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นางสาว งามจิตต์ เจียรกุลประเสริฐ

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SCHWINGER'S METHOD OF COMPUTATIONS IN

SYNCHROTRON RADIATION

Miss Ngarmjit Jearnkulprasert

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ในการแผ่รังสีซินโครตรอน, จำนวนเฉลี่ย <N> ของโฟตอนที่แผ่ออกมาในแต่ละรอบจะกำหนดอย่างแม่นตรงและชัดแจ้ง ซึ่งเกี่ยวข้องกับปริพันธ์มิติเดียว เราต้องการแสดงให้เห็นว่าที่พลังงานสูงจำนวนเฉลี่ยของโฟตอนแผ่ออกมาในแต่ละรอบซึ่งเขียนอยู่ในรูป $5\pi\alpha\big/\sqrt{3(1-\beta^2)}$ และตีพิมพ์มาแล้วหลายครั้ง เช่น หนังสือคู่มือของนักพิสิกส์เชิงอนุภาค: วารสารของฟิสิกส์เชิงอนุภาค มีความไม่แม่นและเชิงเส้นกำกับที่แท้จริงด้วยสัมพันธ์ความผิดพลาด มีค่าถึง 160% สำหรับ β = 0.8 และ 82% สำหรับ β = 0.9 <N> สมการที่ชัดแจ้งได้แก้ไขสูตรเดิมที่พลังงานสูง เพื่อความสมบูรณ์เราได้หาจำนวนโฟตอนเฉลี่ยที่แผ่ออกมาต่อรอบสำหรับที่พลังงานระดับต่ำและพลังงานระดับกลางด้วย ในงานชิ้นนี้ได้เชื่อมั่นอยู่บนงานที่เหลือไว้ของชวิงเกอร์ที่ให้รายละเอียดที่มาสิ่งที่เรียกว่าปริพันธ์กำลังของชวิงเกอร์

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An exact and explicit expression is derived for the mean number <N> of photons emitted per revolution in synchrotron radiation. The latter involves a remarkably simple one-dimensional integral. In particular, we show that the familiar high-energy expression $5\pi\alpha \big/ \sqrt{3(1-\beta^2)}$, printed repeatedly in the literature (e.g., in the "Particle Physicist's Handbook": Review of Particle Physics), is found to be inaccurate and only truly asymptotic with relative errors of 160% (!), 82% (!) for β = 0.8, 0.9, respectively. Our explicit expression for <N> provides a new improved high-energy expression for it to replace the earlier formula. For completeness, representations for <N> are also derived in the low and intermediate energy regimes. Since this work relies heavily on Schwinger's monumental work, a fairly detailed derivation is also provided of the so-called Schwinger power integral which is subsequently used in our work.

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Ngarmjit Jearnkulprasert

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List of Symbols and Some Physical Constants

The following symbols are used throughout this thesis unless otherwise stated. The numerical values of following physical constants are based on The Review of Particle Physics by the Particle Data Group (1998).

c	=	Speed of light in vacuum, and is given by 299792458 m s^{-1}
$\hbar \equiv h/2\pi$	=	Reduced Planck constant, and is given by $1.05457267 \times 10^{-34}$ Js or 6.582122×10^{-22} MeV s
e	=	Electron charged magnitude, which is $1.60217733 \times 10^{-19}$ C or $4.8032068 \times 10^{-10}$ esu
m	=	Electron mass, which is 0.51099906 MeV/c^2 or $9.1093898 \times 10^{-31}$ kg
$\alpha = e^2/\hbar c$	=	Fine-structure constant
ω	=	Photon's energy (frequency)
$\delta(x)$	=	Dirac's delta function
P	=	The power of radiation
$\langle N \rangle$	=	The mean number of photons emitted per revolution
E	=	Electric field intensity
B	=	Magnetic induction vector
A	=	Vector potential
Φ	=	Scalar potential
ρ	=	Charged density

j	=	Current density
a	=	Particle’s acceleration
v	=	Particle’s velocity
S	=	Poynting vector
γ	=	Lorentz factor

Chapter I

Introduction

1.1 Early Historical Review-According to Lea (1978) and Pollock (1983)

1.1.1 Brief Historical Review of the Early Theory of Synchrotron Radiation

The synchrotron radiation is widely recognized as an important research tool in physics, chemistry, biology and medicine. The theory of synchrotron radiation is essential for the construction of synchrotrons and storage rings, and provides an explanation for the source of the nonthermal radiation form magnetic stars, especially pulsars. Theoretically, since its main features are well known, it provides us with a good model for the continuing improvements of the calculational methods and simplification of final results.

As far back as 1898, Liénard (quoted in Pollock, 1983) first pointed out that an electric charged particle moving in a circular path should radiate energy and he calculated the rate of radiation from the centripetal acceleration of an electron. Several years later, Schott (1912) derived expressions for the angular distribution of the radiation from a relativistic electron circulating in a uniform magnetic field as a function of the harmonics of the orbital frequency. Then in the 1940's, Schwinger developed the classical theory of radiation for arbitrary electron trajectories, and showed for the case of circular motion how the radiation was distributed among the harmonics of the orbital frequency. Schwinger who in 1945 (Pollock, 1983) reported detailed calculations on the theory of radiation, and published in 1949, examined further the limits of validity of the classical treatment.

At the same time of the above work in the United States, Soviet scientists had been tackling similar problems. In a 1944 paper Ivanenko and Pomeranchuk predicted considerable radiation losses in circular accelerating motion. This work of Schwinger, Invanenko and Pomeranchuk, and, in particular, the monumental research contribution of Schwinger of 1945, 1949 laid the theoretical foundations of synchrotron radiation. The theoretical work of Schwinger, and its direct experimental impact, was so important that Langmuir (Pollock, 1983; cf. Elder et al. 1947) when he first observed the "spark" of radiation he referred to it as "Schwinger Radiation".

1.1.2 Brief Historical Review of Early Experimental Work in Synchrotron Radiation

When the building of multi-million-volt accelerators began, the problem of radiation loss of accelerated electrons received particular attention. Circular electron accelerators of various designs were proposed, by Slepian in 1922 at Westinghouse, by Wideroe in 1928 in Norway, and by Kerst and Serber in 1941 at the University of Illinois. The first such machine that was successful was the 2.3-MeV betatron that Kerst built at Illinois. In this machine radiation loss from the electrons was so small that it could be neglected. With the building of larger electron accelerators the increase of radiation loss, as the fourth power of the energy of a relativistic electron, became a serious matter. Baldwin (1975) recalls such an occasion at the General Electric Research Labs, in Schenectady, NY., when the 100 MeV betatron was constructed in the mid forties. Prompted by Blewett and colleagues (1946), researchers used a sensitive radio receiver to look for emission at the fundamental and several harmonics of the 57 MHz orbital frequency. They set up receiving aerials external to the betatron vacuum vessel, which was a silver-coated doughnut, and later tried with an internal antenna. "**But**" wrote Baldwin (1975), "**with an opaque doughnut coating, complete concrete radiation shield, and closed minds, we did not see the light, literally or figuratively**". The light was not observed until 1947 when the first observation was made. It occurred in the same laboratory, when a 70 MeV electron synchrotron was constructed under the direction of Herbert Pollock. As Pollock was anxious about electrical breakdown on of his assistants ventured a very quick glimpse, with the aid of a mirror, around the corner of the radiation shield, to make sure there was no sparking. He reported an intense "arc" inside the doughnut was observed, which to everyone's amazement, he says, persisted even after the beam deflection voltage was turned off. The bright glow was, indeed, a portion of the radiation of relativistic electrons at harmonics ~ 10^7 of the orbital frequency. Elder et al. (1947) reported in 1947 this observation of what is now referred to as synchrotron radiation.

As new generations of electron accelerators were born and brought into use for nuclear and high-energy research, the synchrotron radiation was regarded as a hindrance to achieving higher electron energies.

1.2 Purpose of the Thesis and Achievements

The purpose of this thesis is to derive an explicit expression for the mean number <N> of photons emitted per revolution in synchrotron radiation with no approximations made. A remarkably simple one-dimensional integral expression is derived for <N>. We make explicit use of the monumental work of Julian Schwinger (1949, 1976, 1978) in this field. As the latter work turns out to be very important in this project, we start almost from scratch and derive, in the process, the Schwinger power integral for synchrotron radiation.

In particular we show, by using our explicit exact expression derived for <N>, that the familiar high-energy expression $5\pi\alpha/\sqrt{3(1-\beta^2)}$, for the charged particle, printed repeatedly in the literature (e.g., in the "Particle Physicist's Handbook": Review of Particle Physics, 1996, p.75; 1998, p.79) give relative errors as high as 160%(!). As a byproduct of the work we provide a new and much improved high-energy expression for <N> following from our explicit expression, to replace the earlier well-known formula. For completeness, we have also derived low and intermediate energy representations for <N>.

Although the main features of synchrotron radiation have been well known for a long time, there is certainly room for further developments as this thesis certainly provides. Also, although most of research seems to be of experimental nature, several recent papers are of theoretical nature as they have appeared recently in the literature: Hirschmugl, Sagurton and Williams (1991); Lieu, (1978); Milton, deRaad and Tsai, (1981); Orisa, (1982); Tsai, (1978); White, (1981).

A very brief account of this work is being published (Manoukian and Jearnkulprasert, 2000).

1.3 Plan of the Thesis

Chapter II is devoted to a detailed derivation of the total power of radiation in synchrotron radiation. In Chapter III we establish the main result to this work, that is, the explicit exact expression for <N>. This chapter (Section 3.1) also includes a detailed derivation of the Schwinger power integral, which is particularly needed in the subsequent work. Chapter IV deals with providing, respectively, high and low/ intermediate regimes representations for <N>. The concluding Chapter V summarizes our findings. Three mathematical appendices are provided including equations and formulae used throughout the thesis. Some properties of vectors are listed in Appendix A. A list of important integrals used in this thesis is given in Appendix B. Appendix C is devoted to the problem of Laurent expansions of some of explicit functions appearing in this work.

Chapter II

Derivation of the Power of Synchrotron Radiation

The previous chapter has dealt with some of the historical developments of synchrotron radiation. In this chapter, before launching into a general discussion, it is worth providing an elementary derivation of the total rate of radiation, based on Larmor's classical formula for a slowly moving electron, and arguments of relativistic invariance.

2.1 The General Solution of Maxwell's Equation

We start with the Maxwell equations (e.g., Jackson, 1975) for the electromagnetic fields (written in standard notation and in cgs units).

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{B} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}. \qquad (2.1)$$

We have just seen that $\nabla \bullet \mathbf{B} = 0$. It is convenient to set

$$\mathbf{B} = \nabla \times \mathbf{A}. \qquad (2.2)$$

Where **A** is the vector potential. The divergence of **B** is then automatically equal to zero because the divergence of a curl is zero, and

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial (\nabla \times \mathbf{A})}{\partial t}$$

$$\nabla \times (\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) = 0 .$$

Accordingly, we may introduce a scalar Φ and set

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla\Phi .$$

Thus we can rewrite **E** as

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi , \qquad (2.3)$$

where Φ is the so-called the scalar potential. We impose the Lorentz gauge condition

$$\nabla \bullet \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0 . \qquad (2.4)$$

Maxwell's equations can be rewritten as wave equations for the potentials. For the scalar potential, we start from the Maxwell equation for the divergence of **E**

$$\nabla \bullet \mathbf{E} = 4\pi\rho ,$$

$$\nabla \bullet (-\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) = 4\pi\rho ,$$

$$\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla \bullet \mathbf{A} = -4\pi\rho ,$$

$$\nabla^2\Phi + \frac{\partial}{\partial t}(-\frac{1}{c^2}\frac{\partial \Phi}{\partial t}) = -4\pi\rho ,$$

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -4\pi\rho . \qquad (2.5)$$

This is the wave equation for the scalar potential. For the vector potential, we consider the Maxwell equation involving the curl of **B**

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j} ,$$

where we have used Eqs. (2.2), and (2.3). Thus

$$\nabla \times (\nabla \times \mathbf{A}) = \frac{1}{c}\frac{\partial}{\partial t}(-\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) + \frac{4\boldsymbol{p}}{c}\mathbf{j}\,,$$

$$\nabla(\nabla \bullet \mathbf{A}) - \nabla^2 \mathbf{A} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla\Phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) + \frac{4\boldsymbol{p}}{c}\mathbf{j}\,,$$

$$\nabla(-\frac{1}{c}\frac{\partial \Phi}{\partial t}) - \nabla^2 \mathbf{A} = -\frac{1}{c}\frac{\partial}{\partial t}\nabla\Phi - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{4\boldsymbol{p}}{c}\mathbf{j}\,,$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\boldsymbol{p}}{c}\mathbf{j}\,. \qquad (2.6)$$

Now we will find the solution of Eqs. (2.5), (2.6) which is equivalent to finding the solution $\boldsymbol{\Psi}$ of the equation first:

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -S\,. \qquad (2.7)$$

Where *S* is called the source. Of course, *S* represents $4\pi\boldsymbol{r}$ and $4\pi\mathbf{j}/c$ corresponding to $\boldsymbol{\Phi}$ and **A,** respectively.

In free space $\boldsymbol{r}$ and $\mathbf{j}$ are zero and we get that the potentials satisfy the three-dimensional wave equation without sources, whose mathematical form is

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = 0\,. \qquad (2.8)$$

To find the solution of Eq (2.8), we consider first the one-dimensional wave equation:

$$\frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = 0\,. \qquad (2.9)$$

Then one possible solution is a function $\boldsymbol{\Psi}$(x,t) of the form

$\boldsymbol{\Psi}(x,t) = f(x-ct),$ (2.10)

which is, some function of the single variable (*x-ct*). The function *f(x-ct)* represents a "rigid" pattern in x which travels forward positive x at the speed *c*. Sometime it convenient to say that a solution of the one-dimensional wave equation is a function of (*t-x/c*).

Let us show that *f(x-ct)* is a solution of the wave equation. Let $f^{/}$ represents the derivative of *f* with respect to its variable *x* and $f^{//}$ represent the second derivative of *f*. Differentiating Eq. (2.10) with respect to *x*, one obtains

$$\frac{\partial \Psi}{\partial x} = f^{/}(x - ct)\,. \tag{2.11}$$

The second derivative of Ψ with respect to *x* is

$$\frac{\partial^2 \Psi}{\partial x^2} = f^{//}(x - ct)\,. \tag{2.12}$$

And taking derivative of Ψ with respect to *t*, we have

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= f^{/}(x - ct)(-c) \\ \frac{\partial^2 \Psi}{\partial t^2} &= f^{//}(x - ct)(c^2) \end{aligned}\,. \tag{2.13}$$

We see that Ψ does indeed satisfy the one-dimensional wave equation.

We start now describing spherical waves. Suppose we have a function that depends only on the radial distance *r* from a certain origin (a function that is spherically symmetric). Let's call the function Ψ(*r*), where *r* is

$$r = \sqrt{x^2 + y^2 + z^2}\ ,$$

the radial distance from the origin. In order to find out what functions Ψ(r) satisfy the wave equation, we will need an expression for the Laplacian of Ψ. Thus we want to find the sum of the second derivatives of Ψ with respect to *x*, *y* and *z*. We use the notation that $\Psi^{/}(r)$ represents the derivative of Ψ with respect to *r* and $\Psi^{//}(r)$ represents the second derivative of Ψ with respect to *r*.

First, we find the derivatives with respect to x. The first derivative is

$$\frac{\partial\Psi(r)}{\partial x} = \Psi^{/}(r)\frac{\partial r}{\partial x}.$$

The second derivative of Ψ with respect to x is

$$\frac{\partial^2\Psi(r)}{\partial x^2} = \Psi^{//}(r)(\frac{\partial r}{\partial x})^2 +.\Psi^{/}(r)\frac{\partial^2 r}{\partial x^2}.$$

We can evaluate the partial derivatives of r with respect to x from

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial^2 r}{\partial x^2} = \frac{1}{r}(1-\frac{x^2}{r^2}).$$

Thus the second derivative of Ψ with respect to x is

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{x^2}{r^2}\Psi^{//} +.\frac{1}{r}(1-\frac{x^2}{r^2})\Psi^{/}. \tag{2.14}$$

Likewise,

$$\frac{\partial^2\Psi}{\partial y^2} = \frac{y^2}{r^2}\Psi^{//} +\frac{1}{r}(1-\frac{y^2}{r^2})\Psi^{/}, \tag{2.15}$$

$$\frac{\partial^2\Psi}{\partial z^2} = \frac{z^2}{r^2}\Psi^{//} +.\frac{1}{r}(1-\frac{z^2}{r^2})\Psi^{/}. \tag{2.16}$$

The Laplacian is the sum of these three derivatives. Remembering that $x^2+y^2+z^2 = r^2$, one obtains

$$\nabla^2\Psi(r) = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2},$$

$$\nabla^2\Psi(r) = \frac{x^2}{r^2}\Psi^{//} + \frac{1}{r}(1-\frac{x^2}{r^2})\Psi^{/} + \frac{y^2}{r^2}\Psi^{//} + \frac{1}{r}(1-\frac{y^2}{r^2})\Psi^{/} + \frac{z^2}{r^2}\Psi^{//} + \frac{1}{r}(1-\frac{z^2}{r^2})\Psi^{/},$$

$$\nabla^2\Psi(r) = \frac{(x^2+y^2+z^2)}{r^2}\Psi^{//} + \frac{3}{r}\Psi^{/} - \frac{x^2+y^2+z^2}{r^3}\Psi^{/},$$

$$\nabla^2\Psi(r) = \frac{(x^2+y^2+z^2)}{r^2}\Psi^{//} + \frac{3}{r}\Psi^{/} - \frac{1}{r}\Psi^{/},$$

$$\nabla^2\Psi(r) = \Psi^{//}(r) + \frac{2}{r}\Psi^{/}(r)\,. \qquad (2.17)$$

It is often more convenient to write this equation in the following form:

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi)\,. \qquad (2.18)$$

You can see that the right-hand side of Eq. (2.18) is the same as in Eq (2.17).

