left(W_{t_k} - W_{t_{k-1}}\right)^2 \simeq (t_k - t_{k-1})$$

in mean square. Hence in this sense we get

$$\int_0^t (dW_t)^2 = \int_0^t dt$$

or, informally,

$$d[W, W]_t = dW_t dW_t = dt. \tag{A.3}$$

Moreover, by Remark A.4(2) with $X_t = W_t$ and $Y_t = t$ we have the quadratic variation of W_t and t:

$$[W, \cdot]_t = 0$$

since W_t is of continuous path and t is of bounded variation. In other words

$$\int_0^t dW_u du = 0 \text{ or } d[W, \cdot]_t = dW_t dt = 0. \tag{A.4}$$

Similarly it follows from Remark A.4(1), with $X_t = t$, that

$$\int_0^t (dt)^2 = 0 \text{ or } d[\cdot, \cdot]_t = (dt)^2 = 0. \tag{A.5}$$

Theorem A.3. *Almost every Brownian path is of unbounded variation on every interval.*

A.3 Ito Processes and Ito Formulae

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t\geq 0}, P)$ be a filtered probability space satisfying the *usual condition*, $W = (W_t, \mathcal{F}_t^W)_{t\geq 0}$ be a standard Brownian motion and let $f = (f(t,\omega))_{t\geq 0, \omega\in\Omega}$ be a random function.

The stochastic integral

$$\int_S^T f(s,\omega)dW_s(\omega)$$

is defined as a kind of Riemann-Stieltjes sum. That is, first divide the interval $[0,t]$ into n sub-intervals: $S = t_0 \leq t_1 \leq ... \leq t_n = T$. Then choose points $\{\tau_i\}$ for $i = 1, 2, ..., n$, such that τ_i lies in the i-th sub-interval: $t_{i-1} \leq \tau_i \leq t_i$. The stochastic integral is now defined as a limit of partial sums, $I = \lim_{n\to\infty} S_n$ (in some sense) with

$$S_n = \sum_{i=1}^{n} f(\tau_i, \omega) \left[W_{t_i} - W_{t_{i-1}} \right] (\omega).$$

Unlike the Riemann-Stieltjes integral, it does make a difference what points τ_i we choose. The following two choices have turned out to be most useful ones:

1. $\tau_i = t_{i-1}$ (the left end point), which leads to the *Ito integral,* from now on denoted by

   $$\int_S^T f(s,\omega)dW_s(\omega) \qquad \text{(limit in probability)}$$

   and

2. $\tau_i = \frac{(t_i - t_{i-1})}{2}$ (the mid point), which leads to *Stratonovich integral*, denoted by

   $$\int_S^T f(s,\omega)dW_s(\omega) \qquad \text{(limit in probability).}$$

   Throughout this thesis, unless stated, the stochastic integral means *Ito integral.*

We now describe our class of functions for which the Ito integral will be defined:

Definition A.8. Let $V = V(S,T)$ be the class of functions

$$f : [0,\infty) \times \Omega \to \mathbb{R}$$

such that

1. $(t,\omega) \mapsto f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$.

2. $f(t,\omega)$ is $\mathcal{F}_t^W$-measurable.

3. $E \int_S^T f^2(s,\omega)ds < \infty$.

To define the Ito integral

$$I_t(f) = \int_S^T f(s,\omega)dW_s(\omega)$$

for functions $f \in V$, we first define $I_t(f_n)$ for a sequence of simple functions f_n. For each function $f \in V$ can be approximated by such f_n. We then use this to define $\int f dW$ as the limit of $\int f_n dW$ as $f_n \to f$.

In detail, we consider the functions $f_n \in V$ of the form

$$f_n(t,\omega) = \sum_{i=1}^{N} y_{i-1}^{(n)}(\omega) I_{(t_{i-1},t_i]}(t), \quad n = 1,2,3,... \tag{A.6}$$

where I is the indicator function of the set G: $I_G(x) = 1$ if $x \in G$ otherwise $I_G(x) = 0$. Note that since $f_n \in V$, each function $y_i^{(n)}$ must be $\mathcal{F}_{t_i}^W$-measurable. For these functions f_n we define the Ito integral as

$$I_t(f_n) = \sum_{i=1}^{N} y_{i-1}^{(n)}(\omega) \left[W_{t_i} - W_{t_{i-1}}\right](\omega). \tag{A.7}$$

Now we make the following important observations:

Remark A.5. 1. **The Ito isometry**: Let $\varphi \in V$. If $\varphi(t,\omega)$ is bounded and of the form (A.6) then

$$E\left(\int_S^T \varphi(s,\omega)dW_s(\omega)\right)^2 = E\int_S^T \varphi^2(s,\omega)ds. \tag{A.8}$$

2. Let $g \in V$ be bounded and $g(\cdot,\omega)$ continuous for each ω. Then there exist functions $f_n \in V$ of the form (A.6) such that

$$E\int_S^T (g - f_n)^2\, dt \to 0 \text{ as } n \to \infty.$$

3. Let $h \in V$ be bounded. Then there exist bounded functions g_n such that $g_n(\cdot,\omega)$ is continuous for all ω and n, and

$$E\int_S^T (h - g_n)^2\, dt \to 0 \text{ as } n \to \infty.$$

4. Let $f \in V$. Then there exists a sequence $\{h_n\} \subset V$ such that h_n is bounded for all n and

$$E\int_S^T (f - h_n)^2\, dt \to 0 \text{ as } n \to \infty.$$

We are ready to complete the definition of the Ito integral

$$\int_S^T f(s,\omega)dW_s(\omega) \text{ for } f \in V.$$

That is, if $f \in V$ we choose, by Remark A.5(2-4), functions $\varphi_n \in V$ of the form (A.6) such that

$$E\int_S^T (f - \varphi_n)^2\, dt \to 0.$$

Then define

$$I_t(f) := \int_S^T f(s,\omega)dW_s(\omega) := \lim_{n\to\infty}\int_S^T \varphi_n(s,\omega)dW_s(\omega).$$

The limit exists as an element of $L^2(\Omega,\mathcal{F},P)$, since $\left\{\int_S^T \varphi_n(s,\omega)dW_s(\omega)\right\}$ forms a Cauchy sequence in $L^2(\Omega,\mathcal{F},P)$, by Remark A.5(1).

