การสลายตัวของ $N^*(1440)$ ในแบบจำลองแบริออนแบบไฮบริด

นางสาวนพมณี ศุภนาม

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาฟิสิกส์
มหาวิทยาลัยเทคโนโลยีสุรนารี
ปีการศึกษา 2545

ISBN: 974–533–194–5

N^*(1440) DECAYS
IN A HYBRID BARYON MODEL

Miss Nopmanee Supanam

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science in Physics
Suranaree University of Technology
Academic year 2002

ISBN: 974–533–194–5

N^*(1440) DECAYS IN A HYBRID BARYON MODEL

Suranaree University of Technology has approved this thesis submitted in partial fulfillment for a Master's Degree.

Thesis Examining Committee

. . . .
(Col. Dr. Worasit Uchai)
Chairman

. . . .
(Asst. Prof. Dr. Yupeng Yan)
Member (Thesis Advisor)

. . . .
(Assoc. Prof. Dr. Prasart Suebka)
Member

. . . .
(Asst. Prof. Dr. Prapan Manyum)
Member

. . . .
(Dr. Chinorat Kobdaj)
Member

. . . .
(Assoc. Prof. Dr. Tawit Chitsomboon)
Vice Rector for Academic Affairs

. . . .
(Assoc. Prof. Dr. Prasart Suebka)
Dean, Institute of Sciences

นพมณี ศุภนาม: การสลายตัวของ $N^*(1440)$ ในแบบจำลองแบริออนแบบไฮบริด ($N^*(1440)$ DECAYS IN A HYBRID BARYON MODEL) อาจารย์ที่ปรึกษา: ผศ. ดร. ยูเปิ้ง แยน, 83 หน้า. ISBN: 974–533–194–5

การสั่นพ้องโรเปอร์หรือ $N^*(1440)$ คือนิวคลีออนในสภาวะกระตุ้นที่ต่ำสุด ได้มีการอธิบายมากขึ้นนับตั้งแต่ได้มีการค้นพบในปี ค.ศ. 1964 ตามแบบจำลองควาร์กสามตัว การสั่นพ้องโรเปอร์กำหนดว่าเป็นการกระตุ้นในแนวรัศมีของนิวคลีออน อันเนื่องจากมีเลขควอนตัมที่เหมือนกับนิวคลีออน แต่จากการศึกษาในรายละเอียดพบว่า เป็นการยากมากที่จะให้ความหมายของการสั่นพ้องเช่นนี้ว่าเป็นสภาวะที่ประกอบด้วยควาร์กสามตัว เพราะมวลที่ต่ำและค่าคงตัวของการคู่ควบกับนิวคลีออนและเมซอนที่แปลก จากความล้มเหลวของแบบจำลองควาร์กสามตัวจึงได้มีการเสนอแบบจำลองต่างๆ อีกมากมาย แต่ยังไม่มีแบบจำลองใดที่ประสบความสำเร็จเป็นอย่างดี

งานวิจัยนี้ เป็นการศึกษาธรรมชาติของการสั่นพ้องโรเปอร์ผ่านกระบวนการสลายตัว ควบคู่ไปกับข้ออ้างอิงที่ว่าการสั่นพ้องโรเปอร์เป็นสถานะที่ประกอบด้วยควาร์กสามตัว และกลูออนในโมดไฟฟ้าตามขวาง (TE) อีกหนึ่งตัว ได้มีการนำแบบจำลองควาร์ก-กลูออนเชิงไม่สัมพัทธภาพมาใช้ด้วย โดยอธิบายจลนศาสตร์ของ $\bar{q}qG$ ด้วยจุดยอด ${}^3S_1$ ซึ่งคู่ของควาร์ก–แอนติควาร์กเกิดขึ้นจาก (สลายไปเป็น) กลูออน การสร้างฟังก์ชันคลื่นของการสั่นพ้องโรเปอร์คำนึงถึงความอิสระของกลูออนในเชิงไม่สัมพัทธภาพ และคำนวณกระบวนการสลายตัว $N^*(1440) \rightarrow N\pi$ เนื่องจากการพัฒนาวิธีการที่ใช้คำนวณในงานวิจัยนี้เป็นแบบทั่วไป ดังนั้นจึงสามารถประยุกต์ใช้กับการคำนวณโมดของ การสลายตัวอื่นๆ เช่น $N\rho$, $N\pi\pi$ และ $\Delta\pi$ ได้เช่นกัน

สาขาวิชาฟิสิกส์ ลายมือชื่อนักศึกษา . . . .

ปีการศึกษา 2545 ลายมือชื่ออาจารย์ที่ปรึกษา . . . .

NOPMANEE SUPANAM: N^*(1440) DECAYS IN A HYBRID BARYON MODEL
THESIS ADVISOR: ASST. PROF. YUPENG YAN, Ph.D. 83 PP.
ISBN: 974–533–194–5

The Roper resonance N^*(1440), the lowest nucleon excited state, has been subjected to intense discussions since its discovery in 1964. In the three–quark picture the Roper resonance had been commonly assigned to a radial excitation of the nucleon since its quantum numbers are the same as the nucleon's. But detailed studies found that it is very difficult to interpret the resonance as a three quark state due to its low mass and strange coupling constants with nucleon and meson. Because of the failure of the three–quark picture, various other models have also been suggested, but none is very successful.

In this work we study the nature of the Roper resonance via its decay processes. We go along with the argument that the Roper resonance is a state of three quarks and one transverse-electric (TE) gluon. A nonrelativistic quark–gluon model is employed, where the dynamics of $\overline{q}qG$ is described in the effective $^3S_1$ vertex in which a quark-antiquark pair is created (destroyed) from (into) a gluon. The wave function of the Roper resonance is properly constructed to take account into the gluon freedom in the nonrelativistic regime and the decay process $N^*(1440) \rightarrow N\pi$ is evaluated analytically. Since the method developed in the work is very general, it could be applied to evaluate other decay process like $N\rho$, $N\pi\pi$ and $\Delta\pi$ without any problem.

สาขาวิชาฟิสิกส์ ลายมือชื่อนักศึกษา....................................

ปีการศึกษา 2545 ลายมือชื่ออาจารย์ที่ปรึกษา..............................

Acknowledgements

First of all, I am grateful to Asst. Prof. Dr. Yupeng Yan for being my thesis advisor and his guidance throughout this work.

I would like to thank Prof. Dr. Amand Fässler for giving me a chance to go to Germany and for financial support. I would also like to thank Dr. Thomas Gutsche for giving me some important information about the structure of the Roper resonance and his helpful suggestions concerning my thesis when I was in Germany.

I would like to thank Mr. Khanchai Khosonthongkee, Mr. Sampart Cheedket, and Mr. Kem Pumsa-ard for helping me to find some papers in The Universität Tübingen.

I would like to thank Mr. Jessada Tanthanuch for setting up the LaTeX program in my computer.

Special thanks go to Mr. Chrisada Sookdhis, whose LaTeX experience helped me done faster than I would have done by myself and I would like to thank for the daily encouragement and for the good relationship we are sharing.

Finally, I would like to thank my family for supporting me throughout my study and giving me the courage to organize my life.

Nopmanee Supanam

Contents

Contents (Continued)

List of Tables

List of Figures

Chapter I

Introduction

All matter consists of atoms. Atoms in turn are built up by nuclei and electrons orbiting around. Nuclei are bound states of protons and neutrons by the strong interaction. Proton and neutrons possess a further substructure of even smaller constituents, the quarks. Quark has not been observed in experiments as isolated objects, but only as clusters such as mesons (quark-antiquark system) and baryons (three quark system). Quark model of hadrons (baryons and mesons) has made a considerable success and been widely accepted, but it is still a challenge to understand the natures of all the observed baryons and mesons in various quark models.