We wish to consider spherically symmetric fields, which can propagate, as spherical waves, our field quantity must be a function of both *r* and *t*. Now we want to know what function Ψ (*r*, t) are solution of the three-dimension wave equation

$$\nabla^2\Psi(r,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi(r,t) = 0\,. \qquad (2.19)$$

Since Ψ (*r*, *t*) depends only on the spatial coordinates and is also a function of *t*, we should write the derivatives with respect to *r* as partial derivatives. Then the wave equation becomes

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi = 0\,.$$

If this equation is multiplied by *r*, one obtains

$$\frac{\partial^2}{\partial r^2}(r\Psi) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(r\Psi) = 0\,. \qquad (2.20)$$

This equation tells us that the function $r\Psi$ satisfies the one-dimension wave equation in the variable *r*. We know that if $r\Psi$ is a function only of (*r-ct*) then it will be a solution of Eq. (2.20). Thus one obtains that spherical waves must have the form

$r\Psi(r, t)=f(r-ct)$,

$$\Psi(r,t) = \frac{f(t - r/c)}{r}\,. \qquad (2.21)$$

Physically this is a solution of Eq. (2.8) everywhere except right near r=0, where it must be a solution of the complete Eq. (2.7). Let's see how that works. What kind of a source s would give a spherical wave.

Suppose we have a spherical wave and see what happens when r goes to zero. This means that the retardation term $-r/c$ in $f(t-r/c)$ can be neglected. Let f be a smooth equation. Then Ψ becomes

$$\Psi(r,t) = \frac{f(t)}{r} \qquad (r \to 0). \tag{2.22}$$

Therefore Ψ is like a Coulomb field for a charge at the origin that varies with time. Thus if we had a little lump of charge with a density ρ and we know that

$$\Phi = \frac{Q}{r}.$$

Where $Q = \int_V \rho d^3\mathbf{r}$ and Φ satisfies the equation

$$\nabla^2\Phi = -4\pi\rho .$$

Following the same mathematics, it is seen that Ψ of (2.22) satisfies

$$\nabla^2\Psi = -s \qquad (r \to 0), \tag{2.23}$$

where s is related to f by

$$f = \frac{S}{4\pi},$$

with

$$S = \int_V s d^3\mathbf{r}.$$

The only difference is that in the general case s and S can be a function of time.

Now the important thing is that if Ψ satisfies Eq. (2.23) for small r, it also satisfies Eq. (2.7). As we go very close to the origin, the 1/*r* dependence of Ψ causes the space derivatives to become vary large. But the time derivative keeps its same value. So as r goes to zero, the term $\partial^2\Psi/\partial t^2$ in Eq. (2.7) can be neglected in comparison to $\nabla^2\Psi$, and Eq. (2.7) becomes equivalent to Eq.(2.23)

Then the source function *s*(*t*) is localized at the origin and has the total strength

$$S(t) = \int_V s(t) d^3\mathbf{r}\,, \tag{2.24}$$

and the solution of Eq. (2.7) is

$$\Psi(\mathbf{r},t) = \frac{1}{4p}\frac{S(t-r/c)}{r}\,. \tag{2.25}$$

The only effect of the term $\partial^2\Psi/\partial t^2$ in Eq. (2.7) is to introduce the retardation (*t*-*r/c*) in the Coulomb-like potential. The solution corresponding to a spread-out source, may be thought of as the source *s*(**r**,*t*) is made up of the sum of many point sources. Since Eq. (2.7) is linear, the resultant field is the superposition of the field from all of such source elements.

By using Eq. (2.25) one knows that the field dΨ at the point (**r**) and the time *t* from a source element $sd^3\mathbf{r}^/$ at the point ($\mathbf{r}^/$) is given by

$$d\Psi(\mathbf{r},t) = \frac{s(\mathbf{r}^/, t-R/c)d^3\mathbf{r}^/}{4pR}\,,$$

where *R* is the distance from $\mathbf{r}^/$ to **r**. Adding the contribution from all the pieces of the source and integrate over all regions where *s* is not equal to zero, one obtains

$$\Psi(\mathbf{r},t) = \int \frac{s(\mathbf{r}^/, t-R/c)d^3\mathbf{r}^/}{4pR}\,. \tag{2.26}$$

This means the field at **r** at time t is the sum of all spherical waves that leave from the source at $\mathbf{r}^{\prime}$ at the time (*t-R/c*).

From this we obtain the general solution of Maxwell's equations. When we let Ψ represent Φ the scalar potential, the source function becomes $4\pi \boldsymbol{r}$. On the other hand if Ψ represents the vector potential **A**, the corresponding source function is $4\pi\mathbf{j}/c$. Thus one obtains the so-called Liénard-Wiechert solution:

$$\Phi(\mathbf{r},t)=\int\frac{\boldsymbol{r}(\mathbf{r}^{\prime},t-R/c)d^{3}\mathbf{r}^{\prime}}{R}\,, \tag{2.27}$$

$$\mathbf{A}(\mathbf{r},t)=\int\frac{\mathbf{j}(\mathbf{r}^{\prime},t-R/c)d^{3}\mathbf{r}^{\prime}}{cR}\,. \tag{2.28}$$

These potentials are known as the "retarded potentials".

2.2 Electric Field of a Point Charge in Motion

The electric field of an arbitrarily moving point charge can be obtained from the Liénard-Wiechert potential. We now specialize Eqs. (2.27) and (2.28) to the case of a point charge *e* moving along a trajectory **w**(t) (position of charge at time t-see Fig.2-1), for which the charge density and current density are given by

$$\boldsymbol{r}(\mathbf{r},t)=e\boldsymbol{d}[\mathbf{r}-\mathbf{w}(t)]\,, \tag{2.29}$$

$$\mathbf{j}(\mathbf{r},t)=e\mathbf{v}\boldsymbol{d}[\mathbf{r}-\mathbf{w}(t)]\,, \tag{2.30}$$

where **v** is the three-velocity of the charge. The potentials for such a source are given by

$$\Phi(\mathbf{r},t)=\int d^{3}\mathbf{r}^{\prime}dt^{\prime}\,\frac{e}{R}\boldsymbol{d}\left[\mathbf{r}^{\prime}-\mathbf{w}(t^{\prime})\right]\boldsymbol{d}\left[t^{\prime}-t+\frac{R(t^{\prime})}{c}\right], \tag{2.31}$$

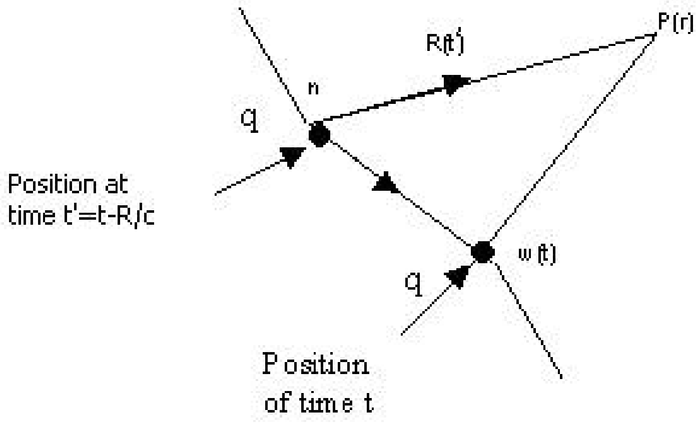

Figure 2-1 The electromagnetic field at time t, of the charge q moving along a trajectory w(t), depends on its position at the retarded time t' = t – R/c

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}'\, dt'\, \frac{e\mathbf{v}(t')}{R}\,\delta\left[\mathbf{r}' - \mathbf{w}(t')\right]\delta\left[t' - t + \frac{R(t')}{c}\right], \tag{2.32}$$

where $R = |\,\mathbf{r} - \mathbf{r}'\,|$ (distance from the retarded position of the charge to the point P).

From the presence of the delta function in Eqs. (2.31) and (2.32), it is clear, that the potentials depend on the retarded position and velocity of the charge, and implies that there will be contributions only when the condition:

$$t' - t + \frac{R(t')}{c} = 0 \qquad \Rightarrow \qquad \frac{R(t')}{c} = t - t', \tag{2.33}$$

is satisfied. This is known as the “light cone” condition.

At this stage, one could carry out the integrations in Eqs. (2.31) and (2.32) and obtain the usual expressions for the Liénard-Wiechert potentials. The electromagnetic fields are then obtained using Eqs. (2.2) and (2.3). However, it turns out to be convenient for our purposes to find $\partial\mathbf{A}/\partial\mathbf{t}$ and $\nabla\Phi$, using Φ and $\mathbf{A}$ as given in (2.31) and (2.32). First, we evaluate $\nabla\Phi$ by using the basic integral result,

$$\int f(\mathbf{x})\boldsymbol{d}(\mathbf{x}-\mathbf{y})d^3\mathbf{x} = f(\mathbf{y}),$$

to can carry out the spatial integrations in (2.31) and obtain

$$\Phi(\mathbf{r},t) = \int dt' \,\frac{e}{R}\boldsymbol{d}(t' - t + \frac{R}{c}),$$

where now $R(t') = |\ \mathbf{r} - \mathbf{w}(t')\ |$. And from $\mathbf{R} = \mathbf{r} - \mathbf{w}(t')$, one obtainds $\frac{d\mathbf{R}}{dt'} = -\frac{d\mathbf{w}(t')}{dt'}$ since $\mathbf{r}$ is held constant. The charged particles velocity is denoted by $\mathbf{v}(t') = \frac{d\mathbf{w}(t')}{dt'}$, hence one obtains $\mathbf{v}(t') = -\frac{d\mathbf{R}(t')}{dt'}$. Therefore,

$$\nabla\Phi(\mathbf{r},t) \equiv \frac{\partial\Phi}{\partial\mathbf{r}} = e\int dt' \left\{\left[-\frac{\mathbf{R}}{R^3}\boldsymbol{d}\left(t' - t + \frac{R}{c}\right)\right] + \frac{\mathbf{R}}{cR^2}\frac{d\boldsymbol{d}}{df}\right\}, \qquad (2.34)$$

where $f(t') = t' - t + R/c$. The first term in the Eq. (2.34) can be simplified by using the formula

$$\boldsymbol{d}[f(t')] = \sum \frac{\boldsymbol{d}(t' - t_0)}{\left|df/dt'\right|_{t'=t_0}},$$

where t_0 stands for the zeros of $f(t')$. This gives T as the solution of $t' = t - R(t')/c$. Thus one obtains

$$\boldsymbol{d}(t' - t + \frac{R}{c}) = \frac{\boldsymbol{d}(t' - T)}{\left|\frac{d}{dt'}\left[t' - t + \frac{R(t')}{c}\right]\right|_{t'=T}},$$

and that

$$\frac{df}{dt'} = \frac{d}{dt'}\left[t' - t + \frac{R(t')}{c}\right] = 1 + \frac{1}{c}\frac{dR}{d\mathbf{R}}\frac{d\mathbf{R}}{dt'} \ ,$$

$$\frac{df}{dt'} = 1 - \frac{\mathbf{v} \bullet \mathbf{R}}{Rc} = 1 - v_R$$

$$\delta(t' - t + \frac{R}{c}) = \frac{\delta(t' - T)}{1 - v_R} \ . \qquad (2.35)$$

For the second term, notice that

$$\frac{d\delta}{df} = (\frac{df}{dt'})^{-1}\frac{d\delta}{dt'} \ . \qquad (2.36)$$

Upon substituting Eqs. (2.35), and (2.36) into Eq. (2.34), one obtains

$$\nabla\Phi(\mathbf{r},t) = e\int dt' \left\{ \left[-\frac{\mathbf{R}}{R^3}\frac{\delta(t' - T)}{(1 - v_R)} \right] + \frac{1}{(1 - v_R)}\frac{\mathbf{R}}{cR^2}\frac{d\delta\left(t' - t + \frac{R}{c}\right)}{dt'} \right\} ,$$

$$\nabla\Phi(\mathbf{r},t) = -\frac{e}{(1 - v_R)}\frac{\mathbf{R}}{R^3} + \frac{e}{(1 - v_R)}\int dt' \frac{1}{(1 - v_R)}\frac{\mathbf{R}}{cR^2}\frac{d\delta(t' - T)}{dt'} \ ,$$

From the property of the Dirac delta function

$$\int_a^b f(t)\frac{d\delta(t - T)}{dt}dt = -f'(T) \ , \qquad (2.37)$$

finally we have

$$\nabla\Phi(\mathbf{r},t) = -\frac{e}{(1 - v_R)}\frac{\mathbf{R}}{R^3} - \frac{e}{(1 - v_R)}\frac{d}{cdt'}\left[\frac{\mathbf{R}}{R^2(1 - v_R)}\right] , \qquad (2.38)$$

where now t' is the retarded time.

Similarly, we do the same thing with the vector potential $\mathbf{A}$ to find $\partial\mathbf{A}/\partial t$. Starting now form Eq. (2.32) and by carrying out the spatial integrations in (2.32) we obtain

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int dt' \, \frac{e\mathbf{v}(t')}{R}\boldsymbol{d}\left(t' - t + \frac{R}{c}\right),$$

$$\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{1}{c}\int dt' \, \frac{e\mathbf{v}(t')}{R}\frac{\partial}{\partial t}\boldsymbol{d}\left(t' - t + \frac{R}{c}\right),$$

$$\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = -\frac{1}{c}\int dt' \, \frac{e\mathbf{v}(t')}{R}\frac{\partial}{\partial f}\boldsymbol{d}\left(t' - t + \frac{R}{c}\right),$$

$$\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = -\frac{1}{c}\int dt' \, \frac{e\mathbf{v}(t')}{R}\left(\frac{df}{dt'}\right)^{-1}\frac{\partial}{\partial t'}\boldsymbol{d}\left(t' - t + \frac{R}{c}\right),$$

$$\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = -\frac{1}{c(1-v_R)}\int dt' \, \frac{1}{(1-v_R)}\frac{e\mathbf{v}(t')}{R}\frac{\partial}{\partial t'}\boldsymbol{d}(t' - T),$$

$$\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{1}{c(1-v_R)}\frac{\partial}{\partial t'}\left(\frac{e\mathbf{v}}{R(1-v_R)}\right). \tag{2.39}$$

Upon substituting Eqs. (2.38), and (2.39) in Eq. (2.3), one obtains for the electric field

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi,$$

$$\mathbf{E} = -\frac{1}{c^2(1-v_R)}\frac{\partial}{\partial t'}\left[\frac{e\mathbf{v}}{R(1-v_R)}\right] + \frac{e}{(1-v_R)}\frac{\mathbf{R}}{R^3} + -\frac{e}{(1-v_R)}\frac{d}{cdt'}\left[\frac{\mathbf{R}}{R^2(1-v_R)}\right],$$

$$\mathbf{E} = \frac{e}{(1-v_R)}\frac{\mathbf{n}}{R^2} + \frac{e}{(1-v_R)}\frac{d}{cdt'}\left[\frac{\mathbf{R} - (\mathbf{v}/c)R}{R^2(1-v_R)}\right]. \tag{2.40}$$

Noting from Eq. (2.33) that

$$\frac{dt'}{dt} = 1 - \frac{dR}{cdt} = (1-v_R)^{-1} \equiv 1 - \frac{\dot{R}}{c}.$$

Eq. (2.40) becomes

$$\mathbf{E} = \frac{e\mathbf{n}}{R^2}\left(1 - \frac{\dot{R}}{c}\right) + e\frac{d}{cdt}\left[\frac{(1-\dot{R}/c)}{R}\left(\mathbf{n} - \frac{\mathbf{v}}{c}\right)\right].$$

And with $t^/$ standing for the retarded time, the required derivatives are

$$\frac{\partial \mathbf{R}}{\partial t} = \frac{\partial}{\partial t}\left[\mathbf{r} - \mathbf{w}(t^/)\right] = -\frac{\partial \mathbf{w}(t^/)}{\partial t} = -\mathbf{v}\left(\frac{\partial t^/}{\partial t}\right),$$

$$\frac{\partial R}{\partial t} = \frac{\partial}{\partial t}\left[c(t - t^/)\right] = c\left(1 - \frac{\partial t^/}{\partial t}\right),$$

or

$$\frac{\partial R}{\partial t} = \frac{\partial}{\partial t}\sqrt{\mathbf{R} \bullet \mathbf{R}} = \mathbf{n} \bullet \left(\frac{\partial \mathbf{R}}{\partial t}\right) = -(\mathbf{n} \bullet \mathbf{v})\left(\frac{\partial t^/}{\partial t}\right),$$

and

$$\frac{\partial \mathbf{n}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\mathbf{R}}{R}\right) = \frac{1}{R}\left[\mathbf{n}(\mathbf{n} \bullet \mathbf{v}) - \mathbf{v}\right]\left(\frac{\partial t^/}{\partial t}\right).$$