We summarize this as follows:

Definition A.9. Let $f \in V(S,T)$. Then the Ito integral of f (from S to T) is defined by

$$\int_S^T f(s,\omega)dW_s(\omega) := \lim_{n\to\infty} \int_S^T \varphi_n(s,\omega)dW_s(\omega) \quad \text{limit in } L^2(\Omega,\mathcal{F},P) \tag{A.9}$$

where $\{\varphi_n\}$ is a sequence of functions of the form (A.6) such that

$$E\int_S^T (f(t,\omega)-\varphi_n(t,\omega))^2\, dt \to 0 \text{ as } n\to\infty. \tag{A.10}$$

Note that such a sequence $\{\varphi_n\}$ satisfying (A.10) exists by Remark A.5(2-4) above. Moreover, by Remark A.5(1) the limit in (A.9) exists and does not depend on the actual choice $\{\varphi_n\}$, as long as (A.10) holds. Furthermore, from Remark A.5(1) and (A.9) we get the following important result.

Corollary A.2. *(The Ito Isometry)*

$$E\left(\int_S^T f(s,\omega)dW_s(\omega)\right)^2 = E\int_S^T f^2(s,\omega)ds \tag{A.11}$$

(see also Steele, 2001 p.85).

Corollary A.3. *If $f(t,\omega), f_n(t,\omega) \in V(S,T)$ for $n = 1,2,...$ and $E\int_S^T (f(t,\omega) - f_n(t,\omega))^2 dt \to 0$ as $n\to\infty$, then*

$$\int_S^T f_n(t,\omega)dW_s(\omega) \to \int_S^T f(t,\omega)dW_s(\omega) \quad \text{in } L^2(\Omega,\mathcal{F},P) \text{ as } n\to\infty.$$

Theorem A.4. *Let $f,g \in V(0,T)$ and let $0 \le S < U < T$. Then*

(i) $\int_S^T f dW_s = \int_S^U f dW_s + \int_U^T f dW_s$ *for almost all* ω

(ii) $\int_S^T (cf+g)\, dW_s = c\int_S^T f dW_s + \int_S^T g dW_s$ *(c constant) for almost all* ω

(iii) $E\int_S^T f dW_s = 0$

(iv) $\int_S^T f dW_s$ *is* $\mathcal{F}_t^W$*-measurable.*

The stochastic integral defined as (A.9) is called *Ito integral.* Moreover, if $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t\geq 0}, P)$ is a filtered probability space and $W = (W_t)_{t\geq 0}$ is a Brownian motion then we define an Ito process as the followings:

Definition A.10. $X = (X_t)_{t\geq 0}$ is an *Ito process* if it can be written as

$$P - a.s. \ \forall t \leq T \quad X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dW_s, \tag{A.12}$$

where X_0 is $\mathcal{F}_0^W$-measurable and $(a_t)_{0\leq t\leq T}$ and $(b_t)_{0\leq t\leq T}$ are $\mathcal{F}_t^W$-adapted processes with

$$\int_0^t |a_s|\, ds < +\infty \quad P - a.s.$$

and

$$\int_0^t |b_s|^2\, ds < +\infty \quad P - a.s.$$

For ease of notation, one uses the following (formal) differential notation in place of the integral notation (A.12):

$$dX_t = a_t(\omega)dt + b_t(\omega)dW_t \tag{A.13}$$

and says that the process $X = (X_t)_{t\geq 0}$ has the *stochastic differential* (A.13). Furthermore, it follows from (A.3), (A.4) and (A.5) that

$$d[X, X]_t = dX_t dX_t = b_t^2(\omega)dt$$

or, in fact the quadratic variation of X is

$$[X, X]_t = \int_0^t b_s^2(\omega)ds.$$

In this context, the martingale property is a very important notion since it relates directly to the notion of arbitrage. In fact, it is easily seen that an *Ito integral* is a martingale. Let us consider the Ito integral:

$$Y_t = \int_0^t f_s(\omega)dW_s.$$

Since $f_t(\omega)$ is $\mathcal{F}_t^W$-adapted then, for $s \leq t$,

$$\begin{aligned} E\left[Y_t \,|\, \mathcal{F}_s^W\right] &= E\left[\int_0^s f_u(\omega)dW_u + \int_s^t f_u(\omega)dW_u \,|\, \mathcal{F}_s^W\right] \\ &= E\left[\int_0^s f_u(\omega)dW_u \,|\, \mathcal{F}_s^W\right] + E\left[\int_s^t f_u(\omega)dW_u \,|\, \mathcal{F}_s^W\right] \\ &= \int_0^s f_u(\omega)dW_u + E\int_s^t f_u(\omega)dW_u \\ &= \int_0^s f_u(\omega)dW_u. \end{aligned}$$

At this point another question may arise that whether the Ito process is a martingale. The following theorem answers this question.

Theorem A.5. *Let $X = (X_t)_{t\geq 0}$ be an Ito process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t\geq 0}, P)$:*

$$X_t = X_0 + \int_0^t a_s(\omega)ds + \int_0^t b_s(\omega)dW_s$$

where the $\mathcal{F}_t^W$-adapted processes $a = (a_t(\omega))_{t\geq 0}$ and $b = (b_t(\omega))_{t\geq 0}$ satisfy the conditions in Definition A.10. Then X is a martingale (relatively to the Brownian filtration $(\mathcal{F}_t^W)_{t\geq 0}$) if and only if $a_t(\omega) = 0$ almost surely, for every $t \geq 0$.