The study of baryon excitation states plays an important role in understanding of the nucleon internal structure, the quark model and hence the nature of the strong interaction. Information is usually extracted from the properties of nucleon excitation state N^*'s, such as their mass spectrum, various production and decay rate(Burkert, 1994). Since the late 1970's, very little has happened in experimental N^* baryon spectroscopy (Moorhouse and Roper, 1974). Considering its importance for the understanding of the baryon structure and for distinguishing various picture of the nonperturbative regime of quantum chromodynamics (QCD), a new generation of experiments on N^* physics with electromagnetic probes has recently been started at new facilities such as Continuous Electron Beam Accelerator Facility (CEBAF) at Jefferson Laboratory (JLAB), Electron Stretcher Accelerator (ELSA) at Bonn, Grenoble Anneau Accelerateur Laser (GRAAL) at Grenoble.

The contribution of the lowest-lying baryon resonance, the $\Delta(1232)$, to a wide range of nuclear phenomena has been extensively studied (Ericson and Weise, 1988). This resonance $(J = 3/2, I = 3/2, P = +1)$ is the dominant feature of the pion-nucleon scattering amplitude at low energy. As such, it strongly influences the creation, propagation and absorption of π in the nuclear medium and acts as an independent degree of freedom of nuclear dynamics at energy scales of the order of a few hundred MeV.

The next baryon resonance, the Roper resonance or $N^*(1440)$, has the same quantum numbers as the nucleon $(J = 1/2, I = 1/2, P = +1)$ and is therefore regarded as its first intrinsic excitation, at an energy of about 500 MeV. The structure of this excitation appears rather complex and its properties could have profound consequences on the understanding of the baryon spectrum. The understanding of the Roper resonance has been a long-standing problem in N^* physics. Its very small branching ratios of electromagnetic decay modes, unusual couplings to the $N\pi$ and $N\sigma$ channels, and its low mass together make it is difficult to identify the resonance as a simple three-quark bound state.

The $N^*(1440)$ is a very wide resonance with a full width of (350 ± 100) MeV while the neighboring nucleon excitation states $N^*(1520)$ and $N^*(1535)$ are twice as narrow (Groom, 2000). It is found that $N^*(1440)$ couples strongly (60-70%) to the π-nucleon channel and significantly (5-10%) to the σ-nucleon (more properly $(\pi-\pi)^{I=0}_{S-wave}$-nucleon) channel (Groom, 2000). There are no data on its coupling to the vector meson-nucleon channels, except for an upper limit of 8% to the ρ-nucleon channel. The branching ratios to radiative final states (0.035-0.048% for $p\gamma$ and 0.009-0.032% for $n\gamma$) are unusually small (compared for example to the branching ratio of 0.52-0.60% for $\Delta \to N\gamma$). The general impression one gets from these data is that the transition of the nucleon to the $N^*(1440)$ (or vice-versa) is induced mainly by scalar fields (π and σ) and very little by vector fields.

We consider the coupling of the $N^*(1440)$ to the $N\pi$ channel. From the partial decay width of the $N^*(1440)$ into the $N\pi$ channel, $\Gamma_{N^* \to N\pi} = (228 \pm 82)$ MeV, one can deduce the values of the coupling constants $g_{\pi NN^*}$ and $f_{\pi NN^*}$ characterizing the strength of the πNN^* pseudoscalar coupling and pseudovector coupling, respectively. We find $g^2_{\pi NN^*}/4\pi = 3.4 \pm 1.2$ and $f^2_{\pi NN^*} = 0.011 \pm 0.004$.

The coupling constant $g_{\sigma NN^*}$ depends on the σ mass and on the width $\Gamma_{\sigma\to\pi\pi}(m^0_\sigma)$ of the σ-meson at the peak of the resonance. The σ-meson of relevance in the many-body problem is the effective degree of freedom accounting for the exchange of two uncorrelated as well as two resonating pions in the scalar-isoscalar channel. It is expected to have mass of the order of 500-550 MeV and to be a broad state. It can be shown that $g_{\sigma NN^*}$ depends weakly on the value of $\Gamma_{\sigma\to\pi\pi}(m^0_\sigma)$ but rather strongly on m^0_σ. The latter effect is a consequence of the coincidence between the σ mass and the difference between the mass of the $N^*(1440)$ and of the nucleon, which determines the phase space limit for the $N^*(1440) \to N\pi\pi$ decay. Fixing $\Gamma_{\sigma\to\pi\pi}(m^0_\sigma) = 250$ MeV, we obtain for example, $g^2_{\sigma NN^*}/4\pi = 0.34 \pm 0.21$

for $m_\sigma^0 = 500$ MeV and $g^2_{\sigma NN^*}/4\pi = 0.56 \pm 0.35$ for $m_\sigma^0 = 550$ MeV.

Comparing the πNN^* and σNN^* coupling constants to the corresponding values for the πNN and σNN vertices, we have

$$\frac{g_{\pi NN^*}}{g_{\pi NN}} \simeq \frac{1}{2} \quad \text{and} \quad \frac{g_{\sigma NN^*}}{g_{\sigma NN}} \simeq \frac{1}{4}.$$

This seems to depart somewhat from the scaling law

$$\frac{g_{\pi NN^*}}{g_{\pi NN}} = \frac{g_{\sigma NN^*}}{g_{\sigma NN}} = \frac{g_{\omega NN^*}}{g_{\omega NN}} = \frac{g_{\rho NN^*}}{g_{\rho NN}},$$

often used on the basis of constituent quark model arguments. There are however large uncertainties.

To have more constraints on the couplings discussed above, it is useful to make meson-exchange models of simple processes in which the Roper resonance is excited and compare to the coupling constants needed to understand the data on these processes to their values derived from the $N^*(1440)$ partial decay widths. One should keep in mind however the limits of such determinations: the exchanged mesons are effective degrees of freedom and meson-baryon vertices involve not only coupling constants but also form factors which may affect significantly the strength of the couplings.

In the simple three-quark picture of baryons the Roper resonance would be the first radial excitation state of the nucleon if one considers only its quantum numbers. But we will find that the resonance possesses a too low mass to be a radial excitation state.

In a simple model where quarks are confined in an oscillator potential whose slope is independent of the flavor quantum numbers, the Hamiltonian of a three-quark system may take the form

$$H = \frac{\vec{p}_1^{\,2}}{2m} + \frac{\vec{p}_2^{\,2}}{2m} + \frac{\vec{p}_3^{\,2}}{2m} + \frac{1}{2}\mu\omega(\vec{r}_2 - \vec{r}_3)^2 + \frac{1}{2}\mu\omega(\vec{r}_1 - \vec{r}_2)^2 + \frac{1}{2}\mu\omega(\vec{r}_3 - \vec{r}_1)^2. \quad (1.1)$$

Here we have supposed all quarks involved have the same mass. Introducing the Jacobi coordinates

$$\vec{\rho} = \frac{(\vec{r}_1 - \vec{r}_2)}{\sqrt{2}}, \quad \vec{\lambda} = \frac{(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)}{\sqrt{6}}, \quad \vec{R}_{cm} = \frac{(\vec{r}_1 + \vec{r}_2 + \vec{r}_3)}{3}, \quad (1.2)$$

and eliminating the center-of-mass motion, the Schödinger equation of the Hamiltonian in eq (1.1) gives the wave function of a three-quark system,

$$\Psi_{N^*(1440)} = \Psi^{\rm spatial}(\vec{\rho}, \vec{\lambda})\Psi^{\rm spin-flavor-color} \quad (1.3)$$

with the spatial wave function $\Psi^{\rm spatial}$ taking the form

$$\Psi^{\rm spatial} = \psi_{n_\rho l_\rho}(\vec{\rho})\psi_{n_\lambda l_\lambda}(\vec{\lambda}), \quad (1.4)$$

where

$$\psi_{n_\rho l_\rho}(\vec{\rho}) = R_{n_\rho l_\rho}(\rho)\mathcal{Y}_{n_\rho l_\rho}(\hat{\rho}), \quad (1.5)$$

$$\psi_{n_\lambda l_\lambda}(\vec{\lambda}) = R_{n_\lambda l_\lambda}(\lambda)\mathcal{Y}_{n_\lambda l_\lambda}(\hat{\lambda}). \quad (1.6)$$

The energy of a state is specified by the quantum number N

$$E_N = \left(N + \frac{3}{2}\right) \quad , \quad N = N_\rho + N_\lambda = (2n_\rho + l_\rho) + (2n_\lambda + l_\lambda), \tag{1.7}$$

and parity $P = (-1)^{l_\rho + l_\lambda}$. The $N = 1$ states have mixed symmetry:

$$\Psi_{11}(\rho) = \psi_{01}(\vec{\rho})\psi_{00}(\vec{\lambda}), \tag{1.8}$$

$$\Psi_{11}(\lambda) = \psi_{00}(\vec{\rho})\psi_{01}(\vec{\lambda}), \tag{1.9}$$

and parity $P = -1$. These states are not corresponding to the parity of the Roper resonance.