With these properties at hand one obtains

$$\mathbf{E} = \frac{e\mathbf{n}}{R^2} - \frac{e\mathbf{n}}{R^2}\frac{\dot{R}}{c} + \frac{e}{c}\frac{d}{dt}\left[\frac{1}{R}\left(\mathbf{n} - \frac{\mathbf{n}\dot{R}}{c} - \frac{\mathbf{v}}{c} + \frac{\mathbf{v}\dot{R}}{c^2}\right)\right],$$

$$\mathbf{E} = \frac{e\mathbf{n}}{R^2} - \frac{e\mathbf{n}}{R^2}\frac{\dot{R}}{c} + \frac{e}{c}\frac{d}{dt}\frac{\mathbf{n}}{R} - \frac{e}{c}\frac{d}{dt}\left(\frac{\mathbf{n}\dot{R}}{cR}\right) + \frac{q}{c^2}\frac{d}{dt}\left[\frac{1}{R}\left(\frac{\mathbf{v}\dot{R}}{c} - \mathbf{v}\right)\right],$$

$$\mathbf{E} = \frac{e\mathbf{n}}{R^2} - \frac{e\mathbf{n}}{R^2}\frac{\dot{R}}{c} + \frac{e}{c}\frac{d}{dt}\frac{\mathbf{n}}{R} - \frac{e}{c}\frac{d}{dt}\left(\frac{\mathbf{n}\dot{R}}{cR}\right) + \frac{q}{c^2}\frac{d}{dt}\left\{\frac{1}{R}\left[-\mathbf{v}\left((1 - \frac{\dot{R}}{c}\right)\right]\right\},$$

$$\mathbf{E} = \frac{e\mathbf{n}}{R^2} - \frac{e\mathbf{n}}{R^2}\frac{\dot{R}}{c} + \frac{e}{c}\frac{d}{dt}\frac{\mathbf{n}}{R} - \frac{e}{c}\frac{d}{dt}\left(\frac{\mathbf{n}\dot{R}}{cR}\right) + \frac{q}{c^2}\frac{d}{dt}\left\{\frac{1}{R}\left[-\mathbf{v}\left(\frac{dt^/}{dt}\right)\right]\right\},$$

$$\mathbf{E}=\frac{e\mathbf{n}}{R^2}-\frac{e\mathbf{n}}{R^2}\frac{\dot{R}}{c}+\frac{e}{c}\frac{d}{dt}\frac{\mathbf{n}}{R}-\frac{e}{c}\frac{d}{dt}\left(\frac{\mathbf{n}\dot{R}}{cR}\right)+\frac{q}{c^2}\frac{d}{dt}\left\{\frac{1}{R}\left[-\mathbf{v}\left(\frac{dt^{/}}{dt}\right)-\mathbf{n}(\mathbf{n}\bullet\mathbf{v})\left(\frac{dt^{/}}{dt}\right)\right.\right.,$$
$$\left.\left.+\mathbf{n}(\mathbf{n}\bullet\mathbf{v})\left(\frac{dt^{/}}{dt}\right)\right]\right\}$$

$$\mathbf{E}=\frac{e\mathbf{n}}{R^2}-\frac{e\mathbf{n}}{R^2}\frac{\dot{R}}{c}+\frac{e}{c}\frac{d}{dt}\frac{\mathbf{n}}{R}-\frac{e}{c}\frac{d}{dt}\left(\frac{\mathbf{n}\dot{R}}{cR}\right)+\frac{q}{c^2}\frac{d}{dt}\left\{\frac{1}{R}\left[-\mathbf{n}(\mathbf{n}\bullet\mathbf{v})\left(\frac{dt^{/}}{dt}\right)\right.\right.,$$
$$\left.\left.+R\frac{1}{R}[\mathbf{n}(\mathbf{n}\bullet\mathbf{v})-\mathbf{v}]\left(\frac{dt^{/}}{dt}\right)\right]\right\}$$

$$\mathbf{E}=\frac{e\mathbf{n}}{R^2}-\frac{e\mathbf{n}}{R^2}\frac{\dot{R}}{c}+\frac{e}{c}\frac{d}{dt}\frac{\mathbf{n}}{R}-\frac{e}{c}\frac{d}{dt}\left(\frac{\mathbf{n}\dot{R}}{cR}\right)+\frac{e}{c^2}\frac{d}{dt}\left[\frac{1}{R}\left(R\frac{d\mathbf{n}}{dt}+\mathbf{n}\frac{dR}{dt}\right)\right],$$

$$\mathbf{E}=\frac{e\mathbf{n}}{R^2}-\frac{e\mathbf{n}}{R^2}\frac{\dot{R}}{c}+\frac{e}{c}\frac{d}{dt}\frac{\mathbf{n}}{R}-\frac{e}{c}\frac{d}{dt}\left(\frac{\mathbf{n}\dot{R}}{cR}\right)+\frac{e}{c^2}\frac{d}{dt}\left[\frac{1}{R}\frac{d}{dt}(R\mathbf{n})\right],$$

$$\mathbf{E}=e\frac{\mathbf{n}}{R^2}+e\frac{R}{c}\frac{d}{dt}\left(\frac{\mathbf{n}}{R^2}\right)+\frac{e}{c^2}\frac{d^2\mathbf{n}}{dt^2}. \qquad (2.42)$$

Thus Eq. (2.40) is related in a rather simple way to Feynman's formula (Feynman, et al., 1989). Feynman also gives a simple interpretation to his formula for the electric field. The first term is the Coulomb field of the charge at its retarded position. The second term takes into account the motion of the charge and is roughly the time rate of change of the retarded Coulomb field multiplied by the time the charge takes to travel from the retarded to the present position. The third term is, in a non-obvious way, a higher-order correction to the electric field.

For to sake of completeness, one also provide the derivation of the usual expression for the electromagnetic field due the charged particle, starting from Eq. (2.40), by defining, in the process, first $K = (1 - v_R)$. All that one has to do is to carry out the time differentiation in Eq. (2.40), noting that

$$\frac{\partial KR}{\partial t^{/}}=\frac{\partial}{\partial t^{/}}\left(R-\frac{\mathbf{v}\bullet\mathbf{R}}{c}\right)=\frac{1}{c}(\mathbf{v}\bullet\mathbf{v}-c\mathbf{n}\bullet\mathbf{v}-\mathbf{R}\bullet\mathbf{a}).$$

From Eq. (2.40)

$$\mathbf{E}=\frac{e}{(1-v_R)}\frac{\mathbf{n}}{R^2}+\frac{e}{(1-v_R)}\frac{d}{cdt^{/}}\left[\frac{\mathbf{R}-(\mathbf{v}/c)R}{R^2(1-v_R)}\right],$$

$$\mathbf{E}=\frac{e}{K}\frac{\mathbf{n}}{R^2}+\frac{e}{K}\frac{d}{cdt^{/}}\left[\frac{\mathbf{n}-(\mathbf{v}/c)}{KR}\right],$$

$$\mathbf{E}=\frac{e}{K}\frac{\mathbf{n}}{R^2}+\frac{e}{K}\frac{d}{cdt^{/}}\left(\frac{\mathbf{n}}{KR}-\frac{\mathbf{v}}{cKR}\right),$$

$$\mathbf{E}=\frac{e}{K}\frac{\mathbf{n}}{R^2}+\frac{e}{cK}\left[\frac{1}{RK}\frac{d\mathbf{n}}{dt^{/}}+\mathbf{n}\frac{d}{dt^{/}}\left(\frac{1}{KR}\right)-\frac{1}{cKR}\frac{d\mathbf{v}}{dt^{/}}-\frac{\mathbf{v}}{c}\frac{d}{dt^{/}}\left(\frac{1}{KR}\right)\right],$$

$$\mathbf{E}=\frac{e}{K}\frac{\mathbf{n}}{R^2}+\frac{e}{cK}\left\{\frac{\mathbf{n}(\mathbf{n}\bullet\mathbf{v})-\mathbf{v}}{R^2K}-\frac{\mathbf{n}}{cK^2R^2}\left[v^2-c(\mathbf{n}\bullet\mathbf{v})-\mathbf{R}\bullet\frac{dv}{dt^{/}}\right]-\frac{1}{cKR}\frac{d\mathbf{v}}{dt^{/}}\right.$$
$$\left.+\frac{\mathbf{v}}{c^2K^2R^2}\left[v^2-c(\mathbf{n}\bullet\mathbf{v})-\mathbf{R}\bullet\frac{dv}{dt^{/}}\right]\right\},$$

$$\mathbf{E}=\frac{e}{K}\frac{\mathbf{n}}{R^2}+\frac{e}{cK}\left\{\frac{\mathbf{n}(\mathbf{n}\bullet\mathbf{v})-\mathbf{v}}{R^2K}-\frac{1}{cK^2R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\left[v^2-c(\mathbf{n}\bullet\mathbf{v})-\mathbf{R}\bullet\frac{dv}{dt^{/}}\right]\right.,$$
$$\left.-\frac{1}{cKR}\frac{d\mathbf{v}}{dt^{/}}\right\}$$

$$\mathbf{E}=\frac{e\mathbf{n}}{K^2R^2}\left(1-\frac{\mathbf{n}\bullet\mathbf{v}}{c}\right)+\frac{e[\mathbf{n}(\mathbf{n}\bullet\mathbf{v})-\mathbf{v}]}{cR^2K^2}-\frac{e}{c^2K^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)v^2$$
$$+\frac{e}{cK^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)(\mathbf{n}\bullet\mathbf{v})+\frac{e}{c^2K^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\mathbf{R}\bullet\frac{dv}{dt^{/}}-\frac{e}{c^2K^2R}\frac{d\mathbf{v}}{dt^{/}},$$

$$\mathbf{E}=\frac{e}{K^2R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)-\frac{e\mathbf{n}(\mathbf{n}\bullet\mathbf{v})}{cK^2R^2}+\frac{e\mathbf{n}(\mathbf{n}\bullet\mathbf{v})}{cK^2R^2}-\frac{e}{c^2K^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)v^2$$
$$+\frac{e}{cK^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)(\mathbf{n}\bullet\mathbf{v})+\frac{e}{c^2K^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\left(\mathbf{R}\bullet\frac{dv}{dt'}\right)-\frac{e}{c^2K^3R}\left(1-\frac{\mathbf{n}\bullet\mathbf{v}}{c}\right)\frac{d\mathbf{v}}{dt'},$$

$$\mathbf{E}=\frac{e}{K^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\left(1-\frac{\mathbf{n}\bullet\mathbf{v}}{c}\right)-\frac{e}{K^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\frac{v^2}{c^2}+\frac{e}{cK^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)(\mathbf{n}\bullet\mathbf{v})$$
$$+\frac{e}{c^2K^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\left(\mathbf{R}\bullet\frac{dv}{dt'}\right)-\frac{e}{c^2K^3R^2}\left[\mathbf{R}\bullet\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\right]\frac{d\mathbf{v}}{dt'},$$

$$\mathbf{E}=\frac{e}{K^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)-\frac{e}{K^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\frac{v^2}{c^2}+\frac{e}{cK^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)(\mathbf{n}\bullet\mathbf{v})$$
$$+\frac{e}{K^3R^2}\left\{\mathbf{R}\times\left[\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\times\frac{dv}{c^2dt'}\right]\right\}-\frac{e}{cK^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)(\mathbf{n}\bullet\mathbf{v}),$$

$$\mathbf{E}=\frac{e}{K^3R^2}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\left(1-\frac{v^2}{c^2}\right)+\frac{e}{K^3R^2}\left\{\mathbf{R}\times\left[\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\times\frac{dv}{c^2dt'}\right]\right\}.$$

The final result can be written in the form

$$\mathbf{E}=\frac{e(1-v^2/c^2)}{R^2(1-v_R)^3}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)+\frac{e}{R^2(1-v_R)^3}\left\{\mathbf{R}\times\left[\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\times\frac{d\mathbf{v}}{c^2dt'}\right]\right\}. \qquad (2.43)$$

For the magnetic field we have

$$\mathbf{B}=e\left[\frac{(\mathbf{v}\times\mathbf{n})(1-\boldsymbol{b}^2)}{cK^3R^2}+\frac{(\mathbf{a}\bullet\mathbf{n})(\mathbf{v}\times\mathbf{n})}{c^3K^3R}+\frac{\mathbf{a}\times\mathbf{n}}{c^2K^2R}\right], \qquad (2.44)$$

as it follows from

$$\mathbf{B}=\mathbf{n}\times\mathbf{E}. \qquad (2.45)$$

The latter expression for **B** involves the unit vector (**n**), from the retarded position of the charge to the observation point. Evidently, the magnetic field of a point charge is always perpendicular to the electric field and to the vector from the retarded point.

One can be see that the first term of E involves the velocity (but not the acceleration) of the particle; it has vector components parallel to $\mathbf{R} = R\mathbf{n}$ and to the velocity **v,** and reduces to the familiar Coulomb field at low speeds. The second term is proportional to the particle's acceleration **a**, and is perpendicular to **R**. Thus the electric field may be decomposed into a velocity component (velocity field) and an acceleration component (acceleration field).

2.3 Total Power radiation by an Accelerated charge- Larmor's Formula

The energy flux associated with the fields of a point charge is given by the Poynting vector

$$\mathbf{S} = \frac{c}{4p}\mathbf{E}\times\mathbf{B} = \frac{c}{4p}[\mathbf{E}\times(\mathbf{n}\times\mathbf{E})] = \frac{c}{4p}[\mathbf{n}E^2 - \mathbf{E}(\mathbf{n}\bullet\mathbf{E})] \,. \qquad (2.46)$$

However this is not all energy flux representing radiation; some of it is just the field energy carried along by the particle as it moves. To calculate the total power radiated by the particle at time $t^{/}$, we draw a sphere radius R (Fig.2-2) and have the particle at its center. Let the appropriate interval

$$t - t^{/} = \frac{R(t^{/})}{c},$$

for radiation correspond to the radius of the sphere. From the section 2.2, both the electric and magnetic fields may by decomposed into a velocity component and an acceleration component. The velocity fields obey inverse-square laws of the distance,

$$\mathbf{E}_v, \mathbf{B}_v \propto \frac{1}{[R^2]}.$$

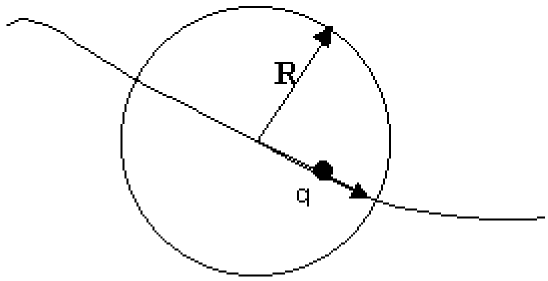

Figure 2-2 Trajectory of the charged particle

On the other hand, the acceleration fields obey inverse-first-power laws,

$$\mathbf{E}_a, \mathbf{B}_a \propto \frac{1}{[R]}.$$

Thus the Poynting vectors associated with the various portions of the fields scale with distance as

$$\mathbf{S}_{vv}[= \frac{c}{4\pi}\mathbf{E}_v \times \mathbf{B}_v] \propto \frac{1}{[R^4]}$$

$$\mathbf{S}_{va}, \mathbf{S}_{av}, \propto \frac{1}{[R^3]}.$$

$$\mathbf{S}_{aa} \propto \frac{1}{[R^2]}$$

Now we consider the integration of the Poynting vector over the surface of the sphere (using $t^{/}$ as the retarded time for all points on the surface of the sphere at time t). Because the area of the sphere is proportional to R^2, thus any term in $\mathbf{S}$ which goes to zero like $1/R^2$ will give finite contribution, but terms like $1/R^3$ and $1/R^4$ will contribute nothing in the limit when R goes to infinity. For this reason only the acceleration field represents radiation. Thus $\mathbf{E}$ will be rewritten only in terms of the radiation term

$$\mathbf{E} = \frac{e}{R^2(1-v_R)^3}\left\{\mathbf{R} \times \left[\left(\mathbf{n} - \frac{\mathbf{v}}{c}\right) \times \frac{d\mathbf{v}}{c^2 dt^{/}}\right]\right\}, \qquad (2.47)$$

(the velocity fields carry energy, and as the charge moves this energy is dragged along-but it's not radiated. In particular, a charge must accelerate in order to radiate.). The radiation field is perpendicular to **R** at any point on the sphere. Thus the second term in the square brackets on the right-hand of Eq. (2.46) is equal to zero. One then obtains the Poynting vector given by

$$\mathbf{S}_{rad} = \frac{c}{4\pi} E_a{}^2 \mathbf{n} .$$

Because the Poynting vector, in magnitude, represents the energy per unit time, per unit area, of radiation emission we set

$$P = -\frac{dE}{dt} = \oint_s \mathbf{S} \bullet d\mathbf{a} = \oint_s r^2 \mid S \mid d\Omega .$$

This means that the power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = r^2 \mid S \mid = \frac{c}{4\pi} \left| r\mathbf{E}_a \right|^2 ,$$

$$\frac{dP}{d\Omega} = \frac{c}{4\pi} \left| \frac{e}{R(1 - v_R)^3} \mathbf{R} \times \left[\left(\mathbf{n} - \frac{\mathbf{v}}{c} \right) \times \frac{d\mathbf{v}}{c^2 dt'} \right] \right|^2 . \qquad (2.48)$$

If the speed of a particle is sufficiently small that it can be neglected in comparison to *c*, then Eq. (2.48) becomes

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} \mid \{ \mathbf{n} \times [\mathbf{n} \times \dot{\mathbf{v}}] \} \mid^2 . \qquad (2.49)$$

If θ is the angle between the acceleration $\dot{\mathbf{v}}$ and **n**, as shown in Fig. 2-3, then power radiated can be written

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} \mid \dot{\mathbf{v}} \mid^2 \sin^2 \theta .$$

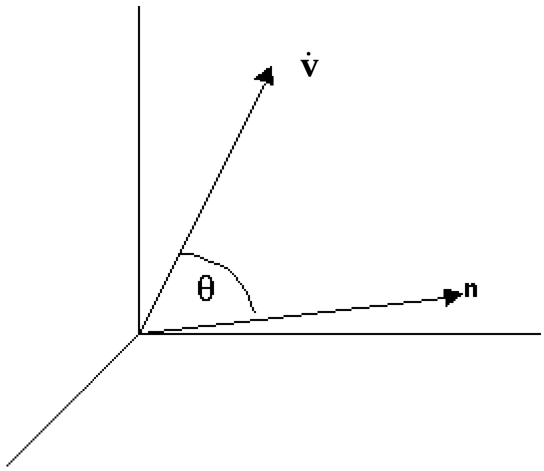

Figure 2-3 Angular dependence of the acceleration

A polar plot of this "$\sin^2\theta$" distribution of radiated power is shown in Fig. 2-4. No power is radiated in the forward or backward direction (it is emitted in a donut shape about the direction of instantaneous acceleration).

