QUANTUM DYNAMICS OF THE STERN-GERLACH EFFECT

Mr. Attasit Rotjanakusol

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พลศาสตร์ควอนตัมของปรากฏการณ์สเทอร์น-เกอร์ลาช

นายอรรถสิทธิ์ โรจนกุศล

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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วิทยานิพนธ์นี้ศึกษาอย่างเป็นระบบทั้งเชิงพลศาสตร์และการวิเคราะห์เกี่ยวกับผลของสเทอร์น-เกอร์ลาช การศึกษาจะพิจารณาใน 5 ประเด็นคือ (1) ศึกษาเชิงกลศาสตร์ควอนตัม (2) รวมผลของการใช้สมการสนาม $\vec{\nabla} \cdot \vec{B} = 0$ (3) กลศาสตร์ควอนตัมของแรงโลเร็นตซ์ (4) การตกกระทบบนฉากของลำอนุภาคในสองมิติแทนที่จะเป็นหนึ่งมิติ และ (5) ผลสหสัมพันธ์ระหว่างตัวแปรเชิงพลศาสตร์ การวิเคราะห์พลศาสตร์เชิงควอนตัมจะอยู่ในอันดับขนาด $|e|/\sqrt{\hbar c} \equiv \sqrt{\alpha}$ สำหรับอิเล็กตรอนและอนุภาคประจุสปิน 1/2 เช่นโปรตอน โดยที่ α คือค่าคงตัวเรียกว่า fine-structure constant และนำไปสู่สมการสำหรับหาโอกาสของความหนาแน่นของอนุภาคที่ไปตกกระทบบนฉาก ณ ตำแหน่งต่างๆ โดยที่สนามแม่เหล็กควบคุมได้ให้สม่ำเสมอตามแนวยาวตามทิศทางเริ่มต้นเฉลี่ยของการเคลื่อนที่ของอนุภาค ในกรณีสนามไม่สม่ำเสมอในแนวยาว สนามแม่เหล็กจะชี้ในระนาบที่ถูกกำหนดให้เป็นแกนควอนไทเซชันของสปินและทิศทางเริ่มต้นเฉลี่ยของการเคลื่อนที่ ในท้ายที่สุดเราได้ทิ้งท้ายให้นักทดลองได้ทำการทดลองผลของสเทอร์น-เกอร์ลาชสำหรับอิเล็กตรอนดังที่ได้บรรยายไว้ในงานนี้

สาขาวิชาฟิสิกส์	ลายมือชื่อนักศึกษา………..…………………………....
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ATTASIT ROTJANAKUSOL: QUANTUM DYNAMICS OF THE STERN-GERLACH EFFECT. THESIS ADVISOR: PROF. EDOUARD B. MANOUKIAN, Ph.D. 166PP. ISBN 974-533-214-3

This Thesis involves the systematic, rigorous, dynamical and analytical studies of the Stern-Gerlach (SG) effect. The studies are included (1) quantum mechanical, and takes into account of (2) the field equation $\vec{\nabla}\cdot\vec{B}=0$, (3) the quantum mechanical counterpart of the Lorentz force, (4) the two, rather than one, dimensional aspect of the beam hitting the observation screen and (5) the rather non-trivial correlations, as is explicitly shown, that occur between the dynamical variables. The quantum dynamical analysis is carried out to the leading order in $|e|/\sqrt{\hbar c}\equiv\sqrt{\alpha}$ for the electron, where α is the fine-structure constant, and for spin 1/2 charged particles (e.g., the proton), in general, and leads to a unitary expression for the probability density on the observation screen, where the magnetic field has a controllable longitudinal uniform component along the initial average direction of propagation of the particle, in addition to a non-uniform, almost longitudinal, magnetic field lying in the plane defined by the quantization axis, in question, of the spin and the initial average direction of propagation. We invite experimentalist to finally carry out the experiment on the SG effect for the electron as described in the bulk of this work.

สาขาวิชาฟิสิกส์ ลายมือชื่อนักศึกษา………..…………………………...

ปีการศึกษา 2545 ลายมือชื่ออาจารย์ที่ปรึกษา……………………………..

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Attasit Rotjanakusol

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Chapter I

Introduction

Many physicists would be surprised to learn that the classic Stern-Gerlach experiment (Gerlach and Stern, 1921,1922a, 1922b) has not yet been carried out (Batelaan, Gay, and Schwendiman, 1997; Dehmelt, 1990; Rabi, 1988; Badurek, Rauch, and Tuppinger, 1986; Estermann, 1975) for the electron. The reason for the obstacle in performing the experiment is that the Lorentz force arising form a transversal magnetic field component to the initial direction of motion of the electron causes a deviation of the particle from its initial path thus causing a blurring (Gerlach and Stern, 1921) of the expected splitting of a beam. It is, by now, well known to every student of physics that the original experiment (Gerlach and Stern, 1921, 1922a, 1922b) was done with a beam of neutral silver atoms. The experiment showed that such a beam splits into components as a consequence of the fact that the silver atoms possess angular momentum thus establishing the quantization of angular momentum.

Regarding the electron and its relationship to the rest of the microscopic physics, it is worth recalling the well known remark (Dehmelt, 1990) made by Albert Einstein stating:‘‘ We know, it would be sufficient to really understand the electron ’’. Regarding the Stern-Gerlach experiment and its role in quantum physics, Julian Schwinger (Schwinger, Englert, and Scully, 1988) states:‘‘ Even today, it is not widely appreciated that the Stern-Gerlach experiment epitomizes the quantum mechanical description of microscopic phenomena’’. The intuitive appeal of the

Stern-Gerlach experiment is about establishing the quantum nature of the microscopic world and of the quantization of angular momentum and spin, by the observation, by classical means, of small spots, but nevertheless of macroscopic extensions, on an observation screen due to a beam splitting by a non-uniform magnetic field acting on spin. The inference (e.g., Wheeler and Zurek, 1983; Brown and Maclay, 1969; Deutsch and Candelas, 1979; Kennedy, Grichley, and Dowker, 1980; Manoukian, 1989, 1987a, 1987b, 1990) about the quantum nature of microscopic physics by such classical means is certainly very satisfactory and quite convincing about the correctness of this monumental theory. Much effort has been done in the literature (Batelaan et al., 1997; Schwinger et al., 1988, Bloom and Erdman, 1962; Scully, Englert, and Schwinger, 1987, Martens and deMuynck, 1993; Englert, Schwinger, and Scully, 1988; Patil, 1998; Platt, 1992; Garraway and Stenholm, 1999; Cruz-Barrios and Gomez-Camacho, 2003) on the theoretical nature of the Stern-Gerlach effect, as of today, an analytical dynamical treatment of the problem which is (1) quantum mechanical, and takes into account (2) the field equation $\vec{\nabla}\cdot\vec{B}=0$, (3) the quantum counterpart of the Lorentz force, (4) the two, rather than one, dimensional aspect of the beam hitting the observation screen and (5) the rather non-trivial correlations that occur, as is explicitly shown, between the dynamical variables describing the intensity distribution.

It is the purpose of this research investigation, as reported in this thesis, to carry out a rigorous theoretical analysis of the Stern-Gerlach effect which takes into account the five indispensable points just mentioned. An analytical dynamical treatment of this effect to the leading order in $|e|/\sqrt{\hbar c}\equiv\sqrt{\alpha}$, for the electron, where α is the fine-structure constant, and for spin 1/2 charged particles (e.g., the proton), in general, is shown to lead to a unitary, i.e., a positive definite, expression for the probability intensity distribution on the observation screen, where the magnetic field has a controllable uniform component along the initial average direction of propagation of the particle, in addition to a non-uniform, almost longitudinal, magnetic field lying in the longitudinal plane defined by the quantization axis, in question, of the spin and in the initial average direction of propagation. With an initially prepared Gaussian wavepacket the analysis leads to a sum of so-called bivariate Gaussian distributions for the probability intensity distribution with a non-zero correlation (Manoukian, 1986). The uniform longitudinal controllable magnetic field, as will be shown, has a dual role in our analysis. Although longitudinal, it reduces effectively the quantum Lorentz force contribution by reducing, in turn, the correlation between the dynamical variables describing the probability density of observation, and also provides a positive definite expression for the latter. Needless to say, the reduction of the effect of the Lorentz force effect is of great importance for carrying out the experiment as mentioned above. Another aspect of a transversal magnetic field perpendicular to the non-uniform component along the quantization axis of the spin, tends to cause a further splitting of the beam in a direction perpendicular to the quatization axis as well. This unpleasant characteristic does not happen in our work by our careful choice of the magnetic field set up. The feasibility of performing the experiment with (an almost) longitudinal non-uniform magnetic field was emphasized several years ago (Brillouin, 1928) and also quite recently (Batelaan et al., 1997). But what our analytical study shows that the presence also of a uniform longitudinal controllable magnetic field yields to a unitary expression for the probability density on the observation screen in addition to reducing the Lorentz force effect.

The thesis is organized as follows. In Chapter II a detailed study is carried out of Gaussian distributions, their free time-developments and the computations of various expectation values of pertinent observables for this work. It also summarizes some of the important properties of bivariate Gaussian distributions with non-zero correlations. Chapter III sets up the Hamiltonian together with the magnetic field in question for the experiment. Most importantly, this chapter is involved in carrying out, explicitly, the time-dependent averages and the correlations for the interacting case needed in Chapter IV. In this latter chapter, we perform the rather difficult task of developing the time-evolution of the Gaussian wavepecket for the interacting case and after a very tedious analysis done the expression for the intensity distribution to be observed on the detection screen in the experiment is obtained. In Chapter V detailed graphical analyses are plotted, based on our analytical expression obtained in Chapter IV and a physical interpretation of the pattern of the beam splitting is given. Chapter VI deals with our conclusion and achievements and also invites experimentalists to finally carry out an experiment on the Stern-Gerlach effect for the electron as discussed in the bulk of this work.

Chapter II

Gaussian Distributions

2.1 The Gaussian Distribution and Its Free Time-Development

As an initial state for the physical process to be analyzed in this work we consider a wavepacket defined in the $\vec{x}$ - description by

$$\Psi_0(\vec{x}) = \frac{1}{(2\pi)^{3/4}\gamma^{3/2}} \exp\left(\frac{i}{\hbar}\vec{p}_0 \cdot \vec{x}\right) \exp\left(-\frac{\vec{x}^2}{4\gamma^2}\right) \tag{2.1}$$

where

$$\vec{p}_0 = (0, p_0, 0) \tag{2.2}$$

taken to be along the x_2 - axis, denoting the average momentum of the particle in question. That is,

$$\left\langle \Psi_0 \middle| \left(-i\hbar\vec{\nabla}\right) \middle| \Psi_0 \right\rangle = \vec{p}_0 \tag{2.3}$$

as is easily checked. This equation together with several other expectation values will be worked out in Section 2.2.

In Eq. (2.1), γ denotes the standard deviation. It provides a measure of the deviation of the position of a particle about its average position, as described by the wavepacket. It is explicitly given by

$$\sqrt{\left\langle \left(\vec{X} - \left\langle \vec{X} \right\rangle \right)^2 \right\rangle} = \gamma \tag{2.4}$$

where $\vec{X}$ denotes the position operator, and where we have used the notation $\left\langle \Psi_0 \middle| \square \middle| \Psi_0 \right\rangle \equiv \left\langle \square \right\rangle$ for convenience. Here we have used the notation γ, rather than the standard symbol σ, in order not to confuse it with the Pauli matrices.

In this section, we are interested in the free time-development of the wavepacket in Eq. (2.1), i.e., of the expression

$$\left[\exp\left(-\frac{i}{\hbar} t H_0 \right) \Psi_0 \right] (\vec{x}) \equiv \Psi_0 (\vec{x}, t) \tag{2.5}$$

where H_0 is the free Hamiltonian

$$H_0 = -\frac{\hbar^2 \nabla^2}{2M} \tag{2.6}$$

for a particle of mass M. The interacting case will be studied in Chapter IV.

By using the integral

$$\int_{-\infty}^{\infty} dk e^{ikx} e^{-k^2\gamma^2} = \frac{\sqrt{\pi}}{\gamma} \exp\left(-\frac{x^2}{4\gamma^2}\right) \tag{2.7}$$

and considering the integral expressions

$$\frac{1}{(2\pi)^{1/4}\gamma^{1/2}} \exp\left(-\frac{x^2}{4\gamma^2}\right) = \frac{\gamma^{1/2}}{\left(2\pi^3\right)^{1/4}} \int_{-\infty}^{\infty} dk e^{ikx} e^{-k^2\gamma^2} \tag{2.8}$$

$$\frac{1}{(2\pi)^{1/4}\gamma^{1/2}} \exp\left(-\frac{x^2}{4\gamma^2}\right) \exp\left(\frac{i}{\hbar} p_0 x\right) = \frac{\gamma^{1/2}}{\left(2\pi^3\right)^{1/4}} \int_{-\infty}^{\infty} dk e^{i\left(k+\frac{p_0}{\hbar}\right)x} e^{-k^2\gamma^2} \tag{2.9}$$

as applied to the wavepacket in Eq. (2.1), we obtain

$$e^{-\frac{i\hbar t}{2M}\left(-\nabla^2\right)} \Psi_0\left(\vec{x}\right) = e^{-\frac{i\hbar t}{2M}\left(-\nabla^2\right)} \left[\frac{\gamma^{3/2}}{\left(2\pi^3\right)^{3/4}} \int_{-\infty}^{\infty} d^3\vec{k} e^{i\left(\vec{k}+\frac{\vec{p}_0}{\hbar}\right)\vec{x}} e^{-\vec{k}^2\gamma^2}\right] \tag{2.10}$$

or

$$\Psi_0\left(\vec{x},t\right) = \frac{\gamma^{3/2}}{\left(2\pi^3\right)^{3/4}} \int_{-\infty}^{\infty} d^3\vec{k} e^{i\left(\vec{k}+\frac{\vec{p}_0}{\hbar}\right)\vec{x}} e^{-\frac{i\hbar t}{2M}\left(\vec{k}+\frac{\vec{p}_0}{\hbar}\right)^2} e^{-\vec{k}^2\gamma^2} \,. \tag{2.11}$$

Upon using the identity

$$i\left(\vec{k}+\frac{\vec{p}_0}{\hbar}\right)\cdot\vec{x}-\frac{i\hbar t}{2M}\left(\vec{k}+\frac{\vec{p}_0}{\hbar}\right)^2-\vec{k}^2\gamma^2$$

$$=i\left(\vec{k}+\frac{\vec{p}_0}{\hbar}\right)\cdot\vec{x}-\frac{i\hbar t}{2M}\left(\vec{k}^2+2\vec{k}\cdot\frac{\vec{p}_0}{\hbar}+\frac{\vec{p}^2{}_0}{\hbar^2}\right)-\vec{k}^2\gamma^2$$

$$=-\frac{i\vec{p}_0^2 t}{2M\hbar}-\vec{k}^2\left(\gamma^2+\frac{iht}{2M}\right)+i\vec{k}\cdot\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)+\frac{i}{\hbar}\vec{p}_0\cdot\vec{x} \qquad (2.12)$$

and setting

$$\Sigma_t^2=\gamma^2+\frac{i\hbar t}{2M} \qquad (2.13)$$

Eq. (2.12) may be rewritten as

$$=-\frac{i\vec{p}_0^2 t}{2M\hbar}-\Sigma_t^2\left[\vec{k}^2-\frac{i\vec{k}}{\Sigma_t^2}\cdot\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)\right]+\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}$$

$$=-\frac{i\vec{p}_0^2 t}{2M\hbar}-\Sigma_t^2\left[\left(\vec{k}-\frac{i}{2\Sigma_t^2}\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)\right)^2+\frac{1}{4\Sigma_t^4}\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)^2\right]+\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}$$

$$=-\frac{i\vec{p}_0^2 t}{2M\hbar}+\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}-\frac{1}{4\Sigma_t^2}\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)^2-\Sigma_t^2\left(\vec{k}-\frac{i}{2\Sigma_t^2}\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)\right)^2. \qquad (2.14)$$

Hence we obtain for Eq. (2.11)

$$\Psi_0\left(\vec{x},t\right)=\frac{\gamma^{3/2}}{\left(2\pi^3\right)^{3/4}}e^{-\frac{i\vec{p}_0^2 t}{2M\hbar}}e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}e^{-\frac{1}{4\Sigma_t^2}\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)^2}\int_{-\infty}^{\infty}d^3\vec{k}e^{-\Sigma_t^2\left(\vec{k}-\frac{i}{2\Sigma_t^2}\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)\right)^2}. \tag{2.15}$$

We now make a change of

$$\vec{k}-\frac{i}{2\Sigma_t^2}\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)\equiv\vec{K} \tag{2.16}$$

to obtain

$$\Psi_0\left(\vec{x},t\right)=\frac{\gamma^{3/2}}{\left(2\pi^3\right)^{3/4}}e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}e^{-\frac{i\vec{p}_0^2 t}{2M\hbar}}e^{-\frac{1}{4\Sigma_t^2}\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)^2}\int_{-\infty}^{\infty}d^3\vec{K}e^{-\Sigma_t^2\vec{K}^2}. \tag{2.17}$$

Finally the integral

$$\int_{-\infty}^{\infty}dxe^{-ax^2}=\sqrt{\frac{\pi}{a}} \tag{2.18}$$

gives

$$\Psi_0(\vec{x},t) = \frac{\gamma^{3/2}}{\left(2\pi^3\right)^{3/4}} e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}} e^{-\frac{i\vec{p}_0^2 t}{2M\hbar}} e^{-\frac{1}{4\Sigma_t^2}\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)^2} \left(\frac{\pi}{\Sigma_t^2}\right)^{3/2}$$

$$= \frac{\gamma^{3/2}}{\left(2\pi^3\right)^{3/4}} e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}} e^{-\frac{i\vec{p}_0^2 t}{2M\hbar}} \exp\left(-\frac{\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)^2}{4\left(\gamma^2+\frac{iht}{2M}\right)}\right)\left(\frac{\pi}{\gamma^2+\frac{iht}{2M}}\right)^{3/2} \quad (2.19)$$

which simplifies to the expression

$$\Psi_0(\vec{x},t) = \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}} e^{-\frac{i\vec{p}_0^2 t}{2M\hbar}}}{(2\pi)^{3/4}\gamma^{3/2}\left(1+\frac{iht}{2M\gamma^2}\right)^{3/2}} \exp\left(-\frac{\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)^2}{4\gamma^2\left(1+\frac{iht}{2M\gamma^2}\right)}\right). \quad (2.20)$$

2.2 Expectation Value of Observables

Throughout this section, we use the notation

$$\int d^3\vec{x}\,\Psi_0^*(\vec{x})\,A\,\Psi_0(\vec{x}) = \langle A\rangle \quad (2.21)$$

for any observable A.

For the subsequent analyses we need to evaluate the expectation values

$$\langle x_i\rangle, \langle p_i\rangle, \langle x_i x_j\rangle, \langle x_i x_j x_k\rangle, \langle x_i p_j\rangle, \langle p_i p_j\rangle, \langle p_i p_j p_k\rangle, \langle x_k p_i p_j\rangle, \langle x_i x_j p_k\rangle. \quad (2.22)$$

The free time-dependent counterparts of the any observable A is given by

$$A(t) = e^{\frac{it}{\hbar}H_0} A e^{-\frac{it}{\hbar}H_0} \tag{2.23}$$

and is referred to as the Heisenberg representation of A. Here H_0 is the free Hamiltonian defined in Eq. (2.6). The expectation values of time-dependent observable will be worked out in Section 3.3.

To the above end, the expectation values in Eq. (2.22) are explicitly evaluated as follows:

$$\begin{aligned}
\langle x_1 \rangle &= \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_1 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \\
&= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_1 e^{-\frac{\vec{x}^2}{2\gamma^2}} \\
&= \int dr d\theta d\phi\, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r \sin\theta \cos\phi e^{-\frac{r^2}{2\gamma^2}} \\
&= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^3 \sin^2\theta \cos\phi e^{-\frac{r^2}{2\gamma^2}} \\
&= 0
\end{aligned} \tag{2.24}$$

where we have used the integral

$$\int_0^{2\pi} d\phi \cos\phi = 0 \tag{2.25}$$

$$\begin{aligned}\langle x_2 \rangle &= \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_2 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \\ &= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_2 e^{-\frac{\vec{x}^2}{2\gamma^2}} \\ &= \int dr d\theta d\phi\, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r \sin\theta \sin\phi e^{-\frac{r^2}{2\gamma^2}} \\ &= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^3 \sin^2\theta \sin\phi e^{-\frac{r^2}{2\gamma^2}} \\ &= 0 \end{aligned} \tag{2.26}$$

since

$$\int_0^{2\pi} d\phi \sin\phi = 0 \tag{2.27}$$

$$\begin{aligned}\langle x_3 \rangle &= \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_3 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \\ &= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_3 e^{-\frac{\vec{x}^2}{2\gamma^2}} \end{aligned}$$

$$= \int dr d\theta d\phi \, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r\cos\theta e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^3 \sin\theta \cos\theta e^{-\frac{r^2}{2\gamma^2}}$$

$$= 0 \tag{2.28}$$

by using the fact that

$$\int_0^{\pi} d\theta \sin\theta \cos\theta = 0 \tag{2.29}$$

therefore

$$\langle x_i \rangle = 0 . \tag{2.30}$$

$$\langle p_1 \rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left(-i\hbar\frac{\partial}{\partial x_1}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left(i\hbar\frac{x_1}{2\gamma^2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2} \langle x_1 \rangle$$

$$= 0 \tag{2.31}$$

$$\langle p_2 \rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left(-i\hbar \frac{\partial}{\partial x_2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left[-i\hbar\left(-\frac{x_2}{2\gamma^2} + \frac{i}{\hbar} p_0\right)\right] \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left(i\hbar \frac{x_2}{2\gamma^2} + p_0\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\langle x_2 \rangle + p_0$$

$$= p_0 \tag{2.32}$$

$$\langle p_3 \rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left(-i\hbar \frac{\partial}{\partial x_3}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left(i\hbar \frac{x_3}{2\gamma^2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\langle x_3 \rangle$$

$$= 0\,. \tag{2.33}$$

Hence we have

$$\langle p_i \rangle = p_0 \delta^{i2} \tag{2.34}$$

which verifies the expression in Eq. (2.3).

We now consider quadratic variables including correlation functions. These are evaluated as follows:

$$\langle x_1 x_1 \rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_1 x_1 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_1 x_1 e^{-\frac{\vec{x}^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r^2 \sin^2\theta \cos^2\phi e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^4 \sin^3\theta \cos^2\phi e^{-\frac{r^2}{2\gamma^2}} \qquad (2.35)$$

and using the integrals

$$\int_0^\pi d\theta \sin^3\theta = \frac{4}{3} \qquad (2.36)$$

$$\int_0^{2\pi} d\phi \cos^2\phi = \pi \qquad (2.37)$$

$$\int_0^\infty dr\, r^4 e^{-\frac{r^2}{2\gamma^2}} = 1\cdot 3\cdot\left(\frac{1}{\gamma^2}\right)^{-5/2}\left(\frac{\pi}{2}\right)^{1/2} = 3\gamma^5\left(\frac{\pi}{2}\right)^{1/2} \qquad (2.38)$$

we obtain

$$\langle x_1 x_1 \rangle = \frac{1}{(2\pi)^{3/2}\gamma^3} \cdot \frac{4}{3}\pi \cdot 3\gamma^5 \left(\frac{\pi}{2}\right)^{1/2}$$

$$= \gamma^2 . \tag{2.39}$$

Similarly, we have

$$\langle x_1 x_2 \rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_1 x_2 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_1 x_2 \, e^{-\frac{\vec{x}^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r\sin\theta\cos\phi r\sin\theta\sin\phi e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^4 \sin^3\theta \sin\phi\cos\phi e^{-\frac{r^2}{2\gamma^2}} \tag{2.40}$$

and using the integral

$$\int_0^{2\pi} d\phi \sin\phi\cos\phi = 0 \tag{2.41}$$

this gives

$$\langle x_1 x_2 \rangle = 0 \tag{2.42}$$

$$\left\langle x_1x_3\right\rangle = \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\,x_1x_3\,\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\,\frac{1}{(2\pi)^{3/2}\gamma^3}\,x_1x_3\,e^{-\frac{\vec{x}^2}{2\gamma^2}}$$

$$= \int drd\theta d\phi\, r^2\sin\theta\,\frac{1}{(2\pi)^{3/2}\gamma^3}\,r\sin\theta\cos\phi r\cos\theta e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int drd\theta d\phi\,\frac{1}{(2\pi)^{3/2}\gamma^3}\,r^4\sin^2\theta\cos\theta\cos\phi e^{-\frac{r^2}{2\gamma^2}} \qquad (2.43)$$

and using the integral in Eq. (2.25) leads to

$$\left\langle x_1x_3\right\rangle = 0 \qquad (2.44)$$

$$\left\langle x_2x_2\right\rangle = \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\,x_2x_2\,\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\,\frac{1}{(2\pi)^{3/2}\gamma^3}\,x_2x_2\,e^{-\frac{\vec{x}^2}{2\gamma^2}}$$

$$= \int drd\theta d\phi\, r^2\sin\theta\,\frac{1}{(2\pi)^{3/2}\gamma^3}\,r^2\sin^2\theta\sin^2\phi e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int drd\theta d\phi\,\frac{1}{(2\pi)^{3/2}\gamma^3}\,r^4\sin^3\theta\sin^2\phi e^{-\frac{r^2}{2\gamma^2}}\,. \qquad (2.45)$$

The integrals

$$\int_0^{2\pi} d\phi \sin^2\phi = \pi \tag{2.46}$$

$$\int_0^{\pi} d\theta \sin^3\theta = \frac{4}{3} \tag{2.47}$$

and the integral in Eq. (2.35), then give

$$\langle x_2 x_2 \rangle = \frac{1}{(2\pi)^{3/2}\gamma^3} \cdot \frac{4}{3}\pi \cdot 3\gamma^5 \left(\frac{\pi}{2}\right)^{1/2}$$

$$= \gamma^2 \tag{2.48}$$

$$\langle x_2 x_3 \rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_2 x_3 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_2 x_3 \, e^{-\frac{\vec{x}^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r \sin\theta \sin\phi r \cos\theta e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^4 \sin^2\theta \cos\theta \sin\phi e^{-\frac{r^2}{2\gamma^2}} \tag{2.49}$$

and since

$$\int_0^{2\pi} d\phi \sin\phi = 0 \tag{2.50}$$

this gives

$$\langle x_2 x_3 \rangle = 0 \tag{2.51}$$

$$\begin{aligned}
\langle x_3 x_3 \rangle &= \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \, x_3 x_3 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \\
&= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_3 x_3 \, e^{-\frac{\vec{x}^2}{2\gamma^2}} \\
&= \int dr d\theta d\phi \, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r^2 \cos^2\theta e^{-\frac{r^2}{2\gamma^2}} \\
&= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^4 \sin\theta \cos^2\theta e^{-\frac{r^2}{2\gamma^2}} .
\end{aligned} \tag{2.52}$$

The integral

$$\int_0^{\pi} d\theta \sin\theta \cos^2\theta = \frac{2}{3} \tag{2.53}$$

and the one in Eq. (2.35) imply that

$$\left\langle x_3 x_3 \right\rangle = \frac{1}{(2\pi)^{3/2}\gamma^3} \cdot \frac{2}{3} \cdot 2\pi \cdot 3\gamma^5 \left(\frac{\pi}{2}\right)^{1/2}$$

$$= \gamma^2 . \tag{2.54}$$

That is, quite generally we may write

$$\left\langle x_i x_j \right\rangle = \gamma^2 \delta^{ij} . \tag{2.55}$$

We also need the following expectation values, where the orders of non-commuting operators are to be noted:

$$\left\langle x_1 p_1 \right\rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_1 \left(-i\hbar \frac{\partial}{\partial x_1}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_1 \left(i\hbar \frac{x_1}{2\gamma^2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2} \left\langle x_1 x_1 \right\rangle$$

$$= \frac{i\hbar}{2} \tag{2.56}$$

$$\left\langle x_1 p_2 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 \left(-i\hbar\frac{\partial}{\partial x_2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 \left(i\hbar\frac{x_2}{2\gamma^2} + p_0\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\left\langle x_1 x_2 \right\rangle + p_0 \left\langle x_1 \right\rangle$$

$$= 0 \tag{2.57}$$

$$\left\langle x_1 p_3 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 \left(-i\hbar\frac{\partial}{\partial x_3}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 \left(i\hbar\frac{x_3}{2\gamma^2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\left\langle x_1 x_3 \right\rangle$$

$$= 0 \tag{2.58}$$

$$\left\langle x_2 p_1 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_2 \left(-i\hbar\frac{\partial}{\partial x_1}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_2 \left(i\hbar\frac{x_1}{2\gamma^2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\left\langle x_2 x_1 \right\rangle$$

$$= 0 \tag{2.59}$$

$$\left\langle x_2 p_2 \right\rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_2 \left(-i\hbar \frac{\partial}{\partial x_2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_2 \left(i\hbar \frac{x_2}{2\gamma^2} + p_0 \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2} \left\langle x_2 x_2 \right\rangle + p_0 \left\langle x_2 \right\rangle$$

$$= \frac{i\hbar}{2} \tag{2.60}$$

$$\left\langle x_2 p_3 \right\rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_2 \left(-i\hbar \frac{\partial}{\partial x_3} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_2 \left(i\hbar \frac{x_3}{2\gamma^2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2} \left\langle x_2 x_3 \right\rangle$$

$$= 0 \tag{2.61}$$

$$\left\langle x_3 p_1 \right\rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_3 \left(-i\hbar \frac{\partial}{\partial x_1} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_3 \left(i\hbar \frac{x_1}{2\gamma^2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2} \left\langle x_3 x_1 \right\rangle$$

$$= 0 \tag{2.62}$$

$$\left\langle x_3 p_2 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_3 \left(-i\hbar\frac{\partial}{\partial x_2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_3 \left(i\hbar\frac{x_2}{2\gamma^2} + p_0\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\left\langle x_3 x_2 \right\rangle + p_0 \left\langle x_3 \right\rangle$$

$$= 0 \tag{2.63}$$

$$\left\langle x_3 p_3 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_3 \left(-i\hbar\frac{\partial}{\partial x_3}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_3 \left(i\hbar\frac{x_3}{2\gamma^2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\left\langle x_3 x_3 \right\rangle$$

$$= \frac{i\hbar}{2}. \tag{2.64}$$

That is, we may write

$$\left\langle x_i p_j \right\rangle = \frac{i\hbar}{2}\delta^{ij}. \tag{2.65}$$

Similarly, we evaluate the following expectation values

$$\left\langle p_1p_1\right\rangle = \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\left(-i\hbar\frac{\partial}{\partial x_1}\right)\left(-i\hbar\frac{\partial}{\partial x_1}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\left(-i\hbar\frac{\partial}{\partial x_1}\right)\left[\left(i\hbar\frac{x_1}{2\gamma^2}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\right]$$

$$= \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\left[\left(i\hbar\frac{x_1}{2\gamma^2}\right)\left(i\hbar\frac{x_1}{2\gamma^2}\right)+\frac{\hbar^2}{2\gamma^2}\right]\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= -\frac{\hbar^2}{4\gamma^4}\left\langle x_1x_1\right\rangle + \frac{\hbar^2}{2\gamma^2}$$

$$= \frac{\hbar^2}{4\gamma^2} \qquad (2.66)$$

$$\left\langle p_1p_2\right\rangle = \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\left(-i\hbar\frac{\partial}{\partial x_1}\right)\left(-i\hbar\frac{\partial}{\partial x_2}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\left(-i\hbar\frac{\partial}{\partial x_1}\right)\left[\left(i\hbar\frac{x_2}{2\gamma^2}+p_0\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\right]$$

$$= \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\left(i\hbar\frac{x_2}{2\gamma^2}+p_0\right)\left(i\hbar\frac{x_1}{2\gamma^2}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= -\frac{\hbar^2}{4\gamma^4}\left\langle x_2x_1\right\rangle + \frac{i\hbar p_0}{2\gamma^2}\left\langle x_1\right\rangle$$

$$= 0 \qquad (2.67)$$

$$\left\langle p_1 p_3 \right\rangle = \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\left(-i\hbar\frac{\partial}{\partial x_1}\right)\left(-i\hbar\frac{\partial}{\partial x_3}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\left(-i\hbar\frac{\partial}{\partial x_1}\right)\left[\left(i\hbar\frac{x_3}{2\gamma^2}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\right]$$

$$= \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\left(i\hbar\frac{x_3}{2\gamma^2}\right)\left(i\hbar\frac{x_1}{2\gamma^2}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= -\frac{\hbar^2}{4\gamma^4}\left\langle x_3 x_1 \right\rangle$$

$$= 0 \tag{2.68}$$

$$\left\langle p_2 p_2 \right\rangle = \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\left(-i\hbar\frac{\partial}{\partial x_2}\right)\left(-i\hbar\frac{\partial}{\partial x_2}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\left(-i\hbar\frac{\partial}{\partial x_2}\right)\left[\left(i\hbar\frac{x_2}{2\gamma^2}+p_0\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\right]$$

$$= \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\left[\left(i\hbar\frac{x_2}{2\gamma^2}+p_0\right)\left(i\hbar\frac{x_2}{2\gamma^2}+p_0\right)+\frac{\hbar^2}{2\gamma^2}\right]\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= -\frac{\hbar^2}{4\gamma^4}\left\langle x_2 x_2 \right\rangle + \frac{i\hbar p_0}{\gamma^2}\left\langle x_2 \right\rangle + p_0^2 + \frac{\hbar^2}{2\gamma^2}$$

$$= p_0^2 + \frac{\hbar^2}{4\gamma^2} \tag{2.69}$$

$$\left\langle p_2 p_3 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left(-i\hbar\frac{\partial}{\partial x_2}\right)\left(-i\hbar\frac{\partial}{\partial x_3}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left(-i\hbar\frac{\partial}{\partial x_2}\right)\left[\left(i\hbar\frac{x_3}{2\gamma^2}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\right]$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left(i\hbar\frac{x_3}{2\gamma^2}\right)\left(i\hbar\frac{x_2}{2\gamma^2}+p_0\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= -\frac{\hbar^2}{4\gamma^4}\left\langle x_3 x_2\right\rangle + \frac{i\hbar p_0}{2\gamma^2}\left\langle x_3\right\rangle$$

$$= 0 \qquad (2.70)$$

$$\left\langle p_3 p_3 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left(-i\hbar\frac{\partial}{\partial x_3}\right)\left(-i\hbar\frac{\partial}{\partial x_3}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left(-i\hbar\frac{\partial}{\partial x_3}\right)\left[\left(i\hbar\frac{x_3}{2\gamma^2}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\right]$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \left[\left(i\hbar\frac{x_3}{2\gamma^2}\right)\left(i\hbar\frac{x_3}{2\gamma^2}\right)+\frac{\hbar^2}{2\gamma^2}\right]\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= -\frac{\hbar^2}{4\gamma^4}\left\langle x_3 x_3\right\rangle + \frac{\hbar^2}{2\gamma^2}$$

$$= \frac{\hbar^2}{4\gamma^2}\,. \qquad (2.71)$$

Thus we may rewrite Eqs. (2.66) - (2.71) in the unified manner:

$$\left\langle p_i p_j \right\rangle = p_0^i p_0^j + \frac{\hbar^2}{4\gamma^2}\delta^{ij} \tag{2.72}$$

where

$$p_0^i = p_0 \delta^{i2}. \tag{2.73}$$

The evaluation of the product of three observable is more involved and the details are worked out below:

$$\begin{aligned}
\left\langle x_1 x_1 x_1 \right\rangle &= \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_1 x_1 x_1 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \\
&= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_1 x_1 x_1 \, e^{-\frac{\vec{x}^2}{2\gamma^2}} \\
&= \int dr d\theta d\phi \, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r^3 \sin^3\theta \cos^3\phi e^{-\frac{r^2}{2\gamma^2}} \\
&= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^5 \sin^4\theta \cos^3\phi e^{-\frac{r^2}{2\gamma^2}} \\
&= 0
\end{aligned} \tag{2.74}$$

since

$$\int_0^{2\pi} d\phi \cos^3\phi = 0 \tag{2.75}$$

$$\langle x_1 x_1 x_2 \rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_1 x_1 x_2 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_1 x_1 x_2 \, e^{-\frac{\vec{x}^2}{2\gamma^2}}$$

$$= \int drd\theta d\phi \, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r^2 \sin^2\theta \cos^2\phi r \sin\theta \sin\phi e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int drd\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^5 \sin^4\theta \sin\phi \cos^2\phi e^{-\frac{r^2}{2\gamma^2}}$$

$$= 0 \tag{2.76}$$

as follows form the integral

$$\int_0^{2\pi} d\phi \sin\phi \cos^2\phi = 0 \tag{2.77}$$

$$\langle x_1 x_1 x_3 \rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_1 x_1 x_3 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_1 x_1 x_3 \, e^{-\frac{\vec{x}^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r^2 \sin^2\theta \cos^2\phi r \cos\theta e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^5 \sin^3\theta \cos\theta \cos^2\phi e^{-\frac{r^2}{2\gamma^2}}$$

$$= 0 \tag{2.78}$$

on account of the integral

$$\int_0^\pi d\theta \sin^3\theta \cos\theta = 0 \tag{2.79}$$

$$\langle x_1 x_2 x_2 \rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_1 x_2 x_2 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_1 x_2 x_2 \, e^{-\frac{\vec{x}^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r \sin\theta \cos\phi r^2 \sin^2\theta \sin^2\phi e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^5 \sin^4\theta \sin^2\phi \cos\phi e^{-\frac{r^2}{2\gamma^2}}$$

$$= 0 \tag{2.80}$$

as a consequence of the integral

$$\int_0^{2\pi} d\phi \sin^2\phi \cos\phi = 0 \tag{2.81}$$

$$\begin{aligned}
\langle x_1 x_3 x_3 \rangle &= \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_1 x_3 x_3 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \\
&= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_1 x_3 x_3 \, e^{-\frac{\vec{x}^2}{2\gamma^2}} \\
&= \int dr d\theta d\phi \, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r \sin\theta \cos\phi r^2 \cos^2\theta e^{-\frac{r^2}{2\gamma^2}} \\
&= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^5 \sin^2\theta \cos^2\theta \cos\phi e^{-\frac{r^2}{2\gamma^2}} \\
&= 0
\end{aligned} \tag{2.82}$$

where we have used the integral

$$\int_0^{2\pi} d\phi \cos\phi = 0 \tag{2.83}$$

$$\begin{aligned}
\langle x_1 x_2 x_3 \rangle &= \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_1 x_2 x_3 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \\
&= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_1 x_2 x_3 \, e^{-\frac{\vec{x}^2}{2\gamma^2}}
\end{aligned}$$

$$= \int dr d\theta d\phi \, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r\sin\theta\cos\phi r\sin\theta\sin\phi r\cos\theta e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^5 \sin^3\theta\cos\theta\sin\phi\cos\phi e^{-\frac{r^2}{2\gamma^2}}$$