Proof. Suppose first that $a_t(\omega) = 0$ almost surely, for all $t \geq 0$. Then

$$X_t - X_0 = \int_0^t b_s(\omega)dW_s$$

and so by the above discussion, the process $(X_t - X_0)_{t\geq 0}$ is a martingale. It follows that the process $(X_t)_{t\geq 0}$ is a martingale by the linearity of conditional expectation.

Conversely, suppose that the process $(X_t)_{t\geq 0}$ is a martingale. Rewrite the process $(X_t)_{t\geq 0}$ as

$$X_t - X_0 - \int_0^t b_s(\omega)dW_s = \int_0^t a_s(\omega)ds.$$

Since the process $\int_0^t b_s(\omega)dW_s$ is also a martingale, again by the linearity of conditional expectation, it follows that

$$\int_0^t a_s(\omega)ds$$

is a martingale. This means that, for any $t, u \geq 0$,

$$\begin{aligned}
\int_0^t a_s(\omega)ds &= E\left[\int_0^{t+u} a_s(\omega)ds \mid \mathcal{F}_t^W\right] \\
&= E\left[\int_0^t a_s(\omega)ds + \int_t^{t+u} a_s(\omega)ds \mid \mathcal{F}_t^W\right]. \qquad \text{(A.14)}
\end{aligned}$$

Since $a_t(\omega)$ is $\mathcal{F}_t^W$-adapted then

$$E\left[\int_0^t a_s(\omega)ds \mid \mathcal{F}_t^W\right] = \int_0^t a_s(\omega)ds.$$

Therefore (A.14) becomes

$$\int_0^t a_s(\omega)ds = \int_0^t a_s(\omega)ds + E\left[\int_t^{t+u} a_s(\omega)ds \mid \mathcal{F}_t^W\right]$$

or

$$E\left[\int_t^{t+u} a_s(\omega)ds \mid \mathcal{F}_t^W\right] = 0.$$

This implies

$$\int_t^{t+u} E\left[a_s(\omega) \mid \mathcal{F}_t^W\right] ds = 0.$$

Dividing through by $\frac{1}{u}, u > 0$ we get

$$\frac{1}{u}\int_t^{t+u} E\left[a_s(\omega) \mid \mathcal{F}_t^W\right] ds = 0.$$

By the *Lebesgue differentiation theorem* (see for example Jones, 1993, p. 456),

$$\lim_{u \to 0} \frac{1}{u}\int_t^{t+u} E\left[a_s(\omega) \mid \mathcal{F}_t^W\right] ds = E\left[a_t(\omega) \mid \mathcal{F}_t^W\right] = a_t(\omega)$$

almost surely, for almost every t. It follows that $a_t(\omega) = 0$ almost surely, for almost every $t \geq 0$. $\square$

Besides the stochastic differential of an Ito process one can also find the stochastic differential of a function. In this section we give formulae for investigating the stochastic differential of a function of a stochastic process which itself has a stochastic differential. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $F(t, x)$

be a function defined on $\mathbb{R}_+ \times \mathbb{R}$ such that $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial x}$ and $\frac{\partial^2 F}{\partial x^2}$ exist. As usual let $W = (W_t, \mathcal{F}_t^W)_{t \geq 0}$ be a standard Brownian motion and $X = (X_t, \mathcal{F}_t^W)_{t \geq 0}$ be an Ito process. Firstly, if $x = W_t$, the process $F = (F(t, W_t))_{t \geq 0}$ has the stochastic differential (see also Oksendal, 1998, Theorem 4.1.2):

$$dF(t, W_t) = \left[\frac{\partial F}{\partial t}(t, W_t) + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(t, W_t) \right] dt + \frac{\partial F}{\partial x}(t, W_t) dW_t. \tag{A.15}$$

Secondly, if $x = X_t$ whose stochastic differential is of the form (A.13), then the stochastic differential of the process $F = (F(t, X_t))_{t \geq 0}$ is of the form

$$dF(t, X_t) = \left[\frac{\partial F}{\partial t}(t, X_t) + a_t \frac{\partial F}{\partial x}(t, X_t) + \frac{1}{2} b_t^2 \frac{\partial^2 F}{\partial x^2}(t, X_t) \right] dt + b_t \frac{\partial F}{\partial x}(t, X_t) dW_t. \tag{A.16}$$

The formulae (A.15) and (A.16) are two forms (among many) of *Ito formulae.* See, e.g., Chapter 4 of Klebaner (1998) for the derivation of the formulae.

A.4 Stochastic Differential Equations

Let $W = (W_t, \mathcal{F}_t^W)_{t \geq 0}$ be a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t \geq 0}, P)$ and $a(t, x)$ and $b(t, x)$ be $\mathcal{F}_t^W$-measurable functions on $\mathbb{R}_+ \times \mathbb{R}$. An equation of the form

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

where $X = (X_t)_{t \geq 0}$ is an unknown process is called a *stochastic differential equation.* This section is devoted to the existence and uniqueness of solutions for stochastic differential equations and solution to linear stochastic differential equations.

In the following theorem, let $X = (X_t)_{t \geq 0}$ be an n-dimensional stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$, $W = (W_t, \mathcal{F}_t^W)_{t \geq 0}$ where $\mathcal{F}_t^W = \sigma(W_s, s \leq t)$ be an m-dimensional standard Brownian motion (recall that $\mathcal{F}_s^W \subset$

$\mathcal{F}_t^W \subset \mathcal{F}, \forall s \leq t)$ and $\mathcal{F}_t^Z = \sigma(Z, W_s, s \leq t)$ where Z is an n-dimensional random variable independent of $\mathcal{F}_t^W$, for all $t \geq 0$ with $E_P |Z|^2 < \infty$.