For the Roper resonance with positive parity, the only possibility is

$$n_\rho = 1 \ , \ n_\lambda = 0 \ , \ l_\rho = l_\lambda = 0 \qquad \text{or} \qquad n_\rho = 0 \ , \ n_\lambda = 1 \ , \ l_\rho = l_\lambda = 0.$$

In this case the Roper resonance has $N = 2$ band in a harmonic oscillator basis. The lightest of these states with a totally symmetrical spatial wave function is usually attributed to the Roper resonance. Its low mass has present some problems for simple three-quark picture as these models are not able to describe the right level ordering of positive and negative parity states (Groom, 2000; Høgaasen and Richard, 1983). Indeed, various quarks models (Liu and Wong, 1983; Isgur and Karl, 1978; Høgaasen and Richard, 1983) met difficulties to explain its mass and electromagnetic couplings.

The N^* is not visible as a well-defined peak in the total pion-nucleon cross section. It is established as a pion-nucleon resonance in the P_{11} channel only through detailed partial wave analyzes (Cutkosky, Forsyth, Hendrick, and Kelly, 1979; Cutkosky and Wang, 1990). In contrast to the negative parity baryon resonance observed in the 1500–1700 MeV range, which can be described by constituent quark models with harmonic confining potentials (Isgur and Karl, 1978). The Roper resonance has been considered a good candidate for a collective excitation and interpreted as a breathing mode of the nucleon in bag models (DeGrand and Rebbi, 1978). A recent coupled-channel calculation (Schütz, Haidenbauer, Speth, and Durso, 1998), involving the πN, $\pi\Delta$ and σN channels, suggest that the $N^*(1440)$ could be explained as a dynamical effect, without an associated genuine three-quark state. It has therefore been suggested to be a gluonic excitation state of the nucleon, i.e., a "hybrid baryon".

The aim of the whole project is to investigate if the Roper resonance could be reasonably interpreted as a bound state of three-quark and one-gluon through studying all its decay modes such as to $N\pi$, $N\pi\pi$, $N\rho$, $\Delta\pi$ and $N\gamma$. The thesis work services as a pioneer study to pave the way for the whole project. We will first construct the wave function of the Roper resonance to properly include the gluon freedom in the nonrelativistic regime, then evaluate the transition amplitude of the process $N^*(1440) \rightarrow N\pi$ in a very general method which could be used, without modification, to other decay channels. In the work we will mainly employ nonrelativistic quark-gluon models where the dynamics of the quark-gluon interaction is described by the effective vertex $^3S_1$ in which a quark-antiquark pair is created/destroyed from/into a gluon and the wave functions of the Roper resonance, nucleon and mesons are nonrelativistic.

This thesis is structured as follows: in chapter II the wave functions of mesons, nucleons, and the Roper resonance are worked out in a quark-gluon model with aid of group theory. In chapter III we introduce the ${}^3S_1$ model for the description of the decay process. The transition amplitude for the decay of the Roper resonance is evaluated in chapter IV. Finally, chapter V gives discussions and conclusions.

Chapter II

Wave Functions

In this chapter we provide some details on how to construct the wave functions of mesons and baryons in the quark model with the aid of group theory. The wave function of the Roper resonance is properly constructed to include the gluon freedom in the nonrelativistic regime, which is a pioneer work as we know.

2.1 Flavor SU(3) Symmetry

The fundamental assumption of the quark model for hadrons is that mesons are quark-antiquark bound states and that baryons are three-quark states. The observed hadrons are eigenstates of the Hamilton operator for the strong interaction H_{st}. We begin by considering a world with only three quarks u, d and s and make the following assumptions:

(1) Flavor university of strong interaction, that is, the strong forces should act in the same way on quarks with different flavors.

(2)Equality of the masses of u, d and s quarks:

$$m_u = m_d = m_s. \tag{2.1}$$

The Hamilton operator of the strong interaction H_{st} is then invariant under SU(3) transformation of the quarks u, d and s. In the framework of the flavor SU(3) symmetry, u, d and s quarks form the fundamental representation of the group. Quark states $|q\rangle$ are transformed according to

$$|q\rangle' = U|q\rangle, \tag{2.2}$$

with

$$U^\dagger U = UU^\dagger = 1, \tag{2.3}$$

$$\det U = 1. \tag{2.4}$$

The unitary, unimodular matrix U can be written in the form

$$U = \exp\left(-i\frac{1}{2}\lambda_i\theta_i\right), \tag{2.5}$$

where λ_i with $i = 1, \ldots, 8$ are linearly independent, hermitian and traceless 3×3 matrices. Conveniently, the matrices are chosen to be the Gell-Mann matrices (Close, 1981)

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \qquad (2.6)$$

2.2 Spin-Flavor Wave Functions of Mesons

In the framework of the flavor SU(3) symmetry, u, d and s quarks and their anti-objects form nine lightest pseudoscalar and nine lightest vector mesons, see Table. 2.1. Those mesons are the ground states of the Hamiltonian of the quark-antiquark strong interaction.

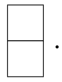
Table 2.1: Meson nonet

	Charge	Strangeness	Examples	
$u\bar{d}$	$+1$	0	π^+	ρ^+
$d\bar{u}$	-1	0	π^-	ρ^-
$u\bar{u}$	0	0	π^0	ρ^0
$d\bar{d}$			η^0	ω^0
$s\bar{s}$			η'^0	ϕ^0
$u\bar{s}$	$+1$	$+1$	K^+	K^{*+}
$d\bar{s}$	0		K^0	K^{*0}
$\bar{u}s$	-1	-1	K^-	K^{*-}
$\bar{d}s$	0		$\bar{K}^0$	$\bar{K}^{*0}$

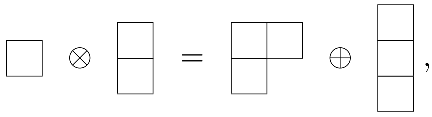
In the language of group theory, the fundamental representation of SU(3) is denoted by the Young tableaux

$$\square\,,$$

while the conjugation representation in which the antiquark states are transformed is depicted by



The Young tableaux for mesons are formed as follows:



with the corresponding dimensions being:

$$\mathbf{3} \otimes \bar{\mathbf{3}} \;=\; \mathbf{8} \oplus \mathbf{1}. \qquad (2.7)$$

The $q\bar{q}$ mesons build a nonet (one singlet and one octet) of the flavor SU(3) group.