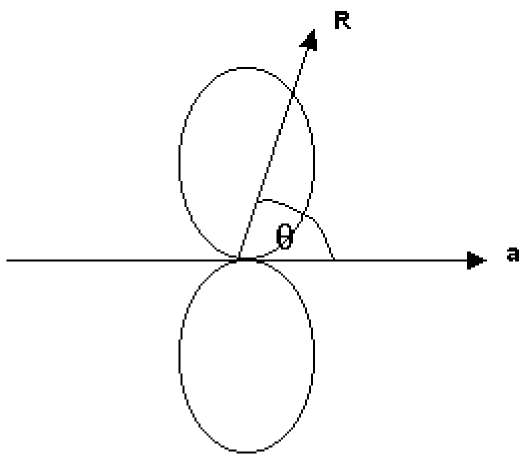

Figure 2-4 Angular dependence of radiation from a slow accelerated charge.

The total radiated power is obtained by integrating over the entire sphere:

$$P = \int_{4\pi} \frac{dP}{d\Omega} d\Omega = \frac{e^2 |\dot{\mathbf{v}}|^2}{4\pi c^3} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^2\theta \, d\theta \, ,$$

or

$$P = \frac{2}{3}\frac{e^2 \,|\, \dot{\mathbf{v}} \,|^2}{c^3}\,. \qquad (2.50)$$

This is the familiar Larmor formula for a nonrelativistic, accelerated charge.

2.4 Total Power Radiation by an Accelerated charge- Larmor's Formula with its Relativistic Generalization

The Larmor's formula for the power radiated by an electron that is instantaneously at rest in its co-moving frame is

$$P = \frac{2}{3}\frac{e^2 \,|\, \dot{\mathbf{v}} \,|^2}{c^3}\,,$$

or it can be written as

$$P = \frac{2}{3}\frac{e^2}{m^2 c^3}\left(\frac{d\mathbf{p}}{dt}\right)^2. \qquad (2.51)$$

Now, radiated energy and elapsed time transform in the same manner under Lorentz transformations, thus the radiated power must be an invariant. We shall have succeeded in deriving a formula for the power radiated by an electron of arbitrary velocity if we can exhibit an invariant that reduces to Eq. (2.51) in the instantaneous rest co-moving system of the electron ($b\rightarrow 0$). To accomplish this, first the time derivative is replaced by the derivative with respect to the invariant proper time. The differential of proper time is defined by

$$ds^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)\,,$$

or

$$ds = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} dt = \frac{dt}{g}, \qquad (2.52)$$

where g = $(1\text{-}v^2/c^2)^{-1/2}$. Secondly, the square of the proper time derivative of the momentum is replaced by the invariant combination

$$\left(\frac{d\mathbf{p}}{dt}\right)^2 \to \left(\frac{d\mathbf{p}}{ds}\right)^2 - \frac{1}{c}\left(\frac{dE}{ds}\right)^2.$$

Hence, as the desired invariant generalization of Eq. (2.51), one obtains

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\left[\left(\frac{d\mathbf{p}}{ds}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{ds}\right)^2\right],$$

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\left[\left(\frac{d\mathbf{p}}{dt}\cdot\frac{dt}{ds}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\cdot\frac{dt}{ds}\right)^2\right],$$

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\left[g^2\left(\frac{d\mathbf{p}}{dt}\right)^2 - \frac{g^2}{c^2}\left(\frac{dE}{dt}\right)^2\right],$$

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\frac{g^2m^2c^4}{m^2c^4}\left[\left(\frac{d\mathbf{p}}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right],$$

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{E}{mc^2}\right)^2\left[\left(\frac{d\mathbf{p}}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right]. \qquad (2.53)$$

The conventional form of this result is obtained on writing

$$\mathbf{p} = \frac{m\mathbf{v}}{(1-b^2)^{1/2}}, \qquad E = \frac{mc^2}{(1-b^2)^{1/2}}, \qquad \hat{\mathbf{a}} = \frac{\mathbf{v}}{c}, \qquad (2.54)$$

and performing the indicated differentiations. Thus, from Eq. (2.53) when substituted in Eq. (2.54) gives

$$P = \frac{2}{3}\frac{e^2}{c^3}\frac{1}{(1-b^2)}\left[\left(\frac{d\mathbf{v}}{dt}\right)^2 - \left(\frac{\mathbf{v}}{c}\times\frac{d\mathbf{v}}{dt}\right)^2\right]. \qquad (2.55)$$

To check that Eq. (2.55) comes form Eq. (2.53), we evaluate the following expressions:

$$\left(\frac{E}{mc^2}\right)^2 = \left(\frac{mc^2}{(1-\boldsymbol{b}^2)^{1/2}}\frac{1}{mc^2}\right)^2 = \frac{1}{1-\boldsymbol{b}^2}, \tag{2.56}$$

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left(\frac{m\mathbf{v}}{(1-\boldsymbol{b}^2)^{1/2}}\right),$$

$$\frac{d\mathbf{p}}{dt} = \frac{m}{(1-\boldsymbol{b}^2)^{1/2}}\frac{d\mathbf{v}}{dt} + m\mathbf{v}\frac{d}{dt}\frac{1}{(1-\boldsymbol{b}^2)^{1/2}},$$

$$\frac{d\mathbf{p}}{dt} = \frac{m}{(1-\boldsymbol{b}^2)^{1/2}}\frac{d\mathbf{v}}{dt} + m\mathbf{v}\left\{\frac{\mathbf{v}}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} \bullet \frac{d\mathbf{v}}{dt}\right\},$$

$$\frac{d\mathbf{p}}{dt} = \frac{m}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}\frac{d\mathbf{v}}{dt} + \frac{m}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}\mathbf{v}\left(\mathbf{v}\bullet\frac{d\mathbf{v}}{dt}\right),$$

$$\left(\frac{d\mathbf{p}}{dt}\right)^2 = \frac{m^2}{(1-\boldsymbol{b}^2)}\left\{\left(\frac{d\mathbf{v}}{dt}\right) + \frac{1}{c^2(1-\boldsymbol{b}^2)}\mathbf{v}\left(\mathbf{v}\bullet\frac{d\mathbf{v}}{dt}\right)\right\}^2,$$

$$\left(\frac{d\mathbf{p}}{dt}\right)^2 = \frac{m^2}{(1-\boldsymbol{b}^2)}\left\{\left(\frac{d\mathbf{v}}{dt}\right)^2 + \frac{2}{c^2(1-\boldsymbol{b}^2)}\left(\mathbf{v}\bullet\frac{d\mathbf{v}}{dt}\right)^2 + \frac{1}{c^4(1-\boldsymbol{b}^2)^2}\left[\mathbf{v}\left(\mathbf{v}\bullet\frac{d\mathbf{v}}{dt}\right)\right]^2\right\}, \tag{2.57}$$

and

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{mc^2}{(1-b^2)^{1/2}}\right] = mc^2\frac{d}{dt}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}},$$

$$\frac{dE}{dt} = \frac{mc^2}{c^2(1-b^2)^{3/2}}\left(\mathbf{v}\bullet\frac{d\mathbf{v}}{dt}\right),$$

$$\frac{dE}{dt} = \frac{m}{(1-b^2)^{3/2}}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right),$$

$$\left(\frac{dE}{dt}\right)^2 = \frac{m^2}{(1-b^2)^3}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2. \tag{2.58}$$

Substituting Eqs. (2.56), (2.57) and (2.58) into Eq. (2.53)

$$P = \frac{2}{3}\frac{m^2 g^2 e^2}{m^2 c^3} g^2 \left\{\left(\frac{d\mathbf{v}}{dt}\right)^2 + \frac{2g^2}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2 + \frac{v^2 g^4}{c^4}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2 - \frac{g^4}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2\right\},$$

$$P = \frac{2}{3}\frac{g^4 e^2}{c^3}\left\{\left(\frac{d\mathbf{v}}{dt}\right)^2 + \frac{2g^2}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2 + \frac{v^2 g^4}{c^4}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2 - \frac{g^4}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2\right\},$$

$$P = \frac{2}{3}\frac{g^6 e^2}{c^3}\left\{\frac{1}{g^2}\left(\frac{d\mathbf{v}}{dt}\right)^2 + \frac{2}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2 + \frac{v^2 g^2}{c^4}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2 - \frac{g^2}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2\right\},$$

$$P = \frac{2}{3}\frac{g^6 e^2}{c^3}\left\{\left[1 - \frac{v^2}{c^2}\right]\left(\frac{d\mathbf{v}}{dt}\right)^2 + \frac{2}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2 + \frac{v^2 g^2}{c^4}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2 - \frac{g^2}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2\right\},$$

$$P = \frac{2}{3}\frac{g^6 e^2}{c^3}\left\{\left(\frac{d\mathbf{v}}{dt}\right)^2 - \frac{v^2}{c^2}\left(\frac{d\mathbf{v}}{dt}\right)^2 + \frac{2}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2 + \frac{v^2 g^2}{c^4}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2 - \frac{g^2}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2\right\},$$

$$P = \frac{2}{3}\frac{g^6 e^2}{c^3}\left\{\left(\frac{d\mathbf{v}}{dt}\right)^2 - \frac{v^2}{c^2}\left(\frac{d\mathbf{v}}{dt}\right)^2 + \frac{2}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2 - \frac{g^2}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2\left(1 - \frac{v^2}{c^2}\right)\right\},$$

$$P = \frac{2}{3}\frac{g^6 e^2}{c^3}\left\{\left(\frac{d\mathbf{v}}{dt}\right)^2 - \frac{v^2}{c^2}\left(\frac{d\mathbf{v}}{dt}\right)^2 + \frac{2}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2 - \frac{1}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2\right\},$$

$$P = \frac{2}{3}\frac{g^6 e^2}{c^3}\left\{\left(\frac{d\mathbf{v}}{dt}\right)^2 - \frac{v^2}{c^2}\left(\frac{d\mathbf{v}}{dt}\right)^2 + \frac{1}{c^2}\left(\mathbf{v} \bullet \frac{d\mathbf{v}}{dt}\right)^2\right\},$$

from Eq. (A.2) in Appendix A and

$$\left(\frac{\mathbf{v}}{c} \times \frac{d\mathbf{v}}{dt}\right) \bullet \left(\frac{\mathbf{v}}{c} \times \frac{d\mathbf{v}}{dt}\right) = \frac{v^2}{c^2}\left(\frac{d\mathbf{v}}{dt}\right)^2 - \left(\frac{\mathbf{v}}{c^2} \bullet \frac{d\mathbf{v}}{dt}\right)^2,$$

leading to Eq. (2.55)

$$P = \frac{2}{3}\frac{e^2 g^6}{c^3}\left[\left(\frac{d\mathbf{v}}{dt}\right)^2 - \left(\frac{\mathbf{v}}{c}\times\frac{d\mathbf{v}}{dt}\right)^2\right].$$

For the circular trajectory for an electron in a synchrotron, in such machines the momentum **p** changes rapidly in direction as the particle rotates, but the change in energy per revolution is small. This means that

$$\left|\frac{d\mathbf{p}}{dt}\right| >> \frac{1}{c}\frac{dE}{dt}.$$

Then the radiated power Eq. (2.53) can be written approximately

$$P = \frac{2}{3}\frac{e^2}{m^2 c^3}\left(\frac{E}{mc^2}\right)^2\left(\frac{d\mathbf{p}}{dt}\right)^2.$$

Now

$$\left(\frac{d\mathbf{p}}{dt}\right)^2 = w_0^{\;2} p^2 = \frac{w_0}{c} b^3 \frac{E^2}{c},$$

where w_0 and R are the instantaneous angular velocity and radius of curvature. Hence,

$$P = \frac{2}{3} w_0 \frac{e^2}{R} b^3 \left(\frac{E}{mc^2}\right)^4.$$

The energy radiated per revolution is then

$$\Delta E = \frac{4}{3} p \left(\frac{e^2}{R}\right) b^3 \left(\frac{E}{mc^2}\right)^4,$$

A useful form of this result is

$$\Delta E_{keV} = 88.5 (E_{GeV})^4 / R_{met},$$

where E_{GeV} is the electron energy in units of 1 GeV = 10^9 eV, R_{met} is the radius of the electron orbit in meters, and ΔE_{keV} is the energy radiated per revolution in units of 1 keV = 10^3 eV.

Chapter III

Explicit Expression for the Mean Number of Photons

The purpose of this chapter is to derive an explicit expression for the mean number <N> of photons emitted per revolution in synchrotron radiation. The resulting expression is a remarkably simple one-dimensional integral. To this end, in Section 3.1, we first provide a derivation of the Schwinger power integral expression (Schwinger, 1949, 1976, 1978). The main expression for <N> is then derived in Section 3.2, by explicitly taking into account the vanishing property of <N> for $\boldsymbol{b} \to 0$. This section also establishes the existence of the integral for <N>.

3.1 The Schwinger Power-Integral Expression

The derivation of the Schwinger power integral expression is based on a consideration of the rate at which the electron does work on the electromagnetic field,

$$-\int \mathbf{j} \bullet \mathbf{E}_{ret} d^3\mathbf{r} .$$

Which can be conveniently divided into two essentially different parts on writing

$$\mathbf{E}_{ret} = \frac{1}{2}(\mathbf{E}_{ret} + \mathbf{E}_{adv}) + \frac{1}{2}(\mathbf{E}_{ret} - \mathbf{E}_{adv}) \tag{3.1}$$

Here $\mathbf{E}_{ret}$ and $\mathbf{E}_{adv}$ are the retarded and advanced electric field intensities generated by the electron charge and current densities, $\boldsymbol{r}$ and $\mathbf{j}$. The first part, derived form the symmetrical combination of $\mathbf{E}_{ret}$ and $\mathbf{E}_{adv}$, changes sign upon reversing the positive sense of the time and therefore represents reactive power. It describes the rate at which the electron stores energy in the electromagnetic field, and inertial effects with which we are not concerned. However, the second is derived from the antisymmetrical combination of $\mathbf{E}_{ret}$ and $\mathbf{E}_{adv}$. The latter remains unchanged upon reversing the positive dense of time, and therefore represents resistive power. Subject to one qualification, it describes the rate of irreversible energy transfer to the electromagnetic field, which is the desired rate of radiation. Included in the second part is a term, which has the form of the time derivative of acceleration-dependent electron energy. The for all realizable accelerations the latter is completely negligible compared to the electron kinetic energy. It will be a simple manner to eliminate these unwanted terms after evaluating the dissipative part. Thus, the power carried away by radiation is, provisionally,

$$P = -\int \mathbf{j} \bullet \mathbf{E} d^3\mathbf{r} \,, \qquad (3.1)$$

with

$$\mathbf{E} = \frac{1}{2}(\mathbf{E}_{ret} - \mathbf{E}_{adv}) = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi \,.$$

The expression of P in terms of the vector and scalar potential, $\mathbf{A}$ and Φ, can be simplified by employing the charge conservation equation

$$\nabla \bullet \mathbf{j} + \frac{\partial \boldsymbol{r}}{\partial t} = 0 \,.$$

Thus,

$$P = \int \mathbf{j} \bullet \left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \nabla \Phi \right) d^3\mathbf{r} \,,$$

$$P = \int \left(\frac{1}{c}\mathbf{j} \bullet \frac{\partial \mathbf{A}}{\partial t} + \mathbf{j} \bullet \nabla \Phi \right) d^3\mathbf{r} \,,$$

$$P = \int \left[\frac{1}{c} \mathbf{j} \bullet \frac{\partial \mathbf{A}}{\partial t} + \nabla \bullet (\mathbf{j}\Phi) - \Phi(\nabla \bullet \mathbf{j}) \right] d^3\mathbf{r} \,,$$

$$P = \int \left[\frac{1}{c} \mathbf{j} \bullet \frac{\partial \mathbf{A}}{\partial t} + \nabla \bullet (\mathbf{j}\Phi) + \Phi \frac{\partial \boldsymbol{r}}{\partial t} \right] d^3\mathbf{r} \,,$$

$$P = \int \left[\frac{1}{c} \mathbf{j} \bullet \frac{\partial \mathbf{A}}{\partial t} + \nabla \bullet (\mathbf{j}\Phi) + \frac{\partial (\Phi \boldsymbol{r})}{\partial t} - \boldsymbol{r} \frac{\partial \Phi}{\partial t} \right] d^3\mathbf{r} \,,$$

$$P = \int \left(\frac{1}{c} \mathbf{j} \bullet \frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{r} \frac{\partial \Phi}{\partial t} \right) d^3\mathbf{r} + \int \frac{\partial (\Phi \boldsymbol{r})}{\partial t} d^3\mathbf{r} + \int \nabla \bullet (\mathbf{j}\Phi) d^3\mathbf{r} \,.$$

Now, $da = r^2 \sin\theta d\theta d\varphi$ and $\mathbf{j} \propto 1/r^2$, $\Phi \propto 1/r$; therefore, the integral vanishes as $1/r$ as r becomes very large. Thus one obtains

$$P = \int \left(\frac{1}{c} \mathbf{j} \bullet \frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{r} \frac{\partial \Phi}{\partial t} \right) d^3\mathbf{r} + \frac{d}{dt} \int \Phi \boldsymbol{r} d^3\mathbf{r} \,.$$

The second term of this formula is of the acceleration energy type and may be discarded. Hence the expression for the radiated power, which still includes unwanted acceleration energy terms, becomes

$$P = \int \left(\frac{1}{c} \mathbf{j} \bullet \frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{r} \frac{\partial \Phi}{\partial t} \right) d^3\mathbf{r} \,. \qquad (3.2)$$

The scalar potentials can be conveniently written as

$$\Phi_{ret,asv}(\mathbf{r}, t) = \int \frac{\boldsymbol{d}\left(t' - t \pm \frac{|\mathbf{r} - \mathbf{r}'|}{c} \right) \boldsymbol{r}(\mathbf{r}', t') d^3\mathbf{r}' dt'}{|\mathbf{r} - \mathbf{r}'|} \,.$$