Theorem A.6 (Existence & Uniqueness Theorem for SDEs). *Let $T > 0$ and $b(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ be $\mathcal{F}-$measurable functions satisfying*

$$|b(t, x)| + |\sigma(t, x)| \leq C (1 + |x|) \,; x \in \mathbb{R}^n, t \in [0, T] \tag{A.17}$$

for some constant C, (where $|\sigma|^2 = \sum |\sigma_{ij}|^2$) and such that

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D |x - y| \,; x, y \in \mathbb{R}^n, t \in [0, T] \tag{A.18}$$

for some constant D. Then the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, t \in [0, T], X_0 = Z \tag{A.19}$$

has a unique t-continuous solution

$$X = (X_t, \mathcal{F}_t^Z)_{t \geq 0} \tag{A.20}$$

with

$$E \int_0^T |X_t|^2 \, dt < \infty. \tag{A.21}$$

Proof. (see Oksendal, 1998, p.66-70)

Remark A.6.

1. The uniqueness of the solution means that if $(X)_{0 \leq t \leq T}$ and $(Y)_{0 \leq t \leq T}$ are two solutions of (A.19) then P-a.s. for all $0 \leq t \leq T$, $X_t = Y_t$.

2. The uniqueness of the theorem above is called *strong* or *pathwise* uniqueness, while *weak* uniqueness simply means that any two solutions are identical in law, i.e. they have the same finite-dimensional distributions.

The following section is one of the most important contents in the context of the AAO (Absence of Arbitrage Opportunity) principle. In the language that the existence of (equivalent) martingale measure is equivalent to the AAO, one needs to understand how a process acts under one (original) measure and how it acts under another measure (equivalent to the original one). Moreover, the measure transform brings the most important theorem (Girsanov's Theorem) saying that under a new (but equivalent) measure a stochastic process with drift can be expressed as a stochastic process without drift and hence is a martingale (under the new measure).

A.5 Measure Transform

Before introducing some forms of The Girsanov's Theorem, the first part of this section begins with how one measure can be transformed to another using the simple example of two normal distributions on the real line.

Recall the probability density of a normal $\mathcal{N}(\mu, \sigma^2)$ random variable:

$$\varphi_{(\mu,\sigma^2)}(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), x \in \mathbb{R},$$

and write

$$\Phi_{(\mu,\sigma^2)}(x) = \int_{-\infty}^{x} \varphi_{(\mu,\sigma^2)}(y)dy, x \in \mathbb{R}$$

for the corresponding distribution function. Consider two pairs (μ_1, σ_1^2) and (μ_2, σ_2^2) of parameters and define

$$f_1(x) = \frac{\varphi_{(\mu_1,\sigma_1^2)}(x)}{\varphi_{(\mu_2,\sigma_2^2)}(x)} \text{ and } f_2(x) = \frac{\varphi_{(\mu_2,\sigma_2^2)}(x)}{\varphi_{(\mu_1,\sigma_1^2)}(x)}.$$

Obviously,

$$\begin{aligned}\Phi_{(\mu_1,\sigma_1^2)}(x) &= \int_{-\infty}^{x} \varphi_{(\mu_1,\sigma_1^2)}(y)dy \\ &= \int_{-\infty}^{x} \varphi_{(\mu_1,\sigma_1^2)}(y)\frac{\varphi_{(\mu_2,\sigma_2^2)}(x)}{\varphi_{(\mu_2,\sigma_2^2)}(x)}dy \\ &= \int_{-\infty}^{x} f_1(y)d\Phi_{(\mu_2,\sigma_2^2)}(y).\end{aligned}$$

Similarly,

$$\Phi_{(\mu_2,\sigma_2^2)}(x) = \int_{-\infty}^{x} f_2(y)d\Phi_{(\mu_1,\sigma_1^2)}(y).$$

The function f_1 and f_2 are called *density function* (or *Radon-Nikodym derivative*) with respect to $\Phi_{(\mu_1,\sigma_1^2)}(x)$ and $\Phi_{(\mu_2,\sigma_2^2)}(x)$, respectively.

For example, suppose that a random variable X has standard normal $\mathcal{N}(\mu,1)$. This presumes a probability P under which X is $\mathcal{N}(\mu,1)$. Recall that the density function of $\mathcal{N}(\mu,1)$ distribution is

$$\varphi_{(\mu,1)}(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2}\right), x \in \mathbb{R},$$

hence,

$$P\left(X \leq x\right) = \Phi_{(\mu,1)}(x) = \int_{-\infty}^{x} \varphi_{(\mu,1)}(y)dy, x \in \mathbb{R}.$$

By the definition of a density function, the probability of a set A on the line is the integral of the density over this set,

$$P_X(A) = P(X \in A) = \int_A \varphi_{(\mu,1)}(x)dx = \int_A dP_X.$$

In infinitesimal notation this relation is often written as

$$dP_X = P_X(dx) = P(X \in dx) = \varphi_{(\mu,1)}(x)dx.$$

Now, define a new probability Q by

$$dQ = e^{-\mu X+\mu^2/2}dP.$$

Then in view of density function

$$\begin{aligned}
Q(X \leq x) &= \int_{-\infty}^{x} e^{-\mu y+\mu^2/2} dP \\
&= \int_{-\infty}^{x} e^{-\mu y+\mu^2/2} \varphi_{(\mu,1)}(y) dy \\
&= \int_{-\infty}^{x} e^{-\mu y+\mu^2/2} \frac{1}{\sqrt{2\pi}} e^{-(y-\mu)^2/2} dy \\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&= \int_{-\infty}^{x} \varphi_{(0,1)}(y) dy.
\end{aligned}$$

We see that $X \sim \mathcal{N}(\mu, 1)$ under P but it has $\mathcal{N}(0,1)$ distribution under Q.