$$= 0 \tag{2.84}$$

as follows form the integral expression

$$\int_0^{2\pi} d\phi \sin\phi\cos\phi = 0 \tag{2.85}$$

$$\langle x_2 x_2 x_2 \rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_2 x_2 x_2 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_2 x_2 x_2 \, e^{-\frac{\vec{x}^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r^3 \sin^3\theta \sin^3\phi e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int dr d\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^5 \sin^4\theta \sin^3\phi e^{-\frac{r^2}{2\gamma^2}}$$

$$= 0 \tag{2.86}$$

as a consequence of

$$\int_0^{2\pi} d\phi \sin^3\phi = 0 \tag{2.87}$$

$$\begin{aligned}
\left\langle x_2 x_2 x_3 \right\rangle &= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_2 x_2 x_3\, \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \\
&= \int d^3\vec{x}\, \frac{1}{(2\pi)^{3/2}\gamma^3}\, x_2 x_2 x_3\, e^{-\frac{\vec{x}^2}{2\gamma^2}} \\
&= \int dr d\theta d\phi\, r^2 \sin\theta\, \frac{1}{(2\pi)^{3/2}\gamma^3}\, r^2 \sin^2\theta \sin^2\phi\, r\cos\theta\, e^{-\frac{r^2}{2\gamma^2}} \\
&= \int dr d\theta d\phi\, \frac{1}{(2\pi)^{3/2}\gamma^3}\, r^5 \sin^3\theta \cos\theta \sin^2\phi\, e^{-\frac{r^2}{2\gamma^2}} \\
&= 0
\end{aligned} \tag{2.88}$$

since

$$\int_0^{\pi} d\theta \sin^3\theta \cos\theta = 0 \tag{2.89}$$

$$\begin{aligned}
\left\langle x_2 x_3 x_3 \right\rangle &= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_2 x_3 x_3\, \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \\
&= \int d^3\vec{x}\, \frac{1}{(2\pi)^{3/2}\gamma^3}\, x_2 x_3 x_3\, e^{-\frac{\vec{x}^2}{2\gamma^2}}
\end{aligned}$$

$$= \int drd\theta d\phi\, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r \sin\theta \sin\phi r^2 \cos^2\theta e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int drd\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^5 \sin^2\theta \cos^2\theta \sin\phi e^{-\frac{r^2}{2\gamma^2}}$$

$$= 0 \qquad (2.90)$$

on account of

$$\int_0^{2\pi} d\phi \sin\phi = 0 \qquad (2.91)$$

$$\langle x_3 x_3 x_3 \rangle = \int d^3\vec{x} \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_3 x_3 x_3 \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x} \frac{1}{(2\pi)^{3/2}\gamma^3} x_3 x_3 x_3\, e^{-\frac{\vec{x}^2}{2\gamma^2}}$$

$$= \int drd\theta d\phi\, r^2 \sin\theta \frac{1}{(2\pi)^{3/2}\gamma^3} r^3 \cos^3\theta e^{-\frac{r^2}{2\gamma^2}}$$

$$= \int drd\theta d\phi \frac{1}{(2\pi)^{3/2}\gamma^3} r^5 \sin\theta \cos^3\theta e^{-\frac{r^2}{2\gamma^2}}$$

$$= 0 \qquad (2.92)$$

by using the integral

$$\int_0^{\pi} d\theta \sin\theta \cos^3\theta = 0\,. \tag{2.93}$$

We may then write for all $i, j, k = 1,2,3$:

$$\left\langle x_i x_j x_k \right\rangle = 0\,. \tag{2.94}$$

Similarly, we evaluate the following expectation values:

$$\begin{aligned}
\left\langle x_1 x_1 p_1 \right\rangle &= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_1 \left(-i\hbar \frac{\partial}{\partial x_1} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \\
&= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_1 \left(i\hbar \frac{x_1}{2\gamma^2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \\
&= \frac{i\hbar}{2\gamma^2} \left\langle x_1 x_1 x_1 \right\rangle \\
&= 0
\end{aligned} \tag{2.95}$$

$$\begin{aligned}
\left\langle x_1 x_1 p_2 \right\rangle &= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_1 \left(-i\hbar \frac{\partial}{\partial x_2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} \\
&= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_1 \left(i\hbar \frac{x_2}{2\gamma^2} + p_0 \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}
\end{aligned}$$

$$= \frac{i\hbar}{2\gamma^2}\left\langle x_1 x_1 x_2 \right\rangle + p_0 \left\langle x_1 x_1 \right\rangle$$

$$= p_0 \gamma^2 \tag{2.96}$$

$$\left\langle x_1 x_1 p_3 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_1 \left(-i\hbar \frac{\partial}{\partial x_3} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_1 \left(i\hbar \frac{x_3}{2\gamma^2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\left\langle x_1 x_1 x_3 \right\rangle$$

$$= 0 \tag{2.97}$$

$$\left\langle x_1 x_2 p_1 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_2 \left(-i\hbar \frac{\partial}{\partial x_1} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_2 \left(i\hbar \frac{x_1}{2\gamma^2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\left\langle x_1 x_2 x_1 \right\rangle$$

$$= 0 \tag{2.98}$$

$$\left\langle x_1 x_2 p_2 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_2 \left(-i\hbar \frac{\partial}{\partial x_2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_2 \left(i\hbar \frac{x_2}{2\gamma^2} + p_0 \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\langle x_1 x_2 x_2\rangle + p_0\langle x_1 x_2\rangle$$

$$= 0 \tag{2.99}$$

$$\langle x_1 x_2 p_3\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_2 \left(-i\hbar\frac{\partial}{\partial x_3}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_2 \left(i\hbar\frac{x_3}{2\gamma^2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\langle x_1 x_2 x_3\rangle$$

$$= 0 \tag{2.100}$$

$$\langle x_1 x_3 p_1\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_3 \left(-i\hbar\frac{\partial}{\partial x_1}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_3 \left(i\hbar\frac{x_1}{2\gamma^2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\langle x_1 x_3 x_1\rangle$$

$$= 0 \tag{2.101}$$

$$\langle x_1 x_3 p_2\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_3 \left(-i\hbar\frac{\partial}{\partial x_2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_1 x_3 \left(i\hbar\frac{x_2}{2\gamma^2} + p_0\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\langle x_1 x_3 x_2\rangle + p_0\langle x_1 x_3\rangle$$

$$= 0 \tag{2.102}$$

$$\langle x_1 x_3 p_3\rangle = \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\,x_1 x_3\left(-i\hbar\frac{\partial}{\partial x_3}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\,x_1 x_3\left(i\hbar\frac{x_3}{2\gamma^2}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\langle x_1 x_3 x_3\rangle$$

$$= 0 \tag{2.103}$$

$$\langle x_2 x_2 p_1\rangle = \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\,x_2 x_2\left(-i\hbar\frac{\partial}{\partial x_1}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\,x_2 x_2\left(i\hbar\frac{x_1}{2\gamma^2}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\langle x_2 x_2 x_1\rangle$$

$$= 0 \tag{2.104}$$

$$\langle x_2 x_2 p_2\rangle = \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\,x_2 x_2\left(-i\hbar\frac{\partial}{\partial x_2}\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\,\frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}\,x_2 x_2\left(i\hbar\frac{x_2}{2\gamma^2}+p_0\right)\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}}e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\left\langle x_2 x_2 x_2 \right\rangle + p_0 \left\langle x_2 x_2 \right\rangle$$

$$= p_0 \gamma^2 \qquad (2.105)$$

$$\left\langle x_2 x_2 p_3 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_2 x_2 \left(-i\hbar \frac{\partial}{\partial x_3}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_2 x_2 \left(i\hbar \frac{x_3}{2\gamma^2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\left\langle x_2 x_2 x_3 \right\rangle$$

$$= 0 \qquad (2.106)$$

$$\left\langle x_2 x_3 p_1 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_2 x_3 \left(-i\hbar \frac{\partial}{\partial x_1}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_2 x_3 \left(i\hbar \frac{x_1}{2\gamma^2}\right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2}\left\langle x_2 x_3 x_1 \right\rangle$$

$$= 0 \qquad (2.107)$$

$$\left\langle x_2 x_3 p_2 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_2 x_3 \left(-i\hbar \frac{\partial}{\partial x_2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_2 x_3 \left(i\hbar \frac{x_2}{2\gamma^2} + p_0 \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2} \left\langle x_2 x_3 x_2 \right\rangle + p_0 \left\langle x_2 x_3 \right\rangle$$

$$= 0 \tag{2.108}$$

$$\left\langle x_2 x_3 p_3 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_2 x_3 \left(-i\hbar \frac{\partial}{\partial x_3} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_2 x_3 \left(i\hbar \frac{x_3}{2\gamma^2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2} \left\langle x_2 x_3 x_3 \right\rangle$$

$$= 0 \tag{2.109}$$

$$\left\langle x_3 x_3 p_1 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_3 x_3 \left(-i\hbar \frac{\partial}{\partial x_1} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}\, x_3 x_3 \left(i\hbar \frac{x_1}{2\gamma^2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2} \left\langle x_3 x_3 x_1 \right\rangle$$

$$= 0 \tag{2.110}$$

$$\left\langle x_3 x_3 p_2 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_3 x_3 \left(-i\hbar \frac{\partial}{\partial x_2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_3 x_3 \left(i\hbar \frac{x_2}{2\gamma^2} + p_0 \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2} \left\langle x_3 x_3 x_2 \right\rangle + p_0 \left\langle x_3 x_3 \right\rangle$$

$$= p_0 \gamma^2 \qquad (2.111)$$

$$\left\langle x_3 x_3 p_3 \right\rangle = \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_3 x_3 \left(-i\hbar \frac{\partial}{\partial x_3} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \int d^3\vec{x}\, \frac{e^{-\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}} x_3 x_3 \left(i\hbar \frac{x_3}{2\gamma^2} \right) \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} e^{-\frac{\vec{x}^2}{4\gamma^2}}$$

$$= \frac{i\hbar}{2\gamma^2} \left\langle x_3 x_3 x_3 \right\rangle$$

$$= 0\,. \qquad (2.112)$$

We may thus conveniently write for all $i, j, k = 1,2,3$:

$$\left\langle x_i x_j p_k \right\rangle = \delta^{ij} p_0 \gamma^2 \delta^{k2}\,. \qquad (2.113)$$

The expressions for the expectation values for the following products then readily follow:

$$\langle x_1 p_1 p_1 \rangle = -\frac{\hbar^2}{4\gamma^4}\langle x_1 x_1 x_1 \rangle + \frac{\hbar^2}{2\gamma^2}\langle x_1 \rangle$$

$$= 0 \tag{2.114}$$

$$\langle x_1 p_1 p_2 \rangle = -\frac{\hbar^2}{4\gamma^4}\langle x_1 x_2 x_1 \rangle + \frac{i\hbar p_0}{2\gamma^2}\langle x_1 x_1 \rangle$$

$$= \frac{ih}{2} p_0 \tag{2.115}$$

$$\langle x_1 p_1 p_3 \rangle = -\frac{\hbar^2}{4\gamma^4}\langle x_1 x_3 x_1 \rangle$$

$$= 0 \tag{2.116}$$

$$\langle x_2 p_1 p_1 \rangle = -\frac{\hbar^2}{4\gamma^4}\langle x_2 x_1 x_1 \rangle + \frac{\hbar^2}{2\gamma^2}\langle x_2 \rangle$$

$$= 0 \tag{2.117}$$

$$\langle x_2 p_1 p_2 \rangle = -\frac{\hbar^2}{4\gamma^4}\langle x_2 x_2 x_1 \rangle + \frac{i\hbar p_0}{2\gamma^2}\langle x_2 x_1 \rangle$$

$$= 0 \tag{2.118}$$

$$\langle x_2 p_1 p_3 \rangle = -\frac{\hbar^2}{4\gamma^4}\langle x_2 x_3 x_1 \rangle$$

$$= 0 \tag{2.119}$$

$$\langle x_3 p_1 p_1 \rangle = -\frac{\hbar^2}{4\gamma^4}\langle x_3 x_1 x_1 \rangle + \frac{\hbar^2}{2\gamma^2}\langle x_3 \rangle$$

$$= 0 \tag{2.120}$$

$$\langle x_3 p_1 p_2 \rangle = -\frac{\hbar^2}{4\gamma^4}\langle x_3 x_2 x_1 \rangle + \frac{i\hbar p_0}{2\gamma^2}\langle x_3 x_1 \rangle$$

$$= 0 \tag{2.121}$$

$$\langle x_3 p_1 p_3 \rangle = -\frac{\hbar^2}{4\gamma^4}\langle x_3 x_3 x_1 \rangle$$

$$= 0 \tag{2.122}$$

$$\langle x_1 p_2 p_2 \rangle = -\frac{\hbar^2}{4\gamma^4}\langle x_1 x_2 x_2 \rangle + \frac{i\hbar p_0}{\gamma^2}\langle x_1 x_2 \rangle + \left(p_0^2 + \frac{\hbar^2}{2\gamma^2} \right)\langle x_1 \rangle$$

$$= 0 \tag{2.123}$$

$$\langle x_2 p_2 p_2 \rangle = -\frac{\hbar^2}{4\gamma^4}\langle x_2 x_2 x_2 \rangle + \frac{i\hbar p_0}{\gamma^2}\langle x_2 x_2 \rangle + \left(p_0^2 + \frac{\hbar^2}{2\gamma^2} \right)\langle x_2 \rangle$$

$$= i\hbar p_0 \tag{2.124}$$

$$\left\langle x_3 p_2 p_2 \right\rangle = -\frac{\hbar^2}{4\gamma^4}\left\langle x_3 x_2 x_2 \right\rangle + \frac{i\hbar p_0}{\gamma^2}\left\langle x_3 x_2 \right\rangle + \left(p_0^2 + \frac{\hbar^2}{2\gamma^2} \right)\left\langle x_3 \right\rangle$$

$$= 0 \tag{2.125}$$

$$\left\langle x_1 p_2 p_3 \right\rangle = -\frac{\hbar^2}{4\gamma^4}\left\langle x_1 x_3 x_2 \right\rangle + \frac{i\hbar p_0}{2\gamma^2}\left\langle x_1 x_3 \right\rangle$$

$$= 0 \tag{2.126}$$

$$\left\langle x_2 p_2 p_3 \right\rangle = -\frac{\hbar^2}{4\gamma^4}\left\langle x_2 x_3 x_2 \right\rangle + \frac{i\hbar p_0}{2\gamma^2}\left\langle x_2 x_3 \right\rangle$$

$$= 0 \tag{2.127}$$

$$\left\langle x_3 p_2 p_3 \right\rangle = -\frac{\hbar^2}{4\gamma^4}\left\langle x_3 x_3 x_2 \right\rangle + \frac{i\hbar p_0}{2\gamma^2}\left\langle x_3 x_3 \right\rangle$$

$$= \frac{i\hbar}{2} p_0 \tag{2.128}$$

$$\left\langle x_1 p_3 p_3 \right\rangle = -\frac{\hbar^2}{4\gamma^4}\left\langle x_1 x_3 x_3 \right\rangle + \frac{\hbar^2}{2\gamma^2}\left\langle x_1 \right\rangle$$

$$= 0 \tag{2.129}$$

$$\left\langle x_2 p_3 p_3 \right\rangle = -\frac{\hbar^2}{4\gamma^4}\left\langle x_2 x_3 x_3 \right\rangle + \frac{\hbar^2}{2\gamma^2}\left\langle x_2 \right\rangle$$

$$= 0 \tag{2.130}$$

$$\left\langle x_3 p_3 p_3 \right\rangle = -\frac{\hbar^2}{4\gamma^4}\left\langle x_3 x_3 x_3 \right\rangle + \frac{\hbar^2}{2\gamma^2}\left\langle x_3 \right\rangle$$

$$= 0 \,. \qquad (2.131)$$

These allow us to write

$$\left\langle x_k p_i p_j \right\rangle = \frac{i\hbar}{2} p_0^i \delta^{jk} + \frac{i\hbar}{2} p_0^i \delta^{ik} \,. \qquad (2.132)$$

For the product of three momentum components, we explicitly have

$$p_1 p_1 p_1 \Psi_0\left(\vec{x}\right) = \left(-i\hbar \frac{\partial}{\partial x_1}\right)\left[\left(-\frac{\hbar^2}{4\gamma^4} x_1 x_1 + \frac{\hbar^2}{2\gamma^2}\right)\Psi_0\left(\vec{x}\right)\right]$$

$$= \left[\left(-\frac{\hbar^2}{4\gamma^4} x_1 x_1 + \frac{\hbar^2}{2\gamma^2}\right)\left(\frac{i\hbar}{2\gamma^2} x_1\right) + \frac{i\hbar^3}{2\gamma^4} x_1\right]\Psi_0\left(\vec{x}\right) \qquad (2.133)$$

$$\left\langle p_1 p_1 p_1 \right\rangle = -\frac{i\hbar^3}{8\gamma^6}\left\langle x_1 x_1 x_1 \right\rangle + \frac{3i\hbar^3}{4\gamma^4}\left\langle x_1 \right\rangle$$

$$= 0 \qquad (2.134)$$

$$p_2 p_2 p_2 \Psi_0\left(\vec{x}\right) = \left(-i\hbar \frac{\partial}{\partial x_2}\right)\left[\left(-\frac{\hbar^2}{4\gamma^4} x_2 x_2 + \frac{i\hbar p_0}{\gamma^2} x_2 + p_0^2 + \frac{\hbar^2}{2\gamma^2}\right)\Psi_0\left(\vec{x}\right)\right]$$

$$= \left[\begin{array}{l} \left(-\frac{\hbar^2}{4\gamma^4} x_2 x_2 + \frac{i\hbar p_0}{\gamma^2} x_2 + p_0^2 + \frac{\hbar^2}{2\gamma^2}\right)\left(\frac{i\hbar}{2\gamma^2} x_2 + p_0\right) \\ + \left(\frac{i\hbar^3}{2\gamma^4} x_2 + \frac{\hbar^2 p_0}{\gamma^2}\right) \end{array}\right]\Psi_0\left(\vec{x}\right) \qquad (2.135)$$

$$=\begin{bmatrix} -\dfrac{i\hbar^3}{8\gamma^6}x_2x_2x_2-\dfrac{\hbar^2 p_0}{2\gamma^4}x_2x_2 \\ +\left(p_0^2+\dfrac{\hbar^2}{2\gamma^2}\right)\left(\dfrac{i\hbar}{2\gamma^2}x_2\right) \\ -\dfrac{\hbar^2 p_0}{4\gamma^4}x_2x_2+\dfrac{i\hbar p_0^2}{\gamma^2}x_2+\left(p_0^2+\dfrac{\hbar^2}{2\gamma^2}\right)p_0 \\ +\dfrac{i\hbar^3}{2\gamma^4}x_2+\dfrac{\hbar^2 p_0}{\gamma^2} \end{bmatrix}\Psi_0\left(\vec{x}\right) \tag{2.136}$$

$$\begin{aligned}\left\langle p_2p_2p_2\right\rangle &= -\frac{i\hbar^3}{8\gamma^6}\left\langle x_2x_2x_2\right\rangle-\frac{3\hbar^2 p_0}{4\gamma^4}\left\langle x_2x_2\right\rangle \\ &\quad+\left[\left(p_0^2+\frac{\hbar^2}{2\gamma^2}\right)\left(\frac{i\hbar}{2\gamma^2}\right)+\frac{i\hbar p_0^2}{\gamma^2}+\frac{i\hbar^3}{2\gamma^4}\right]\left\langle x_2\right\rangle \\ &\quad+p_0^3+\frac{\hbar^2 p_0}{2\gamma^2}+\frac{\hbar^2 p_0}{\gamma^2} \\ &= p_0^3+\frac{3\hbar^2 p_0}{4\gamma^2} \end{aligned} \tag{2.137}$$

$$\begin{aligned} p_3p_3p_3\Psi_0\left(\vec{x}\right) &= \left(-i\hbar\frac{\partial}{\partial x_3}\right)\left[\left(-\frac{\hbar^2}{4\gamma^4}x_3x_3+\frac{\hbar^2}{2\gamma^2}\right)\Psi_0\left(\vec{x}\right)\right] \\ &= \left[\left(-\frac{\hbar^2}{4\gamma^4}x_3x_3+\frac{\hbar^2}{2\gamma^2}\right)\left(\frac{i\hbar}{2\gamma^2}x_3\right)+\frac{i\hbar^3}{2\gamma^4}x_3\right]\Psi_0\left(\vec{x}\right) \end{aligned} \tag{2.138}$$

$$\begin{aligned}\left\langle p_3p_3p_3\right\rangle &= -\frac{i\hbar^3}{8\gamma^6}\left\langle x_3x_3x_3\right\rangle+\frac{3i\hbar^3}{4\gamma^4}\left\langle x_3\right\rangle \\ &= 0 \end{aligned} \tag{2.139}$$

$$p_1 p_1 p_2 \Psi_0(\vec{x}) = \left(-i\hbar \frac{\partial}{\partial x_1}\right)\left[\left(-\frac{\hbar^2}{4\gamma^4} x_1 x_2 + \frac{i\hbar p_0}{2\gamma^2} x_1\right)\Psi_0(\vec{x})\right]$$

$$= \left[\left(-\frac{\hbar^2}{4\gamma^4} x_1 x_2 + \frac{i\hbar p_0}{2\gamma^2} x_1\right)\left(\frac{i\hbar}{2\gamma^2} x_1\right) + \left(\frac{i\hbar^3}{4\gamma^4} x_2 + \frac{\hbar^2 p_0}{2\gamma^2}\right)\right]\Psi_0(\vec{x}) \quad (2.140)$$

$$\langle p_1 p_1 p_2 \rangle = -\frac{i\hbar^3}{8\gamma^6}\langle x_1 x_1 x_2\rangle - \frac{\hbar^2 p_0}{4\gamma^4}\langle x_1 x_1\rangle + \frac{i\hbar^3}{4\gamma^4}\langle x_2\rangle + \frac{\hbar^2 p_0}{2\gamma^2}$$

$$= -\frac{\hbar^2 p_0}{4\gamma^2} + \frac{\hbar^2 p_0}{2\gamma^2}$$

$$= \frac{\hbar^2 p_0}{4\gamma^2} \quad (2.141)$$

$$p_1 p_1 p_3 \Psi_0(\vec{x}) = \left(-i\hbar \frac{\partial}{\partial x_1}\right)\left[\left(-\frac{\hbar^2}{4\gamma^4} x_1 x_3\right)\Psi_0(\vec{x})\right]$$

$$= \left[\left(-\frac{\hbar^2}{4\gamma^4} x_1 x_3\right)\left(\frac{i\hbar}{2\gamma^2} x_1\right) + \frac{i\hbar^3}{4\gamma^4} x_3\right]\Psi_0(\vec{x}) \quad (2.142)$$

$$\langle p_1 p_1 p_3 \rangle = -\frac{i\hbar^3}{8\gamma^6}\langle x_1 x_3 x_1\rangle + \frac{i\hbar^3}{4\gamma^4}\langle x_3\rangle$$

$$= 0 \quad (2.143)$$

$$p_1 p_2 p_2 \Psi_0(\vec{x}) = \left(-i\hbar \frac{\partial}{\partial x_1}\right)\left[\left(-\frac{\hbar^2}{4\gamma^4} x_2 x_2 + \frac{i\hbar p_0}{\gamma^2} x_2 + p_0^2 + \frac{\hbar^2}{2\gamma^2}\right)\Psi_0(\vec{x})\right]$$

$$= \left[\left(-\frac{\hbar^2}{4\gamma^4} x_2 x_2 + \frac{i\hbar p_0}{\gamma^2} x_2 + p_0^2 + \frac{\hbar^2}{2\gamma^2}\right)\left(\frac{i\hbar}{2\gamma^2} x_1\right)\right]\Psi_0(\vec{x}) \quad (2.144)$$

$$\left\langle p_1 p_2 p_2 \right\rangle = -\frac{i\hbar^3}{8\gamma^6}\left\langle x_1 x_2 x_2 \right\rangle - \frac{\hbar^2 p_0}{2\gamma^4}\left\langle x_1 x_2 \right\rangle + \left(p_0^2 + \frac{\hbar^2}{2\gamma^2} \right)\left(\frac{i\hbar}{2\gamma^2} \right)\left\langle x_1 \right\rangle$$

$$= 0 \tag{2.145}$$

$$p_1 p_2 p_3 \Psi_0\left(\vec{x}\right) = \left(-i\hbar \frac{\partial}{\partial x_1} \right)\left[\left(-\frac{\hbar^2}{4\gamma^4} x_3 x_2 + \frac{i\hbar p_0}{2\gamma^2} x_3 \right)\Psi_0\left(\vec{x}\right)\right]$$

$$= \left[\left(-\frac{\hbar^2}{4\gamma^4} x_3 x_2 + \frac{i\hbar p_0}{2\gamma^2} x_3 \right)\left(\frac{i\hbar}{2\gamma^2} x_1 \right)\right]\Psi_0\left(\vec{x}\right) \tag{2.146}$$

$$\left\langle p_1 p_2 p_3 \right\rangle = -\frac{i\hbar^3}{8\gamma^6}\left\langle x_1 x_2 x_3 \right\rangle - \frac{\hbar^2 p_0}{4\gamma^4}\left\langle x_1 x_3 \right\rangle$$

$$= 0 \tag{2.147}$$

$$p_1 p_3 p_3 \Psi_0\left(\vec{x}\right) = \left(-i\hbar \frac{\partial}{\partial x_1} \right)\left[\left(-\frac{\hbar^2}{4\gamma^4} x_3 x_3 + \frac{\hbar^2}{2\gamma^2} \right)\Psi_0\left(\vec{x}\right)\right]$$

$$= \left[\left(-\frac{\hbar^2}{4\gamma^4} x_3 x_3 + \frac{\hbar^2}{2\gamma^2} \right)\left(\frac{i\hbar}{2\gamma^2} x_1 \right)\right]\Psi_0\left(\vec{x}\right) \tag{2.148}$$

$$\left\langle p_1 p_3 p_3 \right\rangle = -\frac{i\hbar^3}{8\gamma^6}\left\langle x_1 x_3 x_3 \right\rangle + \frac{i\hbar^3}{4\gamma^4}\left\langle x_1 \right\rangle$$

$$= 0 \tag{2.149}$$

$$p_2 p_2 p_3 \Psi_0(\vec{x}) = \left(-i\hbar \frac{\partial}{\partial x_2}\right)\left[\left(-\frac{\hbar^2}{4\gamma^4} x_3 x_2 + \frac{i\hbar p_0}{2\gamma^2} x_3\right)\Psi_0(\vec{x})\right]$$

$$= \left[\left(-\frac{\hbar^2}{4\gamma^4} x_3 x_2 + \frac{i\hbar p_0}{2\gamma^2} x_3\right)\left(\frac{i\hbar}{2\gamma^2} x_2 + p_0\right) + \frac{i\hbar^3}{4\gamma^4} x_3\right]\Psi_0(\vec{x}) \qquad (2.150)$$

$$\langle p_2 p_2 p_3 \rangle = -\frac{i\hbar^3}{8\gamma^6}\langle x_3 x_2 x_2 \rangle - \frac{\hbar^2 p_0}{2\gamma^4}\langle x_3 x_2 \rangle + \left(\frac{i\hbar p_0^2}{2\gamma^2} + \frac{i\hbar^3}{4\gamma^4}\right)\langle x_3 \rangle$$

$$= 0 \qquad (2.151)$$

$$p_2 p_3 p_3 \Psi_0(\vec{x}) = \left(-i\hbar \frac{\partial}{\partial x_2}\right)\left[\left(-\frac{\hbar^2}{4\gamma^4} x_3 x_3 + \frac{\hbar^2}{2\gamma^2}\right)\Psi_0(\vec{x})\right]$$

$$= \left[\left(-\frac{\hbar^2}{4\gamma^4} x_3 x_3 + \frac{\hbar^2}{2\gamma^2}\right)\left(\frac{i\hbar}{2\gamma^2} x_2 + p_0\right)\right]\Psi_0(\vec{x}) \qquad (2.152)$$

$$\langle p_2 p_3 p_3 \rangle = -\frac{i\hbar^3}{8\gamma^6}\langle x_2 x_3 x_3 \rangle - \frac{\hbar^2 p_0}{4\gamma^4}\langle x_3 x_3 \rangle + \frac{i\hbar^3}{4\gamma^4}\langle x_2 \rangle + \frac{\hbar^2 p_0}{2\gamma^2}$$

$$= \frac{\hbar^2 p_0}{4\gamma^4}. \qquad (2.153)$$

Therefore

$$\langle p_i p_j p_k \rangle = p_0^i p_0^j p_0^k + \frac{\hbar^2}{4\gamma^2}\left[p_0^i \delta^{jk} + p_0^j \delta^{ik} + p_0^k \delta^{ij}\right]. \qquad (2.154)$$

The expressions for the expectation values obtained may be then conveniently summarized through the equalities:

$$\langle x_i \rangle = 0 \tag{2.155}$$

$$\langle p_i \rangle = p_0 \delta^{i2} \tag{2.156}$$

$$\langle x_i x_j \rangle = \gamma^2 \delta^{ij} \tag{2.157}$$

$$\langle x_i p_j \rangle = \frac{i\hbar}{2} \delta^{ij} \tag{2.158}$$

$$\langle p_i p_j \rangle = p_0^i p_0^j + \frac{\hbar^2}{4\gamma^2} \delta^{ij}, \; p_0^i = p_0 \delta^{i2} \tag{2.159}$$

$$\langle x_i x_j x_k \rangle = 0 \tag{2.160}$$

$$\langle x_i x_j p_k \rangle = \delta^{ij} p_0 \gamma^2 \delta^{k2} \tag{2.161}$$

$$\langle x_k p_i p_j \rangle = \frac{i\hbar}{2} p_0^i \delta^{jk} + \frac{i\hbar}{2} p_0^i \delta^{ik} \tag{2.162}$$

$$\langle p_i p_j p_k \rangle = p_0^i p_0^j p_0^k + \frac{\hbar^2}{4\gamma^2} \left[p_0^i \delta^{jk} + p_0^j \delta^{ik} + p_0^k \delta^{ij} \right]. \tag{2.163}$$

2.3 The Bivariate Correlated Gaussian Distribution

The bivariate Guassian distribution will be also needed in the sequel. Accordingly, we spell out its definition and some of its properties (Manoukian, 1986).

The probability density of the bivariate Guassian distribution is defined by

$$f(\vec{z}) = \frac{1}{2\pi} \sqrt{\det \underset{\sim}{C}} \exp\left[-\frac{1}{2} \sum_{i,j=1}^{2} (z_i - \mu_i) C_{ij} (z_j - \mu_j) \right] \tag{2.164}$$

where $\vec{z} = (z_1, z_2)$ is a two-dimensional vector, $\vec{\mu} = (\mu_1, \mu_2)$, and $\underset{\sim}{C} = \left[C_{ij} \right]$ is a 2×2 matrix such that $\det \underset{\sim}{C} > 0$.

The vector $\vec{\mu}$ and the matrix $\left[C_{ij} \right]$ arise in the following manner:

$$\int_{R^2} d^2\vec{z}\ \vec{z} f(\vec{z}) = \vec{\mu} \tag{2.165}$$

that is, $\vec{\mu}$ is the expectation value of the random variable $\vec{z}$, and

$$\int_{R^2} d^2\vec{z}\ (z_i - \mu_i)(z_j - \mu_j) f(\vec{z}) = \left(C^{-1} \right)_{ij} \tag{2.166}$$

where $\underset{\sim}{C}^{-1}$, referred to as the covariance matrix, is the inverse of the matrix $\underset{\sim}{C}$, which exists since, in particular, we have assumed that $\det \underset{\sim}{C} > 0$. That is, the matrix element $\left(\underset{\sim}{C}^{-1} \right)_{ij}$ provides a measure of the correlation between the variables z_i and z_j. Differently said $\left(\underset{\sim}{C}^{-1} \right)_{ij}$ gives a measure of the deviation of the product $z_i z_j$ about the product of their means $\mu_i \mu_j$. The variance, or the standard deviation square, is a special case of the correlation for the case where $i = j$. We will encounter the above distribution in our analysis of the Stern-Gerlach effect in Chapter IV, where the physical meaning of the variables z_1, z_2 will be spelled out in detail.

Chapter III

The Hamiltonian and Time-Dependent Correlations

3.1 Setting Up the Hamiltonian

We work with the Pauli Hamiltonian defined by

$$H = \frac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2M} - \vec{\mu} \cdot \vec{B} \tag{3.1}$$

where $\vec{p} = -i\hbar\vec{\nabla}$, $\vec{\mu}$ is the magnetic moment

$$\vec{\mu} = \mu\vec{\sigma} \tag{3.2}$$

$$\mu = \frac{q\hbar}{4Mc} g \tag{3.3}$$

and the g - factor, for example, is given by

$$g \simeq 2 \tag{3.4}$$

for the electron and

$$g \simeq 5.59 \tag{3.5}$$

for the proton. The deviation of g form 2 is due to quantum electrodynamics and is referred to as the Schwinger term and is of the order $\alpha/2\pi$, where $\alpha = e^2/\hbar c$ is the fine-structure constant.

In Eq. (3.2), $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, with $\sigma_1, \sigma_2, \sigma_3$ denoting the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.6}$$

In Eq. (3.1), Eq. (3.3), q denotes the charge of the particle and carries its own sign. That is, for the electron $q = -|e|$. $\vec{A}$ is the vector potential and will be conveniently chosen such that

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{3.7}$$

defining the so-called Coulomb gauge. The magnetic field $\vec{B}$ is given by the familiar expression:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{3.8}$$

and must satisfy the field equation

$$\vec{\nabla} \cdot \vec{B} = 0 \,. \tag{3.9}$$

We choose the magnetic field in the form

$$\vec{B} = \left(0, b - \beta x_2, \beta x_3\right) \tag{3.10}$$

where b, β are constants, and note form Eq. (2.1), that $b - \beta x_2$ is along the direction of the average initial momentum of the particle in question (see Eq. (2.2)). We note explicitly that $\vec{B}$, as given in Eq. (3.9), satisfies the field equation Eq. (3.8). [The usual textbook expression $\vec{B} = \left(0, 0, \beta x_3\right)$ is incorrect since it violates the field equation.]

Eq. (3.7) implies that

$$\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = 0 \tag{3.11}$$

$$\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} = b - \beta x_2 \tag{3.12}$$

$$\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = \beta x_3 \tag{3.13}$$

$$\frac{\partial A_3}{\partial x_2} = \frac{\partial A_2}{\partial x_3} \,. \tag{3.14}$$

Upon writing

$$A_3 = x_2 f(x_1, x_3) \tag{3.15}$$

$$A_2 = x_3 g(x_1, x_2) \tag{3.16}$$

we get

$$f(x_1, x_3) = g(x_1, x_2) \tag{3.17}$$

to infer that f and g depend on x_1 only, i.e.,

$$f(x_1, x_3) = g(x_1, x_2) = f\left(x_1\right) \tag{3.18}$$

and

$$A_3 = x_2 f(x_1),\ A_2 = x_3 f(x_1). \tag{3.19}$$

We substitute Eq. (3.18) in Eq. (3.11), Eq. (3.12) to obtain

$$\frac{\partial A_1}{\partial x_3} - x_2 \frac{df(x_1)}{dx_1} = b - \beta x_2 \tag{3.20}$$

$$x_3 \frac{df(x_1)}{dx_1} - \frac{\partial A_1}{\partial x_2} = \beta x_3\,. \tag{3.21}$$

Eq. (3.20) gives

$$\frac{df(x_1)}{dx_1} = \beta \, , \; \frac{\partial A_1}{\partial x_2} = 0 \tag{3.22}$$

$$f\left(x_1\right) = \beta x_1 \, , \; A_1 = F\left(x_1, x_3\right) \tag{3.23}$$

while Eq. (3.19) gives

$$\frac{\partial A_1}{\partial x_3} - x_2 \beta = b - \beta x_2 \tag{3.24}$$

$$\frac{\partial A_1}{\partial x_3} = b \tag{3.25}$$

$$A_1 = b x_3 \, . \tag{3.26}$$

These finally lead to

$$A_1 = b x_3 \tag{3.27}$$

$$A_2 = \beta x_1 x_3 \tag{3.28}$$

$$A_3 = \beta x_1 x_2 \tag{3.29}$$

or in vector form to

$$\vec{A} = (b x_3 , \beta x_1 x_3 , \beta x_1 x_2) \, . \tag{3.30}$$

Before proceeding further, we recall that our initial wavepacket is given by

$$\Psi_0(\vec{x}) = \frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}}{(2\pi)^{3/4}\gamma^{3/2}} \exp\left(-\frac{\vec{x}^2}{4\gamma^2}\right) \tag{3.31}$$

and

$$\left|\Psi_0(\vec{x})\right|^2 = \frac{\exp\left(-\frac{\vec{x}^2}{4\gamma^2}\right)}{(2\pi)^{3/2}\gamma^3} \tag{3.32}$$

with

$$\vec{p}_0 = (0, p_0, 0) \tag{3.33}$$

and

$$\left\langle \Psi_0 \right| \left(-i\hbar\vec{\nabla}\right) \left| \Psi_0 \right\rangle = \vec{p}_0 \tag{3.34}$$

with the latter defining the average initial momentum of a particle.

The experimental set-up is chosen as in figure 3.1 below, where we note

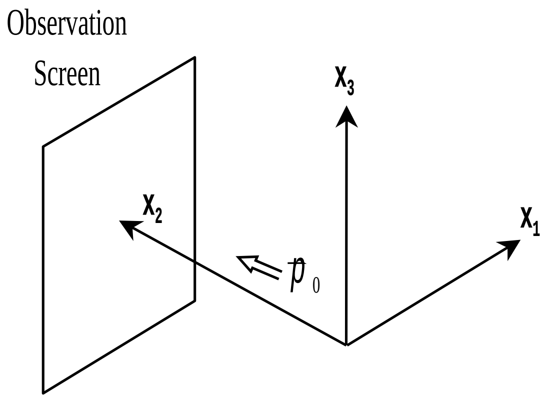

Figure 3.1. Experimental set-up for the study the Stern-Gerlach effect, with $\vec{p}_0$ denoting the average initial momentum of a particle.

that the additional component to the magnetic field to B_3 is chosen along x_2, i.e., defining a longitudinal component, instead of having another transversal component B_1. This is to avoid the splitting of a beam, in addition in the direction to the quantization axis x_3, along the transversal axis x_1. In the component $B_2 = b - \beta x_2$, b is so chosen to minimize the quantum action of the Lorentz force.

We introduce the dimensionless parameter

$$\alpha_q = \frac{|q|^2}{\hbar c} \tag{3.35}$$

and the sign function

$$\varepsilon(q) = \text{sgn}(q) = \begin{cases} 1 & , q > 0 \\ -1 & , q < 0 \end{cases} \tag{3.36}$$

to rewrite the Hamiltonian in Eq. (3.1) as

$$H = H_0 + H_{\mathrm{I}} \tag{3.37}$$

where $H_0 = \frac{\vec{p}^2}{2M}$, and

$$H_{\mathrm{I}} = \varepsilon(q)\sqrt{\alpha_q}\left[-\sqrt{\frac{\hbar}{c}}\vec{A}\cdot\vec{p} + \varepsilon(q)\sqrt{\alpha_q}\frac{\hbar}{2Mc}\vec{A}^2 - \frac{1}{4M}\sqrt{\frac{\hbar^3}{c}}g\vec{\sigma}\cdot\vec{B}\right] \tag{3.38}$$

and we have used the fact that

$$\vec{p}\cdot\vec{A} = \left(\vec{p}\cdot\vec{A}\right) + \vec{A}\cdot\vec{p} = \vec{A}\cdot\vec{p} \tag{3.39}$$

since $\vec{\nabla}\cdot\vec{A} = 0$.