Upon introducing the Fourier integral representation of the delta function δ(t):

$$\boldsymbol{d}(t) = \frac{1}{2\boldsymbol{p}} \int_{-\infty}^{\infty} e^{iwt} d\boldsymbol{w} \,,$$

the potential assumes the form

$$\Phi_{ret,adv}(\mathbf{r},t)=\frac{1}{2\pi}\int\frac{e^{i\omega(t'-t)}e^{\pm i(\omega/c)|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}',t')}{\left|\mathbf{r}-\mathbf{r}'\right|}d^3\mathbf{r}'dt'd\omega,$$

whence

$$\Phi(\mathbf{r},t)=\frac{i}{2\pi}\int\frac{e^{i\omega(t'-t)}\sin\frac{\omega}{c}\left|\mathbf{r}-\mathbf{r}'\right|}{\left|\mathbf{r}-\mathbf{r}'\right|}\,\rho(\mathbf{r}',t')d^3\mathbf{r}'dt'd\omega. \tag{3.3}$$

Similarly,

$$\mathbf{A}_{ret,adv}(\mathbf{r},t)=\int\frac{\delta\left(t'-t\pm\frac{\left|\mathbf{r}-\mathbf{r}'\right|}{c}\right)\frac{1}{c}\mathbf{j}(\mathbf{r}',t')d^3\mathbf{r}'dt'}{\left|\mathbf{r}-\mathbf{r}'\right|},$$

$$\mathbf{A}(\mathbf{r},t)=\frac{i}{2\pi}\int\frac{e^{i\omega(t'-t)}\sin\frac{\omega}{c}\left|\mathbf{r}-\mathbf{r}'\right|}{\left|\mathbf{r}-\mathbf{r}'\right|}\frac{1}{c}\mathbf{j}(\mathbf{r}',t')d^3\mathbf{r}'dt'd\omega. \tag{3.4}$$

The total radiated power, calculated from Eqs. (3.2), (3.3), and (3.4) is

$$\begin{aligned}P(t)=&\int d^3\mathbf{r}\frac{1}{c}\mathbf{j}(r,t)\bullet\frac{\partial}{\partial t}\left[\frac{i}{2\pi}\frac{e^{i\omega(t'-t)}\sin\frac{\omega}{c}\left|\mathbf{r}-\mathbf{r}'\right|}{\left|\mathbf{r}-\mathbf{r}'\right|}\frac{1}{c}\mathbf{j}(\mathbf{r}',t')d^3\mathbf{r}'dt'd\omega\right],\\&-\int d^3\mathbf{r}\rho(\mathbf{r},t)\frac{\partial}{\partial t}\left[\frac{i}{2\pi}\frac{e^{i\omega(t'-t)}\sin\frac{\omega}{c}\left|\mathbf{r}-\mathbf{r}'\right|}{\left|\mathbf{r}-\mathbf{r}'\right|}\rho(\mathbf{r}',t')d^3\mathbf{r}'dt'd\omega\right]\end{aligned}$$

$$\begin{aligned}P(t)=&\int\frac{\omega}{2\pi c^2}\mathbf{j}(r,t)\bullet\mathbf{j}(\mathbf{r}',t')\frac{e^{i\omega(t'-t)}\sin\frac{\omega}{c}\left|\mathbf{r}-\mathbf{r}'\right|}{\left|\mathbf{r}-\mathbf{r}'\right|}d^3\mathbf{r}d^3\mathbf{r}'dt'd\omega\\&-\int\frac{\omega}{2\pi}\rho(\mathbf{r},t)\rho(\mathbf{r}',t')\frac{e^{i\omega(t'-t)}\sin\frac{\omega}{c}\left|\mathbf{r}-\mathbf{r}'\right|}{\left|\mathbf{r}-\mathbf{r}'\right|}d^3\mathbf{r}d^3\mathbf{r}'dt'd\omega\end{aligned},$$

$$P(t) = -\frac{1}{2\pi}\int \left[\rho(\mathbf{r},t)\,\rho(\mathbf{r}',t') - \frac{1}{c^2}\mathbf{j}(r,t)\bullet\mathbf{j}(\mathbf{r}',t') \right],$$

$$\frac{e^{i\omega(t'-t)}\sin\frac{\omega}{c}\left|\mathbf{r}-\mathbf{r}'\right|}{\left|\mathbf{r}-\mathbf{r}'\right|}\, d^3\mathbf{r}\, d^3\mathbf{r}'\, dt'\,\omega d\omega$$

$$P(t) = \int_0^\infty P(\omega,t)\,d\omega .$$

Hence it may be inferred that

$$P(\omega,t) = -\frac{\omega}{\pi}\int \left[\rho(\mathbf{r},t)\,\rho(\mathbf{r}',t') - \frac{1}{c^2}\mathbf{j}(r,t)\bullet\mathbf{j}(\mathbf{r}',t') \right]\cos\omega(t-t'),$$

$$\frac{\sin\frac{\omega}{c}\left|\mathbf{r}-\mathbf{r}'\right|}{\left|\mathbf{r}-\mathbf{r}'\right|}\, d^3\mathbf{r}\, d^3\mathbf{r}'\, dt'$$

$$P(t) = \int_0^\infty d\omega \left\{ -\frac{\omega}{\pi}\int \left[\rho(\mathbf{r},t)\,\rho(\mathbf{r}',t') - \frac{1}{c^2}\mathbf{j}(r,t)\bullet\mathbf{j}(\mathbf{r}',t') \right]\cos\omega(t-t') \frac{\sin\frac{\omega}{c}\left|\mathbf{r}-\mathbf{r}'\right|}{\left|\mathbf{r}-\mathbf{r}'\right|}\, d^3\mathbf{r}\, d^3\mathbf{r}'\, dt' \right\}. \qquad (3.5)$$

The former expression $P(\omega,t)$ represents the power radiated at the time t in a unit angular frequency range about ω.

Thus far, we have dealt with the radiation of an arbitrary charge-current distribution. For a point electron of charge e, located at the variable position **R**(t),

$$\begin{aligned}\rho(\mathbf{r},t) &= e\delta[\mathbf{r}-\mathbf{R}(t)] \\ \mathbf{j}(\mathbf{r},t) &= e\mathbf{v}\delta[\mathbf{r}-\mathbf{R}(t)],\end{aligned} \qquad (3.6)$$

where **v**(t) = d**R**(t)/dt. Substituting Eq. (3.6) into Eq. (3.5) gives

$$P(t) = -\int_0^{\infty} d\omega \left\{ \int \frac{e^2\omega}{\pi} \left\{ \delta[\mathbf{r}-\mathbf{R}(t)]\delta[\mathbf{r}^{/}-\mathbf{R}(t^{/})] - \frac{1}{c^2}[\mathbf{v}(t)\bullet\mathbf{v}(t^{/})]\delta[\mathbf{r}-\mathbf{R}(t)]\delta[\mathbf{r}^{/}-\mathbf{R}(t^{/})] \right\} \cos\omega(t-t^{/}) \frac{\sin\frac{\omega}{c}\left|\mathbf{r}-\mathbf{r}^{/}\right|}{\left|\mathbf{r}-\mathbf{r}^{/}\right|} d^3\mathbf{r}\, d^3\mathbf{r}^{/}\, dt^{/} \right\}$$

Our various formulae can now be simplified by performing the spatial integrations. Hence

$$P(t) = -\int_0^{\infty} d\omega \int_{-\infty}^{\infty} dt^{/} \frac{e^2\omega}{\pi}\left[1-\frac{1}{c^2}[\mathbf{v}(t)\bullet\mathbf{v}(t^{/})]\right]\cos\omega(t-t^{/}) \frac{\sin\frac{\omega}{c}\left|\mathbf{R}(t)-\mathbf{R}(t^{/})\right|}{\left|\mathbf{R}(t)-\mathbf{R}(t^{/})\right|}.$$

By replacing $t^{/} - t = \tau$, the above integral reduces to

$$P(t) = -\int_0^{\infty} d\omega \int_{-\infty}^{\infty} d\tau \frac{e^2\omega}{\pi}\left[1-\frac{1}{c^2}[\mathbf{v}(t)\bullet\mathbf{v}(\tau+t)]\right]\cos\omega(\tau) \frac{\sin\frac{\omega}{c}\left|\mathbf{R}(t)-\mathbf{R}(\tau+t)\right|}{\left|\mathbf{R}(t)-\mathbf{R}(\tau+t)\right|}.$$

The electron is moving with constant speed in a circular path. Since the motion is periodic, the spectrum will consist of harmonics of the rotational angular frequency $\omega_0 = v/R$. The position vector $\mathbf{R}(t)$ is given by $\mathbf{R}(t) = (R\cos\omega_0 t, R\sin\omega_0 t, 0)$ thus

$$\mathbf{v}(t)\bullet\mathbf{v}(t+\tau) = v^2\cos\omega_0\tau,$$

and

$$\left|\mathbf{R}(t+\tau)-\mathbf{R}(t)\right| = 2R\left|\sin\frac{\omega_0\tau}{2}\right|.$$

Hence one obtains

$$P(t) \equiv P = -\int_0^{\infty} d\omega \int_{-\infty}^{\infty} d\tau \frac{e^2\omega}{\pi}\left[1-\frac{v^2\cos\omega_0\tau}{c^2}\right]\cos\omega(\tau)\frac{\sin\left(\frac{\omega}{c}2R\left|\sin\frac{\omega_0\tau}{2}\right|\right)}{2R\left|\sin\frac{\omega_0\tau}{2}\right|},$$

$$P=\int_{0}^{\infty}d\omega\int_{-\infty}^{\infty}d\tau\,\frac{e^{2}\omega}{2\pi R}\left[\frac{v^{2}\cos\omega_{0}\tau}{c^{2}}-1\right]\cos\omega(\tau)\frac{\sin\left(\frac{2R\omega}{c}\left|\sin\frac{\omega_{0}\tau}{2}\right|\right)}{\left|\sin\frac{\omega_{0}\tau}{2}\right|}. \qquad (3.7)$$

Upon defining variables of integrations: $x = \omega_0 \tau$ and $z = \omega/\omega_0$ one obtains

$$dx=\omega_{0}d\tau, \qquad dz=\frac{d\omega}{\omega_{0}}.$$

Thus Eq. (3.7) becomes

$$P=\int_{0}^{\infty}\omega_{0}dz\int_{-\infty}^{\infty}\frac{dx}{\omega_{0}}\frac{e^{2}\omega}{2\pi R}\left[\frac{v^{2}\cos x}{c^{2}}-1\right]\cos\frac{\omega\omega_{0}\tau}{\omega_{0}}\frac{\sin\left(\frac{2R\omega\omega_{0}}{c\omega_{0}}\left|\sin\frac{x}{2}\right|\right)}{\left|\sin\frac{x}{2}\right|},$$

$$P=\int_{0}^{\infty}dz\int_{-\infty}^{\infty}dx\frac{e^{2}\omega}{2\pi R}\left[\frac{v^{2}\cos x}{c^{2}}-1\right]\cos zx\frac{\sin\left(\frac{2zv}{c}\left|\sin\frac{x}{2}\right|\right)}{\left|\sin\frac{x}{2}\right|},$$

which is the Schwinger power integral formula.

3.2 Derivation of the Explicit Expression for <N>

The mean number of photons emitted per revolution is

$$\langle N\rangle=\frac{2\pi\int_{0}^{\infty}\frac{P(\omega)d\omega}{\hbar\omega}}{\omega_{0}}.$$

Thus the expression for <N> is

$$\langle N\rangle = \frac{2\boldsymbol{p}c}{\hbar \boldsymbol{w}_0 c}\int_0^{\infty} dz\int_{-\infty}^{\infty} dx\,\frac{e^2 e^{-izx}}{2\boldsymbol{p}R}\left[\boldsymbol{b}^2\cos x - 1\right]\frac{\sin\left(2\boldsymbol{b}z\sin\frac{x}{2}\right)}{\sin\frac{x}{2}},$$

$$\langle N\rangle = \frac{e^2 c}{\hbar v c}\int_0^{\infty} dz\int_{-\infty}^{\infty} e^{-izx}\left[\boldsymbol{b}^2\cos x - 1\right]\frac{\sin\left(2\boldsymbol{b}z\sin\frac{x}{2}\right)}{\sin\frac{x}{2}},$$

$$\langle N\rangle = \frac{\boldsymbol{a}}{\boldsymbol{b}}\int_0^{\infty} dz\int_{-\infty}^{\infty} dx e^{-izx}\left[\boldsymbol{b}^2\cos x - 1\right]\frac{\sin\left(2\boldsymbol{b}z\sin\frac{x}{2}\right)}{\sin\frac{x}{2}}, \qquad (3.8)$$

where $\boldsymbol{a} = \frac{e^2}{\hbar c}$ is the fine structure constant. Eq. (3.8) may be rewritten as

$$\langle N\rangle = \boldsymbol{a}\int_0^{\infty} dz\int_{-\infty}^{\infty} dx e^{-izx}\left[\boldsymbol{b}^2\cos x - 1\right]\frac{\sin\left(2\boldsymbol{b}z\sin\frac{x}{2}\right)}{\boldsymbol{b}\sin\frac{x}{2}}. \qquad (3.9)$$

Since the integrand in Eq. (3.9) multiplying the exponential factor exp(-izx) is an even function of x, only the real part of the integral in Eq. (3.9) is non-vanishing as one is integrating over the symmetrical region: -∞ < 0 < ∞. It is easily verified that for $\boldsymbol{b}\to 0$, the integrand in Eq. (3.9) goes over to $e^{-izx}2z$. Upon using in the process the elementary integral

$$\int_0^{\infty} dz\int_{-\infty}^{\infty} z e^{-izx} dx = 0,$$

we readily infer that <N>→0 as $\boldsymbol{b}\to 0$ as it should. This latter vanishing property of <N> may be taken explicitly into account by rewriting Eq. (3.9) as

$$\langle N\rangle = \boldsymbol{a}\int_0^{\infty} dz \int_{-\infty}^{\infty} dx e^{-izx} \int_0^{b} d\boldsymbol{r} \left\{ \cos x \frac{\sin\left(2\boldsymbol{b}z\sin\frac{x}{2}\right)}{\sin\frac{x}{2}} - \frac{2z}{\boldsymbol{b}}\left[\cos\left(2z\boldsymbol{r}\sin\frac{x}{2}\right) - 1\right]\right\}. \quad (3.10)$$

In order to evaluate Eq. (3.10), we first integrate over z then over $\boldsymbol{r}$. To this end we note that

$$\int_0^{\infty} dze^{-izx}\cos x \frac{\sin\left(2\boldsymbol{b}z\sin\frac{x}{2}\right)}{\sin\frac{x}{2}} = \frac{\cos x}{2i\sin\frac{x}{2}}\int_0^{\infty} dze^{-izx}\left(e^{2izb\sin\frac{x}{2}} - e^{-2izb\sin\frac{x}{2}}\right),$$

$$\int_0^{\infty} dze^{-izx}\left(e^{2izb\sin\frac{x}{2}} - e^{-2izb\sin\frac{x}{2}}\right) = \frac{1}{i}\int_0^{\infty} dz\left[ie^{z(-ix+2ib\sin\frac{x}{2})} - ie^{z(-ix-2ib\sin\frac{x}{2})}\right].$$

From the table of integrations (Appendix B):

$$-i\int_0^{\infty} dxe^{iax} = \frac{1}{a+i\boldsymbol{e}},$$

where $\boldsymbol{e}\rightarrow +0$. Thus

$$\int_0^{\infty} dz\left(ie^{iz(-x+2b\sin\frac{x}{2})} - ie^{iz(-x-2b\sin\frac{x}{2})}\right) = -\frac{1}{\left(-x+2\boldsymbol{b}\sin\frac{x}{2}\right)+i\boldsymbol{e}} + \frac{1}{\left(-x+2\boldsymbol{b}\sin\frac{x}{2}\right)+i\boldsymbol{e}},$$

$$\begin{aligned}\int_0^{\infty} dz\left[ie^{iz(-x+2b\sin\frac{x}{2})} - ie^{iz(-x-2b\sin\frac{x}{2})}\right] &= -\frac{1}{-x+b+i\boldsymbol{e}} + \frac{1}{-x+b+i\boldsymbol{e}} \\ &= \frac{1}{(-x+i\boldsymbol{e})-b} - \frac{1}{(-x+i\boldsymbol{e})+b}, \\ &= \frac{-x+i\boldsymbol{e}+b+x-i\boldsymbol{e})+b}{(-x+i\boldsymbol{e})^2-b^2} \\ &= \frac{2b}{(-x+i\boldsymbol{e})^2-b^2}\end{aligned}$$

$$\int_0^\infty dz \left[ie^{iz(-x+2b\sin\frac{x}{2})} - ie^{iz(-x-2b\sin\frac{x}{2})} \right] = \frac{2b}{x^2-b^2} = -\frac{2b}{b^2-x^2},$$

where $b = 2\boldsymbol{b}\sin(x/2)$, thus one obtains

$$\frac{\boldsymbol{b}\cos x}{2\sin\frac{x}{2}} \int_0^\infty dz e^{-izx} \left(e^{2izb\sin\frac{x}{2}} - e^{-2izb\sin\frac{x}{2}} \right) = \frac{b\boldsymbol{b}\cos x}{(b^2-x^2)\sin\frac{x}{2}}. \tag{3.11}$$

And for the second term

$$\int_0^\infty dz e^{-izx} 2z\cos(2z\boldsymbol{r}\sin\frac{x}{2}) = \frac{2}{2}\int_0^\infty dz e^{-izx} z\left(e^{i2zr\sin\frac{x}{2}} + e^{-i2zr\sin\frac{x}{2}}\right)$$
$$= \int_0^\infty zdz \left[e^{-iz(-2zr\sin\frac{x}{2}+x)} + e^{-iz(2zr\sin\frac{x}{2}+x)} \right].$$