Remark A.7. If P is a probability such that X is $\mathcal{N}(\mu, 1)$, then $X-\mu$ has $\mathcal{N}(0,1)$ distribution. This is operation on the outcome x itself, and $X-\mu$ has $\mathcal{N}(0,1)$ under the same probability P. But here we change the probability measure P to Q, we leave the outcomes as they are, but assign different law to them.

Let us recall here that if $X = (X_t, \mathcal{F}_t^W)_{t\geq 0}$ is a stochastic process on $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}_s^W \subset \mathcal{F}_t^W \subset \mathcal{F}, \forall s \leq t$, then X is called an Ito process if X is of the form

$$X_t = X_0 + \int_0^t \mu(u,\omega) du + \int_0^t \sigma(u,\omega) dW_u, \tag{A.22}$$

provided that the two integrals are well-defined. The integral form above can be written in the differential notation:

$$dX_t = \mu(t,\omega)dt + \sigma(t,\omega)dW_t,$$

called (Ito) stochastic differential equation. Moreover, if $\mu = 0$ the process X is referred to as *a stochastic process without drift.*

Let $W = (W_t)_{t\geq 0}$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$. One can see that the process of the form

$$\widetilde{W}_t = qt + W_t$$

where q is a constant, is not a standard Brownian motion under that probability P unless $q = 0$. However, if one changes the measure P for another probability measure Q (in which Q is equivalent to P), the Brownian motion with drift can also be viewed as a Brownian motion without drift under the new probability Q. This is the content of the famous Girsanov's Theorems.

Girsanov's theorem is an important tool for the study of the martingale measure for models of financial markets. So we present here some forms of this theorem. In Section 2.3, we will recall also a criterion for free arbitrage as an application of this theorem. Then, in Section **??**, *we will apply it to the problem of free arbitrage for our fractional model.*

- **Girsanov's Theorem**

Since it is known that an Ito process without drift is a martingale (see Section 1.2.3), one is interested in whether a stochastic process with drift can be viewed as a process without drift. Girsanov theorem shows how to change the drift of an Ito stochastic differential equation by changing its law to obtain a process without drift. It says that if we change the drift coefficient of a given Ito process, the law of the process will not change dramatically. In fact, the law of the new process will be absolutely continuous with respect to the law of the original process and one can compute explicitly the Radon-Nikodym derivative.

Now, let us turn back to the Girsanov theorem. Let $W = (W_t)_{t \geq 0}$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t^W$ be the $\sigma-$algebra generated by random variables $W_s; s \leq t$ ($\mathcal{F}_s^W \subset \mathcal{F}_t^W \subset \mathcal{F}, s \leq t$). In other words, $\mathcal{F}_t$ is the smallest $\sigma-$algebra containing all sets of the form

$$\{\omega; W_t(\omega) \in F, t \geq 0\},$$

where $F \subset \mathbb{R}$ is Borel set.

In brief, the *Girsanov theorem I* shows that under some conditions the Ito process of the form

$$Y(t) = \int_0^t a(s,\omega)ds + W(t); Y(0) = 0, t \leq T$$

is a Brownian motion with respect to a new probability Q on $(\Omega, \mathcal{F})$. This result brings the *Girsanov theorem II* which says that the Ito process of the form

$$dY(t) = \beta(t,\omega)dt + \theta(t,\omega)dW(t), t \leq T$$

can be written as

$$dY(t) = \alpha(t,\omega)dt + \theta(t,\omega)d\widetilde{W}(t), t \leq T$$

where $\widetilde{W}(t)$ is a Brownian motion with respect to new probability Q. One can see that under the new driving process $\widetilde{W}(t)$ the drift $\beta(t,\omega)$ of the Ito process $Y(t)$ is changed to $\alpha(t,\omega)$ while the volatility $\theta(t,\omega)$ is unchanged.

Finally, the *Girsanov theorem III* shows that the P-law of the Ito diffusion $X^x(t)$ of the form

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t), t \leq T, X(0) = x$$

and the Q-law of the Ito process $Y^x(t)$ of the form

$$dY(t) = [\gamma(t,\omega) + b(Y(t))]\, dt + \sigma(X(t))dW(t), t \leq T, Y(0) = x$$

are the same. In all cases, the measures P and Q are equivalent (see Section 2.3 for the definition). The precise statements are the followings:

Theorem A.7 (Girsanov Theorem I). *Let $Y(t) \in \mathbb{R}$ be an Ito process of the form*

$$dY(t) = a(t,\omega)dt + dW(t); t \leq T, Y_0 = 0,$$

where $T \leq \infty$ is a given constant and $W(t)$ is Brownian motion. Put

$$M_t = \exp\left(-\int_0^t a(s,\omega)dW_s - \frac{1}{2}\int_0^t a^2(s,\omega)ds\right); t \leq T. \tag{A.23}$$

Assume that $a(s,\omega)$ satisfies Novikov's condition

$$E_P \exp\left(\frac{1}{2}\int_0^t a^2(s,\omega)ds\right) < \infty \tag{A.24}$$

Define the measure Q on $(\Omega, \mathcal{F})$ by

$$dQ(\omega) = M_T(\omega)dP(\omega). \tag{A.25}$$

Then $Y(t)$ is a Brownian motion with respect to probability law Q, for $t \leq T$.