We note that for $|q| = |e|$, $\sqrt{\alpha_q} = \alpha$ defines the fine-structure constant.

To the leading order in $\sqrt{\alpha_q}$, H_{I} in Eq. (3.38) is given by

$$H_{\mathrm{I}} = -\frac{q}{Mc}\left(bx_3p_1 + \beta x_1x_3p_2 + \beta x_1x_2p_3\right) - \mu\left[\sigma_2\left(b - \beta x_2\right) + \beta\sigma_3x_3\right] \tag{3.40}$$

where we have used the expressions for $\vec{A}$ and $\vec{B}$ in Eq. (3.30), Eq. (3.10), respectively.

3.2 Fundamental Commutation Relations

To carry out an analytical quantum dynamical analysis of the Stern-Gerlach effect, we need to evaluate various commutation relations. These are spelled out below.

In particular, we have

$$\left[x_i, p_j\right] = i\hbar\delta_{ij} \tag{3.41}$$

and

$$\begin{aligned}[p_1 p_1, x_1] &= p_1 p_1 x_1 - x_1 p_1 p_1 \\ &= p_1\left(x_1 p_1 - i\hbar\right) - x_1 p_1 p_1 \\ &= x_1 p_1 p_1 - i\hbar p_1 - i\hbar p_1 - x_1 p_1 p_1 \\ &= -2i\hbar p_1 .\end{aligned} \tag{3.42}$$

In the same way we get

$$[p_2 p_2, x_2] = -2i\hbar p_2 \tag{3.43}$$

$$[p_3 p_3, x_3] = -2i\hbar p_3 . \tag{3.44}$$

We also need the following commutation relations:

$$
\begin{aligned}
\left[x_1, p_1 p_2\right] &= x_1 p_1 p_2 - p_1 p_2 x_1 \\
&= x_1 p_1 p_2 - x_1 p_1 p_2 + i\hbar p_2 \\
&= i\hbar p_2
\end{aligned}
\tag{3.45}
$$

and in the same way

$$
\left.
\begin{aligned}
\left[x_2, p_1 p_2\right] &= i\hbar p_1 \\
\left[x_3, p_1 p_3\right] &= i\hbar p_1 \\
\left[x_1, p_1 p_3\right] &= i\hbar p_3 \\
\left[x_3, p_2 p_3\right] &= i\hbar p_2 \\
\left[x_2, p_2 p_3\right] &= i\hbar p_3
\end{aligned}
\right\}
\tag{3.46}
$$

$$
\begin{aligned}
\left[x_1 x_2, p_1 p_2\right] &= x_1 x_2 p_1 p_2 - p_1 p_2 x_1 x_2 \\
&= x_1 x_2 p_1 p_2 - p_1 x_1 \left(x_2 p_2 - i\hbar\right) \\
&= x_1 x_2 p_1 p_2 - x_1 p_1 \left(x_2 p_2 - i\hbar\right) + i\hbar\left(x_2 p_2 - i\hbar\right) \\
&= x_1 x_2 p_1 p_2 - x_1 x_2 p_1 p_2 + i\hbar x_1 p_1 + i\hbar x_2 p_2 + \hbar^2 \\
&= i\hbar x_1 p_1 + i\hbar x_2 p_2 + \hbar^2
\end{aligned}
\tag{3.47}
$$

$$
\left[x_1 x_3, p_1 p_3\right] = i\hbar x_1 p_1 + i\hbar x_3 p_3 + \hbar^2
\tag{3.48}
$$

$$
\begin{aligned}
[p_1 p_1, x_1 x_1] &= p_1 p_1 x_1 x_1 - x_1 x_1 p_1 p_1 \\
&= p_1 \left[x_1 (p_1 x_1) - i\hbar x_1 \right] - x_1 x_1 p_1 p_1 \\
&= p_1 \left[x_1 (x_1 p_1 - i\hbar) - i\hbar x_1 \right] - x_1 x_1 p_1 p_1 \\
&= p_1 [x_1 x_1 p_1 - 2i\hbar x_1] - x_1 x_1 p_1 p_1 \\
&= x_1 p_1 (x_1 p_1) - i\hbar x_1 p_1 - 2i\hbar x_1 p_1 - 2\hbar^2 - x_1 x_1 p_1 p_1 \\
&= x_1 (x_1 p_1 p_1 - i\hbar p_1) - 3i\hbar x_1 p_1 - 2\hbar^2 - x_1 x_1 p_1 p_1 \\
&= x_1 x_1 p_1 p_1 - 4i\hbar x_1 p_1 - 2\hbar^2 - x_1 x_1 p_1 p_1 \\
&= -4i\hbar x_1 p_1 - 2\hbar^2
\end{aligned}
\tag{3.49}
$$

$$
[p_2 p_2, x_2 x_2] = -4i\hbar x_2 p_2 - 2\hbar^2 \tag{3.50}
$$

$$
[p_3 p_3, x_3 x_3] = -4i\hbar x_3 p_3 - 2\hbar^2 \tag{3.51}
$$

$$
\begin{aligned}
[p_1, x_1 x_1] &= p_1 x_1 x_1 - x_1 x_1 p_1 \\
&= x_1 (p_1 x_1) - i\hbar x_1 - x_1 x_1 p_1 \\
&= x_1 (x_1 p_1 - i\hbar) - i\hbar x_1 - x_1 x_1 p_1 p \\
&= x_1 x_1 p_1 - 2i\hbar x_1 - x_1 x_1 p_1 p \\
&= -2i\hbar x_1
\end{aligned}
\tag{3.52}
$$

$$
[p_2, x_2 x_2] = -2i\hbar x_2 \tag{3.53}
$$

$$
[p_3, x_3 x_3] = -2i\hbar x_3 \tag{3.54}
$$

$$
\begin{aligned}
[p_1 p_1, x_1 x_2] &= p_1 p_1 x_1 x_2 - x_1 x_2 p_1 p_1 \\
&= p_1 (x_1 x_2 p_1 - i\hbar x_2) - x_1 x_2 p_1 p_1 \\
&= x_1 x_2 p_1 p_1 - i\hbar x_2 p_1 - i\hbar x_2 p_1 - x_1 x_2 p_1 p_1 \\
&= -2i\hbar x_2 p_1
\end{aligned} \tag{3.55}
$$

$$
\left.
\begin{aligned}
[p_2 p_2, x_1 x_2] &= -2i\hbar x_1 p_2 \\
[p_1 p_1, x_1 x_3] &= -2i\hbar x_3 p_1 \\
[p_3 p_3, x_1 x_3] &= -2i\hbar x_1 p_3 \\
[p_2 p_2, x_2 x_3] &= -2i\hbar x_3 p_2 \\
[p_3 p_3, x_2 x_3] &= -2i\hbar x_2 p_3
\end{aligned}
\right\} \tag{3.56}
$$

$$
\begin{aligned}
[p_1, x_1 x_2] &= p_1 x_1 x_2 - x_1 x_2 p_1 \\
&= x_1 x_2 p_1 - i\hbar x_2 - x_1 x_2 p_1 \\
&= -i\hbar x_2
\end{aligned} \tag{3.57}
$$

$$
\left.
\begin{aligned}
[p_2, x_1 x_2] &= -i\hbar x_1 \\
[p_1, x_1 x_3] &= -i\hbar x_3 \\
[p_3, x_1 x_3] &= -i\hbar x_1 \\
[p_2, x_2 x_3] &= -i\hbar x_3 \\
[p_3, x_2 x_3] &= -i\hbar x_2
\end{aligned}
\right\} \tag{3.58}
$$

3.3 Time-Dependent Expectation Values and Correlations Functions

A time-dependent observable $A(t)$ in the interacting theory is given by

$$A(t) = e^{\frac{i}{\hbar}tH} A e^{-\frac{i}{\hbar}tH} \tag{3.59}$$

where H is the Hamiltonian defined in Eq. (3.1) and A is the corresponding time $t = 0$ observable.

An observable $A(t)$ satisfies Heisenberg's equation of motion

$$\frac{d}{dt} A(t) = \frac{i}{\hbar}\left[H, A(t)\right] \tag{3.60}$$

as follows directly form Eq. (3.59).

If we replace H by H_0 and $A(t)$ by $A_0(t)$, then, we note that

$$\begin{aligned}\frac{d}{dt}\vec{x}_0(t) &= \frac{i}{\hbar}\left[H, \vec{x}_0(t)\right] \\ &= \frac{i}{\hbar} e^{\frac{i}{\hbar}tH_0}\left[H_0, \vec{x}\right] e^{-\frac{i}{\hbar}tH_0} \\ &= \frac{i}{\hbar} e^{\frac{i}{\hbar}tH_0}\left(-\frac{i\hbar\vec{p}}{M}\right) e^{-\frac{i}{\hbar}tH_0} \\ &= \frac{\vec{p}}{M}\end{aligned} \tag{3.61}$$

where we have used the fact that

$$\left[\frac{\vec{p}^2}{2M},\vec{x}\right]=-i\hbar\frac{\vec{p}}{M} \tag{3.62}$$

and that

$$\left[H_0,\vec{p}\right]=0\,. \tag{3.63}$$

We may then integrate Eq. (3.61) to obtain

$$\vec{x}_0=\vec{x}+\frac{\vec{p}}{M}t \tag{3.64}$$

and similarly by using Eq. (3.63),

$$\vec{p}_0\left(t\right)=\vec{p}\,. \tag{3.65}$$

Now we use the interacting Hamiltonian H, with H_{I} given in Eq. (3.40). To the leading order in $\sqrt{\alpha_q}$, we have

$$\left[x_1,p_1\right]=i\hbar$$

$$\left[H,p_1\right]=-\frac{i\hbar q\beta}{Mc}\left(x_3p_2+x_2p_3\right) \tag{3.66}$$

and hence

$$\frac{d}{dt}p_1(t)=\frac{i}{\hbar}\left[-\frac{i\hbar q\beta}{Mc}\left(\left(x_3+\frac{p_3}{M}t\right)p_2+\left(x_2+\frac{p_2}{M}t\right)p_3\right)\right]$$

$$=\frac{q\beta}{Mc}\left(x_3p_2+x_2p_3+\frac{p_2p_3}{M}t+\frac{p_2p_3}{M}t\right) \tag{3.67}$$

which upon integration gives

$$p_1(t)=p_1+\frac{q\beta}{Mc}(x_3p_2+x_2p_3)t+\frac{q\beta}{M^2c}p_2p_3t^2 \tag{3.68}$$

leading to the expectation value

$$\langle p_1(t)\rangle=\langle p_1\rangle+\frac{q\beta}{Mc}(\langle x_3p_2\rangle+\langle x_2p_3\rangle)t+\frac{q\beta}{M^2c}\langle p_2p_3\rangle t^2=0 \tag{3.69}$$

where we have used our earlier results in Eq. (2.155)- Eq. (2.163).

Similarly, we obtain

$$[x_2,p_2]=i\hbar$$

$$[H,p_2]=-\frac{i\hbar q\beta}{Mc}x_1p_3+i\hbar\mu\beta\sigma_2 \tag{3.70}$$

$$\frac{d}{dt}p_2(t)=\frac{i}{\hbar}\left[-\frac{i\hbar q\beta}{Mc}\left(x_1+\frac{p_1}{M}t\right)p_3+i\hbar\mu\beta\sigma_2\right]$$

$$=\frac{q\beta}{Mc}x_1p_3+\frac{q\beta}{M^2c}p_1p_3t-\mu\beta\sigma_2 \qquad (3.71)$$

$$p_2(t)=p_2+\left(\frac{q\beta}{Mc}x_1p_3-\mu\beta\sigma_2\right)t+\frac{q\beta}{2M^2c}p_1p_3t^2 \qquad (3.72)$$

$$\langle p_2(t)\rangle=\langle p_2\rangle+\left(\frac{q\beta}{Mc}\langle x_1p_3\rangle-\mu\beta\langle\sigma_2\rangle\right)t+\frac{q\beta}{2M^2c}\langle p_1p_3\rangle t^2 \qquad (3.73)$$

or

$$\langle p_2(t)\rangle=p_0-\mu\beta\langle\sigma_2\rangle t\,. \qquad (3.74)$$

In a similar fashion we have by detailed calculations:

$$[x_3,p_3]=i\hbar$$

$$[H,p_3]=-\frac{i\hbar q}{Mc}(bp_1+\beta x_1p_2)-i\hbar\mu\beta\sigma_3 \qquad (3.75)$$

$$\frac{d}{dt}p_3(t)=\frac{i}{\hbar}\left[-\frac{i\hbar q}{Mc}\left(bp_1+\beta\left(x_1+\frac{p_1}{M}t\right)p_2\right)-i\hbar\mu\beta\sigma_3\right]$$

$$=\frac{q}{Mc}(bp_1+\beta x_1p_2)+\mu\beta\sigma_3+\frac{q\beta}{M^2c}p_1p_2t \qquad (3.76)$$

$$p_3(t) = p_3 + \left[\frac{q}{Mc}(bp_1 + \beta x_1 p_2) + \mu\beta\sigma_3\right]t + \frac{q\beta}{2M^2c}p_1 p_2 t^2 \tag{3.77}$$

$$\langle p_3(t)\rangle = \langle p_3\rangle + \left[\frac{q}{Mc}\left(b\langle p_1\rangle + \beta\langle x_1 p_2\rangle\right) + \mu\beta\langle\sigma_3\rangle\right]t + \frac{q\beta}{2M^2c}\langle p_1 p_2\rangle t^2 \tag{3.78}$$

$$\langle p_3(t)\rangle = \mu\beta\langle\sigma_3\rangle t \tag{3.79}$$

$$\begin{aligned}
[p_1, x_1] &= -i\hbar \\
[p_1 p_1, x_1] &= -2i\hbar p_1 \\
[H, x_1] &= \frac{1}{2M}(-2i\hbar p_1) - \frac{q}{Mc}(-i\hbar b x_3) \\
&= \frac{-i\hbar}{M}p_1 + \frac{i\hbar qb}{Mc}x_3
\end{aligned} \tag{3.80}$$

$$\begin{aligned}
\frac{d}{dt}x_1(t) &= \frac{i}{h}\left[\frac{-i\hbar}{M}p_1(t) + \frac{i\hbar qb}{Mc}\left(x_3 + \frac{p_3}{M}t\right)\right] \\
&= \frac{1}{M}\left(p_1 + \frac{q\beta}{Mc}(x_3 p_2 + x_2 p_3)t + \frac{q\beta}{M^2c}p_2 p_3 t^2\right) - \frac{qb}{Mc}x_3 - \frac{qb}{M^2c}p_3 t
\end{aligned} \tag{3.81}$$

$$\begin{aligned}
x_1(t) = x_1 &+ \left(\frac{1}{M}p_1 - \frac{qb}{Mc}x_3\right)t \\
&+ \left(\frac{q\beta}{2M^2c}(x_3 p_2 + x_2 p_3) - \frac{qb}{2M^2c}p_3\right)t^2 + \frac{q\beta}{3M^3c}p_2 p_3 t^3
\end{aligned} \tag{3.82}$$

$$\langle x_1(t)\rangle = \langle x_1\rangle + \left(\frac{1}{M}\langle p_1\rangle - \frac{qb}{Mc}\langle x_3\rangle\right)t + \left[\frac{q\beta}{2M^2c}\left(\langle x_3p_2\rangle + \langle x_2p_3\rangle\right) - \frac{qb}{2M^2c}\langle p_3\rangle\right]t^2 + \frac{q\beta}{3M^3c}\langle p_2p_3\rangle t^3 \tag{3.83}$$

$$\langle x_1(t)\rangle = 0 \tag{3.84}$$

$$[p_2, x_2] = -i\hbar$$

$$[p_2p_2, x_2] = -2i\hbar p_2$$

$$[H, x_2] = \frac{1}{2M}(-2i\hbar p_2) - \frac{q}{Mc}(-i\hbar\beta x_1x_3)$$

$$= \frac{-i\hbar}{M}p_2 + \frac{i\hbar q\beta}{Mc}x_1x_3 \tag{3.85}$$

$$\frac{d}{dt}x_2(t) = \frac{i}{\hbar}\left[\frac{-i\hbar}{M}p_2(t) + \frac{i\hbar q\beta}{Mc}\left(x_1 + \frac{p_1}{M}t\right)\left(x_3 + \frac{p_3}{M}t\right)\right]$$

$$= \frac{1}{M}\left[p_2 + \left(\frac{q\beta}{Mc}x_1p_3 - \mu\beta\sigma_2\right)t + \frac{q\beta}{2M^2c}p_1p_3t^2\right] - \frac{q\beta}{Mc}\left(x_1x_3 + \frac{x_3p_1}{M}t + \frac{x_1p_3}{M}t + \frac{p_1p_3}{M^2}t^2\right)$$

$$= \frac{1}{M}p_2 - \frac{q\beta}{Mc}x_1x_3 + \left(\frac{q\beta}{M^2c}x_1p_3 - \frac{\mu\beta}{M}\sigma_2 - \frac{q\beta}{M^2c}x_3p_1 - \frac{q\beta}{M^2c}x_1p_3\right)t + \left(\frac{q\beta}{2M^3c}p_1p_3 - \frac{q\beta}{M^3c}p_1p_3\right)t^2$$

$$= \frac{1}{M}p_2 - \frac{q\beta}{Mc}x_1x_3 - \left(\frac{\mu\beta}{M}\sigma_2 + \frac{q\beta}{M^2c}x_3p_1\right)t - \frac{q\beta}{2M^3c}p_1p_3t^2 \tag{3.86}$$

$$x_2(t) = x_2 + \left(\frac{1}{M}p_2 - \frac{q\beta}{Mc}x_1x_3\right)t$$
$$-\left(\frac{\mu\beta}{2M}\sigma_2 + \frac{q\beta}{2M^2c}x_3p_1\right)t^2 - \frac{q\beta}{6M^3c}p_1p_3t^3 \tag{3.87}$$

$$\langle x_2(t)\rangle = \langle x_2\rangle + \left(\frac{1}{M}\langle p_2\rangle - \frac{q\beta}{Mc}\langle x_1x_3\rangle\right)t$$
$$-\left(\frac{\mu\beta}{2M}\langle\sigma_2\rangle + \frac{q\beta}{2M^2c}\langle x_3p_1\rangle\right)t^2 - \frac{q\beta}{6M^3c}\langle p_1p_3\rangle t^3 \tag{3.88}$$

$$\langle x_2(t)\rangle = \frac{p_0}{M}t - \frac{\mu\beta}{2M}\langle\sigma_2\rangle t^2 \tag{3.89}$$

$$[p_3, x_3] = -i\hbar$$

$$[p_3p_3, x_3] = -2i\hbar p_3$$

$$[H, x_3] = \frac{1}{2M}(-2i\hbar p_3) - \frac{q}{Mc}(-i\hbar\beta x_1x_2)$$
$$= \frac{-i\hbar}{M}p_3 + \frac{i\hbar q\beta}{Mc}x_1x_2 \tag{3.90}$$

$$\frac{d}{dt}x_3(t) = \frac{i}{\hbar}\left[\frac{-i\hbar}{M}p_3(t) + \frac{i\hbar q\beta}{Mc}\left(x_1 + \frac{p_1}{M}t\right)\left(x_2 + \frac{p_2}{M}t\right)\right]$$

$$= \frac{1}{M}\left[p_3 + \left(\frac{q}{Mc}(bp_1 + \beta x_1p_2) + \mu\beta\sigma_3\right)t + \frac{q\beta}{2M^2c}p_1p_2t^2\right]$$
$$-\frac{q\beta}{Mc}\left(x_1x_2 + \frac{x_1p_2}{M}t + \frac{x_2p_1}{M}t + \frac{p_1p_2}{M^2}t^2\right)$$

$$= \frac{1}{M} p_3 - \frac{q\beta}{Mc} x_1 x_2 + \begin{bmatrix} \frac{q}{M^2 c}(bp_1 + \beta x_1 p_2) + \frac{\mu\beta}{M}\sigma_3 \\ -\frac{q\beta}{M^2 c}(x_1 p_2 + x_2 p_1) \end{bmatrix} t$$

$$+ \left(\frac{q\beta}{2M^3 c} p_1 p_2 - \frac{q\beta}{M^3 c} p_1 p_2 \right) t^2$$

$$= \frac{1}{M} p_3 - \frac{q\beta}{Mc} x_1 x_2 + \begin{pmatrix} \frac{qb}{M^2 c} p_1 + \frac{\mu\beta}{M}\sigma_3 \\ -\frac{q\beta}{M^2 c} x_2 p_1 \end{pmatrix} t - \frac{q\beta}{2M^3 c} p_1 p_2 t^2 \quad (3.91)$$

$$x_3(t) = x_3 + \left(\frac{1}{M} p_3 - \frac{q\beta}{Mc} x_1 x_2 \right) t + \begin{pmatrix} \frac{qb}{2M^2 c} p_1 + \frac{\mu\beta}{2M}\sigma_3 \\ -\frac{q\beta}{2M^2 c} x_2 p_1 \end{pmatrix} t^2 - \frac{q\beta}{6M^3 c} p_1 p_2 t^3 \quad (3.92)$$

$$\begin{aligned} \langle x_3(t) \rangle &= \langle x_3 \rangle + \left(\frac{1}{M} \langle p_3 \rangle - \frac{q\beta}{Mc} \langle x_1 x_2 \rangle \right) t \\ &+ \left(\frac{qb}{2M^2 c} \langle p_1 \rangle + \frac{\mu\beta}{2M} \langle \sigma_3 \rangle - \frac{q\beta}{2M^2 c} \langle x_2 p_1 \rangle \right) t^2 - \frac{q\beta}{6M^3 c} \langle p_1 p_2 \rangle t^3 \end{aligned} \quad (3.93)$$

$$\langle x_3(t) \rangle = \frac{\mu\beta}{2M} \langle \sigma_3 \rangle t^2 \quad (3.94)$$

$$[x_1, p_1 p_1] = 2i\hbar p_1$$

$$[H, p_1 p_1] = -\frac{q}{Mc}\left[\beta x_3 p_2 (2i\hbar p_1) + \beta x_2 p_3 (2i\hbar p_1) \right]$$

$$= -\frac{2i\hbar q\beta}{Mc}(x_3 p_1 p_2 + x_2 p_1 p_3) \quad (3.95)$$

$$\frac{d}{dt}\left[p_1(t)p_1(t)\right]=\frac{i}{\hbar}\left[-\frac{2i\hbar q\beta}{Mc}\left[x_3(t)p_1(t)p_2(t)+x_2(t)p_1(t)p_3(t)\right]\right]$$

$$=\frac{2q\beta}{Mc}\left[\left(x_3+\frac{p_3}{M}t\right)p_1p_2+\left(x_2+\frac{p_2}{M}t\right)p_1p_3\right]$$

$$=\frac{2q\beta}{Mc}\left(x_3p_1p_2+x_2p_1p_3+\frac{2p_1p_2p_3}{M}t\right) \tag{3.96}$$

$$p_1(t)p_1(t)=p_1p_1+\frac{2q\beta}{Mc}\left(x_3p_1p_2+x_2p_1p_3\right)t+\frac{2q\beta}{M^2c}p_1p_2p_3t^2 \tag{3.97}$$

$$\left\langle p_1(t)p_1(t)\right\rangle=\left\langle p_1p_1\right\rangle+\frac{2q\beta}{Mc}\left(\left\langle x_3p_1p_2\right\rangle+\left\langle x_2p_1p_3\right\rangle\right)t+\frac{2q\beta}{M^2c}\left\langle p_1p_2p_3\right\rangle t^2 \tag{3.98}$$

$$\left\langle p_1(t)p_1(t)\right\rangle=\frac{\hbar^2}{4\gamma^2} \tag{3.99}$$

$$\left[x_2,p_2p_2\right]=2i\hbar p_2$$

$$\left[H,p_2p_2\right]=-\frac{q}{Mc}\left[\beta x_1p_3\left(2i\hbar p_2\right)\right]+\mu\beta\sigma_2\left(2i\hbar p_2\right)$$

$$=-\frac{2i\hbar q\beta}{Mc}x_1p_2p_3+2i\hbar\mu\beta\sigma_2p_2 \tag{3.100}$$

$$\frac{d}{dt}\left[p_2(t)p_2(t)\right]=\frac{i}{\hbar}\left[-\frac{2i\hbar q\beta}{Mc}x_1(t)p_2(t)p_3(t)+2i\hbar\mu\beta\sigma_2p_2(t)\right]$$

$$=\frac{2q\beta}{Mc}\left(x_1+\frac{p_1}{M}t\right)p_2p_3-2\mu\beta\sigma_2p_2$$

$$= \frac{2q\beta}{Mc}\left(x_1 p_2 p_3 + \frac{p_1 p_2 p_3}{M} t \right) - 2\mu\beta\sigma_2 p_2 \tag{3.101}$$

$$\begin{aligned} p_2(t)\, p_2(t) = p_2 p_2 &+ \left(\frac{2q\beta}{Mc} x_1 p_2 p_3 - 2\mu\beta\sigma_2 p_2 \right) t \\ &+ \frac{q\beta}{M^2 c} p_1 p_2 p_3 t^2 \end{aligned} \tag{3.102}$$

$$\begin{aligned} \left\langle p_2(t)\, p_2(t) \right\rangle = \left\langle p_2 p_2 \right\rangle &+ \left(\frac{2q\beta}{Mc} \left\langle x_1 p_2 p_3 \right\rangle - 2\mu\beta \left\langle \sigma_2 p_2 \right\rangle \right) t \\ &+ \frac{q\beta}{M^2 c} \left\langle p_1 p_2 p_3 \right\rangle t^2 \end{aligned} \tag{3.103}$$

$$\left\langle p_2(t)\, p_2(t) \right\rangle = p_0^2 + \frac{\hbar^2}{4\gamma^2} - 2\mu\beta\, p_0 \left\langle \sigma_2 \right\rangle t \tag{3.104}$$

$$\left[x_3, p_3 p_3 \right] = 2i\hbar p_3$$

$$\left[H, p_3 p_3 \right] = -\frac{q}{Mc}\left[b p_1 (2i\hbar p_3) + \beta x_1 p_2 (2i\hbar p_3) \right] - \mu\beta\sigma_3 (2i\hbar p_3)$$

$$= -\frac{2i\hbar q}{Mc}\left(b p_1 p_3 + \beta x_1 p_2 p_3 \right) - 2i\hbar\mu\beta\sigma_3 p_3 \tag{3.105}$$

$$\frac{d}{dt}\left[p_3(t)\, p_3(t) \right] = \frac{i}{\hbar}\left[-\frac{2i\hbar q}{Mc}\left[b p_1(t)\, p_3(t) + \beta x_1(t)\, p_2(t)\, p_3(t) \right] - 2i\hbar\mu\beta\sigma_3 p_3(t) \right]$$

$$= \frac{2q}{Mc}\left[b p_1 p_3 + \beta \left(x_1 + \frac{p_1}{M} t \right) p_2 p_3 \right] + 2\mu\beta\sigma_3 p_3$$

$$= \frac{2q}{Mc}\left(b p_1 p_3 + \beta x_1 p_2 p_3 \right) + 2\mu\beta\sigma_3 p_3 + \frac{2q\beta}{M^2 c} p_1 p_2 p_3 t \tag{3.106}$$

$$p_3(t)\,p_3(t) = p_3p_3 + \left[\frac{2q}{Mc}\left(bp_1p_3 + \beta x_1p_2p_3\right) + 2\mu\beta\sigma_3p_3\right]t + \frac{q\beta}{M^2c}p_1p_2p_3t^2 \tag{3.107}$$

$$\langle p_3(t)\,p_3(t)\rangle = \langle p_3p_3\rangle + \left[\frac{2q}{Mc}\left(b\langle p_1p_3\rangle + \beta\langle x_1p_2p_3\rangle\right) + 2\mu\beta\langle\sigma_3p_3\rangle\right]t + \frac{q\beta}{M^2c}\langle p_1p_2p_3\rangle t^2 \tag{3.108}$$

$$\langle p_3(t)\,p_3(t)\rangle = \frac{\hbar^2}{4\gamma^2}\,. \tag{3.109}$$

Now we consider time-dependent correlations. Again to the leading order in $\sqrt{\alpha_q}$,

$$\begin{aligned}[H, p_1p_2] &= -\frac{q}{Mc}\left[\beta\left(i\hbar p_2\right)x_3p_2 + \beta\left(i\hbar x_1p_1 + i\hbar x_2p_2 + \hbar^2\right)p_3\right] + \mu\beta\sigma_2\left(i\hbar p_1\right) \\ &= -\frac{i\hbar\beta q}{Mc}\left(x_3p_2p_2 + x_1p_1p_3 + x_2p_2p_3 - i\hbar p_3\right) + i\hbar\mu\beta\sigma_2p_1\end{aligned} \tag{3.110}$$

where we have used the facts that

$$\left.\begin{aligned}[x_1, p_1p_2] &= i\hbar p_2 \\ [x_2, p_1p_2] &= i\hbar p_1\end{aligned}\right\} \tag{3.111}$$

and

$$[x_1x_2, p_1p_2] = i\hbar x_1p_1 + i\hbar x_2p_2 + \hbar^2 . \tag{3.112}$$

Hence we obtain form Eq. (3.110),

$$\begin{aligned}
\frac{d}{dt}\left[p_1(t)p_2(t)\right] &= \frac{i}{\hbar}\left[\begin{matrix} -\dfrac{i\hbar q\beta}{Mc}\begin{pmatrix} x_3(t)p_2(t)p_2(t)+x_1(t)p_1(t)p_3(t) \\ +x_2(t)p_2(t)p_3(t)-i\hbar p_3(t) \end{pmatrix} \\ +i\hbar\mu\beta\sigma_2p_1(t) \end{matrix}\right] \\
&= \frac{q\beta}{Mc}\left[\begin{matrix} \left(x_3+\dfrac{p_3}{M}t\right)p_2p_2+\left(x_1+\dfrac{p_1}{M}t\right)p_1p_3 \\ +\left(x_2+\dfrac{p_2}{M}t\right)p_2p_3-i\hbar p_3 \end{matrix}\right]-\mu\beta\sigma_2p_1 \\
&= \frac{q\beta}{Mc}\left(x_3p_2p_2+x_1p_1p_3+x_2p_2p_3-i\hbar p_3\right)-\mu\beta\sigma_2p_1 \\
&\quad +\frac{q\beta}{Mc}\left(\frac{p_2p_2p_3}{M}+\frac{p_1p_1p_3}{M}+\frac{p_2p_2p_3}{M}\right)t
\end{aligned} \tag{3.113}$$

$$\begin{aligned}
p_1(t)p_2(t) = p_1p_2 &+ \left[\frac{q\beta}{Mc}\begin{pmatrix} x_3p_2p_2+x_1p_1p_3 \\ +x_2p_2p_3-i\hbar p_3 \end{pmatrix}-\mu\beta\sigma_2p_1\right]t \\
&+\frac{q\beta}{2M^2c}\left(p_1p_1p_3+2p_2p_2p_3\right)t^2
\end{aligned} \tag{3.114}$$

$$\begin{aligned}
\left\langle p_1(t)p_2(t)\right\rangle = \left\langle p_1p_2\right\rangle &+ \left[\frac{q\beta}{Mc}\begin{pmatrix} \left\langle x_3p_2p_2\right\rangle+\left\langle x_1p_1p_3\right\rangle \\ +\left\langle x_2p_2p_3\right\rangle-i\hbar\left\langle p_3\right\rangle \end{pmatrix}-\mu\beta\left\langle\sigma_2p_1\right\rangle\right]t \\
&+\frac{q\beta}{2M^2c}\left(\left\langle p_1p_1p_3\right\rangle+2\left\langle p_2p_2p_3\right\rangle\right)t^2
\end{aligned} \tag{3.115}$$

$$\left\langle p_1(t)p_2(t)\right\rangle = 0 . \tag{3.116}$$

In a similar fashion we have

$$\begin{aligned}\left[H, p_1 p_3\right] &= -\frac{q}{Mc}\left[bp_1\left(i\hbar p_1\right) + \beta p_2\left(i\hbar x_1 p_1 + i\hbar x_3 p_3 + \hbar^2\right) + \beta x_2 p_3\left(i\hbar p_3\right)\right] \\ &\quad -\mu\beta\sigma_3\left(i\hbar p_1\right) \\ &= -\frac{i\hbar q}{Mc}\left[bp_1 p_1 + \beta\left(x_1 p_1 p_2 + x_3 p_2 p_3 - i\hbar p_2 + x_2 p_3 p_3\right)\right] \\ &\quad -i\hbar\mu\beta\sigma_3 p_1\end{aligned} \tag{3.117}$$

since

$$\left.\begin{aligned}\left[x_3, p_1 p_3\right] &= i\hbar p_1 \\ \left[x_1, p_1 p_3\right] &= i\hbar p_3\end{aligned}\right\} \tag{3.118}$$

$$\left[x_1 x_3, p_1 p_3\right] = i\hbar x_1 p_1 + i\hbar x_3 p_3 + \hbar^2 \tag{3.119}$$

and

$$\frac{d}{dt}\left[p_1(t) p_3(t)\right] = \frac{i}{\hbar}\begin{bmatrix} -\frac{i\hbar q}{Mc}\begin{pmatrix} bp_1(t) p_1(t) \\ +\beta\begin{pmatrix} x_1(t) p_1(t) p_2(t) + x_3(t) p_2(t) p_3(t) \\ -i\hbar p_2(t) + x_2(t) p_3(t) p_3(t)\end{pmatrix}\end{pmatrix} \\ -i\hbar\mu\beta\sigma_3 p_1(t)\end{bmatrix}$$

$$= \begin{bmatrix} \frac{q}{Mc}\left(bp_1 p_1 + \beta\begin{pmatrix}\left(x_1 + \frac{p_1}{M}t\right) p_1 p_2 + \left(x_3 + \frac{p_3}{M}t\right) p_2 p_3 \\ -i\hbar p_2 + \left(x_2 + \frac{p_2}{M}t\right) p_3 p_3\end{pmatrix}\right) \\ +\mu\beta\sigma_3 p_1\end{bmatrix}$$

$$= \frac{q}{Mc}\left[bp_1p_1 + \beta \begin{pmatrix} x_1p_1p_2 + x_3p_2p_3 \\ +x_2p_3p_3 - i\hbar p_2 \end{pmatrix} \right] + \mu\beta\sigma_3 p_1 + \frac{q\beta}{M^2c}\left(p_1p_1p_2 + p_2p_3p_3 + p_2p_3p_3 \right)t \tag{3.120}$$

$$p_1(t)p_3(t) = p_1p_3 + \left[\frac{q}{Mc}\left(bp_1p_1 + \beta \begin{pmatrix} x_1p_1p_2 + x_3p_2p_3 \\ +x_2p_3p_3 - i\hbar p_2 \end{pmatrix} \right) + \mu\beta\sigma_3 p_1 \right]t + \frac{q\beta}{2M^2c}\left(p_1p_1p_2 + 2p_2p_3p_3 \right)t^2 \tag{3.121}$$

$$\left\langle p_1(t)p_3(t) \right\rangle = \left\langle p_1p_3 \right\rangle + \left[\begin{matrix} \frac{q}{Mc}\left(b\left\langle p_1p_1 \right\rangle + \beta \begin{pmatrix} \left\langle x_1p_1p_2 \right\rangle + \left\langle x_3p_2p_3 \right\rangle \\ + \left\langle x_2p_3p_3 \right\rangle - i\hbar \left\langle p_2 \right\rangle \end{pmatrix} \right) \\ +\mu\beta\left\langle \sigma_3p_1 \right\rangle \end{matrix} \right]t + \frac{q\beta}{2M^2c}\left(\left\langle p_1p_1p_2 \right\rangle + 2\left\langle p_2p_3p_3 \right\rangle \right)t^2$$

$$= \frac{q}{Mc}\left(b\frac{\hbar^2}{4\gamma^2} + \beta\left(\frac{i\hbar}{2}p_0 + \frac{i\hbar}{2}p_0 - i\hbar p_0 \right) \right)t + \frac{q\beta}{2M^2c}\left(\frac{\hbar^2}{4\gamma^2}p_0 + 2\frac{\hbar^2}{4\gamma^2}p_0 \right)t^2 \tag{3.122}$$

$$\left\langle p_1(t)p_3(t) \right\rangle = \frac{\hbar^2 qb}{4\gamma^2 Mc}t + \frac{3\hbar^2 q\beta}{8\gamma^2 M^2c}p_0t^2 \tag{3.123}$$

$$\left[H, p_2p_3 \right] = -\frac{q}{Mc}\left[bp_1(i\hbar p_2) + \beta x_1(i\hbar p_2)p_2 + \beta x_1(i\hbar p_3)p_3 \right] - \mu\left[-\beta\sigma_2(i\hbar p_3) + \beta\sigma_3(i\hbar p_2) \right]$$

$$= -\frac{i\hbar q}{Mc}\left(bp_1p_2 + \beta x_1p_2p_2 + \beta x_1p_3p_3 \right) + i\hbar\mu\beta\left(\sigma_2p_3 - \sigma_3p_2 \right) \tag{3.124}$$

as follows form

$$\left.\begin{aligned}\left[x_3, p_2 p_3\right] = i\hbar p_2 \\ \left[x_2, p_2 p_3\right] = i\hbar p_3\end{aligned}\right\} \tag{3.125}$$

hence

$$\frac{d}{dt}\left[p_2(t)p_3(t)\right] = \frac{i}{\hbar}\begin{bmatrix} -\dfrac{i\hbar q}{Mc}\begin{pmatrix} bp_1(t)p_2(t) \\ +\beta x_1(t)p_2(t)p_2(t) \\ +\beta x_1(t)p_3(t)p_3(t)\end{pmatrix} \\ +i\hbar\mu\beta\left(\sigma_2 p_3(t) - \sigma_3 p_2(t)\right)\end{bmatrix}$$

$$\begin{aligned} &= \frac{q}{Mc}\left(bp_1p_2 + \beta\left(x_1 + \frac{p_1}{M}t\right)p_2p_2 + \beta\left(x_1 + \frac{p_1}{M}t\right)p_3p_3\right) \\ &\quad -\mu\beta\left(\sigma_2 p_3 - \sigma_3 p_2\right) \\ &= \frac{q}{Mc}\left(bp_1p_2 + \beta\begin{pmatrix} x_1p_2p_2 \\ +x_1p_3p_3\end{pmatrix}\right) - \mu\beta\left(\sigma_2p_3 - \sigma_3p_2\right) \\ &\quad + \frac{q\beta}{M^2c}\left(p_1p_2p_2 + p_1p_3p_3\right)t \end{aligned} \tag{3.126}$$

$$\begin{aligned} p_2(t)p_3(t) = p_2p_3 + \begin{bmatrix}\dfrac{q}{Mc}\left(bp_1p_2 + \beta\begin{pmatrix} x_1p_2p_2 \\ +x_1p_3p_3\end{pmatrix}\right) \\ -\mu\beta\left(\sigma_2p_3 - \sigma_3p_2\right)\end{bmatrix}t \\ + \frac{q\beta}{2M^2c}\left(p_1p_2p_2 + p_1p_3p_3\right)t^2 \end{aligned} \tag{3.127}$$