Since each quark or antiquark can be in a spin-up or spin-down state, namely $S_z = \pm\frac{1}{2}$, the two states form a fundamental representation of the SU(2) group in spin space. The representations of mesons in spin space are

$$\mathbf{2} \otimes \bar{\mathbf{2}} \;=\; \mathbf{3} \oplus \mathbf{1}, \qquad (2.8)$$

where the spin wave functions of mesons can take the well-known singlet (spin $S = 0$) or triplet (spin $S = 1$) form. The possible spin-flavor conjugation for mesons are

$$(\mathbf{1}, \mathbf{1}), \;\; (\mathbf{1}, \mathbf{3}), \;\; (\mathbf{8}, \mathbf{1}), \;\; (\mathbf{8}, \mathbf{3}). \qquad (2.9)$$

The nine lightest pseudoscalar and the nine lightest vector mesons which have been observed and their corresponding flavor wave functions are listed in Table.2.2

Table 2.2: Flavor wave functions of the pseudoscalar and vector meson nonets

Pseudoscalar	Vector	Flavor
π^+	ρ^+	$u\bar{d}$
π^-	ρ^-	$d\bar{u}$
π^0	ρ^0	$\frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$
η_1	ω_1	$\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$
η_8	ω_8	$\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$
K^+	K^{*+}	$u\bar{s}$
K^0	K^{*0}	$d\bar{s}$
K^-	K^{*-}	$-s\bar{u}$
$\bar{K}^0$	$\bar{K}^{*0}$	$s\bar{d}$

2.3 Spin-Flavor Wave Functions of Baryons

Baryons are three-quark bound states of the strong interaction in the quark model. The total wave function of a baryon must be antisymmetric since it is a system of three identical fermions. All particles observed are color singlet, that is, the color part of the wave function of a hadron is antisymmetric. Therefore, the spatial-spin-flavor part of the wave function of a baryon must be symmetric. For the baryons in the ground state of the strong interaction Hamiltonian, the spatial wave functions are usually S-states, hence symmetric. The spin-flavor wave function of a baryon in the ground state is required to be symmetric so that its total wave function is antisymmetric.

Taking the SU(3) fundamental representation (uds) and combining it with the SU(2) ($\uparrow\downarrow$) one can form a six-dimensional fundamental representation of SU(6), $u\uparrow$, $d\uparrow$, $s\uparrow$, $u\downarrow$, $d\downarrow$, $s\downarrow$. Physically in the quark model the intrinsic SU(3) degrees of freedom will be multiplied by the SU(2) spin of the quarks. We will quote the following rules for combining states of different permutation symmetry and verify it by writing out the states explicitly. Denoting symmetric, mixed and antisymmetric states by S, M, A respectively, the symmetry properties that arise are shown in the matrix

	S	M	A
S	S	M	A
M	M	S, M, A	M

Recalling that in SU(3) we found $\mathbf{10}_S, \mathbf{8}_M, \mathbf{1}_A$, while in SU(2) $\mathbf{4}_S$ and $\mathbf{2}_M$ emerged, then the above rules imply, for instance, that the $\mathbf{10}$ with spin $\frac{3}{2}$ ($\mathbf{4}$ in SU(2)) will be totally symmetric; the $\mathbf{10}$ with spin $\frac{1}{2}$ ($\mathbf{2}$ in SU(2)) will be totally mixed and so forth.

To classify under SU(6) we collect together those states which are symmetric, then those which are mixed and finally those which are antisymmetric. These are listed below together with their subgroup dimensionalities. The total number of such states is given on the right.

$$S : (\mathbf{10}, \mathbf{4}) + (\mathbf{8}, \mathbf{2}) = \mathbf{56}, \tag{2.10}$$

$$M : (\mathbf{10}, \mathbf{2}) + (\mathbf{8}, \mathbf{4}) + (\mathbf{8}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}) = \mathbf{70}, \tag{2.11}$$

$$A : (\mathbf{8}, \mathbf{2}) + (\mathbf{1}, \mathbf{4}) = \mathbf{20}. \tag{2.12}$$

We can immediately verify these results by using the Young tableaux as

$$\boxed{1} \otimes \boxed{2} \otimes \boxed{3} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} .$$

Table 2.3: Spin-flavor wave functions of a baryon classified according to permutation symmetry

56 (S)	
$(\mathbf{10},\mathbf{4}) : \phi^S\chi^S$	$(\mathbf{8},\mathbf{2}) : (\phi^\rho\chi^\rho + \phi^\lambda\chi^\lambda)/\sqrt{2}$
20 (A)	
$(\mathbf{1},\mathbf{4}) : \phi^A\chi^S$	$(\mathbf{8},\mathbf{2}) : (\phi^\lambda\chi^\rho - \phi^\rho\chi^\lambda)/\sqrt{2}$
70 (ρ)	
$(\mathbf{10},\mathbf{2}) : \phi^S\chi^\rho$	$(\mathbf{8},\mathbf{4}) : \phi^\rho\chi^S$
$(\mathbf{8},\mathbf{2}) : (\phi^\lambda\chi^\rho + \phi^\rho\chi^\lambda)/\sqrt{2}$	$(\mathbf{1},\mathbf{2}) : \phi^A\chi^\rho$
70 (λ)	
$(\mathbf{10},\mathbf{2}) : \phi^S\chi^\lambda$	$(\mathbf{8},\mathbf{4}) : \phi^\lambda\chi^S$
$(\mathbf{8},\mathbf{2}) : (\phi^\rho\chi^\rho - \phi^\lambda\chi^\lambda)/\sqrt{2}$	$(\mathbf{1},\mathbf{2}) : \phi^A\chi^\lambda$

The corresponding dimensions are

$$\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6} \;=\; \mathbf{56}\oplus\mathbf{70}\oplus\mathbf{70}\oplus\mathbf{20}, \tag{2.13}$$

hence the $\mathbf{56}_S$, $\mathbf{70}_{M,S}$, $\mathbf{70}_{M,A}$, and $\mathbf{20}_A$ representation are seen. After this a mixed-symmetric wave function will be labelled by the superscript λ and ρ for mixed-antisymmetric wave function. In table.2.3, we list, for completeness, the spin-flavor wave function of various permutation symmetries.

2.4 The Spatial Wave Functions of Mesons and Baryons

The spatial wave functions of hadrons are very much model dependent since the interaction among quarks is still an open question. What has been commonly accepted is that the interaction must confine the quarks as clusters since experiments have never observed any free quark. The most simplest but well accepted form of the interaction is the harmonic oscillator potential. In this work we will employ the harmonic oscillator approximation for the quark interaction in setting up the quark cluster wave function of the mesons and baryons. The wave functions in the approximation take analytical forms both in coordinate and momentum spaces.

With the oscillator potential

$$V(r) = \frac{1}{2}\mu\omega^2 r^2, \tag{2.14}$$

where μ is the reduced mass of the quark-antiquark pair and r is the relative coordinate, the momentum space wave function for s-wave and p-wave mesons take the form

$$\Phi_s(\vec{p}) = N_s \exp\left(-\frac{1}{2}b^2p^2\right)\frac{1}{\sqrt{4\pi}}, \tag{2.15}$$

$$\Phi_p(\vec{p}) = N_p(bp)\exp\left(-\frac{1}{2}b^2p^2\right)\mathcal{Y}_{1m}(\hat{p}), \tag{2.16}$$

where $\vec{p}$ is the relative momentum with

$$\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2),$$

and

$$N_s = \frac{2b^{3/2}}{\pi^{1/4}},$$

$$N_p = \frac{2(2/3)^{1/2}b^{3/2}}{\pi^{1/4}},$$

with $b^2 = \frac{1}{\mu\omega}$.