Proof. (see Oksendal, 1998, p.154)

Theorem A.8 (Girsanov Theorem II). *Let $Y(t) \in \mathbb{R}$ be an Ito process of the form*

$$dY(t) = \beta(t,\omega)dt + \theta(t,\omega)dW(t); t \leq T \tag{A.26}$$

where $W(t)$ is Brownian motion, $\beta(t,\omega)$ and $\theta(t,\omega)$ are processes on $\mathbb{R}$. Suppose there exist $\mathcal{F}_t-$adapted and square integrable processes $u(t,\omega)$ and $\alpha(t,\omega)$ such that

$$\theta(t,\omega)u(t,\omega) = \beta(t,\omega) - \alpha(t,\omega) \tag{A.27}$$

and assume that $u(t,\omega)$ satisfies Novikov's condition

$$E_P \exp\left(\frac{1}{2}\int_0^t u^2(s,\omega)ds\right) < \infty. \tag{A.28}$$

Put

$$M_t = \exp\left(-\int_0^t u(s,\omega)dW_s - \frac{1}{2}\int_0^t u^2(s,\omega)ds\right); t \leq T \tag{A.29}$$

and

$$dQ(\omega) = M_T(\omega)dP(\omega) \text{ on } \mathcal{F}. \tag{A.30}$$

Then

$$\widehat{W}(t) := \int_0^t u(s,\omega)ds + W(t); t \leq T \tag{A.31}$$

is a Brownian motion with respect to Q *and in terms of* $\widehat{W}(t)$ *the process* $Y(t)$ *has the stochastic integral representation*

$$dY(t) = \alpha(t,\omega)dt + \theta(t,\omega)d\widehat{W}(t). \tag{A.32}$$

Proof. (see Oksendal, 1998, p.156)

Theorem A.9 (Girsanov Theorem III). *Let* $X(t) = X^x(t) \in \mathbb{R}$ *and* $Y(t) = Y^x(t) \in \mathbb{R}$ *be an Ito diffusion and an Ito process, respectively, of the forms*

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t); t \leq T, X(0) = x \tag{A.33}$$

$$dY(t) = [\gamma(t,\omega) + b(Y(t))]\, dt + \sigma(Y(t))dW(t); t \leq T, Y(0) = x \tag{A.34}$$

where the function b,$\sigma : \mathbb{R} \to \mathbb{R}$ *satisfy the conditions of theorem (A.6) and* $\gamma(t,\omega)$ *is* $\mathcal{F}_t-$*adapted and square integrable,* $x \in \mathbb{R}$*. Suppose there exists an* $\mathcal{F}_t-$*adapted and square integrable process* $u(t,\omega)$ *such that*

$$\sigma(Y(t))u(t,\omega) = \gamma(t,\omega)$$

and assume that $u(t,\omega)$ *satisfies Novikov's condition*

$$E_P \exp\left(\frac{1}{2}\int_0^T u^2(s,\omega)ds\right) < \infty.$$

Define M_t*,* Q *and* $\widehat{W}(t)$ *as in (A.29), (A.30) and (A.31). Then*

$$dY(t) = b(Y(t))dt + \sigma(Y(t))d\widehat{W}(t).$$

Therefore, the Q*-law of* $Y^x(t)$ *is the same as the* P*-law of* $X^x(t); t \leq T$*.*

Proof. (see Oksendal, 1998, p.157)

Appendix B

Paper: A Fractional Hull-White Model

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Probability

A Fractional Hull-White Model

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Abstract

In this paper we consider a fractional Hull-White model driven by a fractional Brownian motion. We use an approximate approach to find the solution of this model that exhibits a long-range behavior of the interest.

1. Introduction

It is well-known in mathematical finance that the Hull-White model for interest r_t has the following form

$$dr_t = \big(b(t) - a(t)r_t\big)dt + \sigma(t)dW_t, \tag{1.1}$$

where $a(t)$, $b(t)$ and $\sigma(t)$ are deterministic continuous functions of t and $a(t) > 0$, $\sigma(t) > 0$, and W_t is a standard Brownian motion. This model is very useful in practice of financial markets, it gives also the price of zero-coupon bonds corresponding to each value of the rate r_t.

But each value of r_t can influence upon its behavior in some time range. Correspondingly, the prices of bonds at a time t can have some consequences on their price some time later. In this context, the ordinary Hull-White model is not suitable since its solution is always a Markov process that has no memory.

The purpose of this paper is to introduce a fractional Hull-White model for the interest rate r_t for which the driving process is replaced by a fractional Brownian motion, a process of long memory. A fractional Brownian motion B_t^H, $H \in (0, 1)$, is a centered Gaussian process with the covariance function $R(t, s)$

given by

$$R(t,s) = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right].$$

Notice that if $H = \frac{1}{2}$ the fractional Brownian motion is a standard Brownian motion. However, increments of a fractional Brownian motion are not independent except for the standard Brownian case. Moreover, for $H < \frac{1}{2}$ the increments are negatively correlated and for $H > \frac{1}{2}$ they are positively correlated. In the latter case, B_t^H is a long memory process since the correlation between two observations that are far apart decay to zero very slowly.

From [6] and the references therein, it is known that for $H \in (0,1)$ the fractional Brownian motion B_t^H has a representation as follows:

$$B_t^H = \frac{1}{\Gamma(\alpha+1)}\left[Z_t + \int_0^t (t-s)^\alpha dW_s\right]$$

where $\alpha = H - \frac{1}{2}, \Gamma(\cdot)$ is the gamma function, $(W_t)_{t\geq 0}$ is a standard Brownian motion and

$$Z_t = \int_{-\infty}^0 \left[(t-s)^\alpha - (-s)^\alpha\right] dW_s.$$

Since Z_t is of absolutely continuous trajectories, the long-range property of B_t^H is essentially expressed by the term

$$B_t = \int_0^t (t-s)^\alpha dW_s.$$

Now, let us consider the fractional Hull-White model for the interest r_t of the form:

$$dr_t = \big(b(t) - a(t)r_t\big)dt + \sigma(t)dB_t, \tag{1.2}$$

where B_t is a fractional Brownian motion of Hurst index $H(0 < H < 1)$ defined by

$$B_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s \tag{1.3}$$

and $\sigma(t)$ is a differentiable function of $t \in [0,T], \sigma(t) > 0$. A solution r_t of (1.2) is a long memory process satisfying the following relation:

$$r_t = r_0 + \int_0^t [b(s) - a(s)r_s]ds + \sigma(t)B_t. \tag{1.4}$$

2. Approximate Model

Starting from (1.1) we introduce in this Section a so-called *approximate model*, driven by a semimartingale and we give the solution for this model. And the convergence to the solution of (1.1) will be proved in the next Section.