$$\left\langle p_2(t)\,p_3(t)\right\rangle = \left\langle p_2 p_3\right\rangle + \left[\begin{array}{l} \dfrac{q}{Mc}\left(b\left\langle p_1 p_2\right\rangle + \beta\left(\begin{array}{l}\left\langle x_1 p_2 p_2\right\rangle \\ +\left\langle x_1 p_3 p_3\right\rangle\end{array}\right)\right) \\ -\mu\beta\left(\left\langle\sigma_2 p_3\right\rangle - \left\langle\sigma_3 p_2\right\rangle\right)\end{array}\right] t$$
$$+\frac{q\beta}{2M^2c}\left(\left\langle p_1 p_2 p_2\right\rangle + \left\langle p_1 p_3 p_3\right\rangle\right)t^2 \tag{3.128}$$

$$\left\langle p_2(t)\,p_3(t)\right\rangle = \mu\beta\, p_0\left\langle\sigma_3\right\rangle t \tag{3.129}$$

$$\left[H, x_1 x_1\right] = \frac{1}{2M}\left(-4i\hbar x_1 p_1 - 2\hbar^2\right) - \frac{q}{Mc}\left(b x_3\left(-2i\hbar x_1\right)\right)$$
$$= -\frac{i\hbar}{M}\left(2x_1 p_1 - i\hbar\right) + \frac{2i\hbar q b}{Mc} x_1 x_3 \tag{3.130}$$

as a consequence of the commutation relations

$$\left.\begin{array}{l}\left[p_1 p_1, x_1 x_1\right] = -4i\hbar x_1 p_1 - 2\hbar^2 \\ \left[p_1, x_1 x_1\right] = -2i\hbar x_1\end{array}\right\} \tag{3.131}$$

and

$$\frac{d}{dt}\left[x_1(t)\,x_1(t)\right] = \frac{i}{\hbar}\left[-\frac{i\hbar}{M}\left(2x_1(t)\,p_1(t) - i\hbar\right) + \frac{2i\hbar q b}{Mc} x_1(t)\,x_3(t)\right]$$

$$= \frac{2}{M}\left\{\begin{bmatrix} x_1 + \left(\frac{1}{M}p_1 - \frac{qb}{Mc}x_3\right)t \\ + \left(\frac{q\beta}{2M^2c}\left(x_3p_2 + x_2p_3\right) - \frac{qb}{2M^2c}p_3\right)t^2 \\ + \frac{q\beta}{3M^3c}p_2p_3t^3 \end{bmatrix} \times \left[p_1 + \frac{q\beta}{Mc}\left(x_3p_2 + x_2p_3\right)t + \frac{q\beta}{M^2c}p_2p_3t^2\right]\right\}$$

$$- \frac{i\hbar}{M} - \frac{2qb}{Mc}\left(x_1 + \frac{p_1}{M}t\right)\left(x_3 + \frac{p_3}{M}t\right)$$

$$= \frac{2}{M}\begin{bmatrix} x_1p_1 + \frac{q\beta}{Mc}\left(x_1x_3p_2 + x_1x_2p_3\right)t \\ + \frac{q\beta}{M^2c}x_1p_2p_3t^2 + \frac{p_1p_1}{M}t \\ + \frac{q\beta}{M^2c}\left(x_3p_1p_2 + x_2p_1p_3\right)t^2 \\ + \frac{q\beta}{M^3c}p_1p_2p_3t^3 - \frac{qb}{Mc}x_3p_1t \\ + \left[\frac{q\beta}{2M^2c}\left(x_3p_1p_2 + x_2p_1p_3\right) - \frac{qb}{2M^2c}p_1p_3\right]t^2 \\ + \frac{q\beta}{3M^3c}p_1p_2p_3t^3 \end{bmatrix} \tag{3.132}$$

$$- \frac{i\hbar}{M} - \frac{2qb}{Mc}\left(x_1x_3 + \frac{1}{M}\left(x_1p_3 + x_3p_1\right)t + \frac{p_1p_3}{M^2}t^2\right)$$

$$x_1(t)x_1(t) = x_1x_1 + \frac{2}{M}\begin{bmatrix} x_1p_1t + \frac{q\beta}{2Mc}(x_1x_3p_2 + x_1x_2p_3)t^2 \\ + \frac{q\beta}{3M^2c}x_1p_2p_3t^3 + \frac{p_1p_1}{2M}t^2 \\ + \frac{q\beta}{3M^2c}(x_3p_1p_2 + x_2p_1p_3)t^3 \\ + \frac{q\beta}{4M^3c}p_1p_2p_3t^4 - \frac{qb}{2Mc}x_3p_1t^2 \\ + \left[\frac{q\beta}{6M^2c}(x_3p_1p_2 + x_2p_1p_3) - \frac{qb}{6M^2c}p_1p_3\right]t^3 \\ + \frac{q\beta}{12M^3c}p_1p_2p_3t^4 \end{bmatrix} - \frac{i\hbar}{M}t - \frac{2qb}{Mc}\left(x_1x_3t + \frac{1}{2M}(x_1p_3 + x_3p_1)t^2 + \frac{p_1p_3}{3M^2}t^3\right) \tag{3.133}$$

$$\langle x_1(t)x_1(t)\rangle = \langle x_1x_1\rangle + \frac{2}{M}\begin{bmatrix} \langle x_1p_1\rangle t + \frac{q\beta}{2Mc}(\langle x_1x_3p_2\rangle + \langle x_1x_2p_3\rangle)t^2 \\ + \frac{q\beta}{3M^2c}\langle x_1p_2p_3\rangle t^3 + \frac{1}{2M}\langle p_1p_1\rangle t^2 \\ + \frac{q\beta}{3M^2c}(\langle x_3p_1p_2\rangle + \langle x_2p_1p_3\rangle)t^3 \\ + \frac{q\beta}{4M^3c}\langle p_1p_2p_3\rangle t^4 - \frac{qb}{2Mc}\langle x_3p_1\rangle t^2 \\ + \begin{bmatrix} \frac{q\beta}{6M^2c}(\langle x_3p_1p_2\rangle + \langle x_2p_1p_3\rangle) \\ - \frac{qb}{6M^2c}\langle p_1p_3\rangle \end{bmatrix} t^3 \\ + \frac{q\beta}{12M^3c}\langle p_1p_2p_3\rangle t^4 \end{bmatrix} - \frac{i\hbar}{M}t - \frac{2qb}{Mc}\begin{pmatrix} \langle x_1x_3\rangle t + \frac{1}{2M}(\langle x_1p_3\rangle + \langle x_3p_1\rangle)t^2 \\ + \frac{1}{3M^2}\langle p_1p_3\rangle t^3 \end{pmatrix}$$

$$= \gamma^2 + \frac{2}{M}\left(\frac{i\hbar}{2}t + \frac{1}{2M}\frac{\hbar^2}{4\gamma^2}t^2\right)$$

$$=\gamma^2+\frac{i\hbar}{M}t+\frac{\hbar^2t^2}{4\gamma^2M^2}-\frac{i\hbar}{M}t \qquad (3.134)$$

$$\left\langle x_1\left(t\right)x_1\left(t\right)\right\rangle=\gamma^2+\frac{\hbar^2t^2}{4\gamma^2M^2} \qquad (3.135)$$

$$\left[H,x_2x_2\right]=\frac{1}{2M}\left(-4i\hbar x_2p_2-2\hbar^2\right)-\frac{q}{Mc}\left(\beta x_1x_3\left(-2i\hbar x_2\right)\right)$$

$$=-\frac{i\hbar}{M}\left(2x_2p_2-i\hbar\right)+\frac{2i\hbar q\beta}{Mc}x_1x_2x_3 \qquad (3.136)$$

since

$$\left.\begin{aligned}\left[p_2p_2,x_2x_2\right]&=-4i\hbar x_2p_2-2\hbar^2\\ \left[p_2,x_2x_2\right]&=-2i\hbar x_2\end{aligned}\right\} \qquad (3.137)$$

and

$$\frac{d}{dt}\left[x_2\left(t\right)x_2\left(t\right)\right]=\frac{i}{\hbar}\left[-\frac{i\hbar}{M}\left(2x_2\left(t\right)p_2\left(t\right)-i\hbar\right)+\frac{2i\hbar q\beta}{Mc}x_1\left(t\right)x_2\left(t\right)x_3\left(t\right)\right]$$

$$
=\frac{2}{M}\left\{\begin{bmatrix} x_2+\left(\frac{1}{M}p_2-\frac{q\beta}{Mc}x_1x_3\right)t \\ -\left(\frac{\mu\beta}{2M}\sigma_2+\frac{q\beta}{2M^2c}x_3p_1\right)t^2 \\ -\frac{q\beta}{6M^3c}p_1p_3t^3 \end{bmatrix} \times\left[p_2+\left(\frac{q\beta}{Mc}x_1p_3-\mu\beta\sigma_2\right)t+\frac{q\beta}{2M^2c}p_1p_3t^2\right]\right\}
$$

$$
-\frac{i\hbar}{M}-\frac{2q\beta}{Mc}\left(x_1+\frac{p_1}{M}t\right)\left(x_2+\frac{p_2}{M}t\right)\left(x_3+\frac{p_3}{M}t\right)
$$

$$
=\frac{2}{M}\begin{bmatrix} x_2p_2+\left(\frac{q\beta}{Mc}x_1x_2p_3-\mu\beta\sigma_2x_2\right)t \\ +\frac{q\beta}{2M^2c}x_2p_1p_3t^2+\frac{p_2p_2}{M}t \\ +\left(\frac{q\beta}{M^2c}x_1p_2p_3-\frac{\mu\beta}{M}\sigma_2p_2\right)t^2 \\ +\frac{q\beta}{2M^3c}p_1p_2p_3t^3-\frac{q\beta}{Mc}x_1x_3p_2t \\ -\left(\frac{\mu\beta}{2M}\sigma_2p_2+\frac{q\beta}{2M^2c}x_3p_1p_2\right)t^2 \\ -\frac{q\beta}{6M^3c}p_1p_2p_3t^3 \end{bmatrix}
$$

$$
-\frac{i\hbar}{M}-\frac{2q\beta}{Mc}\begin{pmatrix} x_1x_2x_3 \\ +\left(\frac{x_1x_3p_2}{M}+\frac{x_2x_3p_1}{M}+\frac{x_1x_2p_3}{M}\right)t \\ +\left(\frac{x_3p_1p_2}{M^2}+\frac{x_1p_2p_3}{M^2}+\frac{x_2p_1p_3}{M^2}\right)t^2 \\ +\frac{p_1p_2p_3}{M^3}t^3 \end{pmatrix} \tag{3.138}
$$

$$x_2(t)x_2(t) = x_2x_2 + \frac{2}{M}\begin{bmatrix} x_2p_2t + \left(\frac{q\beta}{2Mc}x_1x_2p_3 - \frac{\mu\beta}{2}\sigma_2x_2\right)t^2 \\ +\frac{q\beta}{6M^2c}x_2p_1p_3t^3 + \frac{p_2p_2}{2M}t^2 \\ +\left(\frac{q\beta}{3M^2c}x_1p_2p_3 - \frac{\mu\beta}{3M}\sigma_2p_2\right)t^3 \\ +\frac{q\beta}{8M^3c}p_1p_2p_3t^4 - \frac{q\beta}{2Mc}x_1x_3p_2t^2 \\ -\left(\frac{\mu\beta}{6M}\sigma_2p_2 + \frac{q\beta}{6M^2c}x_3p_1p_2\right)t^3 \\ -\frac{q\beta}{24M^3c}p_1p_2p_3t^4 \end{bmatrix}$$

$$-\frac{i\hbar}{M}t - \frac{2q\beta}{Mc}\begin{pmatrix} x_1x_2x_3t \\ +\left(\frac{x_1x_3p_2}{2M} + \frac{x_2x_3p_1}{2M} + \frac{x_1x_2p_3}{2M}\right)t^2 \\ +\left(\frac{x_3p_1p_2}{3M^2} + \frac{x_1p_2p_3}{3M^2} + \frac{x_2p_1p_3}{3M^2}\right)t^3 \\ +\frac{p_1p_2p_3}{4M^3}t^4 \end{pmatrix} \qquad (3.139)$$

$$\left\langle x_2(t)x_2(t)\right\rangle = \left\langle x_2x_2\right\rangle + \frac{2}{M}\begin{bmatrix} \left\langle x_2p_2\right\rangle t + \left(\frac{q\beta}{2Mc}\left\langle x_1x_2p_3\right\rangle - \frac{\mu\beta}{2}\left\langle \sigma_2x_2\right\rangle\right)t^2 \\ +\frac{q\beta}{6M^2c}\left\langle x_2p_1p_3\right\rangle t^3 + \frac{1}{2M}\left\langle p_2p_2\right\rangle t^2 \\ +\left(\frac{q\beta}{3M^2c}\left\langle x_1p_2p_3\right\rangle - \frac{\mu\beta}{3M}\left\langle \sigma_2p_2\right\rangle\right)t^3 \\ +\frac{q\beta}{8M^3c}\left\langle p_1p_2p_3\right\rangle t^4 - \frac{q\beta}{2Mc}\left\langle x_1x_3p_2\right\rangle t^2 \\ -\left(\frac{\mu\beta}{6M}\left\langle \sigma_2p_2\right\rangle + \frac{q\beta}{6M^2c}\left\langle x_3p_1p_2\right\rangle\right)t^3 \\ -\frac{q\beta}{24M^3c}\left\langle p_1p_2p_3\right\rangle t^4 \end{bmatrix}$$

$$-\frac{i\hbar}{M}t - \frac{2q\beta}{Mc}\begin{pmatrix} \left\langle x_1x_2x_3\right\rangle t \\ +\begin{pmatrix} \frac{1}{2M}\left\langle x_1x_3p_2\right\rangle + \frac{1}{2M}\left\langle x_2x_3p_1\right\rangle \\ +\frac{1}{2M}\left\langle x_1x_2p_3\right\rangle \end{pmatrix}t^2 \\ +\begin{pmatrix} \frac{1}{3M^2}\left\langle x_3p_1p_2\right\rangle + \frac{1}{3M^2}\left\langle x_1p_2p_3\right\rangle \\ +\frac{1}{3M^2}\left\langle x_2p_1p_3\right\rangle \end{pmatrix}t^3 \\ +\frac{1}{4M^3}\left\langle p_1p_2p_3\right\rangle t^4 \end{pmatrix}$$

$$= \gamma^2 + \frac{2}{M}\begin{bmatrix} \frac{i\hbar}{2}t + \frac{1}{2M}\left(p_0^2 + \frac{\hbar^2}{4\gamma^2}\right)t^2 \\ -\frac{\mu\beta p_0}{3M}\left\langle \sigma_2\right\rangle t^3 - \frac{\mu\beta p_0}{6M}\left\langle \sigma_2\right\rangle t^3 \end{bmatrix} - \frac{i\hbar}{M}t \qquad (3.140)$$

$$\left\langle x_2(t)x_2(t)\right\rangle = \gamma^2 + \frac{1}{M^2}\left(p_0^2 + \frac{\hbar^2}{4\gamma^2}\right)t^2 - \frac{\mu\beta p_0}{M^2}\left\langle \sigma_2\right\rangle t^3 \qquad (3.141)$$

$$[H, x_3x_3] = \frac{1}{2M}\left(-4i\hbar x_3 p_3 - 2\hbar^2\right) - \frac{q}{Mc}\left(\beta x_1 x_2\left(-2i\hbar x_3\right)\right)$$

$$= -\frac{i\hbar}{M}\left(2x_3 p_3 - i\hbar\right) + \frac{2i\hbar q\beta}{Mc} x_1 x_2 x_3 \tag{3.142}$$

$$\left.\begin{aligned}[p_3 p_3, x_3 x_3] &= -4i\hbar x_3 p_3 - 2\hbar^2 \\ [p_3, x_3 x_3] &= -2i\hbar x_3\end{aligned}\right\} \tag{3.143}$$

and

$$\frac{d}{dt}\left[x_3(t)x_3(t)\right] = \frac{i}{\hbar}\left[-\frac{i\hbar}{M}\left(2x_3(t)p_3(t) - i\hbar\right) + \frac{2i\hbar q\beta}{Mc} x_1(t)x_2(t)x_3(t)\right]$$

$$= \frac{2}{M}\left\{\begin{bmatrix} x_3 + \left(\frac{1}{M}p_3 - \frac{q\beta}{Mc}x_1x_2\right)t \\ +\left(\frac{qb}{2M^2c}p_1 + \frac{\mu\beta}{2M}\sigma_3 - \frac{q\beta}{2M^2c}x_2p_1\right)t^2 \\ -\frac{q\beta}{6M^3c}p_1p_2t^3 \end{bmatrix} \times \begin{bmatrix} p_3 + \left[\frac{q}{Mc}\left(bp_1 + \beta x_1 p_2\right) + \mu\beta\sigma_3\right]t \\ +\frac{q\beta}{2M^2c}p_1p_2t^2 \end{bmatrix}\right\}$$

$$-\frac{i\hbar}{M} - \frac{2q\beta}{Mc}\left(x_1 + \frac{p_1}{M}t\right)\left(x_2 + \frac{p_2}{M}t\right)\left(x_3 + \frac{p_3}{M}t\right)$$

$$
=\frac{2}{M}\begin{bmatrix}
x_3p_3+\left(\frac{q}{Mc}\left(bx_3p_1+\beta x_1x_3p_2\right)+\mu\beta\sigma_3x_3\right)t \\
+\frac{q\beta}{2M^2c}x_3p_1p_2t^2+\frac{p_3p_3}{M}t \\
+\left(\frac{q}{M^2c}\left(bp_1p_3+\beta x_1p_2p_3\right)+\frac{\mu\beta}{M}\sigma_3p_3\right)t^2 \\
+\frac{q\beta}{2M^3c}p_1p_2p_3t^3-\frac{q\beta}{Mc}x_1x_2p_3t \\
+\left(\frac{qb}{2M^2c}p_1p_3+\frac{\mu\beta}{2M}\sigma_3p_3-\frac{q\beta}{2M^2c}x_2p_1p_3\right)t^2 \\
-\frac{q\beta}{6M^3c}p_1p_2p_3t^3
\end{bmatrix}
-\frac{i\hbar}{M}-\frac{2q\beta}{Mc}\begin{pmatrix}
x_1x_2x_3 \\
+\left(\frac{x_1x_3p_2}{M}+\frac{x_2x_3p_1}{M}+\frac{x_1x_2p_3}{M}\right)t \\
+\left(\frac{x_3p_1p_2}{M^2}+\frac{x_1p_2p_3}{M^2}+\frac{x_2p_1p_3}{M^2}\right)t^2 \\
+\frac{p_1p_2p_3}{M^3}t^3
\end{pmatrix} \tag{3.144}
$$

$$x_3(t)x_3(t)=x_3x_3+\frac{2}{M}\begin{bmatrix} x_3p_3t+\left(\frac{q}{2Mc}\left(bx_3p_1+\beta x_1x_3p_2\right)+\frac{\mu\beta}{2}\sigma_3x_3\right)t^2 \\ +\frac{q\beta}{6M^2c}x_3p_1p_2t^3+\frac{p_3p_3}{2M}t^2 \\ +\left(\frac{q}{3M^2c}\left(bp_1p_3+\beta x_1p_2p_3\right)+\frac{\mu\beta}{3M}\sigma_3p_3\right)t^3 \\ +\frac{q\beta}{8M^3c}p_1p_2p_3t^4-\frac{q\beta}{2Mc}x_1x_2p_3t^2 \\ +\left(\frac{qb}{6M^2c}p_1p_3+\frac{\mu\beta}{6M}\sigma_3p_3-\frac{q\beta}{6M^2c}x_2p_1p_3\right)t^3 \\ -\frac{q\beta}{24M^3c}p_1p_2p_3t^4 \end{bmatrix}$$

$$-\frac{i\hbar}{M}t-\frac{2q\beta}{Mc}\begin{pmatrix} x_1x_2x_3t \\ +\left(\frac{x_1x_3p_2}{2M}+\frac{x_2x_3p_1}{2M}+\frac{x_1x_2p_3}{2M}\right)t^2 \\ +\left(\frac{x_3p_1p_2}{3M^2}+\frac{x_1p_2p_3}{3M^2}+\frac{x_2p_1p_3}{3M^2}\right)t^3 \\ +\frac{p_1p_2p_3}{4M^3}t^4 \end{pmatrix} \qquad (3.145)$$

$$\left\langle x_3(t)x_3(t)\right\rangle = \left\langle x_3x_3\right\rangle + \frac{2}{M}\left[\begin{array}{l} \left\langle x_3p_3\right\rangle t + \left(\begin{array}{l}\frac{q}{2Mc}\left(b\left\langle x_3p_1\right\rangle + \beta\left\langle x_1x_3p_2\right\rangle\right)\\ +\frac{\mu\beta}{2}\left\langle \sigma_3x_3\right\rangle\end{array}\right)t^2 \\ +\frac{q\beta}{6M^2c}\left\langle x_3p_1p_2\right\rangle t^3 + \frac{1}{2M}\left\langle p_3p_3\right\rangle t^2 \\ +\left(\begin{array}{l}\frac{q}{3M^2c}\left(b\left\langle p_1p_3\right\rangle + \beta\left\langle x_1p_2p_3\right\rangle\right)\\ +\frac{\mu\beta}{3M}\left\langle \sigma_3p_3\right\rangle\end{array}\right)t^3 \\ +\frac{q\beta}{8M^3c}\left\langle p_1p_2p_3\right\rangle t^4 - \frac{q\beta}{2Mc}\left\langle x_1x_2p_3\right\rangle t^2 \\ +\left(\begin{array}{l}\frac{qb}{6M^2c}\left\langle p_1p_3\right\rangle + \frac{\mu\beta}{6M}\left\langle \sigma_3p_3\right\rangle\\ -\frac{q\beta}{6M^2c}\left\langle x_2p_1p_3\right\rangle\end{array}\right)t^3 \\ -\frac{q\beta}{24M^3c}\left\langle p_1p_2p_3\right\rangle t^4 \end{array}\right]$$

$$-\frac{i\hbar}{M}t - \frac{2q\beta}{Mc}\left(\begin{array}{l} \left\langle x_1x_2x_3\right\rangle t \\ +\left(\begin{array}{l}\frac{1}{2M}\left\langle x_1x_3p_2\right\rangle + \frac{1}{2M}\left\langle x_2x_3p_1\right\rangle\\ +\frac{1}{2M}\left\langle x_1x_2p_3\right\rangle\end{array}\right)t^2 \\ +\left(\begin{array}{l}\frac{1}{3M^2}\left\langle x_3p_1p_2\right\rangle + \frac{1}{3M^2}\left\langle x_1p_2p_3\right\rangle\\ +\frac{1}{3M^2}\left\langle x_2p_1p_3\right\rangle\end{array}\right)t^3 \\ +\frac{1}{4M^3}\left\langle p_1p_2p_3\right\rangle t^4 \end{array}\right)$$

$$= \gamma^2 + \frac{2}{M}\left(\frac{i\hbar}{2}t + \frac{\hbar^2t^2}{8M\gamma^2}\right) - \frac{i\hbar}{M} \tag{3.146}$$

$$\left\langle x_3(t)x_3(t)\right\rangle = \gamma^2 + \frac{\hbar^2t^2}{4\gamma^2M^2}. \tag{3.147}$$

Now we use the commutation relations

$$[p_1p_1, x_1x_2] = -2i\hbar x_2 p_1,\ [p_2p_2, x_1x_2] = -2i\hbar x_1 p_2 \tag{3.148}$$

$$[p_1, x_1x_2] = -i\hbar x_2,\ [p_2, x_1x_2] = -i\hbar x_1 \tag{3.149}$$

$$\begin{aligned}[H, x_1x_2] &= \frac{1}{2M}(-2i\hbar x_2 p_1 - 2i\hbar x_1 p_2) - \frac{q}{Mc}\left(bx_3(-i\hbar x_2) + \beta x_1 x_3(-i\hbar x_1)\right) \\ &= -\frac{i\hbar}{M}(x_2p_1 + x_1p_2) + \frac{i\hbar q}{Mc}(bx_2x_3 + \beta x_1x_1x_3)\end{aligned} \tag{3.150}$$

consider the time derivative

$$\begin{aligned}\frac{d}{dt}[x_1(t)x_2(t)] &= \frac{i}{\hbar}\left[-\frac{i\hbar}{M}\begin{pmatrix} x_2(t)p_1(t) \\ +x_1(t)p_2(t)\end{pmatrix} + \frac{i\hbar q}{Mc}\begin{pmatrix} bx_2(t)x_3(t) \\ +\beta x_1(t)x_1(t)x_3(t)\end{pmatrix}\right] \\ &= \frac{1}{M}\left(x_2(t)p_1(t) + x_1(t)p_2(t)\right) \\ &\quad - \frac{q}{Mc}\begin{pmatrix} b\left(x_2 + \frac{p_2}{M}t\right)\left(x_3 + \frac{p_3}{M}t\right) \\ +\beta\left(x_1 + \frac{p_1}{M}t\right)\left(x_1 + \frac{p_1}{M}t\right)\left(x_3 + \frac{p_3}{M}t\right)\end{pmatrix}\end{aligned} \tag{3.151}$$

and the explicit expression

$$x_2(t)\,p_1(t) = \begin{bmatrix} x_2 + \left(\dfrac{1}{M}p_2 - \dfrac{q\beta}{Mc}x_1x_3\right)t \\ -\left(\dfrac{\mu\beta}{2M}\sigma_2 + \dfrac{q\beta}{2M^2c}x_3p_1\right)t^2 - \dfrac{q\beta}{6M^3c}p_1p_3t^3 \end{bmatrix} \times\left[p_1 + \frac{q\beta}{Mc}(x_3p_2 + x_2p_3)t + \frac{q\beta}{M^2c}p_2p_3t^2\right]$$

$$\begin{aligned} &= x_2p_1 + \frac{q\beta}{Mc}(x_2x_3p_2 + x_2x_2p_3)t \\ &\quad + \frac{q\beta}{M^2c}x_2p_2p_3t^2 + \frac{p_1p_2}{M}t \\ &\quad + \frac{q\beta}{M^2c}(x_3p_2p_2 + x_2p_2p_3 - i\hbar p_3)t^2 \\ &\quad + \frac{q\beta}{M^3c}p_2p_2p_3t^3 - \frac{q\beta}{Mc}x_1x_3p_1t \\ &\quad - \left(\frac{\mu\beta}{2M}\sigma_2p_1 + \frac{q\beta}{2M^2c}x_3p_1p_1\right)t^2 - \frac{q\beta}{6M^3c}p_1p_1p_3t^3 \end{aligned} \tag{3.152}$$

to obtain

$$\begin{aligned} \langle x_2(t)\,p_1(t)\rangle &= \langle x_2p_1\rangle + \frac{q\beta}{Mc}\left(\langle x_2x_3p_2\rangle + \langle x_2x_2p_3\rangle\right)t \\ &\quad + \frac{q\beta}{M^2c}\langle x_2p_2p_3\rangle t^2 + \frac{1}{M}\langle p_1p_2\rangle t \\ &\quad + \frac{q\beta}{M^2c}\left(\langle x_3p_2p_2\rangle + \langle x_2p_2p_3\rangle - i\hbar\langle p_3\rangle\right)t^2 \\ &\quad + \frac{q\beta}{M^3c}\langle p_2p_2p_3\rangle t^3 - \frac{q\beta}{Mc}\langle x_1x_3p_1\rangle t \\ &\quad - \left(\frac{\mu\beta}{2M}\langle\sigma_2p_1\rangle + \frac{q\beta}{2M^2c}\langle x_3p_1p_1\rangle\right)t^2 - \frac{q\beta}{6M^3c}\langle p_1p_1p_3\rangle t^3 \end{aligned} \tag{3.153}$$

$$\langle x_2(t)\,p_1(t)\rangle = 0\,. \tag{3.154}$$

Similarly we have,

$$
\begin{aligned}
x_1(t)\,p_2(t) &= \begin{bmatrix} x_1 + \left(\dfrac{1}{M}p_1 - \dfrac{qb}{Mc}x_3\right)t \\ +\left(\dfrac{q\beta}{2M^2c}\left(x_3p_2 + x_2p_3\right) - \dfrac{qb}{2M^2c}p_3\right)t^2 \\ +\dfrac{q\beta}{3M^3c}p_2p_3t^3 \end{bmatrix} \\
&\quad \times \left[p_2 + \left(\dfrac{q\beta}{Mc}x_1p_3 - \mu\beta\sigma_2\right)t + \dfrac{q\beta}{2M^2c}p_1p_3t^2\right] \\
&= x_1p_2 + \left(\dfrac{q\beta}{Mc}x_1x_1p_3 - \mu\beta\sigma_2x_1\right)t \\
&\quad + \dfrac{q\beta}{2M^2c}x_1p_1p_3t^2 + \dfrac{p_1p_2}{M}t \\
&\quad + \left(\dfrac{q\beta}{M^2c}\left(x_1p_1p_3 - i\hbar p_3\right) - \dfrac{\mu\beta}{M}\sigma_2p_1\right)t^2 \\
&\quad + \dfrac{q\beta}{2M^3c}p_1p_1p_3t^3 - \dfrac{qb}{Mc}x_3p_2t \\
&\quad + \left(\dfrac{q\beta}{2M^2c}\left(x_3p_2p_2 + x_2p_2p_3\right) - \dfrac{qb}{2M^2c}p_2p_3\right)t^2 + \dfrac{q\beta}{3M^3c}p_2p_2p_3t^3
\end{aligned}
\tag{3.155}
$$

$$
\begin{aligned}
\left\langle x_1(t)\,p_2(t)\right\rangle &= \left\langle x_1p_2\right\rangle + \left(\dfrac{q\beta}{Mc}\left\langle x_1x_1p_3\right\rangle - \mu\beta\left\langle \sigma_2x_1\right\rangle\right)t \\
&\quad + \dfrac{q\beta}{2M^2c}\left\langle x_1p_1p_3\right\rangle t^2 + \dfrac{1}{M}\left\langle p_1p_2\right\rangle t \\
&\quad + \left(\dfrac{q\beta}{M^2c}\left(\left\langle x_1p_1p_3\right\rangle - i\hbar\left\langle p_3\right\rangle\right) - \dfrac{\mu\beta}{M}\left\langle \sigma_2p_1\right\rangle\right)t^2 \\
&\quad + \dfrac{q\beta}{2M^3c}\left\langle p_1p_1p_3\right\rangle t^3 - \dfrac{qb}{Mc}\left\langle x_3p_2\right\rangle t \\
&\quad + \left(\dfrac{q\beta}{2M^2c}\left(\left\langle x_3p_2p_2\right\rangle + \left\langle x_2p_2p_3\right\rangle\right) - \dfrac{qb}{2M^2c}\left\langle p_2p_3\right\rangle\right)t^2 \\
&\quad + \dfrac{q\beta}{3M^3c}\left\langle p_2p_2p_3\right\rangle t^3
\end{aligned}
\tag{3.156}
$$

$$\left\langle x_1(t)\, p_2(t)\right\rangle = 0\,. \tag{3.157}$$

Therefore

$$\frac{\left\langle x_2(t)\, p_1(t) + x_1(t)\, p_2(t)\right\rangle}{M} = 0\,. \tag{3.158}$$

Upon setting

$$\begin{aligned}
A &= b\left(x_2 + \frac{p_2}{M}t\right)\left(x_3 + \frac{p_3}{M}t\right) + \beta\left(x_1 + \frac{p_1}{M}t\right)\left(x_1 + \frac{p_1}{M}t\right)\left(x_3 + \frac{p_3}{M}t\right) \\
&= b\left(x_2x_3 + \frac{x_2p_3}{M}t + \frac{x_3p_2}{M}t + \frac{p_2p_3}{M^2}t^2\right) \\
&\quad + \beta\left(x_1x_1 + \frac{x_1p_1}{M}t + \frac{x_1p_1}{M}t - \frac{i\hbar}{M}t + \frac{p_1p_1}{M^2}t^2\right)\left(x_3 + \frac{p_3}{M}t\right) \\
&= b\left(x_2x_3 + \frac{x_2p_3}{M}t + \frac{x_3p_2}{M}t + \frac{p_2p_3}{M^2}t^2\right) \\
&\quad + \beta\left(\begin{aligned} & x_1x_1x_3 + \frac{2x_1x_3p_1}{M}t - \frac{i\hbar}{M}x_3t + \frac{x_3p_1p_1}{M^2}t^2 \\ & + \frac{x_1x_1p_3}{M}t + \frac{2x_1p_1p_3}{M^2}t^2 - \frac{i\hbar}{M^2}p_3t^2 + \frac{p_1p_1p_3}{M^3}t^3 \end{aligned}\right)
\end{aligned} \tag{3.159}$$

we obtain

$$\langle A\rangle = b\left(\langle x_2x_3\rangle + \frac{1}{M}\langle x_2p_3\rangle t + \frac{1}{M}\langle x_3p_2\rangle t + \frac{1}{M^2}\langle p_2p_3\rangle t^2\right)$$
$$+\beta\begin{pmatrix}\langle x_1x_1x_3\rangle + \frac{2}{M}\langle x_1x_3p_1\rangle t - \frac{i\hbar}{M}\langle x_3\rangle t + \frac{1}{M^2}\langle x_3p_1p_1\rangle t^2 \\ +\frac{1}{M}\langle x_1x_1p_3\rangle t + \frac{2}{M^2}\langle x_1p_1p_3\rangle t^2 \\ -\frac{i\hbar}{M^2}\langle p_3\rangle t^2 + \frac{1}{M^3}\langle p_1p_1p_3\rangle t^3\end{pmatrix} \tag{3.160}$$

or

$$\langle A\rangle = 0\,. \tag{3.161}$$

Therefore we have

$$\langle x_1(t)x_2(t)\rangle = 0\,. \tag{3.162}$$

By proceeding in a similar manner we get

$$[p_1p_1, x_1x_3] = -2i\hbar x_3p_1\,,\ [p_3p_3, x_1x_3] = -2i\hbar x_1p_3 \tag{3.163}$$

$$[p_1, x_1x_3] = -i\hbar x_3\,,\ [p_3, x_1x_3] = -i\hbar x_1 \tag{3.164}$$

$$\left[H,x_1x_3\right]=\frac{1}{2M}\left(-2i\hbar x_3p_1-2i\hbar x_1p_3\right)-\frac{q}{Mc}\left(bx_3\left(-i\hbar x_3\right)+\beta x_1x_2\left(-i\hbar x_1\right)\right)$$

$$=-\frac{i\hbar}{M}\left(x_3p_1+x_1p_3\right)+\frac{i\hbar q}{Mc}\left(bx_3x_3+\beta x_1x_1x_2\right) \tag{3.165}$$

$$\frac{d}{dt}\left[x_1(t)x_3(t)\right]=\frac{i}{\hbar}\left[-\frac{i\hbar}{M}\begin{pmatrix}x_3(t)p_1(t)\\+x_1(t)p_3(t)\end{pmatrix}+\frac{i\hbar q}{Mc}\begin{pmatrix}bx_3(t)x_3(t)\\+\beta x_1(t)x_1(t)x_2(t)\end{pmatrix}\right]$$

$$=\frac{1}{M}\left(x_3(t)p_1(t)+x_1(t)p_3(t)\right)$$

$$-\frac{q}{Mc}\begin{pmatrix}b\left(x_3+\frac{p_3}{M}t\right)\left(x_3+\frac{p_3}{M}t\right)\\+\beta\left(x_1+\frac{p_1}{M}t\right)\left(x_1+\frac{p_1}{M}t\right)\left(x_2+\frac{p_2}{M}t\right)\end{pmatrix} \tag{3.166}$$

$$x_3(t)p_1(t)=\begin{bmatrix}x_3+\left(\frac{1}{M}p_3-\frac{q\beta}{Mc}x_1x_2\right)t\\+\left(\frac{qb}{2M^2c}p_1+\frac{\mu\beta}{2M}\sigma_3-\frac{q\beta}{2M^2c}x_2p_1\right)t^2-\frac{q\beta}{6M^3c}p_1p_2t^3\end{bmatrix}$$

$$\times\left[p_1+\frac{q\beta}{Mc}\left(x_3p_2+x_2p_3\right)t+\frac{q\beta}{M^2c}p_2p_3t^2\right]$$

$$=x_3p_1+\frac{q\beta}{Mc}\left(x_3x_3p_2+x_2x_3p_3\right)t$$

$$+\frac{q\beta}{M^2c}x_3p_2p_3t^2+\frac{p_1p_3}{M}t$$

$$+\frac{q\beta}{M^2c}\left(x_3p_2p_3-i\hbar p_2+x_2p_3p_3\right)t^2 \tag{3.167}$$

$$+\frac{q\beta}{M^3c}p_2p_3p_3t^3-\frac{q\beta}{Mc}x_1x_2p_1t$$

$$+\left(\frac{qb}{2M^2c}p_1p_1+\frac{\mu\beta}{2M}\sigma_3p_1-\frac{q\beta}{2M^2c}x_2p_1p_1\right)t^2-\frac{q\beta}{6M^3c}p_1p_1p_2t^3$$

$$\begin{aligned}\left\langle x_3(t)\,p_1(t)\right\rangle = &\left\langle x_3 p_1\right\rangle + \frac{q\beta}{Mc}\left(\left\langle x_3 x_3 p_2\right\rangle + \left\langle x_2 x_3 p_3\right\rangle\right)t \\ &+\frac{q\beta}{M^2c}\left\langle x_3 p_2 p_3\right\rangle t^2 + \frac{1}{M}\left\langle p_1 p_3\right\rangle t \\ &+\frac{q\beta}{M^2c}\left(\left\langle x_3 p_2 p_3\right\rangle - i\hbar\left\langle p_2\right\rangle + \left\langle x_2 p_3 p_3\right\rangle\right)t^2 \\ &+\frac{q\beta}{M^3c}\left\langle p_2 p_3 p_3\right\rangle t^3 - \frac{q\beta}{Mc}\left\langle x_1 x_2 p_1\right\rangle t \\ &+\left(\frac{qb}{2M^2c}\left\langle p_1 p_1\right\rangle + \frac{\mu\beta}{2M}\left\langle \sigma_3 p_1\right\rangle - \frac{q\beta}{2M^2c}\left\langle x_2 p_1 p_1\right\rangle\right)t^2 \\ &-\frac{q\beta}{6M^3c}\left\langle p_1 p_1 p_2\right\rangle t^3\end{aligned}$$

$$\begin{aligned}=&\frac{q\beta}{Mc}p_0\gamma^2 t + \frac{q\beta}{M^2c}\frac{i\hbar}{2}p_0 t^2 + \frac{q\beta}{M^2c}\left(\frac{i\hbar}{2}p_0 - i\hbar p_0\right)t^2 \\ &+\frac{q\beta}{M^3c}\frac{\hbar^2}{4\gamma^2}p_0 t^3 + \frac{qb}{2M^2c}\frac{\hbar^2}{4\gamma^2}t^2 - \frac{q\beta}{6M^3c}\frac{\hbar^2}{4\gamma^2}p_0 t^3\end{aligned} \tag{3.168}$$

$$\left\langle x_3(t)\,p_1(t)\right\rangle = \frac{q\beta}{Mc}p_0\gamma^2 t + \frac{qb}{2M^2c}\frac{\hbar^2}{4\gamma^2}t^2 + \frac{5q\beta}{6M^3c}\frac{\hbar^2}{4\gamma^2}p_0 t^3 \tag{3.169}$$

$$\begin{aligned}x_1(t)\,p_3(t) = &\left[\begin{array}{l} x_1 + \left(\frac{1}{M}p_1 - \frac{qb}{Mc}x_3\right)t \\ +\left(\frac{q\beta}{2M^2c}\left(x_3 p_2 + x_2 p_3\right) - \frac{qb}{2M^2c}p_3\right)t^2 + \frac{q\beta}{3M^3c}p_2 p_3 t^3\end{array}\right] \\ &\times\left[p_3 + \left[\frac{q}{Mc}\left(bp_1 + \beta x_1 p_2\right) + \mu\beta\sigma_3\right]t + \frac{q\beta}{2M^2c}p_1 p_2 t^2\right]\end{aligned}$$

$$
\begin{aligned}
&= x_1 p_3 + \left[\frac{q}{Mc}\left(b x_1 p_1 + \beta x_1 x_1 p_2\right) + \mu\beta\sigma_3 x_1 \right] t \\
&\quad + \frac{q\beta}{2M^2 c} x_1 p_1 p_2 t^2 + \frac{p_1 p_3}{M} t \\
&\quad + \left[\frac{q}{M^2 c}\left(b p_1 p_1 + \beta x_1 p_1 p_2 - i\hbar\beta p_2\right) + \frac{\mu\beta}{M}\sigma_3 p_1 \right] t^2 \\
&\quad + \frac{q\beta}{2M^3 c} p_1 p_1 p_2 t^3 - \frac{qb}{Mc} x_3 p_3 t \\
&\quad + \left(\frac{q\beta}{2M^2 c}\left(x_3 p_2 p_3 + x_2 p_3 p_3\right) - \frac{qb}{2M^2 c} p_3 p_3 \right) t^2 \\
&\quad + \frac{q\beta}{3M^3 c} p_2 p_3 p_3 t^3
\end{aligned}
\tag{3.170}
$$

$$
\begin{aligned}
\left\langle x_1(t) p_3(t) \right\rangle &= \left\langle x_1 p_3 \right\rangle + \left[\frac{q}{Mc}\left(b\left\langle x_1 p_1 \right\rangle + \beta\left\langle x_1 x_1 p_2 \right\rangle\right) + \mu\beta\left\langle \sigma_3 x_1 \right\rangle \right] t \\
&\quad + \frac{q\beta}{2M^2 c}\left\langle x_1 p_1 p_2 \right\rangle t^2 + \frac{1}{M}\left\langle p_1 p_3 \right\rangle t \\
&\quad + \left[\frac{q}{M^2 c}\begin{pmatrix} b\left\langle p_1 p_1 \right\rangle + \beta\left\langle x_1 p_1 p_2 \right\rangle \\ -i\hbar\beta\left\langle p_2 \right\rangle \end{pmatrix} + \frac{\mu\beta}{M}\left\langle \sigma_3 p_1 \right\rangle \right] t^2 \\
&\quad + \frac{q\beta}{2M^3 c}\left\langle p_1 p_1 p_2 \right\rangle t^3 - \frac{qb}{Mc}\left\langle x_3 p_3 \right\rangle t \\
&\quad + \left(\frac{q\beta}{2M^2 c}\left(\left\langle x_3 p_2 p_3 \right\rangle + \left\langle x_2 p_3 p_3 \right\rangle\right) - \frac{qb}{2M^2 c}\left\langle p_3 p_3 \right\rangle \right) t^2 \\
&\quad + \frac{q\beta}{3M^3 c}\left\langle p_2 p_3 p_3 \right\rangle t^3 \\
&= \frac{q}{Mc}\left(b\frac{i\hbar}{2} + \beta p_0 \gamma^2 \right) t + \frac{q\beta}{2M^2 c}\frac{i\hbar}{2} p_0 t^2 \\
&\quad + \frac{q}{M^2 c}\left(b\frac{\hbar^2}{4\gamma^2} + \beta\frac{i\hbar}{2} p_0 - i\hbar\beta p_0 \right) t^2 \\
&\quad + \frac{q\beta}{2M^3 c}\frac{\hbar^2}{4\gamma^2} p_0 t^3 - \frac{qb}{Mc}\frac{i\hbar}{2} t \\
&\quad + \left(\frac{q\beta}{2M^2 c}\frac{i\hbar}{2} p_0 - \frac{qb}{2M^2 c}\frac{\hbar^2}{4\gamma^2} \right) t^2 \\
&\quad + \frac{q\beta}{3M^3 c}\frac{\hbar^2}{4\gamma^2} p_0 t^3
\end{aligned}
\tag{3.171}
$$

or

$$\left\langle x_1(t)\,p_3(t)\right\rangle = \frac{q\beta}{Mc}p_0\gamma^2 t + \frac{qb}{2M^2c}\frac{\hbar^2}{4\gamma^2}t^2 + \frac{5q\beta}{6M^3c}\frac{\hbar^2}{4\gamma^2}p_0 t^3 \tag{3.172}$$

and

$$\frac{\left\langle x_3(t)\,p_1(t)\right\rangle + \left\langle x_1(t)\,p_3(t)\right\rangle}{M} = \frac{2q\beta}{M^2c}p_0\gamma^2 t + \frac{qb}{M^3c}\frac{\hbar^2}{4\gamma^2}t^2 + \frac{5q\beta}{3M^4c}\frac{\hbar^2}{4\gamma^2}p_0 t^3. \tag{3.173}$$