For the three-quark objects we assume, as for mesons, that the interaction between quarks are well represented by the harmonic oscillator potential. The Schrödinger equation for the three-quark system takes the form

$$\begin{aligned} E\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3) &= \left(\frac{p_1^2}{2m}+\frac{p_2^2}{2m}+\frac{p_3^2}{2m}\right)\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3) \\ &+ \left[\frac{1}{2}\mu\omega^2(\vec{r}_1-\vec{r}_2)^2+\frac{1}{2}\mu\omega^2(\vec{r}_2-\vec{r}_3)^2\right. \\ &\left.+\frac{1}{2}\mu\omega^2(\vec{r}_3-\vec{r}_1)^2\right]\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3), \end{aligned} \tag{2.17}$$

where the three quarks have the same mass m for simplicity, and $\mu = m/2$. We introduce the Jacobi coordinates

$$\vec{\xi}_1 = \vec{r}_2 - \vec{r}_1, \tag{2.18}$$

$$\vec{\xi}_2 = \vec{r}_3 - \left(\frac{\vec{r}_1+\vec{r}_2}{2}\right), \tag{2.19}$$

$$\vec{R} = \frac{\vec{r}_1+\vec{r}_2+\vec{r}_3}{3}. \tag{2.20}$$

In these new coordinates the Schrödinger equation is rewritten as

$$\begin{aligned} E\Psi(\vec{\xi}_1,\vec{\xi}_2,\vec{R}) &= \left[\frac{1}{2\mu_R}\nabla_R^2+\frac{1}{2\mu_1}\nabla_{\xi_1}^2+\frac{1}{2\mu_2}\nabla_{\xi_2}^2\right]\Psi(\vec{\xi}_1,\vec{\xi}_2,\vec{R}) \\ &+\left[\frac{3}{2}\frac{1}{2}\mu\omega^2\xi_1^2 + 2\frac{1}{2}\mu\omega^2\xi_2^2\right]\Psi(\vec{\xi}_1,\vec{\xi}_2,\vec{R}), \end{aligned} \tag{2.21}$$

where $\mu_R = 3m, \mu_1 = \mu$ and $\mu_2 = 4\mu/3$. The solution for the ground state in the center of mass system is

$$\Psi(\vec{\xi}_1, \vec{\xi}_2) = N_B \exp\left(-\frac{1}{2}\frac{1}{2a^2}\xi_1^2\right) \exp\left(-\frac{1}{2}\frac{2}{3a^2}\xi_2^2\right), \tag{2.22}$$

where $a^2 = 1/(\sqrt{6}\mu\omega)$ and $N_B = 3^{3/4}/(\pi^{3/2}a^3)$. The solution in momentum space is obtained by Fourier transformation as follows:

$$\Psi(\vec{p}_1, \vec{p}_2, \vec{p}_3) = N_{B_p} \exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_1 - \vec{p}_2}{\sqrt{2}}\right)^2\right] \exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_1 + \vec{p}_2 - 2\vec{p}_3}{\sqrt{6}}\right)^2\right], \tag{2.23}$$

where $N_{B_p} = (3^{3/4}a^3)/\pi^{3/2}$.

The root-mean-square radii for mesons and baryons might be defined in terms of the size parameters as follows (Yan, 1994):

For a s-wave meson

$$\begin{aligned} \langle r^2 \rangle_s^{1/2} &= \frac{1}{2}\sqrt{\langle \Phi_s | r^2 | \Phi_s \rangle} \\ &= \frac{1}{2}\sqrt{\frac{3}{2}}b \simeq 0.5 \ \text{fm}. \end{aligned} \tag{2.24}$$

For a p-wave meson

$$\begin{aligned} \langle r^2 \rangle_p^{1/2} &= \frac{1}{2}\sqrt{\langle \Phi_p | r^2 | \Phi_p \rangle} \\ &= \frac{1}{2}\sqrt{\frac{5}{2}}b \simeq 0.64 \ \text{fm}. \end{aligned} \tag{2.25}$$

For the nucleon

$$\begin{aligned}\langle r^2\rangle_N^{1/2} &= \frac{1}{2}\sqrt{\langle\Psi|\xi_1^2|\Psi\rangle} \\ &= a \simeq 0.61 \;\; \text{fm.} \end{aligned} \tag{2.26}$$

Here we have used $a = 3.1$ GeV^{-1} and $b = 4.1$ GeV^{-1} (Maruyama, Furu, and Faessler, 1987; Gutsche, Maruyama, and Faessler, 1989), which are determined by fitting to the nucleon and meson sizes and nucleon-nucleon, nucleon-antinucleon and pion-nucleon reactions.

2.5 The Nucleon Wave Function

In the quark model, a nucleon is composed of three quarks, with totally antisymmetric wave function and should take the form

$$\Phi = \begin{array}{|c|c|c|}\hline & & \\ \hline \end{array}_{\,\text{Spatial}} \oplus \begin{array}{|c|c|c|}\hline & & \\ \hline \end{array}_{\,\text{Spin}-\text{Flavor}} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}_{\,\text{Color}} .$$

The spin-flavor part takes the form:

$$\begin{aligned} |N\rangle^{\text{Spin}-\text{Flavor}} &= \frac{1}{\sqrt{2}}\sum_{J=0,1}\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_J\otimes\frac{1}{2}^{(3)}\right\rangle^{\text{Spin}}_{\frac{1}{2},S_z'} \\ &\cdot\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_J\otimes\frac{1}{2}^{(3)}\right\rangle^{\text{Flavor}}_{\frac{1}{2},T_z'} . \end{aligned} \tag{2.27}$$

The color singlet wave function of the nucleon is given by

$$|N\rangle^{\text{color}} = \frac{1}{\sqrt{6}} \sum_{i,j,k} \epsilon_{ijk} |q_1\rangle_i |q_2\rangle_j |q_3\rangle_k, \tag{2.28}$$

where ϵ_{ijk} is the totally antisymmetric Levi-Civita tensor (see Appendix D). For the harmonic oscillator interaction

$$V(r) = \frac{1}{2}\mu\omega^2 r^2, \tag{2.29}$$

the spatial wave function in momentum space takes the form

$$\begin{aligned} \Psi(\vec{p}_1, \vec{p}_2, \vec{p}_3) &= N_N \exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_1 - \vec{p}_2}{\sqrt{2}}\right)^2\right] \\ &\cdot \exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_1 + \vec{p}_2 - 2\vec{p}_3}{\sqrt{6}}\right)^2\right], \end{aligned} \tag{2.30}$$

where $N_N = (3^{\frac{3}{4}}a^3)/\pi^{\frac{3}{2}}$, with $a = 1/(\sqrt{6}\mu\omega)$.

Then putting all the parts together, one obtains the total wave function of nucleon as

$$\begin{aligned} \Psi_N &= \frac{N_N}{\sqrt{2}} \exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_1 - \vec{p}_2}{\sqrt{2}}\right)^2\right] \exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_1 + \vec{p}_2 - 2\vec{p}_3}{\sqrt{6}}\right)^2\right] \\ &\sum_{J=0,1} \left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_J \otimes \frac{1}{2}^{(3)} \right\rangle^{\text{spin}}_{\frac{1}{2},S'_z} \left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_J \otimes \frac{1}{2}^{(3)} \right\rangle^{\text{flavor}}_{\frac{1}{2},T'_z} \\ &\cdot \frac{1}{\sqrt{6}} \sum_{i,j,k} \epsilon_{ijk} |q_1\rangle_i |q_2\rangle_j |q_3\rangle_k. \end{aligned} \tag{2.31}$$

2.6 The π Meson Wave Function

In the quark-antiquark interaction of harmonic oscillator type, π-meson is s-wave meson and the wave function in momentum space is written as

$$\Psi_{\rm meson}^{\rm s-wave} = N_{\rm meson}^{\rm s-wave} \exp\left(-\frac{1}{2}b^2p^2\right)\frac{1}{\sqrt{4\pi}}, \tag{2.32}$$

where $\vec{p}$ is the relative momentum with $\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$, and $N_{\rm meson}^{\rm s-wave} = 2b^{\frac{3}{2}}/\pi^{\frac{1}{4}}$ with $b^2 = 1/(\mu\omega)$.