This equation can be integrated by parts. Let

$$u = z, \qquad dv_1 = dze^{-iz(-2r\sin\frac{x}{2}+x)}, \qquad dv_2 = dze^{-iz(2r\sin\frac{x}{2}+x)}$$

$$du = dz, \quad v_1 = \frac{ie^{-iz(x-2r\sin\frac{x}{2})}}{x-2\boldsymbol{r}\sin\frac{x}{2}}, \qquad v_2 = \frac{ie^{-iz(x+2r\sin\frac{x}{2})}}{x+2\boldsymbol{r}\sin\frac{x}{2}},$$

then

$$\int_0^\infty zdz \left[e^{-iz(-2zr\sin\frac{x}{2}+x)} + e^{-iz(2zr\sin\frac{x}{2}+x)} \right] = \left[\left. \frac{ize^{-iz(x-2r\sin\frac{x}{2})}}{x-2\boldsymbol{r}\sin\frac{x}{2}} \right|_0^\infty - \int_0^\infty dz \frac{ie^{-iz(x-2r\sin\frac{x}{2})}}{x-2\boldsymbol{r}\sin\frac{x}{2}} \right]$$
$$+ \left[\left. \frac{ize^{-iz(x+2r\sin\frac{x}{2})}}{x+2\boldsymbol{r}\sin\frac{x}{2}} \right|_0^\infty - \int_0^\infty dz \frac{ie^{-iz(x+2r\sin\frac{x}{2})}}{x+2\boldsymbol{r}\sin\frac{x}{2}} \right]$$

$$\int_0^\infty zdz\left[e^{-iz(-2zr\sin\frac{x}{2}+x)}+e^{-iz(2zr\sin\frac{x}{2}+x)}\right]=-\int_0^\infty dz\,\frac{ie^{-iz(x-2r\sin\frac{x}{2})}}{x-2\boldsymbol{r}\sin\frac{x}{2}}-\int_0^\infty dz\,\frac{ie^{-iz(x+2r\sin\frac{x}{2})}}{x+2\boldsymbol{r}\sin\frac{x}{2}}.$$

From the table of integrations (Appendix B):

$$i\int_0^\infty dxe^{-iax}=\frac{1}{a-i\boldsymbol{e}},$$

where $\boldsymbol{e}\to+0$. Thus

$$\int_0^\infty zdz\left[e^{-iz(-2zr\sin\frac{x}{2}+x)}+e^{-iz(2zr\sin\frac{x}{2}+x)}\right]=-\frac{1}{x-c}\frac{1}{x-c-i\boldsymbol{e}}-\frac{1}{x+c}\frac{1}{x+c-i\boldsymbol{e}},$$

$$\int_0^\infty zdz\left[e^{-iz(-2zr\sin\frac{x}{2}+x)}+e^{-iz(2zr\sin\frac{x}{2}+x)}\right]=-\frac{(x+c)(x+c-i\boldsymbol{e})+(x-c)(x-c-i\boldsymbol{e})}{(x^2-a^2)\left[(x-i\boldsymbol{e})^2-a^2\right]}$$

$$=-\frac{x^2+2xc-i\boldsymbol{e}x+c^2-i\boldsymbol{e}c+x^2-2xz-i\boldsymbol{e}x+c^2+i\boldsymbol{e}c}{(x^2-c^2)\left[(x-i\boldsymbol{e})^2-c^2\right]},$$

$$=-\frac{2x^2+2c^2-2i\boldsymbol{e}x}{(x^2-c^2)\left[(x-i\boldsymbol{e})^2-c^2\right]}$$

$$\int_0^\infty dze^{-izx}2z\cos(2z\boldsymbol{r}\sin\frac{x}{2})=-\frac{2x^2+2c^2}{(x^2-c^2)^2},\qquad(3.12)$$

where c = 2$\boldsymbol{r}$sin(x/2). Substituting Eqs. (3.11), and (3.12) into Eq. (3.10), the latter becomes

$$\langle N\rangle=\boldsymbol{a}\int_0^\infty dxF(x),\qquad(3.13)$$

where

$$F(x)=\frac{1}{\boldsymbol{b}}\int_0^b d\boldsymbol{r}\left[\frac{b\boldsymbol{b}\cos x}{(b^2-x^2)\sin\frac{x}{2}}+\frac{2x^2+2c^2}{(x^2-c^2)^2}-\frac{2}{x^2}\right],$$

$$F(x) = \frac{1}{\boldsymbol{b}}\left[\frac{b\boldsymbol{b}^2 \cos x}{(b^2 - x^2)\sin\frac{x}{2}} - \frac{2\boldsymbol{b}}{x^2}\right] + 2\int_0^b d\boldsymbol{r}\frac{x^2 + 4\boldsymbol{r}^2 \sin^2\frac{x}{2}}{(x^2 - 4\boldsymbol{r}^2\sin^2\frac{x}{2})^2}. \qquad (3.14)$$

Considering the integral term

$$\int_0^b d\boldsymbol{r}\frac{x^2 + 4\boldsymbol{r}^2\sin^2\frac{x}{2}}{(x^2 - 4\boldsymbol{r}^2\sin^2\frac{x}{2})^2} = \int_0^b d\boldsymbol{r}\frac{x^2}{(x^2 - e\boldsymbol{r}^2)^2} + \int_0^b d\boldsymbol{r}\frac{e\boldsymbol{r}^2}{(x^2 - e\boldsymbol{r}^2)^2},$$

where $e = 4\sin^2(x/2)$. From the table of integration

$$\int_0^a \frac{x^2}{(b^2 - x^2)^2}dx = \frac{a}{2(b^2 - x^2)} - \frac{1}{4b}\ln\left(\frac{b+a}{b-a}\right),$$

and

$$\int_0^a \frac{1}{(b^2 - x^2)^2}dx = \frac{a}{2b^2(b^2 - x^2)} + \frac{1}{4b^3}\ln\left(\frac{b+a}{b-a}\right).$$

Thus

$$\int_0^b d\boldsymbol{r}\frac{x^2}{(x^2 - e\boldsymbol{r}^2)^2} = \frac{x^2}{e^2}\int_0^b d\boldsymbol{r}\frac{1}{(f^2 - \boldsymbol{r}^2)^2},$$

$$\int_0^b d\boldsymbol{r}\frac{x^2}{(x^2 - e\boldsymbol{r}^2)^2} = \frac{f^2}{e}\left[\frac{\boldsymbol{b}}{2f^2(f^2 - x^2)} + \frac{1}{4f^3}\ln\left(\frac{f+\boldsymbol{b}}{f-\boldsymbol{b}}\right)\right],$$

$$\int_0^b d\boldsymbol{r}\frac{x^2}{(x^2 - e\boldsymbol{r}^2)^2} = \frac{\boldsymbol{b}}{2e(f^2 - x^2)} + \frac{1}{4cf}\ln\left(\frac{f+\boldsymbol{b}}{f-\boldsymbol{b}}\right),$$

and

$$\int_0^b d\boldsymbol{r}\frac{e\boldsymbol{r}^2}{(x^2 - e\boldsymbol{r}^2)^2} = \frac{1}{e}\int_0^b d\boldsymbol{r}\frac{\boldsymbol{r}^2}{(f^2 - \boldsymbol{r}^2)^2},$$

$$\int_0^b d\boldsymbol{r}\frac{e\boldsymbol{r}^2}{(x^2 - e\boldsymbol{r}^2)^2} = \frac{1}{e}\left[\frac{\boldsymbol{b}}{2(f^2 - x^2)} - \frac{1}{4f}\ln\left(\frac{f+\boldsymbol{b}}{f-\boldsymbol{b}}\right)\right],$$

$$\int_0^b d\boldsymbol{r}\frac{e\boldsymbol{r}^2}{(x^2-e\boldsymbol{r}^2)^2}=\frac{\boldsymbol{b}}{2e(f^2-x^2)}-\frac{1}{4ef}\ln\left(\frac{f+\boldsymbol{b}}{f-\boldsymbol{b}}\right),$$

where $f=\dfrac{x}{\sqrt{e}}$. And one obtains

$$\int_0^b d\boldsymbol{r}\frac{x^2}{(x^2-e\boldsymbol{r}^2)^2}+\int_0^b d\boldsymbol{r}\frac{e\boldsymbol{r}^2}{(x^2-e\boldsymbol{r}^2)^2}=\frac{\boldsymbol{b}}{2e(f^2-x^2)}+\frac{1}{4cf}\ln\left(\frac{f+\boldsymbol{b}}{f-\boldsymbol{b}}\right)+\frac{\boldsymbol{b}}{2e(f^2-x^2)}-\frac{1}{4ef}\ln\left(\frac{f+\boldsymbol{b}}{f-\boldsymbol{b}}\right),$$

$$\int_0^b d\boldsymbol{r}\frac{x^2}{(x^2-e\boldsymbol{r}^2)^2}+\int_0^b d\boldsymbol{r}\frac{e\boldsymbol{r}^2}{(x^2-e\boldsymbol{r}^2)^2}=\frac{\boldsymbol{b}}{e(f^2-x^2)}.$$

Substituting this equation into Eq. (3.14) one obtains

$$F(x)=\frac{1}{\boldsymbol{b}}\left[\frac{b\boldsymbol{b}^2\cos x}{(b^2-x^2)\sin\frac{x}{2}}-\frac{2\boldsymbol{b}}{x^2}\right]+2\frac{\boldsymbol{b}}{e(f^2-x^2)},$$

$$F(x)=\frac{1}{\boldsymbol{b}}\left\{\frac{2\boldsymbol{b}^3\cos(x)\sin\left(\frac{x}{2}\right)}{\left[4\boldsymbol{b}^2\sin^2\left(\frac{x}{2}\right)-x^2\right]\sin\frac{x}{2}}-\frac{2\boldsymbol{b}}{x^2}+\frac{2\boldsymbol{b}}{\left[x^2-4\boldsymbol{b}^2\sin^2\left(\frac{x}{2}\right)\right]}\right\},$$

$$F(x)=\frac{1}{\boldsymbol{b}}\left\{\frac{2\boldsymbol{b}^3\cos(x)}{\left[4\boldsymbol{b}^2\sin^2\left(\frac{x}{2}\right)-x^2\right]}-\frac{2\boldsymbol{b}}{x^2}+\frac{2\boldsymbol{b}}{\left[x^2-4\boldsymbol{b}^2\sin^2\left(\frac{x}{2}\right)\right]}\right\},$$

$$F(x)=\frac{1}{\boldsymbol{b}}\left\{\frac{2x^2\boldsymbol{b}^3\cos(x)}{x^2\left[4\boldsymbol{b}^2\sin^2\left(\frac{x}{2}\right)-x^2\right]}-\frac{2\boldsymbol{b}x^2}{x^2\left[4\boldsymbol{b}^2\sin^2\left(\frac{x}{2}\right)-x^2\right]}-\frac{2\boldsymbol{b}(4\boldsymbol{b}^2\sin^2\left(\frac{x}{2}\right)-x^2)}{x^2\left[4\boldsymbol{b}^2\sin^2\left(\frac{x}{2}\right)-x^2\right]}\right\}$$

,

$$F(x)=\frac{1}{\boldsymbol{b}}\left\{\frac{2x^2\boldsymbol{b}^3\cos(x)}{x^2\left[4\boldsymbol{b}^2\sin^2\left(\frac{x}{2}\right)-x^2\right]}-\frac{8\boldsymbol{b}^3\sin^2\left(\frac{x}{2}\right)}{x^2\left[4\boldsymbol{b}^2\sin^2\left(\frac{x}{2}\right)-x^2\right]}\right\},$$

$$F(x)=2\boldsymbol{b}^2\left\{\frac{4\sin^2\left(\frac{x}{2}\right)-x^2\cos(x)}{x^2\left[x^2-4\boldsymbol{b}^2\sin^2\left(\frac{x}{2}\right)\right]}\right\},$$

$$F(x)=2\boldsymbol{b}^2\left\{\frac{\frac{4\sin^2\left(\frac{x}{2}\right)}{x^2}-\cos(x)}{x^2\left[1-\frac{4\boldsymbol{b}^2\sin^2\left(\frac{x}{2}\right)}{x^2}\right]}\right\}.$$

With a change of variable x/2→x thus one obtains

$$F(2x)=\boldsymbol{b}^2\left\{\frac{\frac{\sin^2(x)}{x^2}-\cos(2x)}{2x^2\left[1-\frac{\boldsymbol{b}^2\sin^2(x)}{x^2}\right]}\right\}.$$

Substituting this equation into Eq. (3.13), we finally obtain the remarkably simple expression:

$$\langle N\rangle=\boldsymbol{a}\int_{-\infty}^{\infty}2dx\boldsymbol{b}^2\left\{\frac{\frac{\sin^2(x)}{x^2}-\cos(2x)}{2x^2\left[1-\frac{\boldsymbol{b}^2\sin^2(x)}{x^2}\right]}\right\},$$

$$\langle N \rangle = 2\boldsymbol{a}\boldsymbol{b}^2 \int_0^\infty dx \left\{ \frac{\left(\frac{\sin x}{x}\right)^2 - \cos(2x)}{x^2\left[1 - \boldsymbol{b}^2\left(\frac{\sin x}{x}\right)^2\right]} \right\}. \tag{3.15}$$

The latter may be rewritten as

$$\langle N \rangle = \boldsymbol{a} f(\boldsymbol{b}),$$

where

$$f(\boldsymbol{b}) = 2\boldsymbol{b}^2 \int_0^\infty dx \left\{ \frac{\left(\frac{\sin x}{x}\right)^2 - \cos(2x)}{x^2\left[1 - \boldsymbol{b}^2\left(\frac{\sin x}{x}\right)^2\right]} \right\}. \tag{3.16}$$

There is no question of the existence of the latter integral for all $0 \le \boldsymbol{b} < 1$. We can calculate this integral by using, for example, elementary numerical techniques in the MapleV Program. The table below provides some numerical values of $f(\boldsymbol{b})$:

Table 3.1 Some numerical values of the integral in Eq. (3.16)

$\boldsymbol{b}$	0.2	0.4	0.6	0.8	0.9	0.99
$f(\boldsymbol{b})$	0.1731	0.7694	2.1351	5.7951	11.4003	54.7651

Chapter IV

Explicit Representative Expressions for <N> in Arbitrary Energy Regimes for the Charged Particle

4.1 Explicit Expression in the High-Energy Regime for the Charged Particle-Significant Improvement of the Well Known Formula

We are now interested in the case of a high-energy charged particle, $b \to 1$. To carry out such a study starting from Eq. (3.16), we consider first the Laurent series of $\left(\frac{\sin x}{x}\right)^2$ and $\left(\frac{\sin x}{x}\right)^2 - \cos(2x)$. Eqs. (C.3), and (C.4) in Appendix C provide the Taylor series of $\left(\frac{\sin x}{x}\right)^2$ and $\left(\frac{\sin x}{x}\right)^2 - \cos(2x)$:

$$\left(\frac{\sin x}{x}\right)^2 = 1 - \frac{x^2}{3} + \frac{x^4}{45} - \frac{x^6}{315} + \dots,$$

and

$$\left(\frac{\sin x}{x}\right)^2 - \cos(2x) = \frac{5x^2}{3} - \frac{28x^4}{45} + \frac{3x^6}{35} - \dots.$$

Let

$$f(x) = \left(\frac{\sin x}{x}\right)^2 \approx 1 - \frac{x^2}{3} = 1 + bx^2,$$

and

$$g(x)=\left(\frac{\sin x}{x}\right)^2-\cos(2x)\approx\frac{5x^2}{3}=ax^2$$

where b = -1/3 and a = 5/3. We first rewrite the exact Eq. (3.15) as

$$f(\boldsymbol{b})=2\boldsymbol{b}^2\int_0^\infty dx\left\{\frac{\left(\frac{\sin x}{x}\right)^2-\cos(2x)}{x^2\left[1-\boldsymbol{b}^2\left(\frac{\sin x}{x}\right)^2\right]}\right\}=2\int_0^\infty dx\left\{\frac{\boldsymbol{b}^2 g(x)}{x^2\left[1-\boldsymbol{b}^2 f(x)\right]}\right\}=2G(\boldsymbol{b}),$$

where

$$G(\boldsymbol{b})=\int_0^\infty dx\left\{\frac{\boldsymbol{b}^2 g(x)}{x^2\left[1-\boldsymbol{b}^2 f(x)\right]}\right\},$$

$$\int_0^\infty dx\left\{\frac{\boldsymbol{b}^2 g(x)}{x^2\left[1-\boldsymbol{b}^2 f(x)\right]}\right\}\to\int_0^\infty\frac{dx}{x^2}\left[\frac{a\boldsymbol{b}^2x^2}{(1-\boldsymbol{b}^2-bx^2\boldsymbol{b}^2)}\right]\Bigg|_{b^2=1},$$

with the latter evaluated at $\boldsymbol{b}^2=1$ this equation can be integrated out explicitly thus

$$G(\boldsymbol{b})=\int_0^\infty\frac{dx}{x^2}\left\{\frac{\boldsymbol{b}^2 g(x)}{\left[1-\boldsymbol{b}^2 f(x)\right]}\right\}+\int_0^\infty\frac{dx}{x^2}\left[\frac{a\boldsymbol{b}x^2}{(1-\boldsymbol{b}^2-bx^2\boldsymbol{b}^2)}\right]-\int_0^\infty\frac{dx}{x^2}\left[\frac{a\boldsymbol{b}x^2}{(1-\boldsymbol{b}^2-bx^2\boldsymbol{b}^2)}\right],$$

$$G(\boldsymbol{b})=\int_0^\infty\frac{dx}{x^2}\left\{\frac{\boldsymbol{b}^2 g(x)}{\left[1-\boldsymbol{b}^2 f(x)\right]}\right\}+\int_0^\infty\left[\frac{a\boldsymbol{b}dx}{(1-\boldsymbol{b}^2-bx^2\boldsymbol{b}^2)}\right]-\int_0^\infty\frac{dx}{x^2}\left[\frac{a\boldsymbol{b}x^2}{(1-\boldsymbol{b}^2-bx^2\boldsymbol{b}^2)}\right].\quad(4.1)$$

Consider the second term of this integral, let

$$f_0(\boldsymbol{b})=2\int_0^\infty dx\left[\frac{a\boldsymbol{b}}{(1-\boldsymbol{b}^2-bx^2\boldsymbol{b}^2)}\right],$$

and we have

$$\int_0^\infty dx\left[\frac{a\boldsymbol{b}}{(1-\boldsymbol{b}^2-bx^2\boldsymbol{b}^2)}\right]=\frac{5}{3}\int_0^\infty\frac{\boldsymbol{b}dx}{(1-\boldsymbol{b}^2-bx^2\boldsymbol{b}^2)}=\frac{5}{\boldsymbol{b}}\int_0^\infty\frac{dx}{(x^2+\frac{3}{\boldsymbol{b}^2}-3)}.$$