We set

$$C = b\left(x_3 + \frac{p_3}{M}t\right)\left(x_3 + \frac{p_3}{M}t\right) + \beta\left(x_1 + \frac{p_1}{M}t\right)\left(x_1 + \frac{p_1}{M}t\right)\left(x_2 + \frac{p_2}{M}t\right) \tag{3.174}$$

to obtain

$$\begin{aligned} C &= b\left(x_3x_3 + \frac{x_3p_3}{M}t + \frac{x_3p_3}{M}t - \frac{i\hbar}{M}t + \frac{p_3p_3}{M^2}t^2\right) \\ &\quad + \beta\left(x_1x_1 + \frac{x_1p_1}{M}t + \frac{x_1p_1}{M}t - \frac{i\hbar}{M}t + \frac{p_1p_1}{M^2}t^2\right)\left(x_2 + \frac{p_2}{M}t\right) \\ &= b\left(x_3x_3 + \frac{2x_3p_3}{M}t - \frac{i\hbar}{M}t + \frac{p_3p_3}{M^2}t^2\right) \\ &\quad + \beta\begin{pmatrix} x_1x_1x_2 + \frac{2x_1x_2p_1}{M}t - \frac{i\hbar}{M}x_2t + \frac{x_2p_1p_1}{M^2}t^2 \\ + \frac{x_1x_1p_2}{M}t + \frac{2x_1p_1p_2}{M^2}t^2 - \frac{i\hbar}{M^2}p_2t^2 + \frac{p_1p_1p_2}{M^3}t^3 \end{pmatrix} \end{aligned} \tag{3.175}$$

$$\langle C\rangle = b\left(\langle x_3x_3\rangle + \frac{2}{M}\langle x_3p_3\rangle t - \frac{i\hbar}{M}t + \frac{1}{M^2}\langle p_3p_3\rangle t^2\right)$$
$$+\beta\left(\begin{array}{l}\langle x_1x_1x_2\rangle + \frac{2}{M}\langle x_1x_2p_1\rangle t - \frac{i\hbar}{M}\langle x_2\rangle t + \frac{1}{M^2}\langle x_2p_1p_1\rangle t^2 \\ +\frac{1}{M}\langle x_1x_1p_2\rangle t + \frac{2}{M^2}\langle x_1p_1p_2\rangle t^2 - \frac{i\hbar}{M^2}\langle p_2\rangle t^2 + \frac{1}{M^3}\langle p_1p_1p_2\rangle t^3\end{array}\right)$$

$$= b\left(\gamma^2 + \frac{2}{M}\frac{i\hbar}{2}t - \frac{i\hbar}{M}t + \frac{1}{M^2}\frac{\hbar^2}{4\gamma^2}t^2\right)$$
$$+\beta\left(\frac{1}{M}p_0\gamma^2 t + \frac{2}{M^2}\frac{i\hbar}{2}p_0t^2 - \frac{i\hbar}{M^2}p_0t^2 + \frac{1}{M^3}\frac{\hbar^2}{4\gamma^2}p_0t^3\right) \tag{3.176}$$

$$\langle C\rangle = b\left(\gamma^2 + \frac{\hbar^2}{4\gamma^2M^2}t^2\right) + \beta\left(\frac{1}{M}p_0\gamma^2 t + \frac{\hbar^2}{4\gamma^2M^3}p_0t^3\right) \tag{3.177}$$

$$\frac{d}{dt}\langle x_1(t)x_3(t)\rangle = \frac{2q\beta}{M^2c}p_0\gamma^2 t + \frac{qb}{M^3c}\frac{\hbar^2}{4\gamma^2}t^2 + \frac{5q\beta}{3M^4c}\frac{\hbar^2}{4\gamma^2}p_0t^3$$
$$-\frac{q}{Mc}\left(b\left(\gamma^2 + \frac{\hbar^2}{4\gamma^2M^2}t^2\right) + \beta\left(\frac{1}{M}p_0\gamma^2 t + \frac{\hbar^2}{4\gamma^2M^3}p_0t^3\right)\right)$$
$$= -\frac{qb}{Mc}\gamma^2 + \frac{q\beta}{M^2c}p_0\gamma^2 t + \frac{2q\beta}{3M^4c}\frac{\hbar^2}{4\gamma^2}p_0t^3 \tag{3.178}$$

and hence

$$\langle x_1(t)x_3(t)\rangle = -\frac{qb}{Mc}\gamma^2 t + \frac{q\beta}{2M^2c}p_0\gamma^2t^2 + \frac{q\beta}{24M^4c}\frac{\hbar^2}{\gamma^2}p_0t^4\,. \tag{3.179}$$

Finally we have by the same reasoning

$$[p_2 p_2, x_2 x_3] = -2i\hbar x_3 p_2,\ [p_3 p_3, x_2 x_3] = -2i\hbar x_2 p_3 \tag{3.180}$$

$$[p_2, x_2 x_3] = -i\hbar x_3,\ [p_3, x_2 x_3] = -i\hbar x_2 \tag{3.181}$$

$$\begin{aligned}[H, x_2 x_3] &= \frac{1}{2M}\left(-2i\hbar x_3 p_2 - 2i\hbar x_2 p_3\right) - \frac{q}{Mc}\left(\beta x_1 x_3(-i\hbar x_3) + \beta x_1 x_2(-i\hbar x_2)\right) \\ &= -\frac{i\hbar}{M}\left(x_3 p_2 + x_2 p_3\right) + \frac{i\hbar q\beta}{Mc}\left(x_1 x_3 x_3 + x_1 x_2 x_2\right)\end{aligned} \tag{3.182}$$

$$\begin{aligned}\frac{d}{dt}\left[x_2(t)x_3(t)\right] &= \frac{i}{\hbar}\left[-\frac{i\hbar}{M}\begin{pmatrix} x_3(t)p_2(t) \\ +x_2(t)p_3(t)\end{pmatrix} + \frac{i\hbar q\beta}{Mc}\begin{pmatrix} x_1(t)x_3(t)x_3(t) \\ +x_1(t)x_2(t)x_2(t)\end{pmatrix}\right] \\ &= \frac{1}{M}\left(x_3(t)p_2(t) + x_2(t)p_3(t)\right) \\ &\quad -\frac{q\beta}{Mc}\begin{pmatrix}\left(x_1 + \frac{p_1}{M}t\right)\left(x_3 + \frac{p_3}{M}t\right)\left(x_3 + \frac{p_3}{M}t\right) \\ +\left(x_1 + \frac{p_1}{M}t\right)\left(x_2 + \frac{p_2}{M}t\right)\left(x_2 + \frac{p_2}{M}t\right)\end{pmatrix}\end{aligned} \tag{3.183}$$

$$\begin{aligned}x_3(t)p_2(t) &= \begin{bmatrix} x_3 + \left(\frac{1}{M}p_3 - \frac{q\beta}{Mc}x_1 x_2\right)t \\ +\left(\frac{qb}{2M^2c}p_1 + \frac{\mu\beta}{2M}\sigma_3 - \frac{q\beta}{2M^2c}x_2 p_1\right)t^2 \\ -\frac{q\beta}{6M^3c}p_1 p_2 t^3 \end{bmatrix} \\ &\quad \times\left[p_2 + \left(\frac{q\beta}{Mc}x_1 p_3 - \mu\beta\sigma_2\right)t + \frac{q\beta}{2M^2c}p_1 p_3 t^2\right]\end{aligned}$$

$$
\begin{aligned}
&= x_3 p_2 + \left(\frac{q\beta}{Mc} x_1 x_3 p_3 - \mu\beta\sigma_2 x_3 \right) t \\
&\quad + \frac{q\beta}{2M^2c} x_3 p_1 p_3 t^2 + \frac{p_2 p_3}{M} t \\
&\quad + \left(\frac{q\beta}{M^2c} x_1 p_3 p_3 - \frac{\mu\beta}{M} \sigma_2 p_3 \right) t^2 \\
&\quad + \frac{q\beta}{2M^3c} p_1 p_3 p_3 t^3 - \frac{q\beta}{Mc} x_1 x_2 p_2 t \\
&\quad + \left(\frac{qb}{2M^2c} p_1 p_2 + \frac{\mu\beta}{2M} \sigma_3 p_2 - \frac{q\beta}{2M^2c} x_2 p_1 p_2 \right) t^2 \\
&\quad - \frac{q\beta}{6M^3c} p_1 p_2 p_2 t^3
\end{aligned}
\tag{3.184}
$$

$$
\begin{aligned}
\langle x_3(t) p_2(t) \rangle &= \langle x_3 p_2 \rangle + \left(\frac{q\beta}{Mc} \langle x_1 x_3 p_3 \rangle - \mu\beta \langle \sigma_2 x_3 \rangle \right) t \\
&\quad + \frac{q\beta}{2M^2c} \langle x_3 p_1 p_3 \rangle t^2 + \frac{1}{M} \langle p_2 p_3 \rangle t \\
&\quad + \left(\frac{q\beta}{M^2c} \langle x_1 p_3 p_3 \rangle - \frac{\mu\beta}{M} \langle \sigma_2 p_3 \rangle \right) t^2 \\
&\quad + \frac{q\beta}{2M^3c} \langle p_1 p_3 p_3 \rangle t^3 - \frac{q\beta}{Mc} \langle x_1 x_2 p_2 \rangle t \\
&\quad + \left(\frac{qb}{2M^2c} \langle p_1 p_2 \rangle + \frac{\mu\beta}{2M} \langle \sigma_3 p_2 \rangle - \frac{q\beta}{2M^2c} \langle x_2 p_1 p_2 \rangle \right) t^2 \\
&\quad - \frac{q\beta}{6M^3c} \langle p_1 p_2 p_2 \rangle t^3
\end{aligned}
\tag{3.185}
$$

$$
\langle x_3(t) p_2(t) \rangle = \frac{\mu\beta}{2M} \langle \sigma_3 \rangle p_0 t^2 \tag{3.186}
$$

$$x_2(t)\,p_3(t)=\begin{bmatrix} x_2+\left(\dfrac{1}{M}p_2-\dfrac{q\beta}{Mc}x_1x_3\right)t \\ -\left(\dfrac{\mu\beta}{2M}\sigma_2+\dfrac{q\beta}{2M^2c}x_3p_1\right)t^2-\dfrac{q\beta}{6M^3c}p_1p_3t^3 \end{bmatrix} \times\left[p_3+\left[\frac{q}{Mc}\left(bp_1+\beta x_1p_2\right)+\mu\beta\sigma_3\right]t+\frac{q\beta}{2M^2c}p_1p_2t^2\right]$$

$$\begin{aligned}
&=x_2p_3+\left[\frac{q}{Mc}\left(bx_2p_1+\beta x_1x_2p_2\right)+\mu\beta\sigma_3x_2\right]t \\
&\quad+\frac{q\beta}{2M^2c}x_2p_1p_2t^2+\frac{p_2p_3}{M}t \\
&\quad+\left[\frac{q}{M^2c}\left(bp_1p_2+\beta x_1p_2p_2\right)+\frac{\mu\beta}{M}\sigma_3p_2\right]t^2 \\
&\quad+\frac{q\beta}{2M^3c}p_1p_2p_2t^3-\frac{q\beta}{Mc}x_1x_3p_3t \\
&\quad-\left(\frac{\mu\beta}{2M}\sigma_2p_3+\frac{q\beta}{2M^2c}x_3p_1p_3\right)t^2 \\
&\quad-\frac{q\beta}{6M^3c}p_1p_3p_3t^3
\end{aligned} \tag{3.187}$$

$$\begin{aligned}
\left\langle x_2(t)\,p_3(t)\right\rangle&=\left\langle x_2p_3\right\rangle+\left[\frac{q}{Mc}\left(b\left\langle x_2p_1\right\rangle+\beta\left\langle x_1x_2p_2\right\rangle\right)+\mu\beta\left\langle\sigma_3x_2\right\rangle\right]t \\
&\quad+\frac{q\beta}{2M^2c}\left\langle x_2p_1p_2\right\rangle t^2+\frac{1}{M}\left\langle p_2p_3\right\rangle t \\
&\quad+\left[\frac{q}{M^2c}\left(b\left\langle p_1p_2\right\rangle+\beta\left\langle x_1p_2p_2\right\rangle\right)+\frac{\mu\beta}{M}\left\langle\sigma_3p_2\right\rangle\right]t^2 \\
&\quad+\frac{q\beta}{2M^3c}\left\langle p_1p_2p_2\right\rangle t^3-\frac{q\beta}{Mc}\left\langle x_1x_3p_3\right\rangle t \\
&\quad-\left(\frac{\mu\beta}{2M}\left\langle\sigma_2p_3\right\rangle+\frac{q\beta}{2M^2c}\left\langle x_3p_1p_3\right\rangle\right)t^2 \\
&\quad-\frac{q\beta}{6M^3c}\left\langle p_1p_3p_3\right\rangle t^3
\end{aligned} \tag{3.188}$$

$$\left\langle x_2(t)\,p_3(t)\right\rangle=\frac{\mu\beta}{M}\left\langle\sigma_3\right\rangle t^2 \tag{3.189}$$

and therefore

$$\frac{\left\langle x_3\left(t\right)p_2\left(t\right)+x_2\left(t\right)p_3\left(t\right)\right\rangle}{M}=\frac{3\mu\beta}{2M^2}p_0\left\langle\sigma_3\right\rangle t^2 . \tag{3.190}$$

We set

$$D=\left(x_1+\frac{p_1}{M}t\right)\left(x_3+\frac{p_3}{M}t\right)\left(x_3+\frac{p_3}{M}t\right)+\left(x_1+\frac{p_1}{M}t\right)\left(x_2+\frac{p_2}{M}t\right)\left(x_2+\frac{p_2}{M}t\right) \tag{3.191}$$

to obtain

$$\begin{aligned} D=&\left(x_1+\frac{p_1}{M}t\right)\left(x_3x_3+\frac{x_3p_3}{M}t+\frac{x_3p_3}{M}t-\frac{i\hbar}{M}t+\frac{p_3p_3}{M^2}t^2\right)\\ &+\left(x_1+\frac{p_1}{M}t\right)\left(x_2x_2+\frac{x_2p_2}{M}t+\frac{x_2p_2}{M}t-\frac{i\hbar}{M}t+\frac{p_2p_2}{M^2}t^2\right)\\ =&\, x_1x_3x_3+\frac{2x_1x_3p_3}{M}t-\frac{i\hbar}{M}x_1t+\frac{x_1p_3p_3}{M^2}t^2\\ &+\frac{x_3x_3p_1}{M}t+\frac{2x_3p_1p_3}{M^2}t^2-\frac{i\hbar}{M^2}p_1t^2+\frac{p_1p_3p_3}{M^3}t^3\\ &+x_1x_2x_2+\frac{2x_1x_2p_2}{M}t-\frac{i\hbar}{M}x_1t+\frac{x_1p_2p_2}{M^2}t^2\\ &+\frac{x_2x_2p_1}{M}t+\frac{2x_2p_1p_2}{M^2}t^2-\frac{i\hbar}{M^2}p_1t^2+\frac{p_1p_2p_2}{M^3}t^3 \end{aligned} \tag{3.192}$$

$$\begin{aligned}\langle D\rangle &= \langle x_1x_3x_3\rangle + \frac{2}{M}\langle x_1x_3p_3\rangle t - \frac{i\hbar}{M}\langle x_1\rangle t + \frac{1}{M^2}\langle x_1p_3p_3\rangle t^2 \\ &+ \frac{1}{M}\langle x_3x_3p_1\rangle t + \frac{2}{M^2}\langle x_3p_1p_3\rangle t^2 - \frac{i\hbar}{M^2}\langle p_1\rangle t^2 + \frac{1}{M^3}\langle p_1p_3p_3\rangle t^3 \\ &+ \langle x_1x_2x_2\rangle + \frac{2}{M}\langle x_1x_2p_2\rangle t - \frac{i\hbar}{M}\langle x_1\rangle t + \frac{1}{M^2}\langle x_1p_2p_2\rangle t^2 \\ &+ \frac{1}{M}\langle x_2x_2p_1\rangle t + \frac{2}{M^2}\langle x_2p_1p_2\rangle t^2 - \frac{i\hbar}{M^2}\langle p_1\rangle t^2 + \frac{1}{M^3}\langle p_1p_2p_2\rangle t^3\end{aligned} \tag{3.193}$$

$$\langle D\rangle = 0 \tag{3.194}$$

and hence

$$\frac{d}{dt}\langle x_2(t)x_3(t)\rangle = \frac{3\mu\beta}{2M^2}p_0\langle\sigma_3\rangle t^2 \tag{3.195}$$

$$\langle x_2(t)x_3(t)\rangle = \frac{\mu\beta}{2M^2}p_0\langle\sigma_3\rangle t^3. \tag{3.196}$$

We may then summarize our time-dependent expectation values and the correlation functions for the interacting theory to the leading order in $\sqrt{\alpha_q}$ to be given by

$$\left.\begin{aligned}\langle p_1(t)\rangle &= 0 \\ \langle p_2(t)\rangle &= p_0 - \mu\beta\langle\sigma_2\rangle t \\ \langle p_3(t)\rangle &= \mu\beta\langle\sigma_3\rangle t\end{aligned}\right\} \tag{3.197}$$

$$\left.\begin{aligned}
&\left\langle x_1(t)\right\rangle = 0\\
&\left\langle x_2(t)\right\rangle = \frac{p_0}{M}t - \frac{\mu\beta}{2M}\left\langle\sigma_2\right\rangle t^2\\
&\left\langle x_3(t)\right\rangle = \frac{\mu\beta}{2M}\left\langle\sigma_3\right\rangle t^2
\end{aligned}\right\} \tag{3.198}$$

$$\left.\begin{aligned}
&\left\langle p_1(t)p_1(t)\right\rangle = \frac{\hbar^2}{4\gamma^2}\\
&\left\langle p_2(t)p_2(t)\right\rangle = p_0^2 + \frac{\hbar^2}{4\gamma^2} - 2\mu\beta p_0\left\langle\sigma_2\right\rangle t\\
&\left\langle p_3(t)p_3(t)\right\rangle = \frac{\hbar^2}{4\gamma^2}\\
&\left\langle p_1(t)p_2(t)\right\rangle = 0\\
&\left\langle p_1(t)p_3(t)\right\rangle = \frac{\hbar^2 qb}{4\gamma^2 Mc}t + \frac{3\hbar^2 q\beta}{8\gamma^2 M^2 c}p_0 t^2\\
&\left\langle p_2(t)p_3(t)\right\rangle = \mu\beta p_0\left\langle\sigma_3\right\rangle t
\end{aligned}\right\} \tag{3.199}$$

$$\left.\begin{aligned}
&\left\langle x_1(t)x_1(t)\right\rangle = \gamma^2 + \frac{\hbar^2 t^2}{4\gamma^2 M^2}\\
&\left\langle x_2(t)x_2(t)\right\rangle = \gamma^2 + \frac{1}{M^2}\left(p_0^2 + \frac{\hbar^2}{4\gamma^2}\right)t^2 - \frac{\mu\beta p_0}{M^2}\left\langle\sigma_2\right\rangle t^3\\
&\left\langle x_3(t)x_3(t)\right\rangle = \gamma^2 + \frac{\hbar^2 t^2}{4\gamma^2 M^2}\\
&\left\langle x_1(t)x_2(t)\right\rangle = 0\\
&\left\langle x_1(t)x_3(t)\right\rangle = -\frac{qb}{Mc}\gamma^2 t + \frac{q\beta}{2M^2 c}p_0\gamma^2 t^2 + \frac{q\beta}{24M^4 c}\frac{\hbar^2}{\gamma^2}p_0 t^4\\
&\left\langle x_2(t)x_3(t)\right\rangle = \frac{\mu\beta}{2M^2}p_0\left\langle\sigma_3\right\rangle t^3
\end{aligned}\right\} \tag{3.200}$$

Chapter IV

Quantum Dynamical Development of the Stern-Gerlach Effect

4.1 Initial Conditions and the Density Matrix

The dynamics is most elegantly described in terms of the density operator. At $t = 0$, it is given by

$$\rho = w_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} |\Psi\rangle\langle\Psi| \begin{pmatrix} 1 & 0 \end{pmatrix} + w_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} |\Psi\rangle\langle\Psi| \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{4.1}$$

where $|\Psi\rangle$ is our initial wavepacket state in Eq. (2.1) in the $\vec{x}$- description. We have also used spin states $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Here the weights w_+, w_- satisfy the normalization condition

$$w_+ + w_- = 1 \tag{4.2}$$

and for

$$w_+ = w_- = \frac{1}{2} \tag{4.3}$$

we have an unpolarized beam of electrons. For $w_+ = 1$, for example, we have a polarized beam of electrons with spins along the $+x_3$ - axis.

For $t > 0$, the density operator $\rho(t)$, is given by

$$\rho(t) = w_+ e^{-\frac{it}{\hbar}H} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \| \Psi \rangle \langle \Psi | (1 \quad 0) e^{\frac{it}{\hbar}H} + w_- e^{-\frac{it}{\hbar}H} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \| \Psi \rangle \langle \Psi | (0 \quad 1) e^{\frac{it}{\hbar}H} \tag{4.4}$$

since $e^{-\frac{it}{\hbar}H} \begin{pmatrix} w_+ \\ w_- \end{pmatrix} \| \Psi \rangle$ defines a time developed state.

The probability density at time t, in the $\vec{x}$-description, is then given by

$$f(\vec{x}, t) = \langle \vec{x} | \rho(t) | \vec{x} \rangle \tag{4.5}$$

satisfying the normalization condition

$$\int_{R^3} d^3\vec{x} \; f(\vec{x}) = 1 . \tag{4.6}$$

The probability density of observation on the screen (see Figure 3.1.) is then given by

$$f\left(x_1, x_3; t\right) = \int_{-\infty}^{\infty} dx_2 \left\langle \Psi \right| \rho\left(t\right) \left| \Psi \right\rangle$$

$$= w_+ \int_{-\infty}^{\infty} dx_2 \left| \left\langle \Psi \right| e^{-\frac{it}{\hbar}H} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left| \Psi \right\rangle \right|^2 + w_- \int_{-\infty}^{\infty} dx_2 \left| \left\langle \Psi \right| e^{-\frac{it}{\hbar}H} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left| \Psi \right\rangle \right|^2 \qquad (4.7)$$

which we will evaluate in the sequel.

4.2 Fundamental Commutation Relations Involving the Interaction Hamiltonian

We recall the structure of the interaction Hamiltonian adopted

$$H_{\mathrm{I}} = -\frac{q}{Mc}\left(bx_3 p_1 + \beta x_1 x_3 p_2 + \beta x_1 x_2 p_3\right) - \mu\left[\sigma_2\left(b - \beta x_2\right) + \beta\sigma_3 x_3\right]. \qquad (4.8)$$

To perform the dynamics of the system, we need various commutation relations of H_{I} with the kinetic energy operator $\frac{\vec{p}^{\,2}}{2M}$.

To the above end, we have

$$p_1 p_1 H_{\mathrm{I}} = p_1 \begin{bmatrix} -\frac{q}{Mc}\begin{pmatrix} bx_3 p_1 p_1 + \beta x_1 x_3 p_1 p_2 - i\hbar\beta x_3 p_2 \\ +\beta x_1 x_2 p_1 p_3 - i\hbar\beta x_2 p_3 \end{pmatrix} \\ -\mu\left[\sigma_2\left(b - \beta x_2\right) p_1 + \beta\sigma_3 x_3 p_1\right] \end{bmatrix}$$

$$= -\frac{q}{Mc}\begin{pmatrix} bx_3 p_1 p_1 p_1 + \beta x_1 x_3 p_1 p_1 p_2 - i\hbar\beta x_3 p_1 p_2 - i\hbar\beta x_3 p_1 p_2 \\ +\beta x_1 x_2 p_1 p_1 p_3 - i\hbar\beta x_2 p_1 p_3 - i\hbar\beta x_2 p_1 p_3 \end{pmatrix}$$
$$-\mu\left[\sigma_2\left(b - \beta x_2\right) p_1 p_1 + \beta\sigma_3 x_3 p_1 p_1\right]$$

$$= -\frac{q}{Mc}\left(bx_3 p_1 + \beta x_1 x_3 p_2 + \beta x_1 x_2 p_3\right) p_1 p_1$$
$$-\mu\left[\sigma_2\left(b-\beta x_2\right)+\beta\sigma_3 x_3\right] p_1 p_1 + \frac{2i\hbar q\beta}{Mc}\left(x_3 p_1 p_2 + x_2 p_1 p_3\right)$$

$$= H_1 p_1 p_1 + \frac{2i\hbar q\beta}{Mc}\left(x_3 p_1 p_2 + x_2 p_1 p_3\right) \tag{4.9}$$

hence

$$\left[p_1 p_1, H_1\right] = \frac{2i\hbar q\beta}{Mc}\left(x_3 p_1 p_2 + x_2 p_1 p_3\right) \tag{4.10}$$

$$p_2 p_2 H_1 = p_2 \begin{bmatrix} -\frac{q}{Mc}\left(bx_3 p_1 p_2 + \beta x_1 x_3 p_2 p_2 + \beta x_1 x_2 p_2 p_3 - i\hbar\beta x_1 p_3\right) \\ -\mu\left[\sigma_2\left(b-\beta x_2\right) p_2 + i\hbar\beta\sigma_2 + \beta\sigma_3 x_3 p_2\right] \end{bmatrix}$$

$$= -\frac{q}{Mc}\begin{pmatrix} bx_3 p_1 p_2 p_2 + \beta x_1 x_3 p_2 p_2 p_2 + \beta x_1 x_2 p_2 p_2 p_3 \\ -i\hbar\beta x_1 p_2 p_3 - i\hbar\beta x_1 p_2 p_3 \end{pmatrix}$$
$$-\mu\left[\sigma_2\left(b-\beta x_2\right) p_2 p_2 + i\hbar\beta\sigma_2 p_2 + i\hbar\beta\sigma_2 p_2 + \beta\sigma_3 x_3 p_2 p_2\right]$$

$$= -\frac{q}{Mc}\left(bx_3 p_1 + \beta x_1 x_3 p_2 + \beta x_1 x_2 p_3\right) p_2 p_2 + \frac{2i\hbar q\beta}{Mc} x_1 p_2 p_3$$
$$-\mu\left[\sigma_2\left(b-\beta x_2\right)+\beta\sigma_3 x_3\right] p_2 p_2 - 2i\hbar\mu\beta\sigma_2 p_2$$

$$= H_1 p_2 p_2 + \frac{2i\hbar q\beta}{Mc} x_1 p_2 p_3 - 2i\hbar\mu\beta\sigma_2 p_2 \tag{4.11}$$

or

$$\left[p_2 p_2, H_1\right] = \frac{2i\hbar q\beta}{Mc} x_1 p_2 p_3 - 2i\hbar\mu\beta\sigma_2 p_2 \tag{4.12}$$

$$p_3 p_3 H_{\mathrm{I}} = p_3 \begin{bmatrix} -\dfrac{q}{Mc}\left(bx_3 p_1 p_3 - i\hbar b p_1 + \beta x_1 x_3 p_2 p_3 - i\hbar\beta x_1 p_2 + \beta x_1 x_2 p_3 p_3\right) \\ -\mu\left[\sigma_2\left(b - \beta x_2\right) p_3 + \beta\sigma_3 x_3 p_3 - i\hbar\beta\sigma_3\right] \end{bmatrix}$$

$$= -\frac{q}{Mc}\begin{pmatrix} bx_3 p_1 p_3 p_3 - i\hbar b p_1 p_3 - i\hbar b p_1 p_3 + \\ \beta x_1 x_3 p_2 p_3 p_3 - i\hbar\beta x_1 p_2 p_3 - i\hbar\beta x_1 p_2 p_3 + \beta x_1 x_2 p_3 p_3 p_3 \end{pmatrix}$$
$$-\mu\left[\sigma_2\left(b - \beta x_2\right) p_3 p_3 + \beta\sigma_3 x_3 p_3 p_3 - i\hbar\beta\sigma_3 p_3 - i\hbar\beta\sigma_3 p_3\right]$$

$$= -\frac{q}{Mc}\left(bx_3 p_1 + \beta x_1 x_3 p_2 + \beta x_1 x_2 p_3\right) p_3 p_3$$
$$+\frac{2i\hbar q b}{Mc} p_1 p_3 + \frac{2i\hbar q\beta}{Mc} x_1 p_2 p_3$$
$$-\mu\left[\sigma_2\left(b - \beta x_2\right) + \beta\sigma_3 x_3\right] p_3 p_3 + 2i\hbar\mu\beta\sigma_3 p_3$$

$$= H_1 p_3 p_3 + \frac{2i\hbar q}{Mc}\left(b p_1 p_3 + \beta x_1 p_2 p_3\right) + 2i\hbar\mu\beta\sigma_3 p_3 \tag{4.13}$$

therefore

$$\left[p_3 p_3, H_{\mathrm{I}}\right] = \frac{2i\hbar q}{Mc}\left(b p_1 p_3 + \beta x_1 p_2 p_3\right) + 2i\hbar\mu\beta\sigma_3 p_3. \tag{4.14}$$

Accordingly, we obtain

$$\left[\frac{\vec{p}^2}{2M}, H_{\mathrm{I}}\right] = \frac{1}{2M}\begin{bmatrix} \dfrac{2i\hbar q\beta}{Mc}\left(x_3 p_1 p_2 + x_2 p_1 p_3\right) \\ +\dfrac{2i\hbar q\beta}{Mc} x_1 p_2 p_3 - 2i\hbar\mu\beta\sigma_2 p_2 \\ +\dfrac{2i\hbar q}{Mc}\left(b p_1 p_3 + \beta x_1 p_2 p_3\right) + 2i\hbar\mu\beta\sigma_3 p_3 \end{bmatrix} \tag{4.15}$$

or

$$\left[\frac{\vec{p}^2}{2M}, H_{\mathrm{I}}\right] = \frac{i\hbar q\beta}{M^2c} x_3 p_1 p_2 + \frac{i\hbar q}{M^2c}\left(\beta x_2 + b\right) p_1 p_3$$
$$+\frac{2i\hbar q\beta}{M^2c} x_1 p_2 p_3 - \frac{i\hbar\mu\beta}{M}\sigma_2 p_2 + \frac{i\hbar\mu\beta}{M}\sigma_3 p_3 \quad . \qquad (4.16)$$

To obtain further commutation relations of the operator resulting in Eq. (4.16) with the kinetic energy operator, we note that

$$p_1 p_1 \left[\frac{\vec{p}^2}{2M}, H_{\mathrm{I}}\right] = p_1 \begin{pmatrix} \frac{i\hbar q\beta}{M^2c} x_3 p_1 p_1 p_2 + \frac{i\hbar q}{M^2c}\left(\beta x_2 + b\right) p_1 p_1 p_3 \\ +\frac{2i\hbar q\beta}{M^2c}\left(x_1 p_1 p_2 p_3 - i\hbar p_2 p_3\right) \\ -\frac{i\hbar\mu\beta}{M}\sigma_2 p_1 p_2 + \frac{i\hbar\mu\beta}{M}\sigma_3 p_1 p_3 \end{pmatrix}$$

$$= \frac{i\hbar q\beta}{M^2c} x_3 p_1 p_1 p_1 p_2 + \frac{i\hbar q}{M^2c}\left(\beta x_2 + b\right) p_1 p_1 p_1 p_3$$
$$+\frac{2i\hbar q\beta}{M^2c}\left(x_1 p_1 p_1 p_2 p_3 - i\hbar p_1 p_2 p_3 - i\hbar p_1 p_2 p_3\right)$$
$$-\frac{i\hbar\mu\beta}{M}\sigma_2 p_1 p_1 p_2 + \frac{i\hbar\mu\beta}{M}\sigma_3 p_1 p_1 p_3$$

$$= \begin{pmatrix} \frac{i\hbar q\beta}{M^2c} x_3 p_1 p_2 + \frac{i\hbar q}{M^2c}\left(\beta x_2 + b\right) p_1 p_3 \\ +\frac{2i\hbar q\beta}{M^2c}\left(x_1 p_2 p_3\right) - \frac{i\hbar\mu\beta}{M}\sigma_2 p_2 + \frac{i\hbar\mu\beta}{M}\sigma_3 p_3 \end{pmatrix} p_1 p_1$$
$$+\frac{4\hbar^2 q\beta}{M^2c} p_1 p_2 p_3$$

$$= \left[\frac{\vec{p}^2}{2M}, H_{\mathrm{I}}\right] p_1 p_1 + \frac{4\hbar^2 q\beta}{M^2c} p_1 p_2 p_3 \qquad (4.17)$$

$$\left[p_1 p_1, \left[\frac{\vec{p}^2}{2M}, H_{\mathrm{I}}\right]\right] = \frac{4\hbar^2 q\beta}{M^2c} p_1 p_2 p_3 \qquad (4.18)$$

$$p_2 p_2\left[\frac{\vec{p}^2}{2M}, H_{\mathrm{I}}\right] = p_2\begin{pmatrix} \frac{i\hbar q\beta}{M^2c} x_3 p_1 p_2 p_2 + \frac{i\hbar q}{M^2c}(\beta x_2 + b) p_1 p_2 p_3 \\ + \frac{\hbar^2 q\beta}{M^2c} p_1 p_3 + \frac{2i\hbar q\beta}{M^2c} x_1 p_2 p_2 p_3 \\ - \frac{i\hbar\mu\beta}{M}\sigma_2 p_2 p_2 + \frac{i\hbar\mu\beta}{M}\sigma_3 p_2 p_3 \end{pmatrix}$$

$$= \frac{i\hbar q\beta}{M^2c} x_3 p_1 p_2 p_2 p_2 + \frac{i\hbar q}{M^2c}(\beta x_2 + b) p_1 p_2 p_2 p_3 + \frac{\hbar^2 q\beta}{M^2c} p_1 p_2 p_3 + \frac{\hbar^2 q\beta}{M^2c} p_1 p_2 p_3 + \frac{2i\hbar q\beta}{M^2c} x_1 p_2 p_2 p_2 p_3 - \frac{i\hbar\mu\beta}{M}\sigma_2 p_2 p_2 p_2 + \frac{i\hbar\mu\beta}{M}\sigma_3 p_2 p_2 p_3$$

$$= \begin{pmatrix} \frac{i\hbar q\beta}{M^2c} x_3 p_1 p_2 + \frac{i\hbar q}{M^2c}(\beta x_2 + b) p_1 p_3 \\ + \frac{2i\hbar q\beta}{M^2c} x_1 p_2 p_3 - \frac{i\hbar\mu\beta}{M}\sigma_2 p_2 + \frac{i\hbar\mu\beta}{M}\sigma_3 p_3 \end{pmatrix} p_2 p_2 + \frac{2\hbar^2 q\beta}{M^2c} p_1 p_2 p_3$$

$$= \left[\frac{\vec{p}^2}{2M}, H_{\mathrm{I}}\right] p_2 p_2 + \frac{2\hbar^2 q\beta}{M^2c} p_1 p_2 p_3 \tag{4.19}$$

$$\left[p_2 p_2, \left[\frac{\vec{p}^2}{2M}, H_{\mathrm{I}}\right]\right] = \frac{2\hbar^2 q\beta}{M^2c} p_1 p_2 p_3 \tag{4.20}$$

$$p_3 p_3\left[\frac{\vec{p}^2}{2M}, H_{\mathrm{I}}\right] = p_3\begin{pmatrix} \frac{i\hbar q\beta}{M^2c}(x_3 p_1 p_2 p_3 - i\hbar p_1 p_2) + \frac{i\hbar q}{M^2c}(\beta x_2 + b) p_1 p_3 p_3 \\ + \frac{2i\hbar q\beta}{M^2c} x_1 p_2 p_3 p_3 - \frac{i\hbar\mu\beta}{M}\sigma_2 p_2 p_3 + \frac{i\hbar\mu\beta}{M}\sigma_3 p_3 p_3 \end{pmatrix}$$

$$= \frac{i\hbar q\beta}{M^2 c}\left(x_3 p_1 p_2 p_3 p_3 - i\hbar p_1 p_2 p_3 - i\hbar p_1 p_2 p_3\right)$$
$$+ \frac{i\hbar q}{M^2 c}\left(\beta x_2 + b\right) p_1 p_3 p_3 p_3 + \frac{2i\hbar q\beta}{M^2 c} x_1 p_2 p_3 p_3 p_3$$
$$- \frac{i\hbar\mu\beta}{M}\sigma_2 p_2 p_3 p_3 + \frac{i\hbar\mu\beta}{M}\sigma_3 p_3 p_3 p_3$$

$$= \begin{pmatrix} \frac{i\hbar q\beta}{M^2 c} x_3 p_1 p_2 + \frac{i\hbar q}{M^2 c}\left(\beta x_2 + b\right) p_1 p_3 \\ + \frac{2i\hbar q\beta}{M^2 c} x_1 p_2 p_3 - \frac{i\hbar\mu\beta}{M}\sigma_2 p_2 + \frac{i\hbar\mu\beta}{M}\sigma_3 p_3 \end{pmatrix} p_3 p_3$$
$$+ \frac{2\hbar^2 q\beta}{M^2 c} p_1 p_2 p_3$$

$$= \left[\frac{\vec{p}^2}{2M}, H_{\mathrm{I}}\right] p_3 p_3 + \frac{2\hbar^2 q\beta}{M^2 c} p_1 p_2 p_3 \tag{4.21}$$

$$\left[p_3 p_3, \left[\frac{\vec{p}^2}{2M}, H_{\mathrm{I}}\right]\right] = \frac{2\hbar^2 q\beta}{M^2 c} p_1 p_2 p_3 \tag{4.22}$$

thus obtaining finally

$$\left[\frac{\vec{p}^2}{2M}, \left[\frac{\vec{p}^2}{2M}, H_{\mathrm{I}}\right]\right] = \frac{1}{2M}\left(\frac{4\hbar^2 q\beta}{M^2 c} p_1 p_2 p_3 + \frac{2\hbar^2 q\beta}{M^2 c} p_1 p_2 p_3 + \frac{2\hbar^2 q\beta}{M^2 c} p_1 p_2 p_3\right)$$
$$= \frac{4\hbar^2 q\beta}{M^3 c} p_1 p_2 p_3 . \tag{4.23}$$

Since the components of the momenta commute we have

$$\left[\frac{\vec{p}^2}{2M},\ldots,\left[\frac{\vec{p}^2}{2M},H_\mathrm{I}\right]\right]=0 \tag{4.24}$$

if the latter involves more than two factor of $\frac{\vec{p}^2}{2M}$. We also note

$$\left[H_\mathrm{I},\left[\frac{\vec{p}^2}{2M},H_\mathrm{I}\right]\right]=0 \tag{4.25}$$

is of higher order in $\sqrt{\alpha_q}$.