The spin-flavor part of the π-meson can be written in form:

$$|\pi\rangle^{\rm spin-flavor} = \left|\left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)\right\rangle_{00}^{\rm spin} \cdot \left|\left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)\right\rangle_{1t_z}^{\rm flavor}. \tag{2.33}$$

The color singlet wave function of π-meson is

$$|\pi\rangle^{\rm color} = \frac{1}{\sqrt{3}}\sum_{l=1}^{3}|\bar{q}_1\rangle_l|q_2\rangle_l. \tag{2.34}$$

The π-meson wave function is then

$$\begin{aligned}\Psi_\pi &= N_\pi \exp\left[-\frac{1}{8}b^2\left(\vec{p}_1 - \vec{p}_2\right)^2\right]\left|\left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)\right\rangle_{00}^{\rm spin} \\ &\cdot\left|\left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)\right\rangle_{1t_z}^{\rm flavor} \frac{1}{\sqrt{3}}\sum_{l=1}^{3}|\bar{q}_1\rangle_l|q_2\rangle_l,\end{aligned} \tag{2.35}$$

where $N_\pi = N_\pi^{\rm s-wave}/\sqrt{4\pi} = (b/\pi)^{3/2}$.

2.7 The Roper Resonance Wave Function

Hybrid resonances (q^3G and $q\bar{q}G$) have been studied mainly in bag model. A general conclusion of those studies is that the hybrid states should have their rooms in nature. It has also been concluded that the confined gluon might be both TE (transverse electric) and TM (transverse magnetic) modes, and a TE mode is the lowest eigenmode. Considering its low mass, we presume that the Roper resonance is composed of three valence quarks and a TE gluon, denoted by q^3G.

Three quarks states and hybrid states may have the same quantum numbers; a study of the spectroscopic assignments will therefore not be sufficient to discriminate between the q^3 and q^3G states. A hybrid state is excited in the spin-flavor space, and has an SU(6) spin-flavor wave function orthogonal to that of the nucleon, where as the spin-flavor wave function of a radial excited state is identical to that of the nucleon. The gluon is in the color octet representation of SU(3) (Perkins, 1986). In analogy with the (flavor) octet of mesons in Table.2.2 we can write the color-anticolor states of the 8 gluons as follows:

$$r\bar{b}, r\bar{g}, b\bar{g}, b\bar{r}, g\bar{r}, g\bar{b}, \frac{r\bar{r} - b\bar{b}}{\sqrt{2}}, \frac{r\bar{r} + b\bar{b} - 2g\bar{g}}{\sqrt{6}}. \tag{2.36}$$

With 3 colors and 3 anticolors, we expect $3^2 = 9$ combinations, but one of these is a color singlet and has to be excluded.

Let ϕ, χ and ψ denote flavor, spin, and color wave functions for three quarks and let superscripts S, ρ, λ and a denote the permutation symmetry (S/a is totally symmetric/antisymmetric under any exchange among the three quarks, and λ/ρ is symmetric/antisymmetric under the exchange of the first two quarks). The quantum number of the q^3G states are dictated mainly by the requirements that the three-quark state transform as a color octet. The totally antisymmetric q^3G states are explicitly (Li, Burkert, and Li, 1992; Li, 1991)

$$\begin{aligned} \left|{}^2N_g\right\rangle &= \frac{1}{2}\left[\left(\phi^\rho\chi^\rho - \phi^\lambda\chi^\lambda\right)\psi^\rho - \left(\phi^\rho\chi^\lambda + \phi^\lambda\chi^\rho\right)\psi^\lambda\right] \otimes |G\rangle\,, \quad (2.37) \\ \left|{}^4N_g\right\rangle &= \frac{1}{\sqrt{2}}\left[\left(\phi^\lambda\psi^\rho - \phi^\rho\psi^\lambda\right)\chi^S\right] \otimes |G\rangle\,, \quad (2.38) \end{aligned}$$

where superscript 2 and 4 denote the total quark spins as $2S+1$. In the spin-flavor-color wave functions $|^2N_g\rangle$ and $|^4N_g\rangle$ above, the color components of the three-quark core take the form

$$\psi^\rho_\alpha = \frac{1}{2}\sum_{i,j,k,l} |q_3\rangle_i \lambda^\alpha_{ij} \cdot \epsilon_{jkl}|q_1\rangle_k|q_2\rangle_l, \quad (2.39)$$

$$\psi^\lambda_\alpha = \frac{1}{2}\sum_{i,j,k,l} (|q_1\rangle_i|q_2\rangle_j + |q_1\rangle_j|q_2\rangle_i)\lambda^\alpha_{il} \cdot \epsilon_{jkl}|q_3\rangle_k, \quad (2.40)$$

where λ^a are the Gell-Mann matrices. One may write $|^4N_g\rangle$ and $|^2N_g\rangle$ explicitly

$$\begin{aligned} |^4N_g\rangle = \frac{1}{\sqrt{2}}\Bigg[&\left|\left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_1 \otimes \frac{1}{2}^{(3)}\right\rangle_{\frac{3}{2},m_{123}} \otimes \\ &\left(\mathcal{Y}_{1m_1}(\vec{p}_1+\vec{p}_2+\vec{p}_3-\vec{p}_4)\otimes \mathbf{e}_{1m_s}\right)_{1m_2'}\Bigg]^{\text{spin}}_{\frac{1}{2},S_z''} \\ &\Bigg\{\Bigg[\left|\left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_1 \otimes \frac{1}{2}^{(3)}\right\rangle^{\text{flavor}}_{\frac{1}{2},T_z''} \frac{1}{\sqrt{8}}\sum_\alpha \psi^\rho_\alpha |g\rangle^{\text{color}}_\alpha\Bigg] \\ &- \Bigg[\left|\left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_0 \otimes \frac{1}{2}^{(3)}\right\rangle^{\text{flavor}}_{\frac{1}{2},T_z''} \frac{1}{\sqrt{8}}\sum_\alpha \psi^\lambda_\alpha |g\rangle^{\text{color}}_\alpha\Bigg]\Bigg\}, \quad (2.41) \end{aligned}$$

$$
\begin{aligned}
|^2N_g\rangle &= \frac{1}{2}\sum_{J_{12}}\Bigg\{\Bigg((-1)^{J_{12}}\Bigg[\Bigg|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\Bigg\rangle_{\frac{1}{2},m_{123}}\otimes \\
&\qquad \Big(\mathcal{Y}_{1m_1}(\vec{p}_1+\vec{p}_2+\vec{p}_3-\vec{p}_4)\otimes\mathbf{e}_{1m_s}\Big)_{1m_2'}\Bigg]^{\text{spin}}_{\frac{1}{2},S_z''} \\
&\qquad \Bigg|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\Bigg\rangle^{\text{flavor}}_{\frac{1}{2},T_z''}\frac{1}{\sqrt{8}}\sum_\alpha\psi^\rho_\alpha|g\rangle^{\text{color}}_\alpha\Bigg) \\
&\quad -\Bigg(\Bigg[\Bigg|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{1-J_{12}}\otimes\frac{1}{2}^{(3)}\Bigg\rangle_{\frac{1}{2},m_{123}}\otimes \\
&\qquad \Big(\mathcal{Y}_{1m_1}(\vec{p}_1+\vec{p}_2+\vec{p}_3-\vec{p}_4)\otimes\mathbf{e}_{1m_s}\Big)_{1m_2'}\Bigg]^{\text{spin}}_{\frac{1}{2},S_z''} \\
&\qquad \Bigg|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\Bigg\rangle^{\text{flavor}}_{\frac{1}{2},T_z''}\frac{1}{\sqrt{8}}\sum_\alpha\psi^\lambda_\alpha|g\rangle^{\text{color}}_\alpha\Bigg)\Bigg\}. \quad (2.42)
\end{aligned}
$$

The total wave function of the Roper resonance is the linear combination of the $|^4N_g\rangle$ component and the $|^2N_g\rangle$ one, taking the form

$$
\Psi^{N^*(1440)} = \Psi^{\text{spatial}}_{N^*(1440)}\left[A|^2N_g\rangle + B|^4N_g\rangle\right], \quad (2.43)
$$

with

$$
A^2 + B^2 = 1. \quad (2.44)
$$

In the approximation of the harmonic oscillator interaction among quarks and gluon, the spatial part $\Psi^{\text{spatial}}_{N^*(1440)}$ may take the form

$$
\begin{aligned}
\Psi^{\text{spatial}}_{N^*(1440)} &= N_{N^*}\exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_1-\vec{p}_2}{\sqrt{2}}\right)^2\right]\exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_1+\vec{p}_2-2\vec{p}_3}{\sqrt{6}}\right)^2\right] \\
&\quad \exp\left[-\frac{1}{8}c^2\left(\vec{p}_1+\vec{p}_2+\vec{p}_3-\vec{p}_4\right)^2\right], \quad (2.45)
\end{aligned}
$$

where $N_{N^*} = 3^{1/4}2^{3/2}a^3c^{5/2}/\pi^{7/4}$.