The equation (1.2) is rewritten again as

$$\begin{aligned} dr_t =&(b(t) - a(t)r_t)dt + \sigma(t)dB_t, \quad 0 \le t \le T \\ r_{t(t=0)} =&r_0, \end{aligned} \tag{2.1}$$

where $B_t = \int_0^t (t-s)^\alpha dW_s, \quad -\frac{1}{2} < \alpha < \frac{1}{2}$ (2.2)
and r_0 is a given square integrable random variable. Now define, for every $\varepsilon > 0$, a process B_t^ε as follows:

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s. \tag{2.3}$$

We can notice that

$$\begin{aligned} \int_0^t \int_0^s (s-u+\varepsilon)^{\alpha-1} dW_u ds &= \int_0^t \int_u^t (s-u+\varepsilon)^{\alpha-1} ds dW_u \\ &= \frac{1}{\alpha}\left[\int_0^t (t-u+\varepsilon)^\alpha dW_u - \varepsilon^\alpha \int_0^t dW_u\right] \\ &= \frac{1}{\alpha}\left[B_t^\varepsilon - \varepsilon^\alpha W_t\right]. \end{aligned}$$

By the above computation we get

$$B_t^\varepsilon = \alpha \int_0^t \varphi_s^\varepsilon ds + \varepsilon^\alpha W_t \tag{2.4}$$

where

$$\varphi_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{(\alpha-1)} dW_s.$$

Since $\alpha \int_0^t \varphi_s^\varepsilon ds$ is of bounded variation and $\varepsilon^\alpha W_t$ is a martingale, then B_t^ε is a semimartingale. This result was stated in [6].

Furthermore, from [6] it was also proved that B_t^ε converges to B_t in $L^2(\Omega)$ uniformly with respect to $t \in [0,T]$. We now consider an approximate model defined for each $\varepsilon > 0$ as follows:

$$\begin{aligned} dr_t^\varepsilon =&\big(b(t) - a(t)r_t^\varepsilon\big)dt + \sigma(t)dB_t^\varepsilon, \quad 0 \le t \le T \\ r_{t(t=0)}^\varepsilon =&r_0, \end{aligned} \tag{2.5}$$

where B_t^ε is defined by (2.3) and the random variable r_0 is exactly equal to the value given in the initial condition of (2.1). From (2.4) we have

$$dB_t^\varepsilon = \alpha\varphi_t^\varepsilon dt + \varepsilon^\alpha dW_t.$$

Hence, after substituting dB_t^ε by its above-mentioned expression, (2.5) becomes

$$\begin{aligned} dr_t^\varepsilon &= \left(b(t) + \alpha\sigma(t) - a(t)r_t^\varepsilon\right) dt + \varepsilon^\alpha \sigma(t) dW_t \\ r_{t(t=0)}^\varepsilon &= r_0. \end{aligned} \tag{2.5'}$$

Theorem 1. *Suppose that $a(t)$ and $b(t)$ are continuous functions on $[0,T]$. Then for each $\varepsilon > 0$ there exists a unique solution r_t^ε for (2.5′) given by*

$$r_t^\varepsilon = e^{-\int_0^t a(s)ds}\left[r_0 + \varepsilon^\alpha \int_0^t \sigma(s) e^{\int_0^s a(u)du} dW_s + \int_0^t \big(b(u) + \sigma(u)\alpha\varphi_u^\varepsilon\big) e^{\int_0^u a(s)ds} du\right]. \tag{2.6}$$

Proof. We split the equation (2.5′) into two equations:

$$\begin{aligned} dr_t^{(1)} &= - a(t) r_t^{(1)} dt + \sigma(t)\varepsilon^\alpha dW_t, \quad 0 \le t \le T \\ r_{t(t=0)}^{(1)} &= r_0^{(1)} \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} dr_t^{(2)} &= \big(b(t) - a(t) r_t^{(2)}\big) dt + \sigma(t)\alpha\varphi_t^\varepsilon dt, \\ r_{t(t=0)}^{(2)} &= r_0^{(2)}, \end{aligned} \tag{2.8}$$

where $r_t^{(1)} + r_t^{(2)} = r_t^\varepsilon$ satisfies (2.5′) and $r_0^{(1)}$ and $r_0^{(2)}$ are two square integrable random variables such that $r_0^{(1)} + r_0^{(2)} = r_0^\varepsilon$ (given initial condition). We see that (2.7) is a linear stochastic differential equation of the form

$$dr(t) = \big(\alpha(t) + \beta(t) r(t)\big) dt + \big(\gamma(t) + \delta(t) r(t)\big) dW_t.$$

It is known that if coefficients $\alpha, \beta, \gamma, \delta$ are continuous functions of t, then the existence and uniqueness for solution of (2.7) are assured. Moreover, its solution has the form (see, [3] for example)

$$r(t) = U(t)\left[r(0) + \int_0^t \frac{\alpha(s) - \delta(s)\gamma(s)}{U(s)} ds + \int_0^t \frac{\gamma(s)}{U(s)} dW_s\right]$$

where

$$U(t) = U(0)\exp\left[\int_0^t \left(\beta(s) - \frac{1}{2}\delta^2(s)\right)ds + \int_0^t \delta(s)dW_s\right].$$