4.3 Quantum Dynamical Evolution of the System

With $\exp(-itH/\hbar)$ as the time-evolution operator, the following expectation values of the Heisenberg operators in the state Eq. (2.1), relevant to the observation screen, to the leading order in $\sqrt{\alpha_q}$, are readily obtained:

$$\langle x_1(t)\rangle = 0 \tag{4.26}$$

$$\langle x_3(t)\rangle = \frac{\mu\beta}{2M}\sigma_3 t^2 \tag{4.27}$$

and the important non-trivial correlation occurring between the dynamical variables $x_1(t),\ x_3(t)$:

$$\left\langle \left(x_1(t)-\left\langle x_1(t)\right\rangle\right)\left(x_3(t)-\left\langle x_3(t)\right\rangle\right)\right\rangle = -\frac{qbt\gamma^2}{Mc}+\frac{q\beta p_0 t^2\gamma^2}{2M^2c}+\frac{q\beta p_0 t^4\hbar^2}{24M^4\gamma^2 c} \tag{4.28}$$

with

$$\left(\left\langle \left(x_1(t)-\left\langle x_1(t)\right\rangle\right)^2\right\rangle\right)^{1/2} = \left(\left\langle \left(x_3(t)-\left\langle x_3(t)\right\rangle\right)^2\right\rangle\right)^{1/2} = \gamma(t) \tag{4.29}$$

and

$$\gamma(t)=\gamma\left(1+\frac{\hbar^2 t^2}{4M^2\gamma^4}\right)^{1/2}. \tag{4.30}$$

We use a variation of the Baker-Campbell-Hausdorff formula which states that if

$$\left[B,\left[A,B\right]\right]=0 \tag{4.31}$$

$$\left[B,\left[A,\left[A,B\right]\right]\right]=0 \tag{4.32}$$

$$\left[A,\left[A,\left[A,B\right]\right]\right]=0 \tag{4.33}$$

for two operators A and B, then

$$e^{A+B}=e^{\left(\frac{1}{2}[A,B]+\frac{1}{6}[A,[A,B]]\right)}e^B e^A. \tag{4.34}$$

We set

$$A = -\frac{it}{\hbar}\frac{\vec{p}^2}{2M} \tag{4.35}$$

$$B = -\frac{it}{\hbar}H_{\mathrm{I}} \tag{4.36}$$

to obtain

$$\begin{aligned}\exp\left(-\frac{it}{\hbar}H\right) &= \exp\left(-\frac{it}{\hbar}\left(\frac{\vec{p}^2}{2M}+H_{\mathrm{I}}\right)\right)\\ &= \exp\left(-\frac{t^2}{2\hbar^2}\left[\frac{\vec{p}^2}{2M},H_{\mathrm{I}}\right]\right)\exp\left(\frac{2it^3q\beta}{3\hbar M^3c}p_1p_2p_3\right)\\ &\quad\exp\left(-\frac{it}{\hbar}H_{\mathrm{I}}\right)\exp\left(-\frac{it}{\hbar}H_0\right)\end{aligned} \tag{4.37}$$

where $H_0 = \frac{\vec{p}^2}{2M}$.

Accordingly, for the time-developed state we have to evaluate the expression

$$\begin{aligned}\exp\left(-\frac{it}{\hbar}H\right)\Psi(\vec{x}) &= \exp\left(-\frac{t^2}{2\hbar^2}\left[\frac{\vec{p}^2}{2M},H_{\mathrm{I}}\right]\right)\exp\left(\frac{2it^3q\beta}{3\hbar M^3c}p_1p_2p_3\right).\\ &\quad\exp\left(-\frac{it}{\hbar}H_{\mathrm{I}}\right)\exp\left(-\frac{it}{\hbar}H_0\right)\Psi(\vec{x})\end{aligned} \tag{4.38}$$

But from Eq. (2.20),

$$\Psi_0(\vec{x},t)=\frac{e^{\frac{i}{\hbar}\vec{p}_0\cdot\vec{x}}e^{-\frac{i\vec{p}_0^2 t}{2M\hbar}}}{(2\pi)^{3/4}\gamma^{3/2}\left(1+\frac{iht}{2M\gamma^2}\right)^{3/2}}\exp\left(-\frac{\left(\vec{x}-\frac{\vec{p}_0}{M}t\right)^2}{4\gamma^2\left(1+\frac{iht}{2M\gamma^2}\right)}\right). \tag{4.39}$$

For the subsequent analysis, we set

$$F=\frac{1}{4\gamma^2\left(1+\frac{i\hbar t}{2M\gamma^2}\right)}. \tag{4.40}$$

To carry out the time-evolution operation on Ψ, we use the identities

$$e^{ia\frac{p}{\hbar}}f(x)=f(x+a) \tag{4.41}$$

$$\begin{aligned}e^{ia\frac{p}{\hbar}}f(x)g(x)&=f(x+a)e^{ia\frac{p}{\hbar}}g(x)\\&=f(x+a)g(x+a)\end{aligned} \tag{4.42}$$

$$e^{a\frac{p}{\hbar}}f(x)=e^{i(-ia)\frac{p}{\hbar}}f(x)=f(x-ia). \tag{4.43}$$

Hence to the leading order in the exponential we have

$$\exp\left(-\frac{it}{\hbar}H_{\mathrm{I}}\right)\exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-x_1^2F\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)\exp\left(-x_3^2F\right)\equiv\boxed{1} \tag{4.44}$$

where H_{I} is defined by Eq. (3.40) and we set

$$\exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-x_1^2F\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)\exp\left(-x_3^2F\right)=f\left(x_1,x_2,x_3\right). \qquad (4.45)$$

In details, we have

$$\boxed{1}=\exp\left(\frac{it}{\hbar}\mu\left(\sigma_2\left(b-\beta x_2\right)+\sigma_3\beta x_3\right)\right)$$
$$\exp\left(\frac{it}{\hbar}\frac{q}{Mc}\left(bx_3p_1+\beta x_1x_3p_2+\beta x_1x_2p_3\right)\right)f\left(x_1,x_2,x_3\right)$$

$$=\exp\left(\frac{it}{\hbar}\mu\left(\sigma_2\left(b-\beta x_2\right)+\sigma_3\beta x_3\right)\right)\exp\left(\frac{it}{\hbar}\frac{q}{Mc}\left(bx_3p_1+\beta x_1x_3p_2\right)\right)$$
$$\exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-x_1^2F\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-\left(x_3+\frac{tq\beta}{Mc}x_1x_2\right)^2F\right)$$

$$=\exp\left(\frac{it}{\hbar}\mu\left(\sigma_2\left(b-\beta x_2\right)+\sigma_3\beta x_3\right)\right)\exp\left(\frac{it}{\hbar}\frac{q}{Mc}\left(bx_3p_1+\beta x_1x_3p_2\right)\right)$$
$$\exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-x_1^2F\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-x_3^2F\right)\exp\left(-\frac{2tq\beta}{Mc}x_1x_2x_3F\right)$$

$$=\exp\left(\frac{it}{\hbar}\mu\left(\sigma_2\left(b-\beta x_2\right)+\sigma_3\beta x_3\right)\right)\exp\left(\frac{it}{\hbar}\frac{q}{Mc}\left(bx_3p_1\right)\right)$$
$$\exp\left(\frac{ip_0}{\hbar}\left(x_2+\frac{tq\beta}{Mc}x_1x_3\right)\right)\exp\left(-x_1^2F\right)$$
$$\exp\left(-\left(x_2+\frac{tq\beta}{Mc}x_1x_3-\frac{p_0}{M}t\right)^2F\right)\exp\left(-x_3^2F\right)$$
$$\exp\left(-\frac{2tq\beta}{Mc}x_1\left(x_2+\frac{tq\beta}{Mc}x_1x_3\right)x_3F\right)$$

$$= \exp\left(\frac{it}{\hbar}\mu\left(\sigma_2\left(b-\beta x_2\right)+\sigma_3\beta x_3\right)\right)\exp\left(\frac{it}{\hbar}\frac{q}{Mc}\left(bx_3p_1\right)\right)$$
$$\exp\left(\frac{ip_0}{\hbar}\left(x_2+\frac{tq\beta}{Mc}x_1x_3\right)\right)\exp\left(-x_1^2F\right)$$
$$\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)\exp\left(-\frac{2tq\beta}{Mc}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F\right)$$
$$\exp\left(-x_3^2F\right)\exp\left(-\frac{2tq\beta}{Mc}x_1x_2x_3F\right)$$

$$= \exp\left(\frac{it}{\hbar}\mu\left(\sigma_2\left(b-\beta x_2\right)+\sigma_3\beta x_3\right)\right)\exp\left(\frac{ip_0}{\hbar}\left(x_2+\frac{tq\beta}{Mc}\left(x_1+\frac{tqb}{Mc}x_3\right)x_3\right)\right)$$
$$\exp\left(-\left(x_1+\frac{tqb}{Mc}x_3\right)^2F\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-\frac{2tq\beta}{Mc}\left(x_1+\frac{tqb}{Mc}x_3\right)\left(x_2-\frac{p_0}{M}t\right)x_3F\right)$$
$$\exp\left(-x_3^2F\right)\exp\left(-\frac{2tq\beta}{Mc}\left(x_1+\frac{tqb}{Mc}x_3\right)x_2x_3F\right)$$

$$= \exp\left(\frac{it}{\hbar}\mu\left(\sigma_2\left(b-\beta x_2\right)+\sigma_3\beta x_3\right)\right)\exp\left(\frac{ip_0}{\hbar}x_2\right)$$
$$\exp\left(\frac{itq\beta p_0}{\hbar Mc}x_1x_3\right)\exp\left(-x_1^2F\right)$$
$$\exp\left(-\frac{2tqb}{Mc}x_1x_3F\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-\frac{2tq\beta}{Mc}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F\right)\exp\left(-x_3^2F\right)$$
$$\exp\left(-\frac{2tq\beta}{Mc}x_1x_2x_3F\right)$$

$$= f\left(x_1,x_2,x_3\right)\exp\left(\frac{it}{\hbar}\mu\left(\sigma_2\left(b-\beta x_2\right)+\sigma_3\beta x_3\right)\right)$$
$$\exp\left(\frac{itq\beta p_0}{\hbar Mc}x_1x_3\right)\exp\left(-\frac{2tqb}{Mc}x_1x_3F\right)$$
$$\exp\left(-\frac{2tq\beta}{Mc}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F\right)\exp\left(-\frac{2tq\beta}{Mc}x_1x_2x_3F\right)$$

$$= f\left(x_1, x_2, x_3\right)\exp\begin{bmatrix} \frac{it}{\hbar}\mu\left(\sigma_2\left(b-\beta\frac{p_0}{M}t\right)+\sigma_3\beta x_3\right) \\ -\frac{it\mu\beta\sigma_2}{\hbar}\left(x_2-\frac{p_0}{M}t\right)+\frac{itq\beta p_0}{\hbar Mc}x_1x_3 \\ -\frac{2tqb}{Mc}x_1x_3F-\frac{2tq\beta}{Mc}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F \\ -\frac{2tq\beta}{Mc}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F-\frac{2tq\beta}{Mc}\frac{p_0}{M}tx_1x_3F \end{bmatrix} \tag{4.46}$$

$$\boxed{1} = f\left(x_1, x_2, x_3\right)\exp\begin{bmatrix} \frac{it}{\hbar}\mu\left(\sigma_2\left(b-\beta\frac{p_0}{M}t\right)+\sigma_3\beta x_3\right) \\ -\frac{it\mu\beta\sigma_2}{\hbar}\left(x_2-\frac{p_0}{M}t\right)+\frac{itq\beta p_0}{\hbar Mc}x_1x_3 \\ -\frac{2tqb}{Mc}x_1x_3F-\frac{4tq\beta}{Mc}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F \\ -\frac{2t^2q\beta p_0}{M^2c}x_1x_3F \end{bmatrix} \tag{4.47}$$

$$\exp\frac{2it^3q\beta\hbar^2}{3M^3c}\frac{p_1}{\hbar}\frac{p_2}{\hbar}\frac{p_3}{\hbar}f\left(x_1, x_2, x_3\right) \equiv \boxed{2} \tag{4.48}$$

where the operator on the left-hand side of Eq. (4.48) gives

$$\boxed{2} = \exp\left(\frac{ip_0}{\hbar}\left(x_2+\frac{2t^3q\beta\hbar^2}{3M^3c}\frac{p_1}{\hbar}\frac{p_3}{\hbar}\right)\right)\exp\left(-\left(x_2+\frac{2t^3q\beta\hbar^2}{3M^3c}\frac{p_1}{\hbar}\frac{p_3}{\hbar}-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-\left(x_1+\frac{2t^3q\beta\hbar^2}{3M^3c}\frac{p_2}{\hbar}\frac{p_3}{\hbar}\right)^2F\right)\exp\left(-\left(x_3+\frac{2t^3q\beta\hbar^2}{3M^3c}\frac{p_1}{\hbar}\frac{p_2}{\hbar}\right)^2F\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(\frac{ip_0}{\hbar}\frac{2t^3q\beta\hbar^2}{3M^3c}\frac{p_1}{\hbar}\frac{p_3}{\hbar}\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-\frac{4t^3q\beta\hbar^2}{3M^3c}\left(x_2-\frac{p_0}{M}t\right)\frac{p_1}{\hbar}\frac{p_3}{\hbar}F\right)$$
$$\exp\left(-x_1^2F\right)\exp\left(-\frac{4t^3q\beta\hbar^2}{3M^3c}x_1\frac{p_2}{\hbar}\frac{p_3}{\hbar}F\right)$$
$$\exp\left(-x_3^2F\right)\exp\left(-\frac{4t^3q\beta\hbar^2}{3M^3c}x_3\frac{p_1}{\hbar}\frac{p_2}{\hbar}F\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)\exp\left(\frac{ip_0}{\hbar}\frac{2t^3q\beta\hbar^2}{3M^3c}\frac{p_1}{\hbar}\frac{p_3}{\hbar}\right)$$
$$\exp\left(-\left(x_1+\frac{4it^3q\beta\hbar^2}{3M^3c}\left(x_2-\frac{p_0}{M}t\right)\frac{p_3}{\hbar}F\right)^2F\right)$$
$$\exp\left(-\left(x_3+\frac{4it^3q\beta\hbar^2}{3M^3c}x_1\frac{p_2}{\hbar}F\right)^2F\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)\exp\left(\frac{ip_0}{\hbar}\frac{2t^3q\beta\hbar^2}{3M^3c}\frac{p_1}{\hbar}\frac{p_3}{\hbar}\right)$$
$$\exp\left(-x_1^2F\right)\exp\left(-\frac{8it^3q\beta\hbar^2}{3M^3c}x_1\left(x_2-\frac{p_0}{M}t\right)\frac{p_3}{\hbar}F^2\right)$$
$$\exp\left(-x_3^2F\right)\exp\left(-\frac{8it^3q\beta\hbar^2}{3M^3c}x_1x_3\frac{p_2}{\hbar}F^2\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-\left(x_1+\frac{p_0}{\hbar}\frac{2t^3q\beta\hbar^2}{3M^3c}\frac{p_3}{\hbar}\right)^2F\right)\exp\left(\frac{ip_0}{\hbar}\frac{2t^3q\beta\hbar^2}{3M^3c}\frac{p_1}{\hbar}\frac{p_3}{\hbar}\right)$$
$$\exp\left(-\left(x_3-\frac{8t^3q\beta\hbar^2}{3M^3c}x_1\left(x_2-\frac{p_0}{M}t\right)F^2\right)^2F\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2 F\right)$$
$$\exp\left(-x_1^2F\right)\exp\left(-\frac{p_0}{\hbar}\frac{4t^3q\beta\hbar^2}{3M^3c}x_1\frac{p_3}{\hbar}F\right)$$
$$\exp\left(-\left(x_3+\frac{p_0}{\hbar}\frac{2t^3q\beta\hbar^2}{3M^3c}\frac{p_1}{\hbar}F\right)^2 F\right)$$
$$\exp\left(\frac{16t^3q\beta\hbar^2}{3M^3c}x_1\left(x_2-\frac{p_0}{M}t\right)\left(x_3+\frac{p_0}{\hbar}\frac{2t^3q\beta\hbar^2}{3M^3c}\frac{p_1}{\hbar}F\right)F^3\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2 F\right)$$
$$\exp\left(-x_1^2F\right)\exp\left(-\frac{p_0}{\hbar}\frac{4t^3q\beta\hbar^2}{3M^3c}x_1\frac{p_3}{\hbar}F\right)$$
$$\exp\left(-x_3^2F\right)\exp\left(-\frac{p_0}{\hbar}\frac{4t^3q\beta\hbar^2}{3M^3c}x_3\frac{p_1}{\hbar}F^2\right)$$
$$\exp\left(\frac{16t^3q\beta\hbar^2}{3M^3c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^3\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2 F\right)$$
$$\exp\left(-x_1^2F\right)\exp\left(-\left(x_3+\frac{p_0}{\hbar}\frac{4it^3q\beta\hbar^2}{3M^3c}x_1F\right)^2 F\right)$$
$$\exp\left(-\frac{p_0}{\hbar}\frac{4t^3q\beta\hbar^2}{3M^3c}x_1\frac{p_3}{\hbar}F\right)$$
$$\exp\left(\frac{16t^3q\beta\hbar^2}{3M^3c}\left(x_1+\frac{p_0}{\hbar}\frac{4it^3q\beta\hbar^2}{3M^3c}x_3F^2\right)\left(x_2-\frac{p_0}{M}t\right)x_3F^3\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-x_1^2F\right)\exp\left(-x_3^2F\right)\exp\left(-\frac{8ip_0t^3q\beta\hbar}{3M^3c}x_1x_3F^2\right)$$
$$\exp\left(-\frac{p_0}{\hbar}\frac{4t^3q\beta\hbar^2}{3M^3c}x_1\frac{p_3}{\hbar}F\right)$$
$$\exp\left(\frac{16t^3q\beta\hbar^2}{3M^3c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^3\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-x_1^2F\right)\exp\left(-x_3^2F\right)\exp\left(-\frac{8ip_0t^3q\beta\hbar}{3M^3c}x_1x_3F^2\right)$$
$$\exp\left(\frac{16t^3q\beta\hbar^2}{3M^3c}x_1\left(x_2-\frac{p_0}{M}t\right)\left(x_3+\frac{p_0}{\hbar}\frac{4it^3q\beta\hbar^2}{3M^3c}x_1F\right)F^3\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-x_1^2F\right)\exp\left(-x_3^2F\right)\exp\left(-\frac{8ip_0t^3q\beta\hbar}{3M^3c}x_1x_3F^2\right) \tag{4.49}$$
$$\exp\left(\frac{16t^3q\beta\hbar^2}{3M^3c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^3\right)$$

$$\boxed{2} = f\left(x_1,x_2,x_3\right)\exp\left(-\frac{8ip_0t^3q\beta\hbar}{3M^3c}x_1x_3F^2\right)$$
$$\exp\left(\frac{16t^3q\beta\hbar^2}{3M^3c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^3\right) \tag{4.50}$$

$$\exp\left(-\frac{t^2}{2\hbar^2}\left[\frac{\vec{p}^2}{2M},H_{\mathrm{I}}\right]\right)f\left(x_1,x_2,x_3\right)$$

$$=\exp\begin{pmatrix}-\frac{i\hbar q\beta t^2}{2M^2c}x_3\frac{p_1}{\hbar}\frac{p_2}{\hbar}-\frac{i\hbar qt^2}{2M^2c}\left(\beta x_2+b\right)\frac{p_1}{\hbar}\frac{p_3}{\hbar}\\ -\frac{i\hbar q\beta t^2}{M^2c}x_1\frac{p_2}{\hbar}\frac{p_3}{\hbar}+\frac{i\mu\beta t^2}{2M}\sigma_2\frac{p_2}{\hbar}-\frac{i\mu\beta t^2}{2M}\sigma_3\frac{p_3}{\hbar}\end{pmatrix}f\left(x_1,x_2,x_3\right)\equiv\boxed{3} \tag{4.51}$$

thus defining the expression in question denoted by $\boxed{3}$. Also introducing

$$\boxed{A}=\exp\left(-\frac{i\mu\beta t^2}{2M}\sigma_3\frac{p_3}{\hbar}\right)f\left(x_1,x_2,x_3\right) \tag{4.52}$$

we have

$$\boxed{A}=\exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)\exp\left(-x_1^2F\right)$$
$$\exp\left(-\left(x_3-\frac{\mu\beta t^2}{2M}\sigma_3\right)^2F\right)$$

$$=\exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)\exp\left(-x_1^2F\right)$$
$$\exp\left(-x_3^2F\right)\exp\left(\frac{\mu\beta t^2}{M}\sigma_3x_3F\right)$$

$$=f\left(x_1,x_2,x_3\right)\exp\left(\frac{\mu\beta t^2}{M}\sigma_3x_3F\right). \tag{4.53}$$

Similarly for

$$\boxed{B} = \exp\left(\frac{i\mu\beta t^2}{2M}\sigma_2\frac{p_2}{\hbar}\right)f\left(x_1,x_2,x_3\right) \tag{4.54}$$

we obtain

$$\begin{aligned}
\boxed{B} &= \exp\left(\frac{ip_0}{\hbar}\left(x_2+\frac{\mu\beta t^2}{2M}\sigma_2\right)\right)\exp\left(-\left(x_2+\frac{\mu\beta t^2}{2M}\sigma_2-\frac{p_0}{M}t\right)^2F\right)\\
&\quad\exp\left(-x_1^2F\right)\exp\left(-x_3^2F\right)\\
&= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(\frac{ip_0}{\hbar}\frac{\mu\beta t^2}{2M}\sigma_2\right)\\
&\quad\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)\exp\left(-\frac{\mu\beta t^2}{M}\sigma_2\left(x_2-\frac{p_0}{M}t\right)F\right)\\
&\quad\exp\left(-x_1^2F\right)\exp\left(-x_3^2F\right)\\
&= f\left(x_1,x_2,x_3\right)\exp\left(\frac{ip_0\mu\beta t^2}{2\hbar M}\sigma_2\right)\exp\left(-\frac{\mu\beta t^2}{M}\sigma_2\left(x_2-\frac{p_0}{M}t\right)F\right)
\end{aligned} \tag{4.55}$$

and for

$$\boxed{C} = \exp\left(-\frac{i\hbar q\beta t^2}{M^2c}x_1\frac{p_2}{\hbar}\frac{p_3}{\hbar}\right)f\left(x_1,x_2,x_3\right) \tag{4.56}$$

we have

$$\boxed{C} = \exp\left(\frac{ip_0}{\hbar}\left(x_2 - \frac{\hbar q\beta t^2}{M^2 c}x_1\frac{p_3}{\hbar}\right)\right)\exp\left(-\left(x_2 - \frac{\hbar q\beta t^2}{M^2 c}x_1\frac{p_3}{\hbar} - \frac{p_0}{M}t\right)^2 F\right)$$

$$\exp\left(-x_1^2 F\right)\exp\left(-\left(x_3 - \frac{\hbar q\beta t^2}{M^2 c}x_1\frac{p_2}{\hbar}\right)^2 F\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\frac{ip_0}{\hbar}\frac{\hbar q\beta t^2}{M^2 c}x_1\frac{p_3}{\hbar}\right)$$

$$\exp\left(-\left(x_2 - \frac{p_0}{M}t\right)^2 F\right)\exp\left(\frac{2\hbar q\beta t^2}{M^2 c}x_1\left(x_2 - \frac{p_0}{M}t\right)\frac{p_3}{\hbar}F\right)$$

$$\exp\left(-x_1^2 F\right)\exp\left(-x_3^2 F\right)\exp\left(\frac{2\hbar q\beta t^2}{M^2 c}x_1 x_3\frac{p_2}{\hbar}F\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2 - \frac{p_0}{M}t\right)^2 F\right)$$

$$\exp\left(-x_1^2 F\right)\exp\left(-\frac{ip_0}{\hbar}\frac{\hbar q\beta t^2}{M^2 c}x_1\frac{p_3}{\hbar}\right)$$

$$\exp\left(-\left(x_3 - \frac{2i\hbar q\beta t^2}{M^2 c}x_1\left(x_2 - \frac{p_0}{M}t\right)F\right)^2 F\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2 - \frac{p_0}{M}t\right)^2 F\right)$$

$$\exp\left(-x_1^2 F\right)\exp\left(-\frac{ip_0}{\hbar}\frac{\hbar q\beta t^2}{M^2 c}x_1\frac{p_3}{\hbar}\right)$$

$$\exp\left(-x_3^2 F\right)\exp\left(\frac{4i\hbar q\beta t^2}{M^2 c}x_1\left(x_2 - \frac{p_0}{M}t\right)x_3 F^2\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2 - \frac{p_0}{M}t\right)^2 F\right)$$

$$\exp\left(-x_1^2 F\right)\exp\left(-\left(x_3 - \frac{p_0 q\beta t^2}{M^2 c}x_1\right)^2 F\right)$$

$$\exp\left(\frac{4i\hbar q\beta t^2}{M^2 c}x_1\left(x_2 - \frac{p_0}{M}t\right)\left(x_3 - \frac{p_0 q\beta t^2}{M^2 c}x_1\right)F^2\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-x_1^2F\right)\exp\left(-x_3^2F\right)\exp\left(\frac{2p_0q\beta t^2}{M^2c}x_1x_3F\right)$$
$$\exp\left(\frac{4i\hbar q\beta t^2}{M^2c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^2\right)$$

$$= f\left(x_1,x_2,x_3\right)\exp\left(\frac{2p_0q\beta t^2}{M^2c}x_1x_3F\right)\exp\left(\frac{4i\hbar q\beta t^2}{M^2c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^2\right). \quad (4.57)$$

Also the definition

$$\boxed{D} = \exp\left(-\frac{i\hbar qt^2}{2M^2c}\left(\beta x_2+b\right)\frac{p_1}{\hbar}\frac{p_3}{\hbar}\right)f\left(x_1,x_2,x_3\right) \quad (4.58)$$

leads to

$$\boxed{D} = \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-\left(x_1-\frac{\hbar qt^2}{2M^2c}\left(\beta x_2+b\right)\frac{p_3}{\hbar}\right)^2F\right)$$
$$\exp\left(-\left(x_3-\frac{\hbar qt^2}{2M^2c}\left(\beta x_2+b\right)\frac{p_1}{\hbar}\right)^2F\right)$$

$$
\begin{aligned}
&= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2 F\right)\\
&\quad \exp\left(-x_1^2F\right)\exp\left(\frac{\hbar q t^2}{M^2c}x_1\left(\beta x_2+b\right)\frac{p_3}{\hbar}F\right)\\
&\quad \exp\left(-x_3^2F\right)\exp\left(\frac{\hbar q t^2}{M^2c}\left(\beta x_2+b\right)x_3\frac{p_1}{\hbar}F\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2 F\right)\exp\left(-x_1^2F\right)\\
&\quad \exp\left(-\left(x_3-\frac{i\hbar q t^2}{M^2c}x_1\left(\beta x_2+b\right)F\right)^2 F\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2 F\right)\exp\left(-x_1^2F\right)\\
&\quad \exp\left(-x_3^2F\right)\exp\left(\frac{2i\hbar q t^2}{M^2c}x_1\left(\beta x_2+b\right)x_3F^2\right)
\end{aligned}
$$

$$
= f\left(x_1,x_2,x_3\right)\exp\left(\frac{2i\hbar q t^2}{M^2c}\begin{pmatrix}\beta x_1\left(x_2-\frac{p_0}{M}t\right)x_3\\ +bx_1x_3+\beta\frac{p_0}{M}tx_1x_3\end{pmatrix}F^2\right)
$$

$$
\begin{aligned}
&= f\left(x_1,x_2,x_3\right)\exp\left(\frac{2i\hbar q t^2}{M^2c}\left(\beta\frac{p_0}{M}t+b\right)x_1x_3F^2\right)\\
&\quad \exp\left(\frac{2i\hbar q\beta t^2}{M^2c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^2\right)
\end{aligned}. \tag{4.59}
$$

For the expression

$$
\boxed{E} = \exp\left(-\frac{i\hbar q\beta t^2}{2M^2c}x_3\frac{p_1}{\hbar}\frac{p_2}{\hbar}\right)f\left(x_1,x_2,x_3\right) \tag{4.60}
$$

we have

$$\boxed{E} = \exp\left(\frac{ip_0}{\hbar}\left(x_2 - \frac{\hbar q\beta t^2}{2M^2c}x_3\frac{p_1}{\hbar}\right)\right)\exp\left(-\left(x_2 - \frac{\hbar q\beta t^2}{2M^2c}x_3\frac{p_1}{\hbar} - \frac{p_0}{M}t\right)^2 F\right)$$
$$\exp\left(-\left(x_1 - \frac{\hbar q\beta t^2}{2M^2c}x_3\frac{p_2}{\hbar}\right)^2 F\right)\exp\left(-x_3^2 F\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(\left(-\frac{ip_0}{\hbar}\frac{\hbar q\beta t^2}{2M^2c}x_3\frac{p_1}{\hbar}\right)\right)$$
$$\exp\left(-\left(x_2 - \frac{p_0}{M}t\right)^2 F\right)\exp\left(\frac{\hbar q\beta t^2}{M^2c}\left(x_2 - \frac{p_0}{M}t\right)x_3\frac{p_1}{\hbar}F\right)$$
$$\exp\left(-x_1^2 F\right)\exp\left(\frac{\hbar q\beta t^2}{M^2c}x_1x_3\frac{p_2}{\hbar}F\right)\exp\left(-x_3^2 F\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2 - \frac{p_0}{M}t\right)^2 F\right)\exp\left(\left(-\frac{ip_0q\beta t^2}{2M^2c}x_3\frac{p_1}{\hbar}\right)\right)$$
$$\exp\left(-\left(x_1 - \frac{i\hbar q\beta t^2}{M^2c}\left(x_2 - \frac{p_0}{M}t\right)x_3F\right)^2 F\right)\exp\left(-x_3^2 F\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2 - \frac{p_0}{M}t\right)^2 F\right)\exp\left(\left(-\frac{ip_0q\beta t^2}{2M^2c}x_3\frac{p_1}{\hbar}\right)\right)$$
$$\exp\left(-x_1^2 F\right)\exp\left(\frac{2i\hbar q\beta t^2}{M^2c}x_1\left(x_2 - \frac{p_0}{M}t\right)x_3F^2\right)\exp\left(-x_3^2 F\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2 - \frac{p_0}{M}t\right)^2 F\right)\exp\left(-\left(x_1 - \frac{p_0q\beta t^2}{2M^2c}x_3\right)^2 F\right)$$
$$\exp\left(\frac{2i\hbar q\beta t^2}{M^2c}\left(x_1 - \frac{p_0q\beta t^2}{2M^2c}x_3\right)\left(x_2 - \frac{p_0}{M}t\right)x_3F^2\right)\exp\left(-x_3^2 F\right)$$

$$= \exp\left(\frac{ip_0}{\hbar}x_2\right)\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)$$
$$\exp\left(-x_1^2F\right)\exp\left(\frac{p_0q\beta t^2}{M^2c}x_1x_3F\right)$$
$$\exp\left(\frac{2i\hbar q\beta t^2}{M^2c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^2\right)\exp\left(-x_3^2F\right)$$

$$= f\left(x_1,x_2,x_3\right)\exp\left(\frac{p_0q\beta t^2}{M^2c}x_1x_3F\right)\exp\left(\frac{2i\hbar q\beta t^2}{M^2c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^2\right) \quad (4.61)$$

leading for the expression in Eq. (4.51) the result

$$\boxed{3} = f\left(x_1,x_2,x_3\right)\exp\left(\frac{\mu\beta t^2}{M}\sigma_3x_3F\right)$$
$$\exp\left(\frac{ip_0\mu\beta t^2}{2\hbar M}\sigma_2\right)\exp\left(-\frac{\mu\beta t^2}{M}\sigma_2\left(x_2-\frac{p_0}{M}t\right)F\right)$$
$$\exp\left(\frac{2p_0q\beta t^2}{M^2c}x_1x_3F\right)\exp\left(\frac{4i\hbar q\beta t^2}{M^2c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^2\right)$$
$$\exp\left(\frac{2i\hbar qt^2}{M^2c}\left(\beta\frac{p_0}{M}t+b\right)x_1x_3F^2\right)\exp\left(\frac{2i\hbar q\beta t^2}{M^2c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^2\right)$$
$$\exp\left(\frac{p_0q\beta t^2}{M^2c}x_1x_3F\right)\exp\left(\frac{2i\hbar q\beta t^2}{M^2c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^2\right)$$

$$= f\left(x_1,x_2,x_3\right)\exp\left(\frac{\mu\beta t^2}{M}\sigma_3x_3F\right)$$
$$\exp\left(\frac{ip_0\mu\beta t^2}{2\hbar M}\sigma_2\right)\exp\left(-\frac{\mu\beta t^2}{M}\sigma_2\left(x_2-\frac{p_0}{M}t\right)F\right)$$
$$\exp\left(\frac{3p_0q\beta t^2}{M^2c}x_1x_3F\right)\exp\left(\frac{8i\hbar q\beta t^2}{M^2c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^2\right). \quad (4.62)$$
$$\exp\left(\frac{2i\hbar qt^2}{M^2c}\left(\beta\frac{p_0}{M}t+b\right)x_1x_3F^2\right)$$

Collecting the above terms together, we obtain

$$
\begin{aligned}
e^{-\frac{it}{\hbar}H}\Psi\left(\vec{x}\right) = {} & \exp\left(\frac{it}{\hbar}\mu\left(\sigma_2\left(b-\beta\frac{p_0}{M}t\right)+\sigma_3\beta x_3\right)\right) \\
& \exp\left(-\frac{it\mu\beta\sigma_2}{\hbar}\left(x_2-\frac{p_0}{M}t\right)\right)\exp\left(\frac{itq\beta p_0}{\hbar Mc}x_1x_3\right) \\
& \exp\left(-\frac{2tqb}{Mc}x_1x_3F\right)\exp\left(-\frac{4tq\beta}{Mc}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F\right) \\
& \exp\left(-\frac{2t^2q\beta p_0}{M^2c}x_1x_3F\right)\exp\left(-\frac{8ip_0t^3q\beta\hbar}{3M^3c}x_1x_3F^2\right) \\
& \exp\left(\frac{16t^3q\beta\hbar^2}{3M^3c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^3\right)\exp\left(\frac{\mu\beta t^2}{M}\sigma_3x_3F\right) \\
& \exp\left(\frac{ip_0\mu\beta t^2}{2\hbar M}\sigma_2\right)\exp\left(-\frac{\mu\beta t^2}{M}\sigma_2\left(x_2-\frac{p_0}{M}t\right)F\right) \\
& \exp\left(\frac{3p_0q\beta t^2}{M^2c}x_1x_3F\right)\exp\left(\frac{8i\hbar q\beta t^2}{M^2c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^2\right) \\
& \exp\left(\frac{2i\hbar qt^2}{M^2c}\left(\beta\frac{p_0}{M}t+b\right)x_1x_3F^2\right) \\
& \exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)\exp\left(-\left(x_1^2+x_3^2\right)F\right)
\end{aligned}
$$

which simplifies to

$$e^{-\frac{it}{\hbar}H}\Psi(\vec{x})=\begin{bmatrix}\exp\left(-\left(x_2-\frac{p_0}{M}t\right)^2F\right)\exp\left(-\frac{4tq\beta}{Mc}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F\right)\\ \exp\left(\frac{16q\beta\hbar^2t^3}{3M^3c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^3\right)\\ \exp\left(\frac{8i\hbar q\beta t^2}{M^2c}x_1\left(x_2-\frac{p_0}{M}t\right)x_3F^2\right)\end{bmatrix}$$

$$\begin{bmatrix}\exp\left(\frac{it\mu}{\hbar}\sigma_2\left(b-\beta\frac{p_0}{M}t\right)\right)\exp\left(-\frac{it\mu\beta\sigma_2}{\hbar}\left(x_2-\frac{p_0}{M}t\right)\right)\\ \exp\left(\frac{ip_0\mu\beta t^2}{2\hbar M}\sigma_2\right)\exp\left(-\frac{\mu\beta t^2}{M}\sigma_2\left(x_2-\frac{p_0}{M}t\right)F\right)\end{bmatrix}$$

$$\begin{bmatrix}\exp\left(\frac{itq\beta p_0}{\hbar Mc}x_1x_3\right)\exp\left(\frac{p_0q\beta t^2}{M^2c}x_1x_3F\right)\\ \exp\left(-\frac{2tqb}{Mc}x_1x_3F\right)\exp\left(-\frac{8ip_0t^3q\beta\hbar}{3M^3c}x_1x_3F^2\right)\\ \exp\left(\frac{\mu\beta t^2}{M}\sigma_3x_3F\right)\exp\left(\frac{2i\hbar qt^2}{M^2c}\left(\beta\frac{p_0}{M}t+b\right)x_1x_3F^2\right)\\ \exp\left(\frac{it\mu\beta}{\hbar}\sigma_3x_3\right)\exp\left(-\left(x_1^2+x_3^2\right)F\right)\end{bmatrix}. \quad (4.63)$$

4.4 Explicit Analytical Expression for the Intensity Distribution

Here we have to evaluate the expression

$$\begin{aligned}\langle\vec{x}|\rho(t)|\vec{x}\rangle=w_+&\left[e^{-\frac{it}{\hbar}H}\begin{pmatrix}1\\0\end{pmatrix}\Psi(\vec{x})\right]^\dagger\left[e^{-\frac{it}{\hbar}H}\begin{pmatrix}1\\0\end{pmatrix}\Psi(\vec{x})\right]\\ +w_-&\left[e^{-\frac{it}{\hbar}H}\begin{pmatrix}0\\1\end{pmatrix}\Psi(\vec{x})\right]^\dagger\left[e^{-\frac{it}{\hbar}H}\begin{pmatrix}0\\1\end{pmatrix}\Psi(\vec{x})\right]\end{aligned} \quad (4.64)$$

and then integrate over x_2 to obtain the probability density in question in Eq. (4.7).