Then, the total wave function of the Roper resonance can be written as

$$\begin{aligned}
\Psi_{N^*(1440)} &= N_{N^*}\exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_1-\vec{p}_2}{\sqrt{2}}\right)^2\right] \\
&\exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_1+\vec{p}_2-2\vec{p}_3}{\sqrt{6}}\right)^2\right] \\
&\exp\left[-\frac{1}{8}c^2\left(\vec{p}_1+\vec{p}_2+\vec{p}_3-\vec{p}_4\right)^2\right] \\
&\mathcal{Y}_{1m_1}(\vec{p}_1+\vec{p}_2+\vec{p}_3-\vec{p}_4)\left[A|^2N_g'\rangle+B|^4N_g'\rangle\right], \qquad (2.46)
\end{aligned}$$

with

$$\begin{aligned}
|^2N_g'\rangle &= \frac{1}{2}\sum_{m_{123},m_1} C\left(\frac{1}{2}\frac{1}{2}1;S_z''m_{123}(S_z''-m_{123})\right) \\
&C\left(111;(S_z''-m_{123})m_1(S_z''-m_{123}-m_1)\right) \\
&\sum_{J_{12}}\left\{\left(\left[\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\right\rangle_{\frac{1}{2},m_{123}}\mathbf{e}_{1m_s}\right]^{\text{spin}}_{\frac{1}{2},S_z''}\right.\right. \\
&\quad\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\right\rangle^{\text{flavor}}_{\frac{1}{2},T_z''}(-1)^{J_{12}}\frac{1}{\sqrt{8}}\sum_a\psi_a^\rho|g\rangle_a^{\text{color}}\Bigg) \\
&\quad-\left(\left[\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{1-J_{12}}\otimes\frac{1}{2}^{(3)}\right\rangle_{\frac{1}{2},m_{123}}\mathbf{e}_{1m_s}\right]^{\text{spin}}_{\frac{1}{2},S_z''}\right. \\
&\quad\left.\left.\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\right\rangle^{\text{flavor}}_{\frac{1}{2},T_z''}\frac{1}{\sqrt{8}}\sum_\alpha\psi_\alpha^\lambda|g\rangle_\alpha^{\text{color}}\right)\right\}, \qquad (2.47)
\end{aligned}$$

and

$$
\begin{aligned}
|^4N_g'\rangle &= \frac{1}{\sqrt{2}}\sum_{m_{123},m_1} C\left(\frac{1}{2}\frac{3}{2}1;S_z''m_{123}(S_z''-m_{123})\right) \\
&\quad C\,(111;(S_z''-m_{123})m_1(S_z''-m_{123}-m_1)) \\
&\quad \left(\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_1\otimes\frac{1}{2}^{(3)}\right\rangle^{\text{spin}}_{\frac{3}{2},m_{123}}\mathbf{e}_{1m_s}\right) \\
&\quad \left\{\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_1\otimes\frac{1}{2}^{(3)}\right\rangle^{\text{flavor}}_{\frac{1}{2},T_z''}\frac{1}{\sqrt{8}}\sum_\alpha\psi^\rho_\alpha|g\rangle^{\text{color}}_\alpha\right. \\
&\quad \left.-\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_0\otimes\frac{1}{2}^{(3)}\right\rangle^{\text{flavor}}_{\frac{1}{2},T_z''}\frac{1}{\sqrt{8}}\sum_\alpha\psi^\lambda_\alpha|g\rangle^{\text{color}}_\alpha\right\}, \qquad (2.48)
\end{aligned}
$$

where the first Clebsch-Gordon coefficients in both eq. (2.47) and (2.48) come from the coupling between the spin of the three-quark core and the total angular momentum of the TE gluon, and the second from the coupling between the spin and orbital angular momenta of the gluon.

Chapter III

The Interaction Operators

In this chapter, we provide some detail on how to construct the interaction operator for the decay of the Roper resonance. First, we discuss the relevant $q\bar{q}g$ dynamics, as defined in the so-called $^3S_1$ model. Then we work out the interaction operator of our model in the nonrelativistic approximation.

The dynamics of a $q\bar{q}g$ vertex is effectively described by vector $^3S_1$ interaction. We start with the quark-antiquark-gluon vertex, which can be deduced from the interaction Lagrangian density

$$\mathcal{L}_{int} = \bar{\psi} g \gamma^\mu A^\alpha_\mu \frac{\lambda^\alpha}{2} \psi, \tag{3.1}$$

where g is the strong coupling constant, $\frac{\lambda^\alpha}{2}$ is the SU(3) generators, A^α_μ is the gluon fields ($\alpha = 1, \ldots, 8$) and ψ is the quark field. The gluon field A^α_μ might be rewritten as

$$A^\alpha_\mu = A^\alpha A_\mu,$$

with A^α responsible for only the color sector. For free quarks, ψ are just the four-momentum eigensolutions of Dirac's equation (see Appendix C), taking the form

$$\psi = u(\vec{p}) e^{ip\cdot x}, \tag{3.2}$$

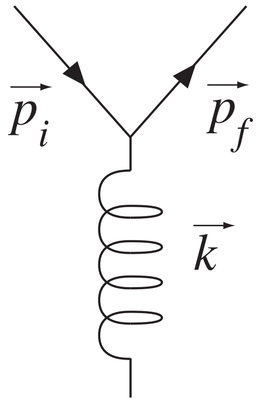


Figure 3.1: Diagram for the quark-antiquark-gluon vertex.

where u is a four-component spinor independent of x.

For the transition, depicted in Fig. 3.1, of a quark with momentum $\vec{p}_i$ into an antiquark with momentum $\vec{p}_f$ and a gluon with Lorentz index μ and color label α $(= 1, \ldots, 8)$, the Lagrangian is simply

$$\mathcal{L}_{int} = \bar{v}(\vec{p}_f) e^{ip_f \cdot x} g \gamma^\mu A_\mu A^\alpha \frac{\lambda^\alpha}{2} u(\vec{p}_i) e^{ip_i \cdot x}, \tag{3.3}$$

where the Dirac spinors for the quark (u_i) and antiquark (v_f) are defined as

$$u_i(\vec{p}_i, \vec{\sigma}_i) = \sqrt{\frac{E_i + m}{2m}} \begin{pmatrix} \chi_i \\ \frac{\vec{\sigma}_i \cdot \vec{p}_i}{E_i + m} \chi_i \end{pmatrix}, \tag{3.4}$$

$$v_f(\vec{p}_f, \vec{\sigma}_f) = \sqrt{\frac{E_f + m}{2m}} \begin{pmatrix} \frac{\vec{\sigma}_f \cdot \vec{p}_f}{E_f + m} \bar{\chi}_f \\ \bar{\chi}_f \end{pmatrix}, \tag{3.5}$$

with

$$\bar{v}_f = v^\dagger \gamma^0,$$

where χ and $\bar{\chi}$ are 2-component spinors respectively for quark and antiquark, defined as:

$$\chi(\text{spin up}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi(\text{spin down}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\bar{\chi}(\text{spin up}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \bar{\chi}(\text{spin down}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{3.6}$$