Let

$$a = \sqrt{\frac{3}{\boldsymbol{b}^2} - 3}\,.$$

Therefore from the well know integral (Eq. (B.9)), in Appendix B we have,

$$\frac{5}{\boldsymbol{b}}\int_0^\infty \frac{dx}{(x^2 + \frac{3}{\boldsymbol{b}^2} - 3)} = \frac{5}{2\boldsymbol{b}}\frac{\boldsymbol{p}}{\sqrt{\frac{3}{\boldsymbol{b}^2} - 3}} = \frac{5\boldsymbol{p}}{2\sqrt{3}\sqrt{1-\boldsymbol{b}^2}}\,.$$

Thus

$$f_0(\boldsymbol{b}) = \frac{5\boldsymbol{p}}{\sqrt{3}\sqrt{1-\boldsymbol{b}^2}}\,, \qquad (4.2)$$

and Eq. (4.1) becomes

$$G(\boldsymbol{b}) = \int_0^\infty \frac{dx}{x^2}\left\{\left[\frac{\boldsymbol{b}^2 g(x)}{\left[1-\boldsymbol{b}^2 f(x)\right]}\right] - \left[\frac{a\boldsymbol{b}x^2}{(1-\boldsymbol{b}^2 - bx^2\boldsymbol{b}^2)}\right]\right\} + \frac{5\boldsymbol{p}}{2\sqrt{3}\sqrt{1-\boldsymbol{b}^2}}\,. \qquad (4.3)$$

Now that we have our exact expression, the next step is to consider only the first integral in Eq. (4.3). Let

$$h(x) = \left\{\frac{\boldsymbol{b}^2\left[\left(\frac{\sin x}{x}\right)^2 - \cos(2x)\right]}{\left[1-\boldsymbol{b}^2\left(\frac{\sin x}{x}\right)^2\right]}\right\} - \frac{5\boldsymbol{b}x^2}{3\left[1-\boldsymbol{b}^2(1-\frac{x^2}{3})\right]}\,. \qquad (4.4)$$

Setting first $\boldsymbol{b}$ = 1, the first and second terms above are equal, respectively, to $\dfrac{\left(\frac{\sin x}{x}\right)^2 - \cos(2x)}{\left[1-\left(\frac{\sin x}{x}\right)^2\right]}$ and 5. Therefore, we will add and subtract these two terms to

Eq. (4.4). Accordingly, we can rewrite Eq. (4.4) as

$$h(x)=\left[\frac{\boldsymbol{b}^2\left[\left(\frac{\sin x}{x}\right)^2-\cos(2x)\right]}{(1-\boldsymbol{b}^2\left(\frac{\sin x}{x}\right)^2)}\right]-\frac{5\boldsymbol{b}x^2}{3\left[1-\boldsymbol{b}^2(1-\frac{x^2}{3})\right]}+\frac{\left(\frac{\sin x}{x}\right)^2-\cos(2x)}{\left[1-\left(\frac{\sin x}{x}\right)^2\right]},$$

$$-\frac{\left(\frac{\sin x}{x}\right)^2-\cos(2x)}{\left[1-\left(\frac{\sin x}{x}\right)^2\right]}+5-5$$

$$G(\boldsymbol{b})=\int_0^\infty\frac{dx}{x^2}\left\{\frac{\boldsymbol{b}^2\left[\left(\frac{\sin x}{x}\right)^2-\cos(2x)\right]}{(1-\boldsymbol{b}^2\left(\frac{\sin x}{x}\right)^2)}-\frac{\left(\frac{\sin x}{x}\right)^2-\cos(2x)}{(1-\left(\frac{\sin x}{x}\right)^2)}-\frac{5\boldsymbol{b}x^2}{3\left[1-\boldsymbol{b}^2(1-\frac{x^2}{3})\right]}+5\right\}$$

$$+\int_0^\infty\frac{dx}{x^2}\left[\frac{\left(\frac{\sin x}{x}\right)^2-\cos(2x)}{(1-\left(\frac{\sin x}{x}\right)^2)}-5\right]+\frac{5\boldsymbol{p}}{2\sqrt{3}\sqrt{1-\boldsymbol{b}^2}}. \qquad (4.5)$$

Let

$$a_0=2\int_0^\infty\frac{dx}{x^2}\left[\frac{\left(\frac{\sin x}{x}\right)^2-\cos(2x)}{(1-\left(\frac{\sin x}{x}\right)^2)}-5\right].$$

This integral can be evaluated numerically. Using, for example, the elementary MapleV Program, yields:

a_0 = -9.5580. (4.6)

Eq. (4.4) becomes

$$G(\beta)=\int_0^\infty \frac{dx}{x^2}\left\{\frac{\beta^2 g(x)}{\left[1-\beta^{2)} f(x)\right]}-\frac{g(x)}{\left[1-f(x)\right]}-\frac{5\beta x^2}{3\left[1-\beta^2(1-\frac{x^2}{3})\right]}+5\right\}. \quad (4.7)$$
$$-4.77899+\frac{5\pi}{2\sqrt{3}\sqrt{1-\beta^2}}$$

Consider the following combination:

$$\frac{\beta^2 g(x)}{\left[1-\beta^{2)} f(x)\right]}-\frac{g(x)}{\left[1-f(x)\right]}=\frac{\beta^2 g(x)-\beta^2 g(x) f(x)-g(x)+\beta^2 g(x) f(x)}{\left[1-\beta^2 f(x)\right]\left[1-f(x)\right]},$$

$$\frac{\beta^2 g(x)}{\left[1-\beta^{2)} f(x)\right]}-\frac{g(x)}{\left[1-f(x)\right]}=\frac{(\beta^2-1)g(x)}{\left[1-\beta^2 f(x)\right]\left[1-f(x)\right]}. \quad (4.8)$$

The remaining part of the integrand in Eq. (4.7) may be rewritten as:

$$\frac{-5\beta x^2}{3\left[1-\beta^2(1-\frac{x^2}{3})\right]}+5=\frac{15\left(1-\beta^2+\frac{\beta^2 x^2}{3}\right)-5\beta x^2}{3(1-\beta^2)+\beta^2 x^2},$$

$$=\frac{15\left(1-\beta^2+\frac{\beta^2 x^2}{3}\right)-5\beta x^2}{3(1-\beta^2)+\beta^2 x^2}\,\frac{(1+\beta)}{(1+\beta)},$$

$$=\frac{15\left(1-\beta^2+\frac{\beta^2 x^2}{3}\right)-5\beta x^2+15\beta\left(1-\beta^2+\frac{\beta^2 x^2}{3}\right)-5\beta^2 x^2}{3(1-\beta^2)(1+\beta)+\beta^2 x^2(1+\beta)},$$

$$=\frac{15-15\beta^2+5\beta^2 x^2-5\beta x^2+15\beta-15\beta^3+5\beta^3 x^2-5\beta^2 x^2}{3(1-\beta^2)(1+\beta)+\beta^2 x^2(1+\beta)},$$

$$=\frac{15-15\beta^2-5\beta x^2+15\beta-15\beta^3+5\beta^3 x^2}{3(1-\beta^2)(1+\beta)+\beta^2 x^2(1+\beta)},$$

$$\frac{-5\beta x^2}{3\left[1-\beta^2(1-\frac{x^2}{3})\right]}+5=\frac{15(1-\beta^2)+15\beta(1-\beta^2)-5\beta x^2(1-\beta^2)}{3(1-\beta^2)(1+\beta)+\beta^2x^2(1+\beta)},$$

$$\frac{-5\beta x^2}{3\left[1-\beta^2(1-\frac{x^2}{3})\right]}+5=\frac{-5(1-\beta^2)[\beta x^2-3(1-\beta)}{3(1-\beta^2)(1+\beta)+\beta^2x^2(1+\beta)}. \tag{4.9}$$

Thus, from Eqs. (4.8) and (4.9) the Eq. (4.7) becomes

$$G(\beta)=\int_0^\infty\frac{dx}{x^2}\left\{\frac{(\beta^2-1)g(x)}{\left[1-\beta^2 f(x)\right]\left[1-f(x)\right]}-\frac{5(1-\beta^2)[\beta x^2-3(1-\beta)}{3(1-\beta^2)(1+\beta)+\beta^2x^2(1+\beta)}\right\} -4.77899+\frac{5\pi}{2\sqrt{3}\sqrt{1-\beta^2}}. \tag{4.10}$$

We add and subtract

$$\int_0^\infty(1-\beta^2)\frac{dx}{x^2}\frac{9}{10\beta}\frac{x^2}{\left[\frac{3(1-\beta^2)}{\beta^2}+x^2\right]},$$

to Eq. (4.10), Thus Eq. (4.10) becomes

$$G(\beta)=-(1-\beta^2)\int_0^\infty\frac{dx}{x^2}\left\{\frac{g(x)}{\left[1-\beta^2 f(x)\right]\left[1-f(x)\right]}+\frac{5[\beta x^2-3(1-\beta)]}{3(1-\beta^2)(1+\beta)+\beta^2x^2(1+\beta)}-\frac{9}{10\beta}\frac{x^2}{\left[\frac{3(1-\beta^2)}{\beta^2}+x^2\right]}\right\}-\int_0^\infty(1-\beta^2)\frac{dx}{x^2}\frac{9}{10\beta}\frac{x^2}{\left[\frac{3(1-\beta^2)}{\beta^2}+x^2\right]}+\frac{a_0}{2}+\frac{f_0}{2}. \tag{4.11}$$

The second integral may be explicitly integrated out as before. Let

$$f_1(\beta)=2(1-\beta^2)\int_0^\infty\frac{dx}{x^2}\frac{9}{10\beta}\frac{x^2}{\left[\frac{3(1-\beta^2)}{\beta^2}+x^2\right]},$$

and

$$(1-\boldsymbol{b}^2)\int_0^\infty \frac{dx}{x^2}\frac{9}{10\boldsymbol{b}}\frac{x^2}{\left[\frac{3(1-\boldsymbol{b}^2)}{\boldsymbol{b}^2}+x^2\right]}=(1-\boldsymbol{b}^2)\frac{9}{10\boldsymbol{b}}\int_0^\infty \frac{dx}{\left[\frac{3(1-\boldsymbol{b}^2)}{\boldsymbol{b}^2}+x^2\right]}.$$

In the integral formula (B.9), let

$$a=\sqrt{\frac{3(1-\boldsymbol{b}^2)}{\boldsymbol{b}^2}}.$$

Thus, we have

$$(1-\boldsymbol{b}^2)\frac{9}{10\boldsymbol{b}}\int_0^\infty \frac{dx}{\left[\frac{3(1-\boldsymbol{b}^2)}{\boldsymbol{b}^2}+x^2\right]}=(1-\boldsymbol{b}^2)\frac{9}{10\boldsymbol{b}}\left(\frac{\boldsymbol{p}}{2\sqrt{\frac{3(1-\boldsymbol{b}^2)}{\boldsymbol{b}^2}}}\right).$$

And

$$f_1(\boldsymbol{b})=\sqrt{1-\boldsymbol{b}^2}\,\frac{3\sqrt{3}\boldsymbol{p}}{10}. \qquad (4.13)$$

Therefore Eq. (4.11) becomes

$$G(\boldsymbol{b})=-(1-\boldsymbol{b}^2)\int_0^\infty \frac{dx}{x^2}\left\{\frac{g(x)}{\left[1-\boldsymbol{b}^2 f(x)\right]\left[1-f(x)\right]}+\frac{5[\boldsymbol{b}x^2-3(1-\boldsymbol{b})]}{3(1-\boldsymbol{b}^2)(1+\boldsymbol{b})+\boldsymbol{b}^2x^2(1+\boldsymbol{b})}\right.$$
$$\left.-\frac{9}{10\boldsymbol{b}}\frac{x^2}{\left[\frac{3(1-\boldsymbol{b}^2)}{\boldsymbol{b}^2}+x^2\right]}\right\}-\sqrt{1-\boldsymbol{b}^2}\,\frac{3\sqrt{3}\boldsymbol{p}}{20}+\frac{a_0}{2}+\frac{f_0}{2},$$

and from Eq. (3.16) we conclude that

$f(\boldsymbol{b}) = f_0(\boldsymbol{b}) + a_0 + f_1(\boldsymbol{b}) + \boldsymbol{e}(\boldsymbol{b})$,

this is an exact equation, where

$$f_0(\boldsymbol{b}) = \frac{5\boldsymbol{p}}{\sqrt{3}\sqrt{1-\boldsymbol{b}^2}},$$

a_0 = -9.5580,

$$f_1(\boldsymbol{b}) = \sqrt{1-\boldsymbol{b}^2}\,\frac{3\sqrt{3}\boldsymbol{p}}{10},$$

and

$$\boldsymbol{e}(\boldsymbol{b}) = -2(1-\boldsymbol{b}^2)\int \frac{dx}{x^2}\left[g_1(x,\boldsymbol{b}) + g_2(x,\boldsymbol{b}) + g_3(x,\boldsymbol{b})\right], \qquad (4.14)$$

with

$$g_1(x,\boldsymbol{b}) = \frac{\left[\left(\frac{\sin x}{x}\right)^2 - \cos(2x)\right]}{\left[1-\boldsymbol{b}^2\left(\frac{\sin x}{x}\right)^2\right]}\frac{1}{\left[1-\left(\frac{\sin x}{x}\right)^2\right]},$$

$$g_2(x,\boldsymbol{b}) = \frac{5[\boldsymbol{b}x^2 - 3(1-\boldsymbol{b})]}{3(1-\boldsymbol{b}^2)(1+\boldsymbol{b}) + \boldsymbol{b}^2x^2(1+\boldsymbol{b})},$$

$$g_3(x,\boldsymbol{b}) = -\frac{9}{10\boldsymbol{b}}\frac{x^2}{\left[\frac{3(1-\boldsymbol{b}^2)}{\boldsymbol{b}^2} + x^2\right]}.$$

For $\boldsymbol{b}$ = 1, the integrand in $\boldsymbol{e}(\boldsymbol{b})$ reduces to

$$\frac{\left(\frac{\sin x}{x}\right)^2 - \cos(2x)}{\left[1-\left(\frac{\sin x}{x}\right)^2\right]^2} - \frac{15}{x^2} + \frac{8}{5},$$

and form Eq. (C.5), in Appendix C, we have

$$\frac{\left(\dfrac{\sin z}{z}\right)^2 - \cos(2z)}{\left[1-\left(\dfrac{\sin z}{z}\right)^2\right]^2} - \frac{15}{z^2} + \frac{8}{5} = O(z^2)\ .$$

Thus

$$\int_0^\infty \frac{dx}{x^2}\left\{\frac{\left(\dfrac{\sin x}{x}\right)^2 - \cos(2x)}{\left[1-\left(\dfrac{\sin x}{x}\right)^2\right]^2} - \frac{15}{x^2} + \frac{8}{5}\right\}_{b=1}\quad ,$$

exists. Also

$$\boldsymbol{e}(\boldsymbol{b}) = O(\sqrt{1-\boldsymbol{b}^2})\ ;\boldsymbol{b} \to 1. \qquad (4.15)$$

Because of the (1 - $\boldsymbol{b}^2$) factor multiplying the integral in (4.14), one may naively expect that $\boldsymbol{e}(\boldsymbol{b})$ vanishes like $O(1 - \boldsymbol{b}^2)$ for $\boldsymbol{b} \to 1$. The careful analysis given above, however, shows that it vanishes like $O(\sqrt{1-\boldsymbol{b}^2})$ as indicated. That is, at high energies we may write

$$\langle N\rangle \cong \frac{5\boldsymbol{pa}}{\sqrt{3(1-\boldsymbol{b}^2)}} + a_0\boldsymbol{a}\ . \qquad (4.16)$$

The asymptotic constant a_0 is overwhelmingly large in magnitude. It is the important contribution that survives in the limit $\boldsymbol{b} \to 1$ beyond the $1\big/\sqrt{1-\boldsymbol{b}^2}$ term. The relative errors in Eq. (4.16) are quite satisfactory some of which are given in the table below.

Table 4.1 Relative errors of the mean number of photons in the high-energy regime according to our representation in Eq. (4.16)

b	0.8	0.9	0.99
Relative error	4.11%	1.34%	0.063%

These relative errors are to be compared with the well-known relative errors of the well-known formula printed repeatedly in the literature. These corresponding relative errors are given in the table below.

Table 4.2 Relative errors of the well known formula

b	0.8	0.9	0.99
Relative error	160% (!)	82% (!)	17%

Our novel high-energy expression is a definite replacement to the well-known formula as it provides a significant improvement to the latter.