To the above end we use the identities:

$$\sigma_3\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},\ \sigma_3\begin{pmatrix}0\\1\end{pmatrix}=-\begin{pmatrix}0\\1\end{pmatrix} \tag{4.65}$$

$$\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\i\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\-i\end{pmatrix} \tag{4.66}$$

$$\begin{pmatrix}0\\1\end{pmatrix}=-\frac{i}{2}\begin{pmatrix}1\\i\end{pmatrix}+\frac{i}{2}\begin{pmatrix}1\\-i\end{pmatrix} \tag{4.67}$$

$$\sigma_2\begin{pmatrix}1\\i\end{pmatrix}=\begin{pmatrix}1\\i\end{pmatrix},\ \sigma_2\begin{pmatrix}1\\-i\end{pmatrix}=-\begin{pmatrix}1\\-i\end{pmatrix} \tag{4.68}$$

and the orthogonality of the spinors $\begin{pmatrix}1 & i\end{pmatrix}$, $\begin{pmatrix}1 & -i\end{pmatrix}$. Also we note that

$$F(t)+F^*(t)=\frac{1}{2\gamma^2(t)} \tag{4.69}$$

$$iF(t)+\left(iF(t)\right)^*=\frac{1}{4\gamma^2(t)}\frac{\hbar t}{M\gamma^2} \tag{4.70}$$

$$iF^2(t)+\left(iF^2(t)\right)^*=\frac{1}{8\gamma^4(t)}\frac{\hbar t}{M\gamma^2} \tag{4.71}$$

and we set

$$x_2'=x_2-\frac{p_0}{M}t \tag{4.72}$$

$$F=\frac{1}{4\gamma^2\left(1+\frac{i\hbar t}{2M\gamma^2}\right)}\equiv F(t). \tag{4.73}$$

From Eqs. (4.63), (4.65)-(4.73), we obtain, up to a normalization factor, the following expression for the x_2- integrand in Eq. (4.7):

$$\exp\left(-\frac{1}{2\gamma^2(t)}\left[x_2'^2 + a(t)x_1 x_2' x_3\right]\right)$$
$$\left[\exp\left(\frac{t^2}{2\gamma^2(t)}\frac{\mu\beta}{M}x_2'\right) + \exp\left(-\frac{t^2}{2\gamma^2(t)}\frac{\mu\beta}{M}x_2'\right)\right] f(x_1, x_3; t) \quad (4.74)$$

where $a(t)$, of order $\sqrt{\alpha_q}$, is a function of t only, and up to a mutiplicative time-dependent constant,

$$\begin{aligned} f(x_1, x_3; t) &\equiv w_+ f_+(x_1, x_3; t) + w_- f_-(x_1, x_3; t) \\ &= w_+ \exp\left(-\frac{1}{2\gamma^2(t)}\left[x_1^2 + x_3^2 - \frac{t^2}{M}\mu\beta x_3 - \frac{x_i A_{ij} x_j}{\gamma^2(t)}\right]\right) \\ &\quad + w_- \exp\left(-\frac{1}{2\gamma^2(t)}\left[x_1^2 + x_3^2 + \frac{t^2}{M}\mu\beta x_3 - \frac{x_i A_{ij} x_j}{\gamma^2(t)}\right]\right). \end{aligned} \quad (4.75)$$

A summation over the repeated indices $i, j = 1,3$ in Eq. (4.75) is understood, and

$$A_{13} = A_{31} = -\frac{qbt\gamma^2}{Mc} + \frac{q\beta p_0 t^2 \gamma^2}{2M^2 c} + \frac{q\beta p_0 t^4 \hbar^2}{24 M^4 \gamma^2 c}, A_{11} = A_{33} = 0. \quad (4.76)$$

The later expression in Eq. (4.76) is to be compared with the correlation of the dynamical variables $x_1(t)$, $x_3(t)$ in Eq. (4.28).

In reference to the x_2 - integral in Eq. (4.7), we have, with

$$b(t) = \frac{\mu\beta t^2}{M} \tag{4.77}$$

and for the shifted x_2' - integral,

$$\begin{aligned}&\int_{-\infty}^{\infty} dx_2' \exp\left(-\frac{1}{2\gamma^2(t)}\left(x_2'^2 + \left[a(t)x_1x_3 \pm b(t)\right]x_2'\right)\right)\\&= \sqrt{2\pi}\gamma(t)\exp\left(\frac{1}{8\gamma^2(t)}\left[a(t)x_1x_3 \pm b(t)\right]^2\right)\end{aligned} \tag{4.78}$$

where $\left[a(t)x_1x_3 \pm b(t)\right]^2$ is necessarily of a higher order correction in $\sqrt{\alpha_q}$.

Accordingly, for the probability density $f(x_1, x_3; t)$ we obtain the preliminary expression given in Eq. (4.75). Upon setting

$$\langle g\rangle_t^{\pm} = \int dx_1 dx_3 g(x_1, x_3) f_{\pm}(x_1, x_3; t) \tag{4.79}$$

$$\langle g\rangle_0^{\pm} = \langle g\rangle^{\pm} \tag{4.80}$$

where $\gamma(t)$ as defined in Eq. (4.30), we note that any significant correction to the one derived in Eq. (4.75) consistent with the following constraints, as dictated by the explicit expectation values in Eqs. (4.26)-(4.29), by normalizability and positivity, are easily obtained:

1) $\langle x_1 \rangle_t^{\pm} = 0$ + higher orders

2) $\langle x_3 \rangle_t^{\pm} = \dfrac{\mu\beta t^2}{2M} \langle \sigma_3 \rangle^{\pm}$ + higher orders

3) $\sqrt{\langle x_1^2 \rangle_t^{\pm}} = \gamma(t)$ + higher orders

4) $\left(\langle x_3^2 \rangle_t^{\pm} - \left(\langle x_3^2 \rangle_t^{\pm} \right)^2 \right)^{1/2} = \gamma(t)$ + higher orders

5) $\left\langle \left(x_1 - \langle x_1 \rangle_t^{\pm} \right) \left(x_3 - \langle x_3 \rangle_t^{\pm} \right) \right\rangle_t^{\pm} = A_{13}$ + higher orders

6) $\int dx_1 dx_3 f(x_1, x_3; t) = 1$

7) $f(x_1, x_3; t)$ is real and positive

where A_{13} is given in Eq. (4.28), Eq. (4.76), and higher orders stand relative to the parameter $\sqrt{\alpha_q}$.

To satisfy, in the process, constraint 2) (see also Eq. (4.27)), we multiply the right-hand side of Eq. (4.75) by an overall normalizing factor $\exp\left(-\dfrac{\left(\mu\beta t^2 / 2M \right)^2}{2\gamma^2(t)} \right)$ giving

$$\begin{aligned} f(x_1, x_3; t) \propto\ & w_+ \exp\left(-\frac{1}{2\gamma^2(t)} \left[x_1^2 + \left(x_3 - \frac{\mu\beta t^2}{2M} \right)^2 - \frac{x_i A_{ij} x_j}{\gamma^2(t)} \right] \right) \\ & + w_- \exp\left(-\frac{1}{2\gamma^2(t)} \left[x_1^2 + \left(x_3 + \frac{\mu\beta t^2}{2M} \right)^2 - \frac{x_i A_{ij} x_j}{\gamma^2(t)} \right] \right) \end{aligned} . \tag{4.81}$$

Consistency with the constraints 1) – 6) necessarily gives

$$f\left(x_1,x_3;t\right)=\frac{\sqrt{\det \underset{\sim}{C}}}{2\pi}\begin{bmatrix} w_+ \exp\left(-\frac{1}{2}\left(x_i - x_{i0}\right)C^{ij}\left(x_j - x_{j0}\right)\right) \\ +w_- \exp\left(-\frac{1}{2}\left(x_i + x_{i0}\right)C^{ij}\left(x_j + x_{j0}\right)\right) \end{bmatrix} \tag{4.82}$$

where $\left[\underset{\sim}{C}\right]=\left[C^{ij}\right]$, $C^{11}=C^{33}=\dfrac{1}{\gamma^2\left(t\right)}$, $i,j=1,3$,

$$C^{13}=C^{31}=\frac{1}{\gamma^4\left(t\right)}\left(\frac{qbt\gamma^2}{Mc}-\frac{q\beta p_0 t^2\gamma^2}{2M^2c}-\frac{q\beta p_0 t^4\hbar^2}{24M^4\gamma^2 c}\right) \tag{4.83}$$

$$x_{i0}=\frac{\mu\beta}{2M}t^2\delta_{i3} \tag{4.84}$$

and $w_+ = w_- = \dfrac{1}{2}$ for an unpolarized beam.

The probability density in Eq. (4.82) is a sum of bivariate normal distributions (Manoukian (1986)) and

$$\left[\Sigma^{ij}\right]=\left[\left[\underset{\sim}{C}^{-1}\right]^{ij}\right] \tag{4.85}$$

is the so-called covariance matrix describing the correlation between x_1 and x_3 on the screen for $i \neq j$. $\underset{\sim}{\Sigma}$ is a measure of dispersion in all directions in the $\left(x_1,x_3\right)$- plane. The multiplicative factor $\sqrt{\det \underset{\sim}{C}}\big/2\pi$ is the standard normalization factor.

Finally, the constrain 7) implies that $\det \underset{\sim}{C} > 0$, i.e., it leads to a positivity requirement. This in turn implies that we should have

$$\frac{|q|t}{Mc}\left| b - \frac{\beta p_0}{2M}t - \frac{\beta p_0 t^3 \hbar^2}{24 M^3 \gamma^4} \right| < 1 + \frac{\hbar^2 t^2}{4M^2\gamma^4}\ . \tag{4.86}$$

In reference to this inequality consider first the case with $b = 0$, i.e., the constraint

$$C < 1 + \frac{\hbar^2 t^2}{4M^2\gamma^4} \tag{4.87}$$

with

$$C = \frac{|q|\beta p_0 t^2}{2M^2 c}\left(1 + \frac{\hbar^2 t^2}{12 M^2 \gamma^4} \right). \tag{4.88}$$

By setting,

$$\Delta z = \frac{|\mu|\beta t^2}{2M} \tag{4.89}$$

$$\frac{p_0}{M}t = L \tag{4.90}$$

with the latter denoting the macroscopic distance from the particle's initial center of the wavepacket to the observation screen, we may rewrite C as

$$C = \frac{4L}{|g|}\frac{M}{\hbar}\frac{\Delta z}{t}\left(1+\frac{\hbar^2 t^2}{12M^2\gamma^4}\right). \tag{4.91}$$

For the electron with $\Delta z \sim 10^{-3}m$, $t \sim 10^{-6}s$, $L \sim 1\,m$, $\gamma < 10^{-3}m$

$$C \sim 1.73\times10^{7}\left(1+\frac{1.12\times10^{-21}}{\gamma^4}\right) \tag{4.92}$$

which is a very large number and the positivity constraint in Eq. (4.87) cannot be satisfied. On the other hand, the uniform magnetic field $(0,b,0)$ may a priori be set at

$$b = \frac{\beta}{2}L \tag{4.93}$$

defined simply in terms of the non-uniform magnetic field gradient $\partial B_2/\partial x_2 = -\beta = -\partial B_3/\partial x_3$ (see Eq. (3.10)) and the distance to the observation screen L, independently of any of the details of the spin ½ charged particle considered and of the (initial) spread γ. [The uniform magnetic field component b may be, of course, chosen so that $C^{13} = 0$, but this would mean to choose a different uniform magnetic field for every different charged particle, and a different spread γ, and would not be physically as interesting.] The matrix elements in Eq. (4.83) then simply become

$$C^{13} = C^{31} = -\varepsilon(q)\frac{1}{3|g|}\frac{\Delta z}{\gamma}\frac{L}{\gamma}\frac{\hbar t}{M}\frac{1}{\gamma^4(t)} \tag{4.94}$$

and the positivity constraint

$$\frac{1}{3|g|}\frac{\Delta z}{\gamma}\frac{L}{\gamma}\frac{\hbar t}{M\gamma^2} < 1 + \frac{\hbar^2 t^2}{4M^2\gamma^4} \tag{4.95}$$

is readily satisfied. For example, for the electron with $\Delta z = 10^{-3}\,m$, $L = 0.7\,m$, $\gamma = 0.6\times10^{-3}\,m$, $t = 10^{-6}\,s$, corresponding to an initial average speed of $7\times10^5\,m/s$, a magnetic field gradient $\beta = 1.96\times10^2\,T/m$, and a uniform longitudinal magnetic field $b = 68.6\,T$ [at present, the desired magnetic field can not be produced], the left-hand side of Eq. (4.95) is ≈ 0.1.

A detailed graphical analysis of the probability density $f(x_1, x_3; t)$ in Eq. (4.82) with the matrix C as given in Eq. (4.94) will be carried out for charged as well as neutral particles in the next chapter.

Chapter V

Analysis of the Intensity Distribution on the Observation Screen

We recall the explicit expression for the intensity distribution obtained in Chapter IV,

$$f\left(x_1,x_3;t\right)=\frac{\sqrt{\det \underset{\sim}{C}}}{2\pi}\left[\begin{array}{l} w_+ \exp\left(-\frac{1}{2}\left(x_i-x_{i0}\right)C^{ij}\left(x_j-x_{j0}\right)\right) \\ +w_- \exp\left(-\frac{1}{2}\left(x_i+x_{i0}\right)C^{ij}\left(x_j+x_{j0}\right)\right)\end{array}\right] \tag{5.1}$$

where

$$C^{11}=C^{33}=\frac{1}{\gamma^2\left(t\right)} \tag{5.2}$$

and

$$C^{13}=C^{31}=\frac{1}{\gamma^4\left(t\right)}\left(\frac{qbt\gamma^2}{Mc}-\frac{q\beta p_0 t^2\gamma^2}{2M^2c}-\frac{q\beta p_0 t^4\hbar^2}{24M^4\gamma^2 c}\right) \tag{5.3}$$

$$\Delta z=\frac{\left|\mu\right|\beta t^2}{2M}\,. \tag{5.4}$$

In figure 5.1, we plot the probability density in the absence of the magnetic field, i.e., for $\vec{B}=0$, $t=10^{-6}s$, $\gamma=0.6\times10^{-3}m$, and the corresponding one for $\vec{B}\neq0$, with $\beta=1.96\times10^{2}\,T/M$ is plotted in figure 5.2, for the electron for an unpolarized beam. The probability density for other values of the parameters are in figures 5.3-5.12. This is also plotted for the uncharged case, respectively, in figures 5.13-5.18 for direct comparison.

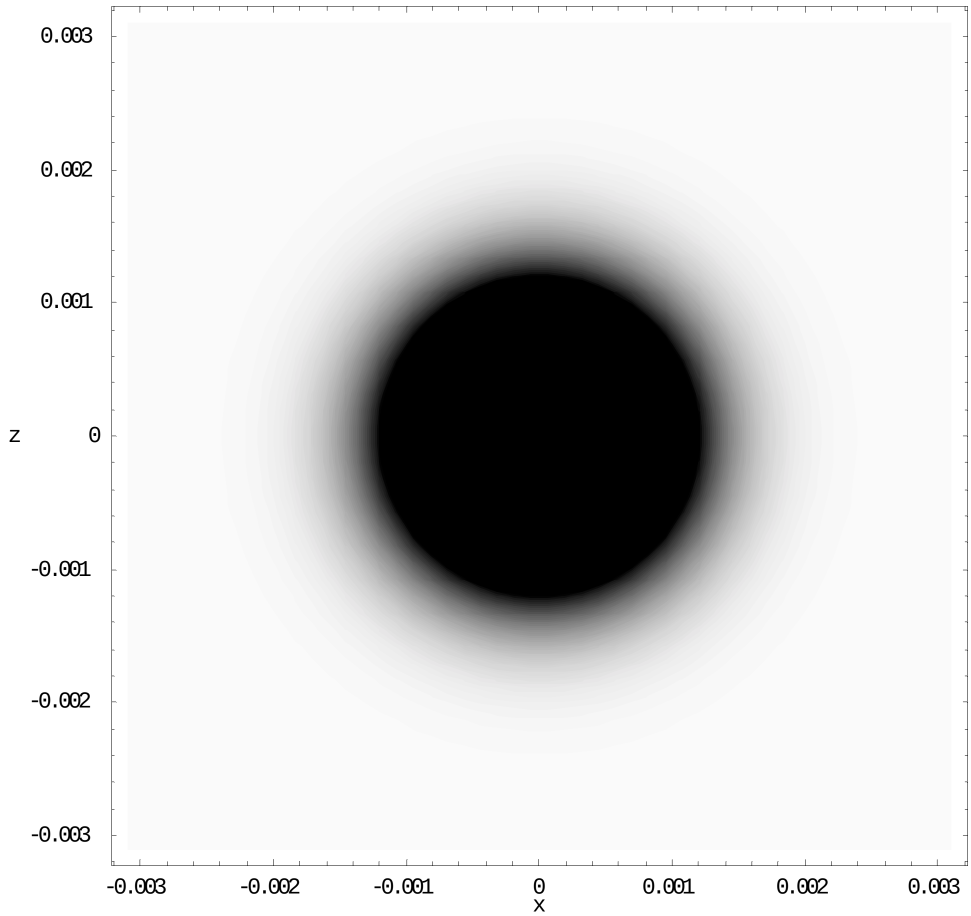

Figure 5.1. Plot of the density $f(x_1,x_3;t)$ for $\vec{B}=0$, $\gamma=0.65\times10^{-3}m$, $t=10^{-6}s$

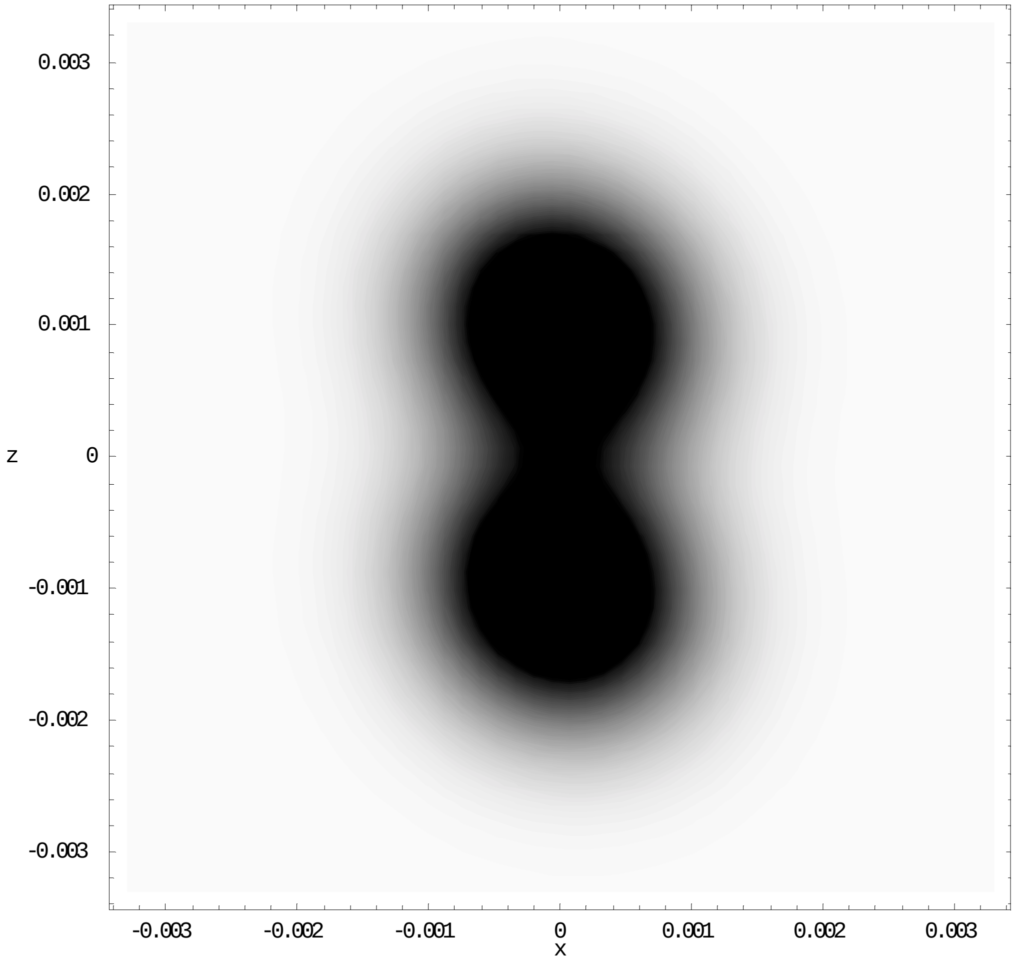

Figure 5.2. Plot of the density $f\left(x_1, x_3; t\right)$ for electron, based on Eq. (4.82), Eq. (4.93), Eq. (4.94) with $\Delta z = 10^{-3}\,m$, $\gamma = 0.65\times10^{-3}\,m$, $t = 10^{-6}\,s$, $L = 0.7\,m$.

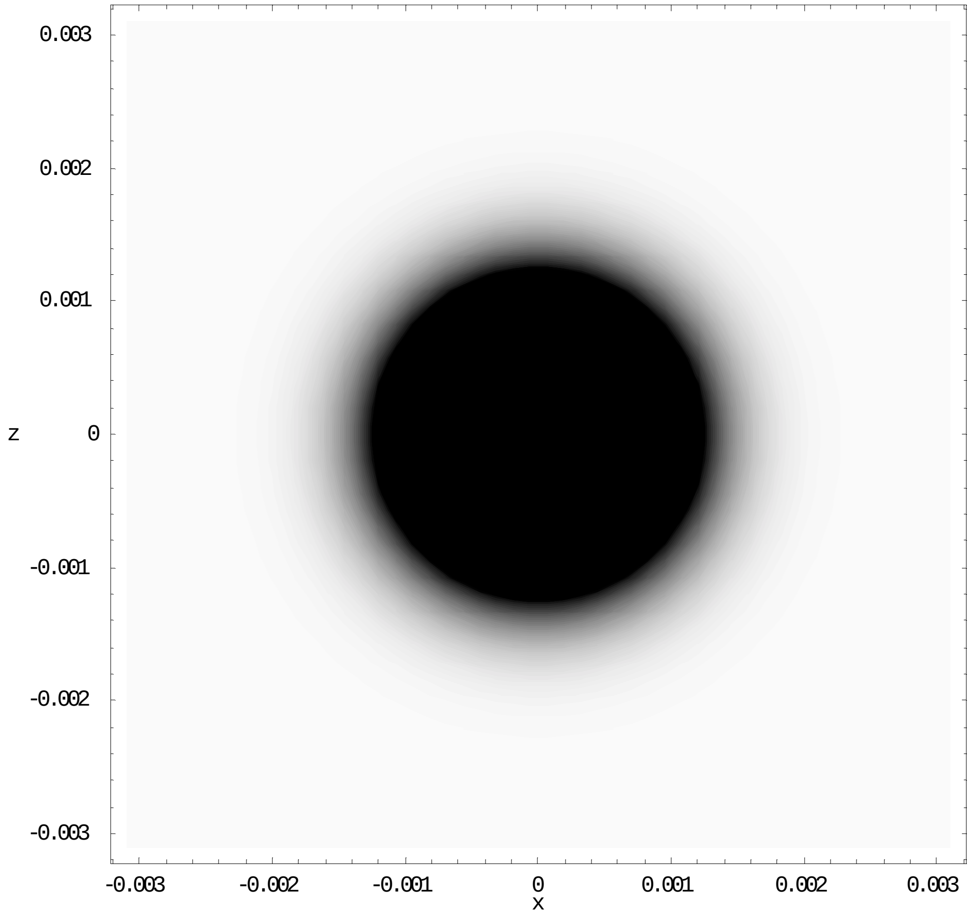

Figure 5.3. Plot of the density $f\left(x_1, x_3; t\right)$ for $\vec{B} = 0$, $\gamma = 0.6\times10^{-3}m$, $t = 10^{-6}s$

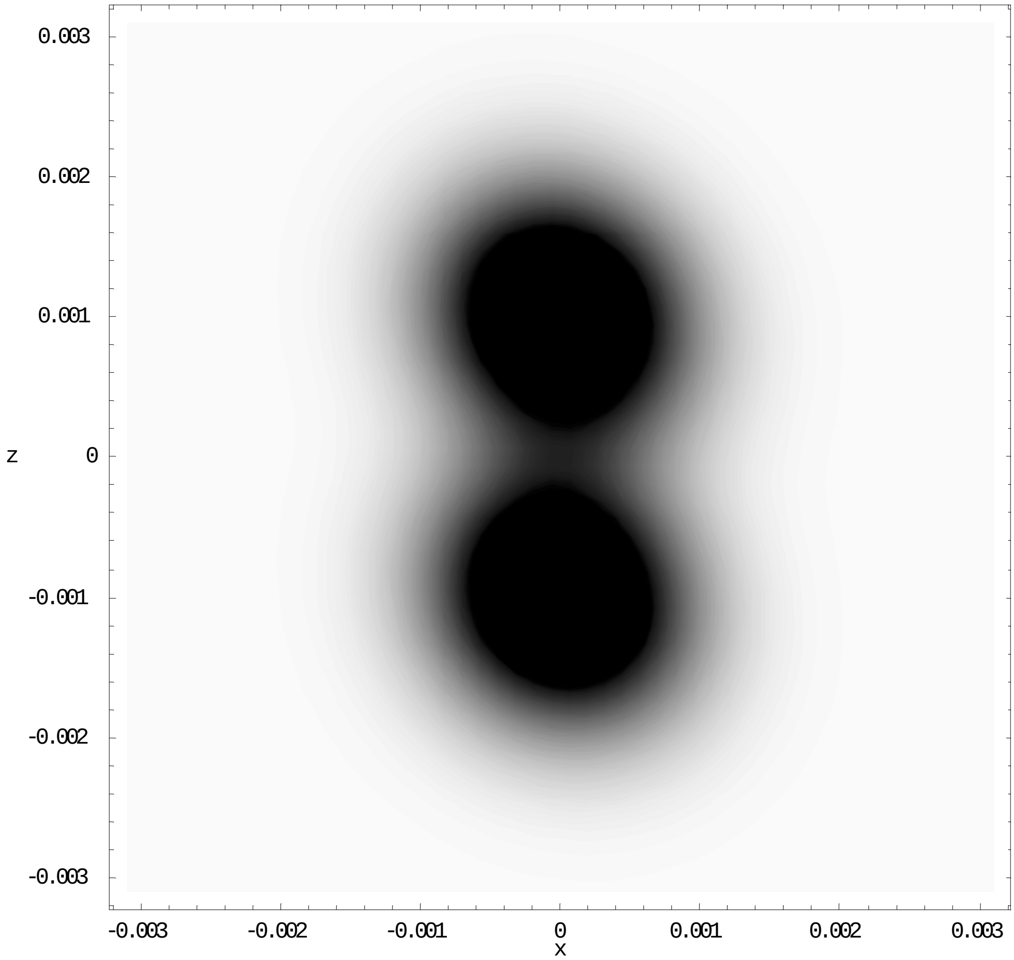

Figure 5.4. Plot of the density $f\left(x_1, x_3; t\right)$ for electron, based on Eq. (4.82), Eq. (4.93), Eq. (4.94) with $\Delta z = 10^{-3} m$, $\gamma = 0.6\times10^{-3} m$, $t = 10^{-6} s$, $L = 0.7\, m$.

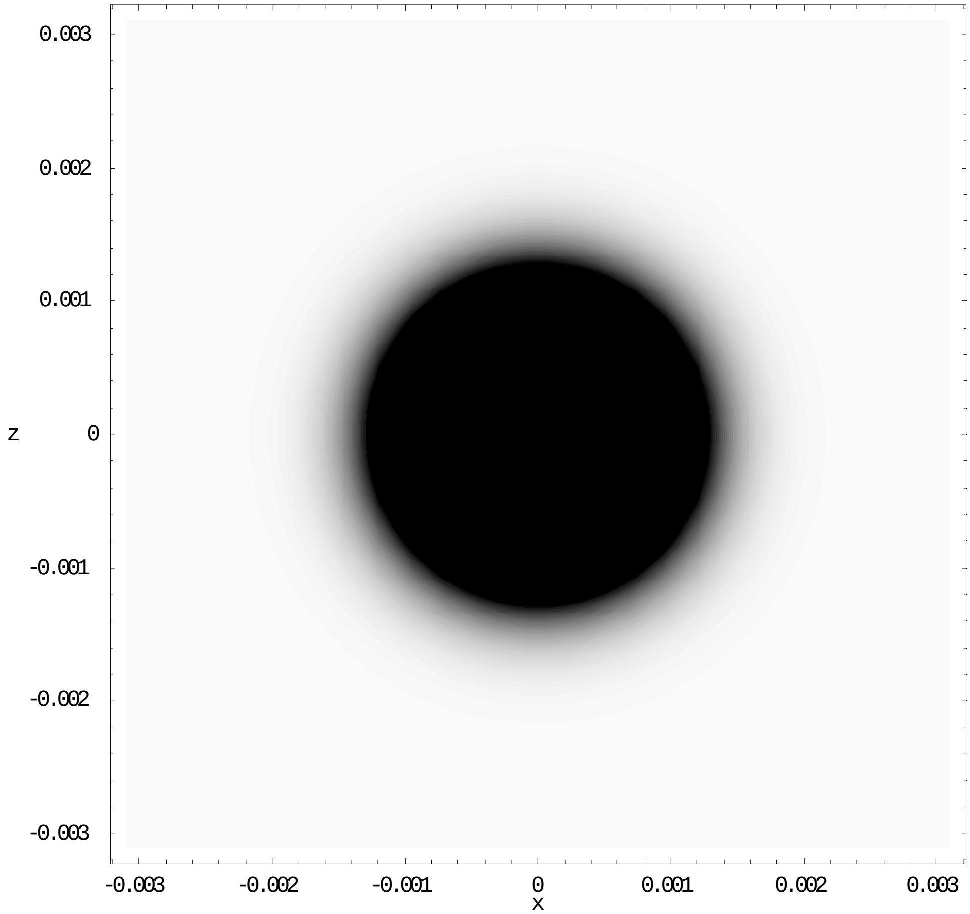

Figure 5.5. Plot of the density $f\left(x_1, x_3; t\right)$ for $\vec{B}=0$, $\gamma = 0.55\times10^{-3}m$, $t = 10^{-6}s$

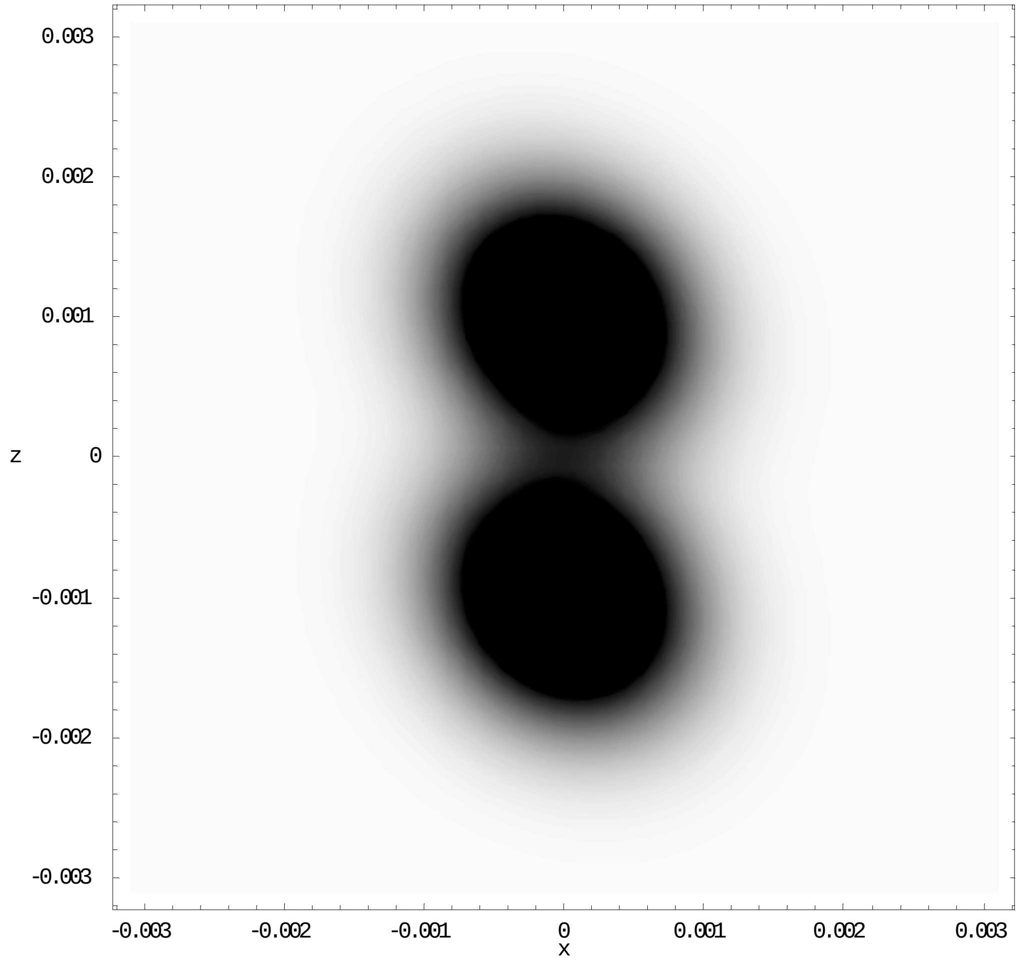

Figure 5.6. Plot of the density $f\left(x_1,x_3;t\right)$ for electron, based on Eq. (4.82), Eq. (4.93), Eq. (4.94) with $\Delta z=10^{-3}\,m$, $\gamma=0.55\times10^{-3}\,m$, $t=10^{-6}\,s$, $L=0.7\,m$.