Here we have $\alpha(t) = 0, \beta(t) = -a(t), \gamma(t) = \sigma(t)\varepsilon^\alpha$, and $\delta(t) = 0$. Then, with $U(0) = 1$,

$$r_t^{(1)} = e^{-\int_0^t a(s)ds}\left[r_0^{(1)} + \varepsilon^\alpha \int_0^t \sigma(s)e^{\int_0^s a(r)dr}dW_s\right]. \tag{2.9}$$

Now let us consider the equation (2.8) that can be rewritten in the form:

$$\frac{dr_t^{(2)}}{dt} + a(t)r_t^{(2)} = b(t) + \sigma(t)\alpha\varphi_t^\varepsilon, \quad 0 \le t \le T,$$

This is an ordinary linear differential equation whose solution can be given by:

$$r_t^{(2)} = e^{-\int_0^t a(s)ds}\left[r_0^{(2)} + \int_0^t (b(u) + \sigma(u)\alpha\varphi_u^\varepsilon)e^{\int_0^t a(s)ds}du\right]. \tag{2.10}$$

Combining (2.9) and (2.10) yields the expression (2.6) for the solution r_t^ε of (2.5′).

3. Convergence

We note that the equation (1.2) is a fractional linear stochastic differential equation whose solution is defined by (1.4). Under the regularity assumptions on $a(t)$ and $b(t)$, it is easy to verify that there exists such a unique solution for (1.2). Denote this solution by r_t and suppose that r_t^ε is the solution of the corresponding approximate model (2.5′). Thus r_t and r_t^ε satisfy the following equations:

$$dr_t = (b(t) - a(t)r_t)dt + \sigma(t)dB_t, \quad 0 \le t \le T$$

$$dr_t^\varepsilon = (b(t) - a(t)r_t^\varepsilon)dt + \sigma(t)dB_t^\varepsilon, \quad 0 \le t \le T.$$

Theorem 2. *r_t^ε converges to r_t in $L^2(\Omega)$ uniformly with respect to $t \in [0,T]$ as $\varepsilon \to 0$.*

Proof. We have

$$r_t - r_t^\varepsilon = -\int_0^t a(s)\,(r_s - r_s^\varepsilon)ds + \sigma(t)(B_t - B_t^\varepsilon) - \int_0^t (B_s - B_s^\varepsilon)d\sigma(s). \tag{3.1}$$

Denote by $\|\cdot\|$ the norm in $L^2(\Omega)$. Then

$$\|r_t - r_t^\varepsilon\| \le \int_0^t |a(s)|\,\|r_s - r_s^\varepsilon\|ds + |\sigma(t)|\|B_t - B_t^\varepsilon\|$$

$$+ \|B_t - B_t^\varepsilon\|\,|\sigma(t) - \sigma(0)|\,. \tag{3.2}$$

Since $a(t)$ and $\sigma(t)$ are continuous, hence bounded, on $[0,T]$ then there exist positive constants M_1, M_2 such that

$$|a(t)| \leq M_1 = \max_{0\leq t\leq T} |a(t)| \text{ and } |\sigma(t)| \leq M_2 = \max_{0\leq t\leq T} |\sigma(t)|.$$

Hence,

$$\|r_t - r_t^\varepsilon\| \leq M_1 \int_0^t \|r_s - r_s^\varepsilon\| ds + 2M_2 \|B_t - B_t^\varepsilon\|. \tag{3.3}$$

We know that $B_t^\varepsilon \to B_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0,T]$ and we have also the following estimate (see [6])

$$\|B_t - B_t^\varepsilon\|^2 = E\,|B_t - B_t^\varepsilon|^2 \leq C(\alpha)\,\varepsilon^{1+2\alpha}, \tag{3.4}$$

where $C(\alpha)$ is a constant depending only on α. Then

$$\|B_t - B_t^\varepsilon\| \leq K(\alpha)\,\varepsilon^{\frac{1}{2}+\alpha}, \tag{3.5}$$

where $K(\alpha) = \sqrt{C(\alpha)}$. It follows from (3.2), (3.3) and (3.4) that

$$\|r_t - r_t^\varepsilon\| \leq M_1 \int_0^t \|r_s - r_s^\varepsilon\| ds + M\,\varepsilon^{\frac{1}{2}+\alpha}, \tag{3.6}$$

where $M = 2M_2K(\alpha)$. A standard application of Gronwall Lemma will give us:

$$\|r_t - r_t^\varepsilon\| \leq e^{M_1 t} M\,\varepsilon^{\frac{1}{2}+\alpha}. \tag{3.7}$$

Hence

$$\sup_{0\leq t\leq T} \|r_t - r_t^\varepsilon\| \leq e^{M_1 T} M\,\varepsilon^{\frac{1}{2}+\alpha} \to 0 \tag{3.8}$$

as $\varepsilon \to 0$. The proof of Theorem 3.1 is thus complete.

Remarks

1. It is known that a fractional stochastic dynamical system driven by a fractional Brownian motion exhibits a long-range behavior of system states. In spite of the fact that the interest rate is in general a short rate, its behavior in some considerably long time has no more Markov property. In this context, a fractional Hull-White model is needed to understand realistic dynamics of interest rate.

2. The interest rate is strictly related to bond prices. And as we know, for a bond market driven by a fractional Brownian motion, in general, the absence of arbitrage opportunity can not be guaranteed [5]. But by the approximate approach given by authors of [6], the fractional bond price model can be approximated by a model driven by a semimartingale where there is no more any arbitrage opportunity.

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