4.2 Explicit Expression in the Low and Intermediate Regimes for the Charged Particle

In the case of low and intermediate energies of the charged particle, the particle velocity is relatively small compared to that of light. From Eq. (3.17) when $b << 1$ we can be expanded

$$\frac{1}{1-b^2\left(\frac{\sin x}{x}\right)^2} \equiv \frac{1}{1-z^2} = 1 + z^2 + z^4 + z^6 + \ldots,$$

where $z \equiv b\,(\sin x/x)$. Therefore from Eq. (3.16) we have

$$f(b) = 2b^2 \int_0^\infty \frac{dx}{x^2}\left[\left(\frac{\sin x}{x}\right)^2 - \cos(2x)\right] \sum_{k=0}^{5}\left[b^2\left(\frac{\sin x}{x}\right)^2\right]^k$$
$$+\, 2b^2 \int_0^\infty \frac{dx}{x^2}\left[\left(\frac{\sin x}{x}\right)^2 - \cos(2x)\right]\left\{\frac{1}{1-b^2\left(\frac{\sin x}{x}\right)^2} - \sum_{k=0}^{5}\left[b^2\left(\frac{\sin x}{x}\right)^2\right]^k\right\} \quad (4.17)$$

From Eq. (4.17) we may write

$f(b) = g(b) + \tilde{e}(b)$

this is an exact equation, where

$$g(b) = 2b^2 \int_0^\infty \frac{dx}{x^2}\left[\left(\frac{\sin x}{x}\right)^2 - \cos(2x)\right] \sum_{k=0}^{5}\left[b^2\left(\frac{\sin x}{x}\right)^2\right]^k . \quad (4.18)$$

This integral is readily calculated numerically and gives

$g(b) = 4.1888b^2 + 3.351b^4 + 2.8125b^6 + 2.4678b^8 + 2.2241b^{10}$,

and

$$\tilde{e}(b) = 2b^2 \int_0^\infty \frac{dx}{x^2}\left[\left(\frac{\sin x}{x}\right)^2 - \cos(2x)\right]\left\{\frac{1}{1-b^2\left(\frac{\sin x}{x}\right)^2} - \sum_{k=0}^{5}\left[b^2\left(\frac{\sin x}{x}\right)^2\right]^k\right\}. \quad (4.19)$$

Similarly, Eq. (4.19) is calculated numerically, and leads to the following table:

Table 4.3 Errors in the mean number of photons for low and intermediate energy regimes for the charged particle according to Eq. (4.19)

b	0.2	0.4	0.6	0.7	0.75
$e(b)$	0.1409	0.2547	0.4707	-0.1382	-0.1936

The relative errors in Eq. (4.16) (calculated by MapleV Program) are more than satisfactory as shown below in the following table:

Table 4.4 Relative errors of mean number of photons for low and intermediate energy regimes for the charged particle according to Eq. (4.19)

b	0.2	0.4	0.6	0.7	0.75
Relative error	0.00019	0.005%	0.31%	0.5%	3.1%

Chapter V

Conclusion

We derived an exact and explicit and remarkably simple expression for the mean number <N> of photons emitted per revolution in synchrotron radiation. The latter is given by the elementary integral

$$\langle N \rangle = 2\alpha\beta^2 \int_0^\infty dx \left\{ \frac{\left(\dfrac{\sin x}{x}\right)^2 - \cos(2x)}{x^2\left[1-\beta^2\left(\dfrac{\sin x}{x}\right)^2\right]} \right\}.$$

It was shown that the familiar high-energy expression $5\pi\alpha \big/ \sqrt{3(1-\beta^2)}$ printed repeatedly in the literature (e.g., Review of Particle Physics (the “Particle Physicist’s Handbook”), 1996, p.75, 1998, p.79), is found to be inaccurate and only truly asymptotic as it provides relative errors of 160%, 82%, and 17% for β = 0.8, 0.9, and 0.99, respectively.

Our explicit expression provides a much-improved high-energy expression for <N>, over the familiar one, and is given by:

$$\langle N \rangle \cong \frac{5\pi\alpha}{\sqrt{3(1-\beta^2)}} + a_0\alpha\,, \qquad \text{(high-energy regime)}$$

where a_0 = -9.5580 and is overwhelmingly large in magnitude. It is the important contribution that survives in the limit $\beta \to 1$ beyond the $1\big/\sqrt{1-\beta^2}$ term. The relative errors of the above representation are only 4.11% (!) for β = 0.8, 1.34% for β = 0.9, and 0.063% for b = 0.99 providing a much significant improvement over the familiar expression.

For completeness, we have also provided a representation for <N> in low and intermediate energy regimes given by

$$<N> \cong a\{4.1888b^2 + 3.351b^4 + 2.8125b^6 + 2.4678b^8 + 2.2241b^{10}\}$$

The relative errors of the above are satisfactory with 0.19×10^{-3} %, 0.31%, and 3.1% for b = 0.2, 0.4, and 0.7.

This thesis concludes an important chapter in the theory of synchrotron radiation.

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Appendices

Appendix A

Vector Analysis Formulae

We collect here, for easy reference, some (e.g., Spiegel, 1990) vector identities, vector differentiations and integration formulae which have been useful in writing up the thesis.

A.1 Vector algebra

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \qquad \text{(A.1)}$$

$$(\mathbf{A} \times \mathbf{B}) \bullet (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \qquad \text{(A.2)}$$

$$\mathbf{A} \times \mathbf{B} \times \mathbf{C} \times \mathbf{D} = [(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{D}]\mathbf{C} - [(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}]\mathbf{D} \qquad \text{(A.3)}$$

$$\mathbf{A} \times [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] = (\mathbf{B} \cdot \mathbf{D})(\mathbf{A} \times \mathbf{C}) + (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \times \mathbf{D}) \qquad \text{(A.4)}$$

A.2 Vector differential operators

$$\mathbf{grad}(\psi\varphi) = \psi\, \mathbf{grad}\, \varphi + \varphi\, \mathbf{grad}\, \psi \qquad \text{(A.5)}$$

$$\mathbf{grad}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times \mathbf{curl}\, \mathbf{B} + \mathbf{B} \times \mathbf{curl}\, \mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \qquad \text{(A.6)}$$

$$\mathrm{div}\, (\psi\mathbf{A}) = \psi\, \mathrm{div}\, \mathbf{A} + \mathbf{A}\, \mathbf{grad}\psi \qquad \text{(A.7)}$$

$$\mathrm{div}\, (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \mathbf{curl}\, \mathbf{A} - \mathbf{A} \cdot \mathbf{curl}\, \mathbf{B} \qquad \text{(A.8)}$$

$$\mathbf{curl}\, (\psi\mathbf{A}) = \psi\, \mathbf{curl}\, \mathbf{A} - \mathbf{A} \times \mathbf{grad}\, \psi \qquad \text{(A.9)}$$

$$\mathbf{curl}\, (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\, \mathrm{div}\, \mathbf{B} - \mathbf{B}\, \mathrm{div}\, \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \qquad \text{(A.10)}$$

$$\mathbf{curl}\, \mathbf{curl}\, \mathbf{A} = \mathbf{grad}\, \mathrm{div}\, \mathbf{A} - \nabla^2\mathbf{A} \qquad \text{(A.11)}$$

$$\mathbf{curl}\, \mathbf{grad}\, \varphi = 0 \qquad \text{(A.12)}$$

div **curl A** = 0 (A.13)

A.3 Integral Theorems

The divergence theorem (or Gauss' Theorem) states that

$$\int_V div\mathbf{A}\, dV = \oint_S \mathbf{A} \bullet \mathbf{n}\, da \qquad (A.14)$$

where the closed surface S bounds the volume V; **n** is the unit vector in the direction of the outward normal.

Stokes' Theorem states that

$$\int_S \mathbf{curlA} \bullet \mathbf{n}\, da = \oint_l \mathbf{A} \bullet d\mathbf{l} \qquad (A.15)$$

where the closed line l bounds the open surface S; the positive sense of traversing l is such that the right-hand screw direction is parallel to **n**.

Appendix B

Useful Integral formulae

The following integrals have been of utmost importance leading to the explicit expression for <N>:

$$\int f(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})d^3\mathbf{x} = f(\mathbf{y})\,, \tag{B.1}$$

$$\int_a^b f(t)\frac{d\delta(t-T)}{dt}dt = -f^{/}(T)\,, \qquad a < T < b \tag{B.2}$$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}d\omega \tag{B.3}$$

$$\int_0^{\infty} dz \int_{-\infty}^{\infty} ze^{-izx}dx = 0 \tag{B.4}$$

$$-i\int_0^{\infty} dxe^{iax} = \frac{1}{a+i\varepsilon} \tag{B.5}$$

$$i\int_0^{\infty} dxe^{-iax} = \frac{1}{a-i\varepsilon} \tag{B.6}$$

$$\int_0^a \frac{x^2}{(b^2-x^2)^2}dx = \frac{a}{2(b^2-x^2)} - \frac{1}{4b}\ln\left(\frac{b+a}{b-a}\right) \tag{B.7}$$

$$\int_0^a \frac{1}{(b^2-x^2)^2}dx = \frac{a}{2b^2(b^2-x^2)} + \frac{1}{4b^3}\ln\left(\frac{b+a}{b-a}\right) \tag{B.8}$$

$$\int_0^{\infty} \frac{dx}{\left(x^2+a^2\right)^n} = \frac{\pi(2n-3)!!}{2a^{2n-1}(2n-2)!!} \tag{B.9}$$

Appendix C

Laurent Expansions of Some Singular Functions

In this appendix, we provide a very brief account of Laurent expansions which has applications to the function for <N> in the high-energy regime. One frequently encounters functions that are analytic in an annular region, say, of inner radius r and outer radius R, as shown in Fig. C.1.

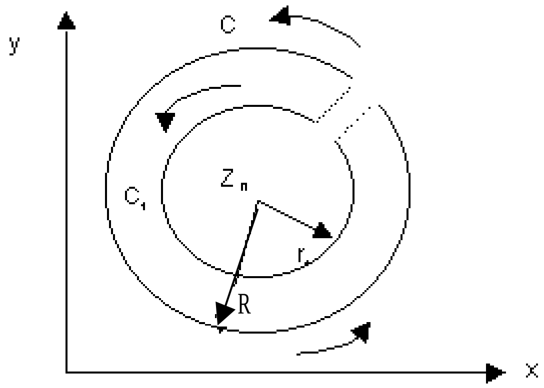

Figure C-1. The complex z-plane.

For a function that fails to be analytic (singular) at a point z_0, we cannot develop a Taylor expansion at that point. Then Taylor's formula no longer applies, and we need a new type of series, known as a Laurent series, consisting of positive and negative integer powers of $z - z_0$ and being convergent in some annulus (bounded by two circles with center z_0) in which $f(z)$ is analytic. $f(z)$ may have singular points not only outside the large circle (as for a Taylor series) but also inside the smaller circle.

C.1 The Laurent Theorem

If $f(z)$ is analytic on two concentric circles C_1 and C_2 with center z_0 and in the annulus between them, then $f(z)$ can be represented by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots. \quad \text{(C.1)}$$

The coefficients of this Laurent series are given by the integrals

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t - z_0)^{n+1}} dt \\ b_n &= \frac{1}{2\pi i} \oint_C (t - z_0)^{n-1} f(t) dt \end{aligned} . \quad \text{(C.2)}$$

In the Laurent expansion of a function $f(z)$, the first part $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is called the analytic part (regular part) which is similar to the Taylor's series of the function and the remainder $\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ is known as the principal part. The second summation is of chief interest in our work, since it contains the singularity.

But the Laurent series coefficients need not come out from evaluation of contour integrals (which may be very intractable). Other techniques such as ordinary series expansions may provide the coefficients.

C.2 Some Examples of Laurent Expansions

Laurent series about the indicated singularity for some important functions are worked out:

(1) $\dfrac{e^z}{(z-1)^2}$; $z = 1$.

Let $z - 1 = u$.

Then

$$f(z) = \frac{e^{1+u}}{u^2} = e\frac{e^u}{u^2},$$

$$f(z) = \frac{e}{u^2}\left[1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \ldots\right],$$

$$f(z) = \frac{e}{(z-1)^2} + \frac{e}{(z-1)} + \frac{e}{2!} + \frac{e(z-1)}{3!} + \frac{e(z-1)^2}{4!} + \ldots.$$

Here z=1 is a pole of order 2

(2) $\dfrac{1}{(z-a)(z-b)}$; $0 < |a| < |z| < |b|$.

$$f(z) = \frac{1}{(z-a)(z-b)},$$

$$f(z) = \frac{1}{a-b}\left[\frac{1}{z-a} + \frac{1}{b-z}\right],$$

$$f(z) = \frac{1}{a-b}\left[\frac{1}{z\left(1-\frac{a}{z}\right)} + \frac{1}{b\left(1-\frac{z}{b}\right)}\right],$$

$$f(z) = \frac{1}{a-b}\left[\frac{\left(1-\frac{a}{z}\right)^{-1}}{z} + \frac{\left(1-\frac{z}{b}\right)^{-1}}{b}\right],$$

$$f(z)=\frac{1}{a-b}\left[\frac{1+\frac{a}{z}+\frac{a^2}{z^2}+\frac{a^3}{z^3}+\dots}{z}+\frac{1+\frac{z}{b}+\frac{z^2}{b^2}+\frac{z^3}{b^3}+\dots}{b}\right],$$

$$f(z)=\frac{1}{a-b}\left[\dots+\frac{a^3}{z^4}+\frac{a^2}{z^3}+\frac{a}{z^2}+\frac{1}{z}+\frac{1}{b}+\frac{z}{b^2}+\frac{z^2}{b^3}+\frac{z^3}{b^4}+\dots\right].$$

Hence $f(z)$ had an essential singularity at $z = 0$

(3) $\left(\frac{\sin z}{z}\right)^2; \quad z=0.$

$$f(z)=\left(\frac{\sin z}{z}\right)^2,$$

form

$$\sin(z)=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots,$$

$$f(z)=\frac{1}{z^2}\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}\dots\right]^2,$$

$$f(z)=1-\frac{z^2}{6}+\frac{z^4}{36}-\frac{z^2}{6}+\frac{z^4}{120}+\frac{z^4}{120}-\frac{z^6}{5040}-\frac{z^6}{5040}-\frac{z^6}{720}-\frac{z^6}{720}\dots,$$

$$f(z)=1-\frac{z^2}{3}+\frac{2z^4}{45}-\frac{z^6}{135}+\dots, \qquad \text{(C.3)}$$

Thus about the points $z = 0$ the Laurent series contains no terms of the principal part. Hence $z = 0$ is a removal singularity of $f(z)$.

(4) $\left(\frac{\sin z}{z}\right)^2-\cos(2z); \quad z=0.$

$$f(z)=\left(\frac{\sin z}{z}\right)^2-\cos(2z),$$

from

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots,$$

and

$$\cos(2z) = 1 - 2z^2 + \frac{2z^4}{3} - \frac{4z^6}{45} + \ldots,$$

thus

$$\left(\frac{\sin z}{z}\right)^2 - \cos(2z) = \frac{5z^2}{3} - \frac{28z^4}{45} + \frac{3z^6}{35} - \ldots. \qquad \text{(C.4)}$$

Thus about the points $z = 0$ the Laurent series has no principal part. That is, $z = 0$ is a removal singularity of $f(z)$.

(5) $\dfrac{\left(\frac{\sin z}{z}\right)^2 - \cos(2z)}{(1 - \left(\frac{\sin z}{z}\right)^2)^2}$; $\quad z = 0$.

$$f(z) = \frac{\left(\frac{\sin z}{z}\right)^2 - \cos(2z)}{(1 - \left(\frac{\sin z}{z}\right)^2)^2},$$

and from Eqs. (C.3) and (C.4) we know the series expressions of $\left(\frac{\sin z}{z}\right)^2$ and $\left(\frac{\sin z}{z}\right)^2 - \cos(2z)$:

$$1 - \left(\frac{\sin z}{z}\right)^2 = \frac{z^2}{3} - \frac{2z^4}{45} + \frac{z^6}{135} - \ldots,$$

$$\left[1 - \left(\frac{\sin z}{z}\right)^2\right]^2 = \left(\frac{z^2}{3} - \frac{2z^4}{45} + \frac{z^6}{135} - \ldots\right)\left(\frac{z^2}{3} - \frac{2z^4}{45} + \frac{z^6}{135} - \ldots\right),$$

$$\left[1-\left(\frac{\sin z}{z}\right)^2\right]^2=\frac{z^4}{9}-\frac{2z^6}{135}+\frac{z^8}{945}-\frac{z^6}{135}+\frac{4z^8}{2025}+\frac{z^8}{945}+\ldots,$$

$$\left[1-\left(\frac{\sin z}{z}\right)^2\right]^2=\frac{z^4}{9}-\frac{4z^6}{135}+\ldots,$$

Thus,

$$\frac{\left(\frac{\sin z}{z}\right)^2-\cos(2z)}{\left[1-\left(\frac{\sin z}{z}\right)^2\right]^2}=\frac{\frac{5z^2}{3}-\frac{28z^4}{45}+\frac{3z^6}{35}-\ldots}{z^4(\frac{1}{9}-\frac{4z^2}{135}+\ldots)},$$

$$\frac{\left(\frac{\sin z}{z}\right)^2-\cos(2z)}{\left[1-\left(\frac{\sin z}{z}\right)^2\right]^2}=\frac{\frac{9}{z^4}\left(\frac{5z^2}{3}-\frac{28z^4}{45}+\frac{3z^6}{35}-\ldots\right)}{(1-\frac{4z^2}{15}+\ldots)},$$

$$\frac{\left(\frac{\sin z}{z}\right)^2-\cos(2z)}{\left[1-\left(\frac{\sin z}{z}\right)^2\right]^2}=\frac{\frac{9}{z^4}\left(\frac{5z^2}{3}-\frac{28z^4}{45}+\frac{3z^6}{35}-\ldots\right)}{(1-\frac{4z^2}{15}+\ldots)},$$

$$\frac{\left(\frac{\sin z}{z}\right)^2-\cos(2z)}{\left[1-\left(\frac{\sin z}{z}\right)^2\right]^2}=\left(\frac{15}{z^2}-\frac{28}{5}+\ldots\right)\left(1+\frac{4z^2}{15}+\ldots\right),$$

$$\frac{\left(\frac{\sin z}{z}\right)^2 - \cos(2z)}{\left[1-\left(\frac{\sin z}{z}\right)^2\right]^2} = \frac{15}{z^2} - \frac{8}{5} + O(z^2)\ . \qquad \text{(C.5)}$$

The Laurent series differs from the Taylor series by the obvious feature of negative powers of (z - z_0). For this reason the Laurent series will always diverge at least at $z = z_0$ and perhaps as far out to some distance.

Biography

Ngarmjit Jearnkulprasert was born on 26th August 1977 in Bangkok. She went to study at St. Francissevia Convent, where she graduated with a High School Diploma in 1993. After that she went to study in the Department of Physics, Faculty of Science, at Mahidol University, where she graduated with a B.Sc degree Physics in 1998. In 1998, she decided to study for a Master’s degree in the School of Physics, Institute of Science, Suranaree University of Technology.