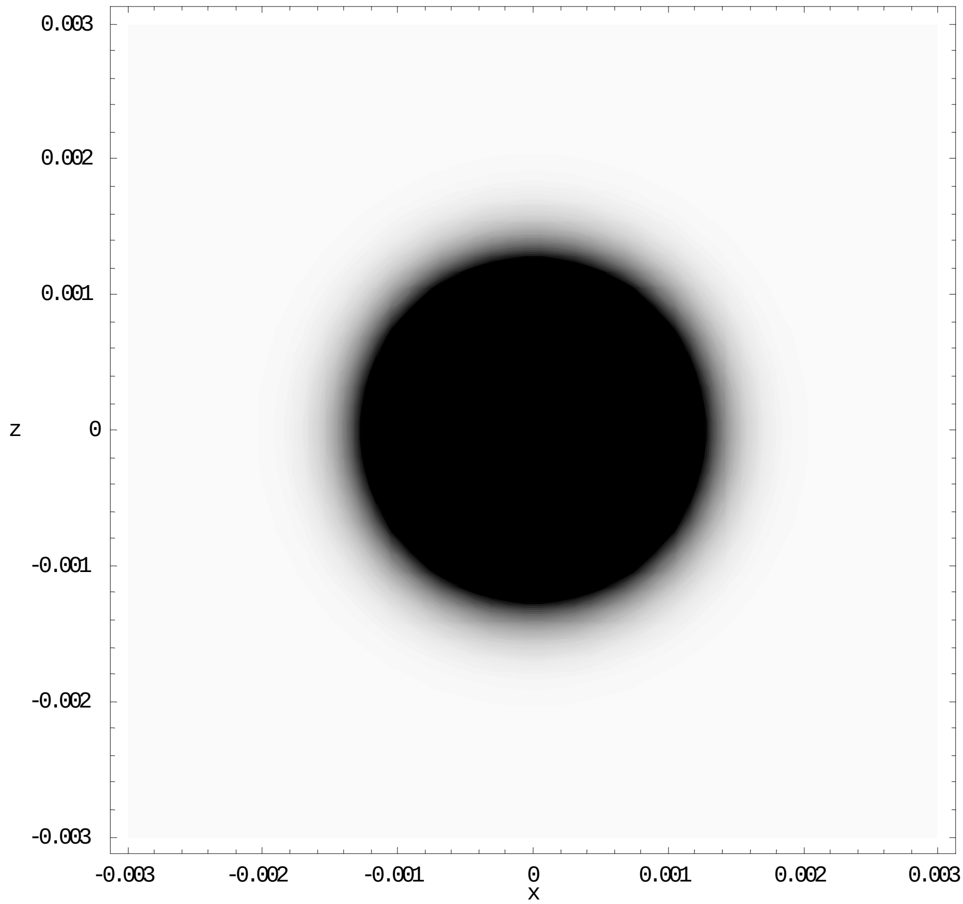

Figure 5.7. Plot of the density $f\left(x_1, x_3; t\right)$ for $\vec{B} = 0$, $\gamma = 0.50 \times 10^{-3} m$, $t = 10^{-6} s$

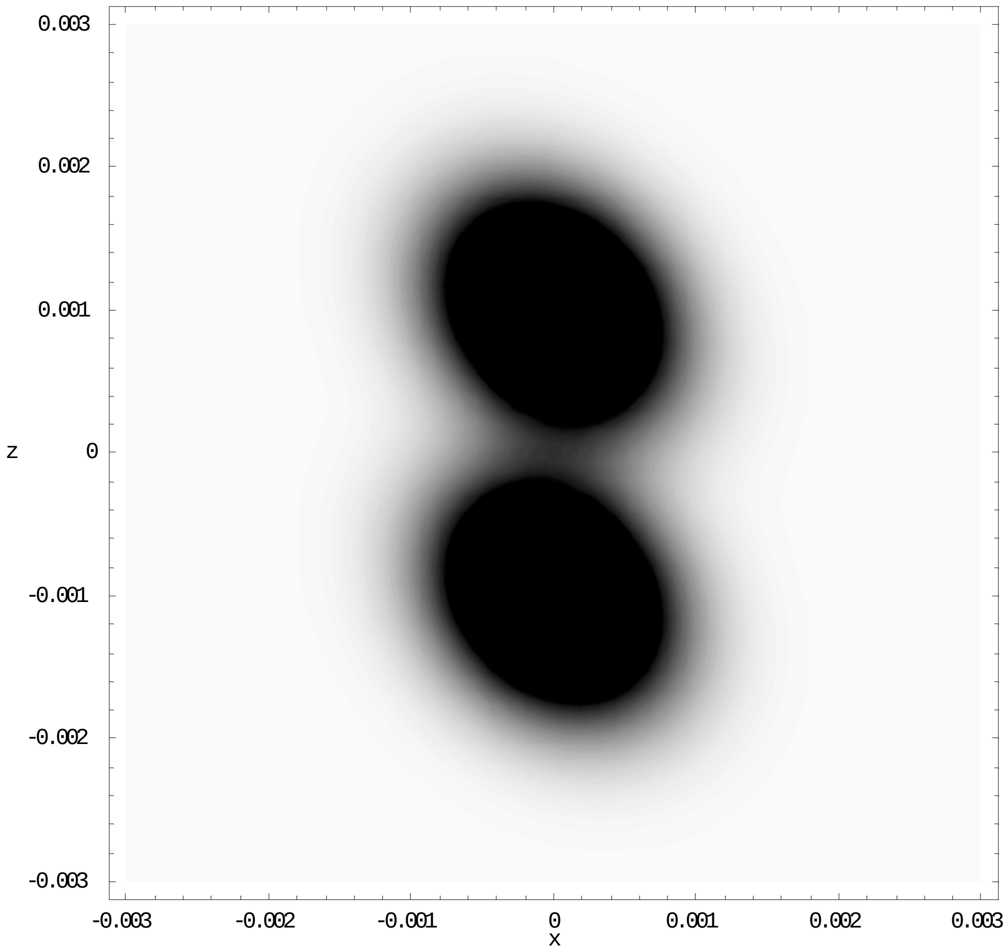

Figure 5.8. Plot of the density $f(x_1, x_3; t)$ for electron, based on Eq. (4.82), Eq. (4.93), Eq. (4.94) with $\Delta z = 10^{-3} m$, $\gamma = 0.50 \times 10^{-3} m$, $t = 10^{-6} s$, $L = 0.7\, m$.

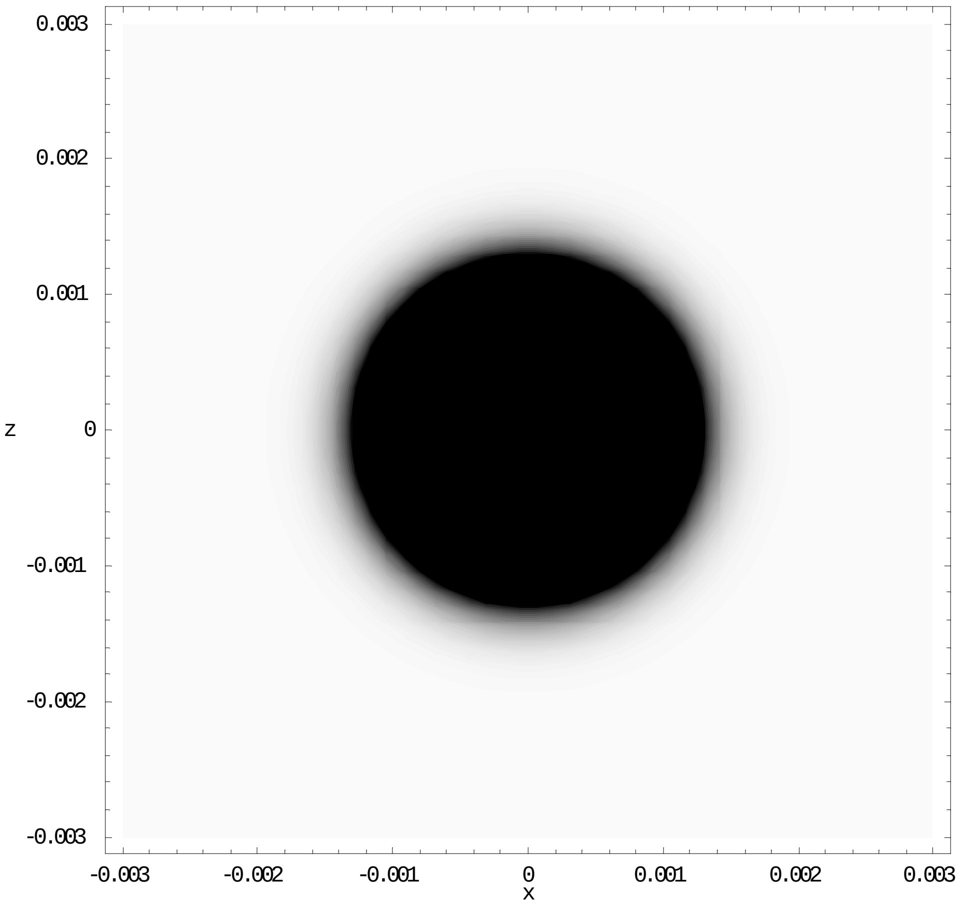

Figure 5.9. Plot of the density $f\left(x_1, x_3; t\right)$ for $\vec{B}=0$, $\gamma=0.45\times10^{-3}m$, $t=10^{-6}s$

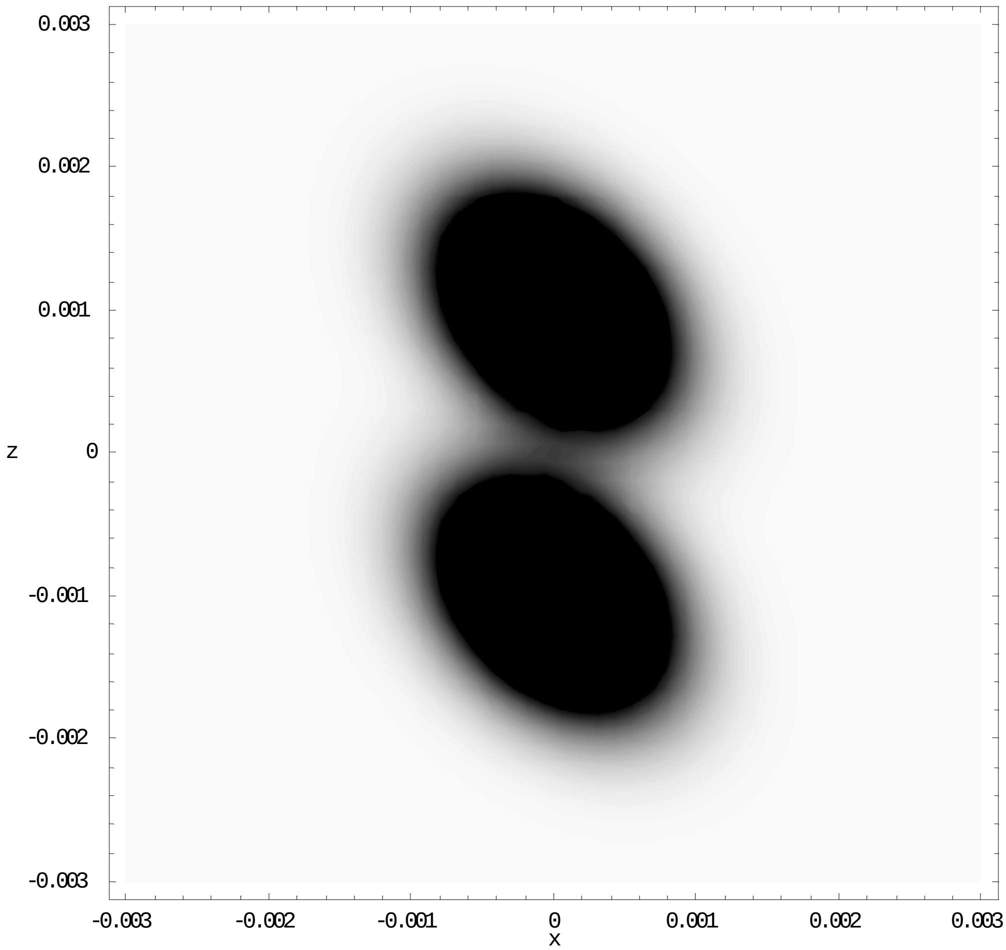

Figure 5.10. Plot of the density $f\left(x_1, x_3; t\right)$ for electron, based on Eq. (4.82), Eq. (4.93), Eq. (4.94) with $\Delta z = 10^{-3}\,m$, $\gamma = 0.45\times10^{-3}\,m$, $t = 10^{-6}\,s$, $L = 0.7\,m$.

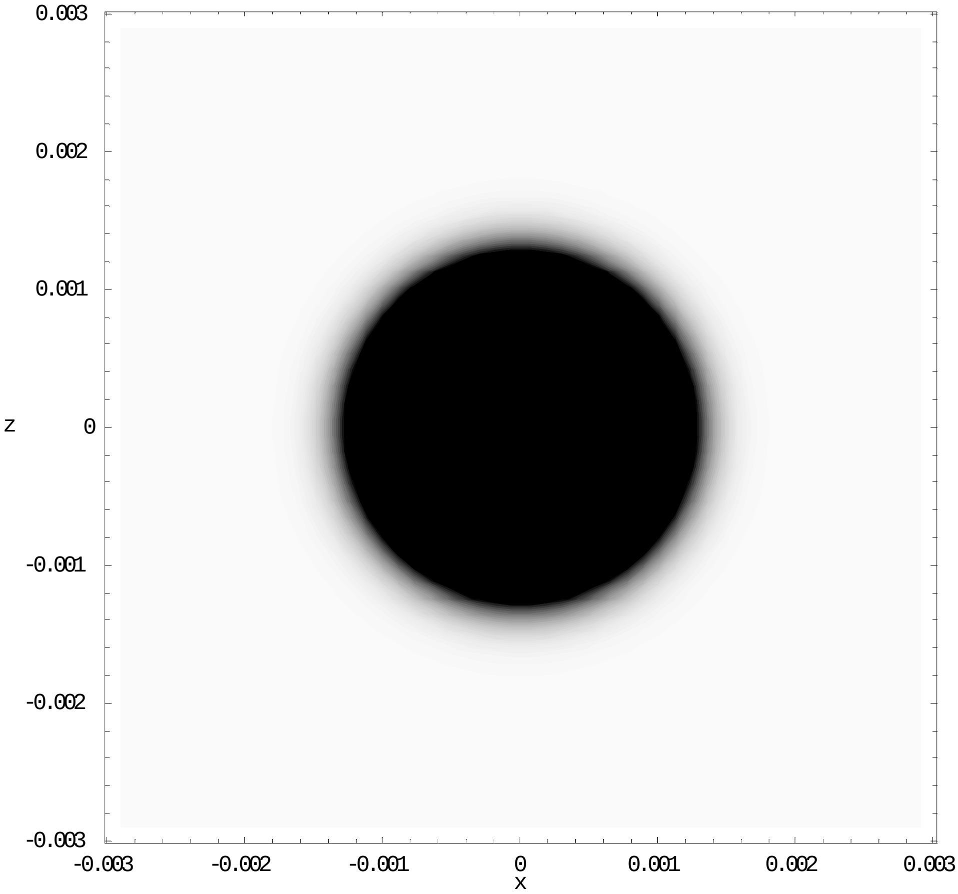

Figure 5.11. Plot of the density $f(x_1, x_3; t)$ for $\vec{B} = 0$, $\gamma = 0.40 \times 10^{-3} m$, $t = 10^{-6} s$

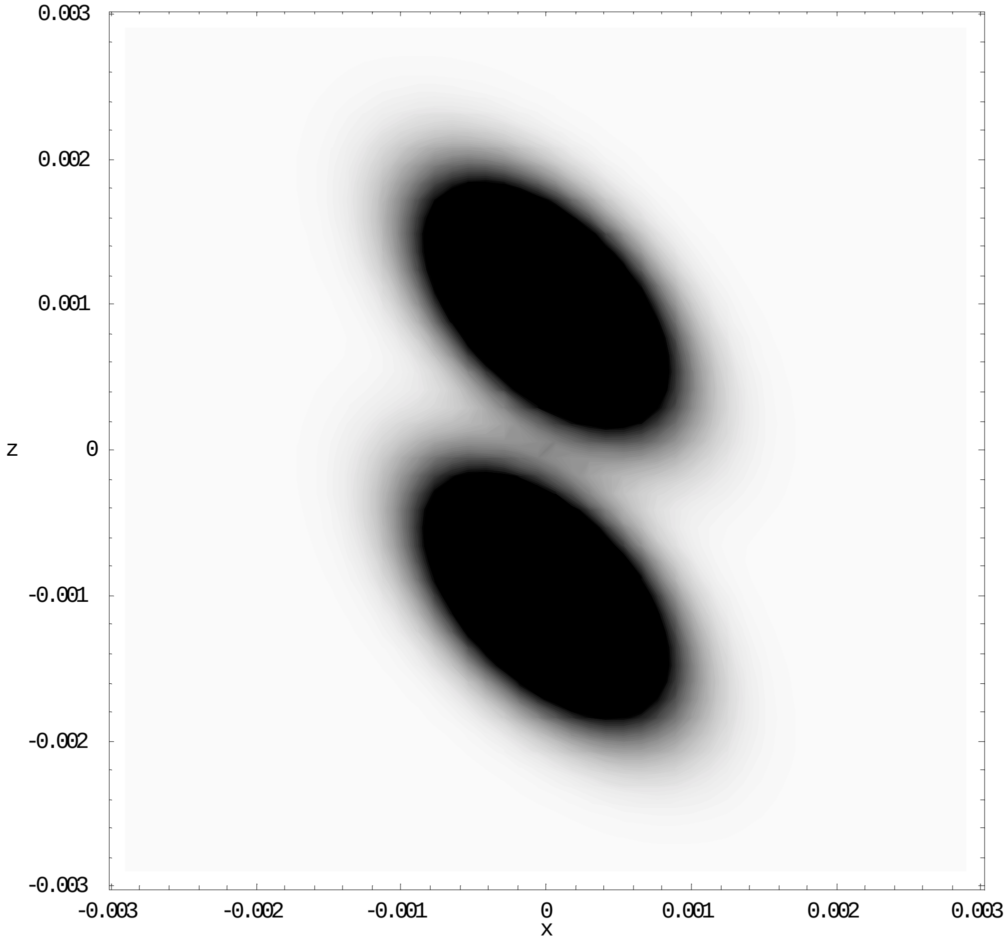

Figure 5.12. Plot of the density $f(x_1, x_3; t)$ for electron, based on Eq. (4.82), Eq. (4.93), Eq. (4.94) with $\Delta z = 10^{-3}\,m$, $\gamma = 0.40\times10^{-3}\,m$, $t = 10^{-6}\,s$, $L = 0.7\,m$.

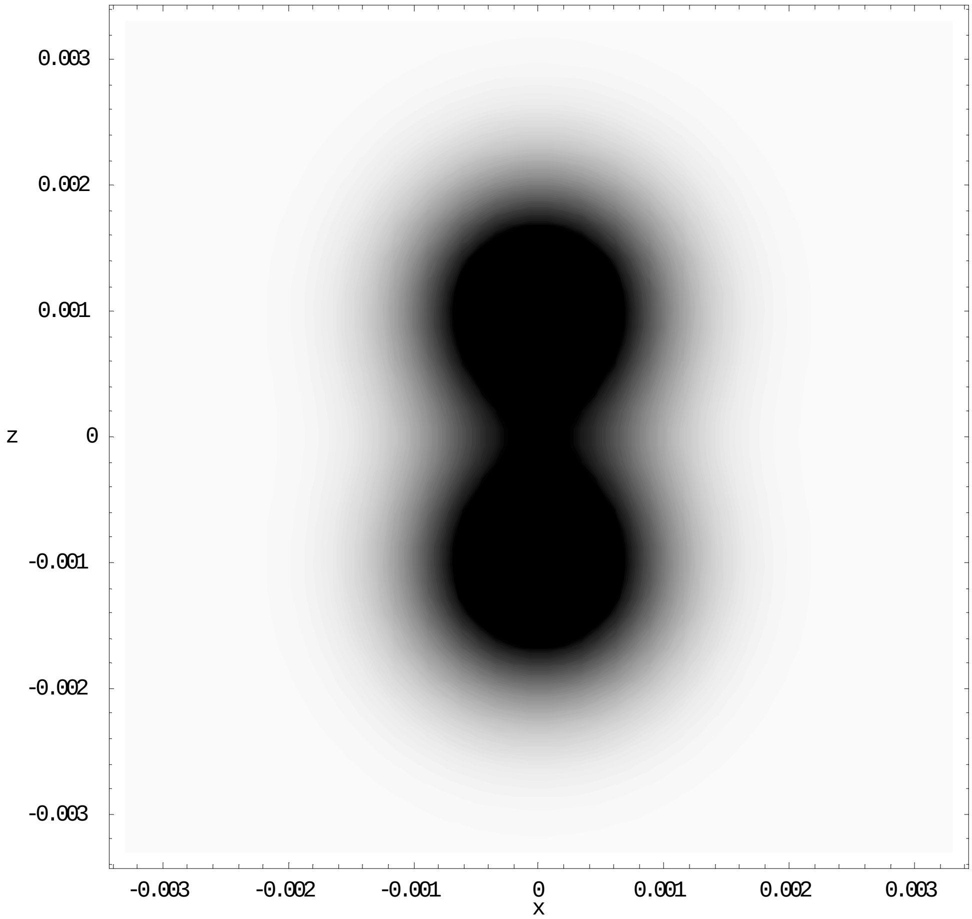

Figure 5.13. Plot of the density $f(x_1, x_3; t)$ for uncharged particles, based on Eq. (4.82), Eq. (4.83), for $|x_0| = 10^{-3} m$, $\gamma = 0.65 \times 10^{-3} m$, $t = 10^{-6} s$.

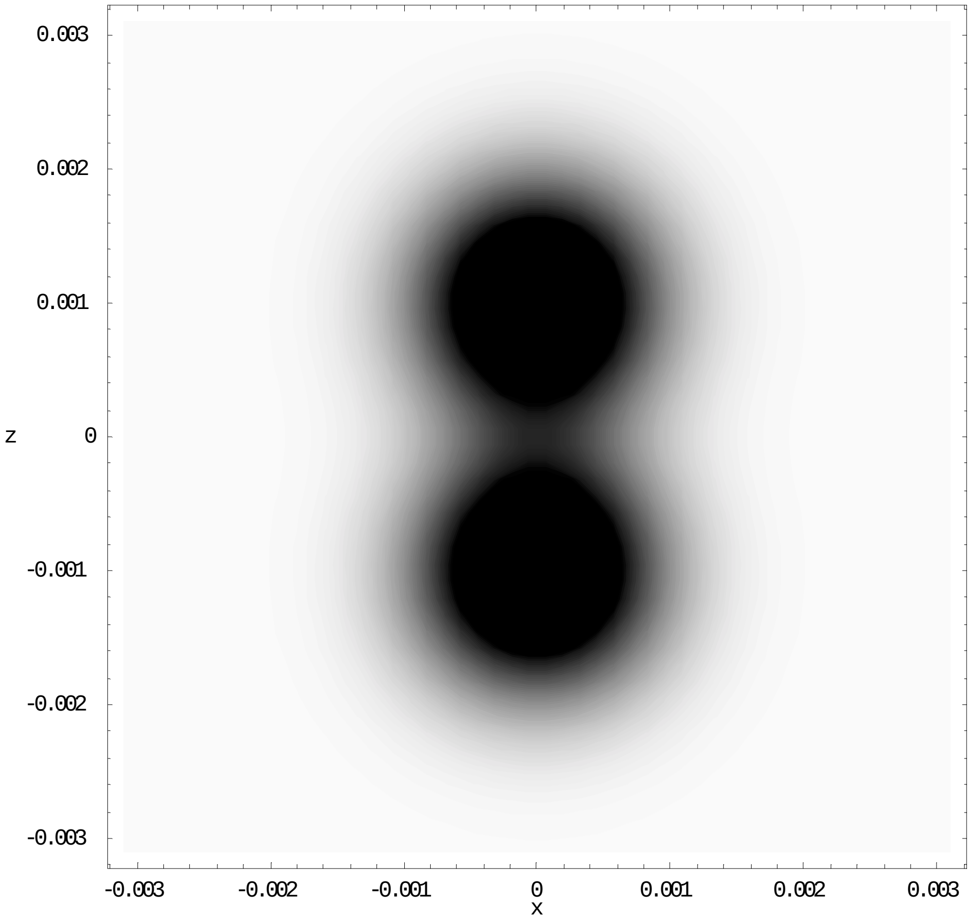

Figure 5.14. Plot of the density $f(x_1, x_3; t)$ for uncharged particles, based on Eq. (4.82), Eq. (4.83), for $|x_0| = 10^{-3} m$, $\gamma = 0.6 \times 10^{-3} m$, $t = 10^{-6} s$.

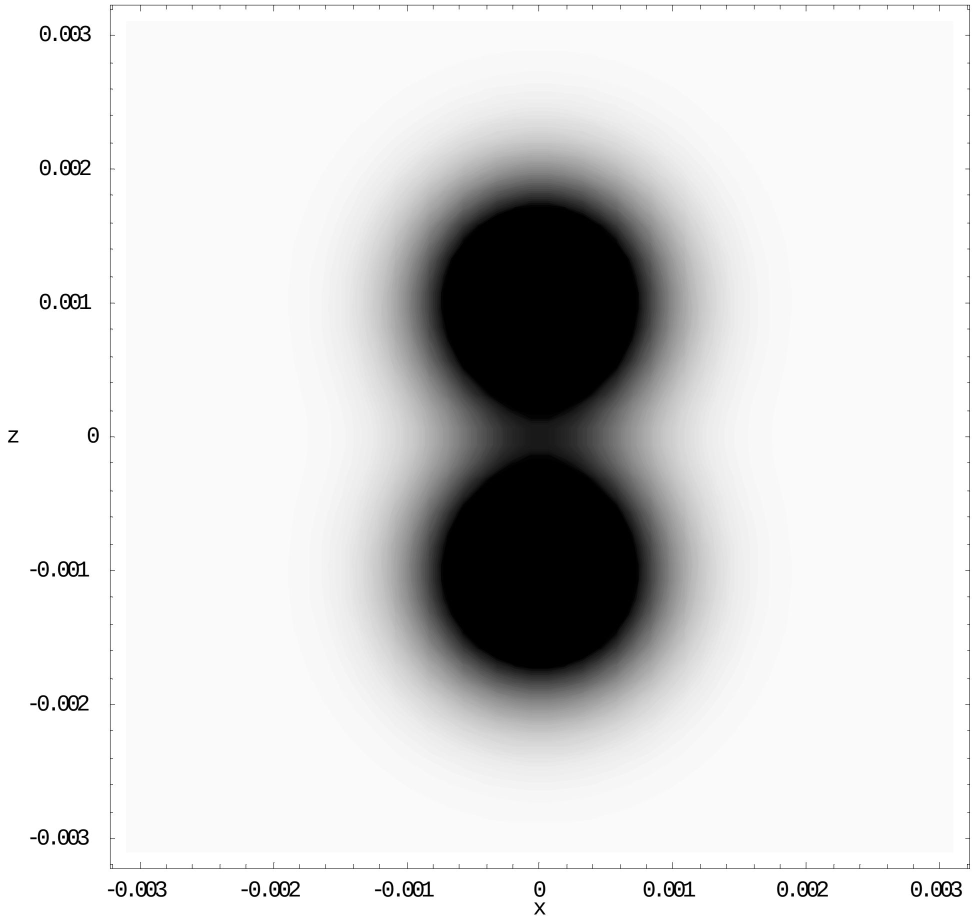

Figure 5.15. Plot of the density $f\left(x_1, x_3; t\right)$ for uncharged particles, based on Eq. (4.82), Eq. (4.83), for $\left|x_0\right| = 10^{-3}\,m$, $\gamma = 0.55\times10^{-3}\,m$, $t = 10^{-6}\,s$.

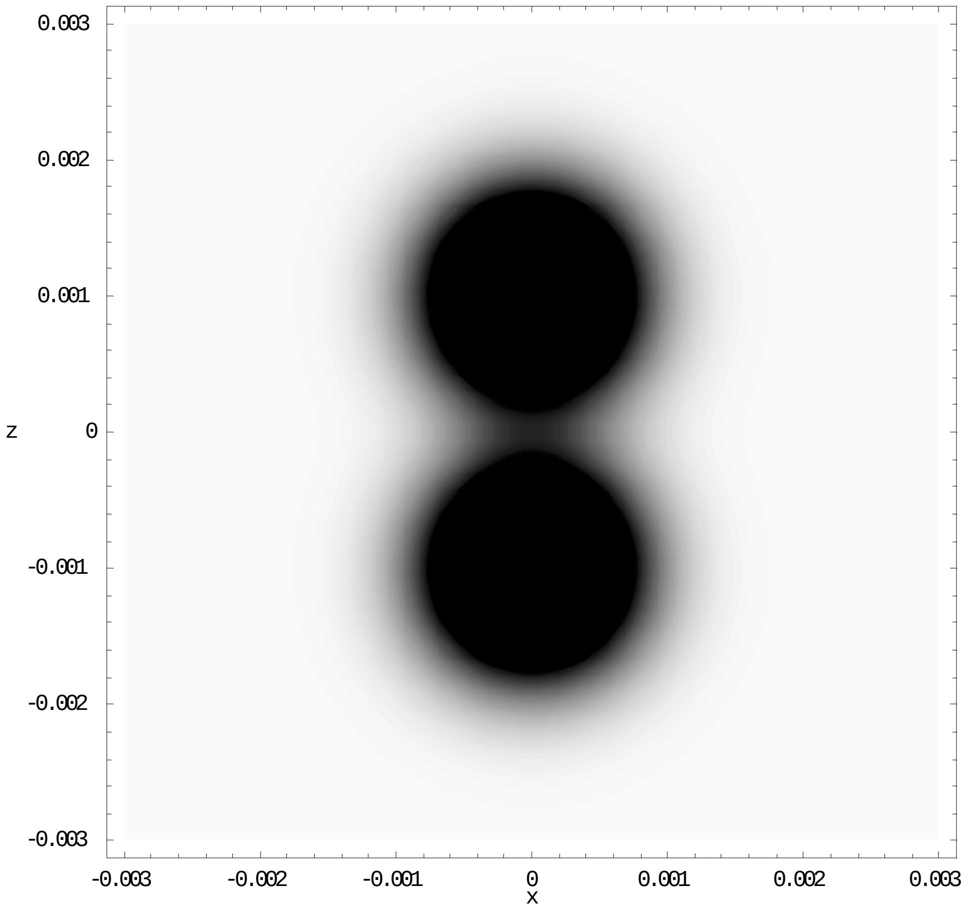

Figure 5.16. Plot of the density $f\left(x_1, x_3; t\right)$ for uncharged particles, based on Eq. (4.82), Eq. (4.83), for $\left|x_0\right| = 10^{-3}\,m$, $\gamma = 0.50\times10^{-3}\,m$, $t = 10^{-6}\,s$.

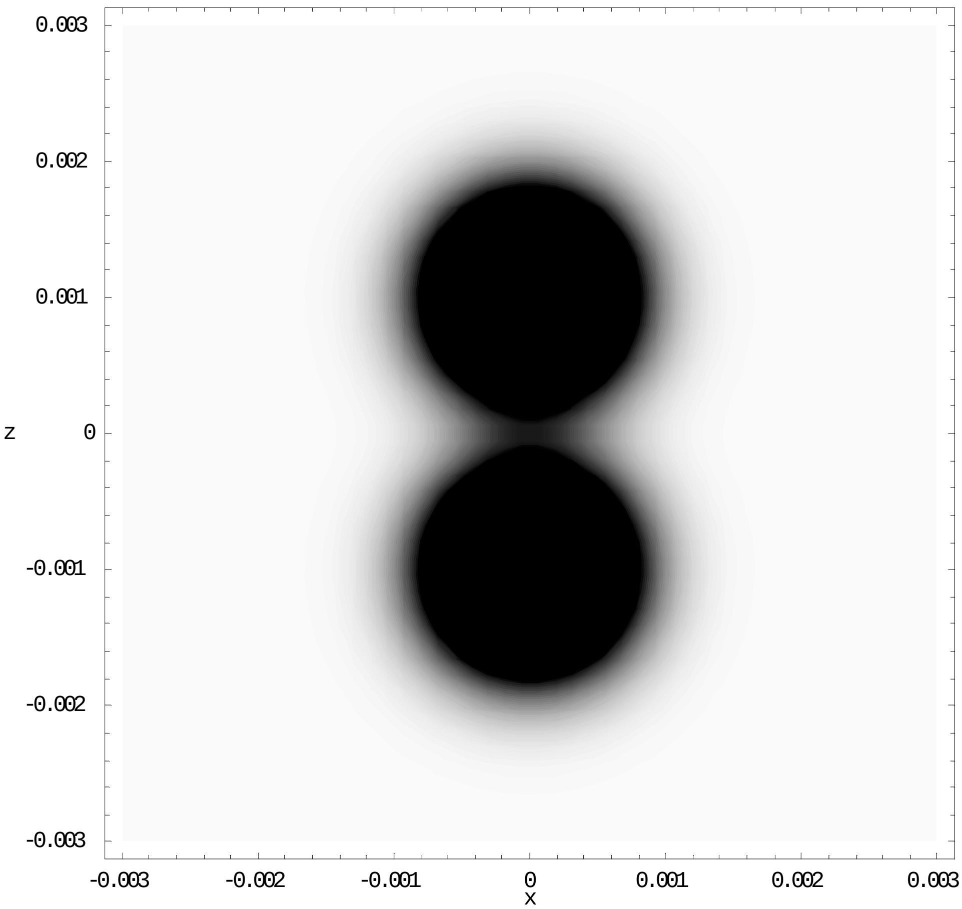

Figure 5.17. Plot of the density $f\left(x_1, x_3; t\right)$ for uncharged particles, based on Eq. (4.82), Eq. (4.83), for $|x_0| = 10^{-3}m$, $\gamma = 0.45\times10^{-3}m$, $t = 10^{-6}s$.

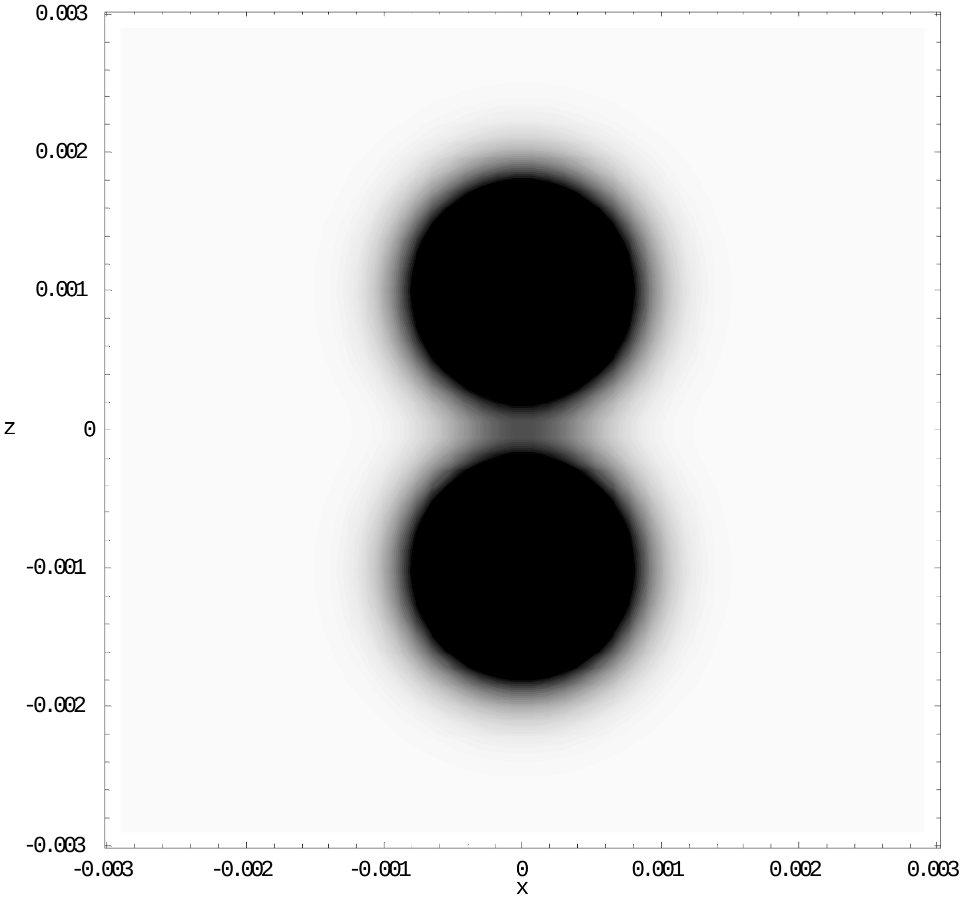

Figure 5.18. Plot of the density $f\left(x_1, x_3; t\right)$ for uncharged particles, based on Eq. (4.82), Eq. (4.83), for $\left|x_0\right| = 10^{-3}\,m$, $\gamma = 0.40 \times 10^{-3}\,m$, $t = 10^{-6}\,s$.

In every case a clear splitting of the beam along the quantization axis is observed. The asymmetry with elongations in the second and the fourth quadrants in figures 5.2, 5.4, 5.6, 5.8, 5.10, 5.12 for the electron, as a negatively charged particle, are easy to understand. For the electron $\varepsilon(q) = -1$, $C^{13} = C^{31} > 0$ and the probability density gets, respectively, positive contributions for $x_3 > x_{30}$, $x_1 < 0$ and $x_3 < -x_{30}$, $x_1 > 0$. Exactly the opposite happens for a positively charged particle with elongations in the first and third quadrants. The formal physical argument for this asymmetry is that it arises as a consequence of the direction of the Loentz force, as determined by the so-called right-hand rule, on a charged particle as applied to the transverse part of the non-uniform magnetic field. For an uncharged particle, the symmetry is clearly restored as observed in figures 5.13-5.18. That is, form the pattern of the beam splitting observed, in addition to the confirmation of the Stren-Gerlach effect, one would also determine the nature of the particle whether it is positively charged, negatively charged or uncharged.

Chapter VI

Conclusion

We have carried out a rigorous quantum dynamical analytical treatment of the Stern-Gerlach effect and as seen form the bulk of this work, the computations turned out to be quite tedious. The analysis was based on the following important contributions taken into account: As mentioned above, (1) it is a quantum mechanical treatment, it takes into account (2) the important field equation $\vec{\nabla}\cdot\vec{B}=0$, (3) the quantum mechanical counterpart of the Lorentz force, (4) the two, rather than one, dimensional aspect of the beam hitting the observation screen, (5) the rather non-trivial correlations that occur, as we have shown to exist, between the dynamical variables. The analytical study was based on a leading order treatment in $\sqrt{\alpha}$, where α is the fine-structure constant, for spin 1/2 particles, including the electron, and shows to lead to a unitary, i.e., a positive definite, expression for the probability intensity distribution on the observation screen, where the magnetic field has a controllable uniform component along the initial average direction of propagation of the particles, in addition to a non-uniform, almost longitudinal, magnetic field lying in the longitudinal plane defined by the quantization axis, in question, of the spin and the initial average direction of propagation. With an initially prepared Gaussian wavepacket, the intensity distribution on the observation screen was shown to be given by a sum of bivariate Gaussian distributions with a non-zero correlation and the corresponding expression in question is spelled out in Eqs. (5.1)-(5.4). The uniform longitudinal controllable magnetic field had a dual role in the analysis. Although longitudinal, it reduces effectively the quantum Lorentz force contribution, which has otherwise caused difficulty in performing the experiment, and in turn, reduces the correlations between the dynamical variables and also provides a positive definite expression for the probability density which is an indispensable property for the physical interpretation of quantum phenomena. The clear splitting of a beam on the observation screen obtained, also shows an asymmetry of beam patterns observed which indicate whether the particle is positively or negatively charged. For an uncharged particle, the symmetry of the patterns is restored, again indicating that the particle involved in the experiment is uncharged. An experiment of the Stern-Gerlach effect for the electron (or, e.g., the proton) as described in the bulk of this research is certainly worth carrying out and, needless to say, is a challenging one to perform. We hope that this work will initiate experimentalists to finally perform such an experiment of the Stern-Gerlach effect for the electron as we have described, analyzed in this work and have finally derived the corresponding analytical pertinent expressions above for this effect for the first time.

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Curriculum Vitae

Mr. Attasit Rotjanakusol was born on July 20th, 1976 in Surin, Northeast Thailand. He graduated with a High School Diploma from Surawittayakarn in 1994. After that he received the DPST scholarship to study in the Department of Physics, Faculty of Science, of Khon Kaen University and graduated with a B.Sc. degree in 1999. After that he decided to study for his Master's degree in the School of Physics, Institute of Science of the Suranaree University of Technology.