The vector potential A_μ is given by a plane wave:

$$A_\mu(x, \vec{k}) = \varepsilon_\mu(\vec{k}) N_k \left(e^{-ik\cdot x} + e^{ik\cdot x}\right), \tag{3.7}$$

where $\varepsilon_\mu(\vec{k})$ is the polarization vector of gluon and N_k is the normalization constant. The general polarization vectors of gluon is given by

$$\varepsilon_\mu(\vec{k}) = \left(\frac{\vec{k}\cdot\vec{\varepsilon}}{m_g}\ ,\ \vec{\varepsilon} + \frac{\vec{k}\cdot\vec{\varepsilon}}{m_g(k_0 + m_g)}\vec{k}\right). \tag{3.8}$$

We choose only one direction of the gluon then eq.(3.7) is reduced to

$$A_\mu(x, \vec{k}) = \varepsilon_\mu(\vec{k}) N_k \left(e^{-ik\cdot x}\right). \tag{3.9}$$

Substituting eq.(3.9) in eq.(3.3) gives

$$\begin{aligned} \mathcal{L}_{int} &= \bar{v}(\vec{p}_f) e^{ip_f\cdot x} g\gamma^\mu \varepsilon_\mu(\vec{k}) A^\alpha \frac{\lambda^\alpha}{2} e^{ip_i\cdot x} e^{-ik\cdot x} u(\vec{p}_i) \\ &= \bar{v}(\vec{p}_f) g\gamma^\mu \varepsilon_\mu(\vec{k}) A^\alpha \frac{\lambda^\alpha}{2} e^{i(p_i+p_f-k)\cdot x} u(\vec{p}_i). \end{aligned} \tag{3.10}$$

The interaction Lagrangian in eq.(3.10) can be written out explicitly using the following representation of the Dirac 4×4 matrices

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \tag{3.11}$$

where $\sigma^i (i = 1, 2, 3)$ is the indicated Pauli matrix, $\mathbf{1}$ denotes the 2×2 unit matrix, and $\mathbf{0}$ is the 2×2 matrix of zeroes.

Now

$$\gamma^\mu \varepsilon_\mu = \gamma^0 \varepsilon_0 - \vec{\gamma} \cdot \vec{\varepsilon}_\mu, \tag{3.12}$$

hence

$$\begin{aligned}
\mathcal{L}_{int} &= \bar{v}(\vec{p}_f) g \left(\gamma^0 \varepsilon_0 - \vec{\gamma} \cdot \vec{\varepsilon}_\mu\right) A^\alpha \frac{\lambda^\alpha}{2} e^{i(p_i + p_f - k)\cdot x} u(\vec{p}_i) \\
&= \bar{v}(\vec{p}_f) g \gamma^0 \left(\frac{\vec{k} \cdot \vec{\varepsilon}}{m_g}\right) A^\alpha \frac{\lambda^\alpha}{2} e^{i(p_i + p_f - k)\cdot x} u(\vec{p}_i) \\
&\quad - \bar{v}(\vec{p}_f) g \vec{\gamma} \cdot \left(\vec{\varepsilon} + \frac{\vec{k} \cdot \vec{\varepsilon}}{m_g(k_0 + m_g)} \vec{k}\right) A^\alpha \frac{\lambda^\alpha}{2} e^{i(p_i + p_f - k)\cdot x} u(\vec{p}_i) \\
&= \bar{v}(\vec{p}_f) g \gamma^0 \left(\frac{\vec{k} \cdot \vec{\varepsilon}}{m_g}\right) A^\alpha \frac{\lambda^\alpha}{2} e^{i(p_i + p_f - k)\cdot x} u(\vec{p}_i) \\
&\quad - \bar{v}(\vec{p}_f) g \vec{\gamma} \cdot \vec{\varepsilon} A^\alpha \frac{\lambda^\alpha}{2} e^{i(p_i + p_f - k)\cdot x} u(\vec{p}_i) \\
&\quad - \bar{v}(\vec{p}_f) g \left(\frac{\vec{k} \cdot \vec{\varepsilon}}{m_g(k_0 + m_g)}\right) (\vec{\gamma} \cdot \vec{k}) A^\alpha \frac{\lambda^\alpha}{2} e^{i(p_i + p_f - k)\cdot x} u(\vec{p}_i). \\
&= \sqrt{\frac{E_f + m}{2m}} \sqrt{\frac{E_i + m}{2m}} \bar{\chi}_f{}^\dagger \left[\frac{\vec{\sigma}_f \cdot \vec{p}_f}{E_f + m} + \frac{\vec{\sigma}_i \cdot \vec{p}_i}{E_i + m} - \left(\frac{\vec{\sigma}_f \cdot \vec{p}_f}{E_f + m}\right) \left(\frac{\vec{\sigma}_i \cdot \vec{p}_i}{E_i + m}\right) \vec{\sigma} \cdot \vec{\varepsilon} \right. \\
&\quad - \vec{\sigma} \cdot \vec{\varepsilon} - \left(\frac{\vec{\sigma}_f \cdot \vec{p}_f}{E_f + m}\right) \left(\frac{\vec{\sigma}_i \cdot \vec{p}_i}{E_i + m}\right) \left(\frac{\vec{k} \cdot \vec{\varepsilon}}{m_g(k_0 + m_g)}\right) \vec{\sigma} \cdot \vec{k} \\
&\quad \left. - \vec{\sigma} \cdot \vec{k} \right] g A^\alpha \frac{\lambda^\alpha}{2} e^{i(p_i + p_f - k)\cdot x} \chi_i.
\end{aligned} \tag{3.13}$$

The interaction is given by

$$W_{fi} = \int \mathcal{L}_{int} d^4x = \bar{\chi_f}^\dagger V_{fi} \chi_i, \quad (3.14)$$

with

$$V_{fi} = \sqrt{\frac{E_f+m}{2m}}\sqrt{\frac{E_i+m}{2m}}\left\{\frac{\vec{\sigma}_f\cdot\vec{p}_f}{E_f+m}+\frac{\vec{\sigma}_i\cdot\vec{p}_i}{E_i+m}-\left(\frac{\vec{\sigma}_f\cdot\vec{p}_f}{E_f+m}\right)\left(\frac{\vec{\sigma}_i\cdot\vec{p}_i}{E_i+m}\right)\vec{\sigma}\cdot\vec{\varepsilon} -\vec{\sigma}\cdot\vec{\varepsilon}-\left(\frac{\vec{\sigma}_f\cdot\vec{p}_f}{E_f+m}\right)\left(\frac{\vec{\sigma}_i\cdot\vec{p}_i}{E_i+m}\right)\left(\frac{\vec{k}\cdot\vec{\varepsilon}}{m_g(k_0+m_g)}\right)\vec{\sigma}\cdot\vec{k} -\vec{\sigma}\cdot\vec{k}\right\} gA^\alpha\frac{\lambda^\alpha}{2}\delta(\vec{p}_i+\vec{p}_f-\vec{k}). \quad (3.15)$$

In the nonrelativistic quark model, the quark-antiquark-gluon transition operator corresponds to the nonrelativistic quark-antiquark-gluon interaction of lowest order QCD, where the created $q\bar{q}$ carries the quantum number $^{2S+1}L_J = {}^3S_1$. In the nonrelativistic approximation, namely $E \simeq m$, $k_0 \simeq m_g$, and $|\vec{p}_i| = |\vec{p}_f| = |\vec{k}| \simeq 0$, we have

$$W_{fi} = \bar{\chi_f}^\dagger V_{fi}\chi_i, \quad (3.16)$$

with

$$V_{fi} = -gA^\alpha\frac{\lambda^\alpha}{2}\vec{\sigma}\cdot\vec{\varepsilon}\delta(\vec{p}_i+\vec{p}_f-\vec{k}) = gA^\alpha\frac{\lambda^\alpha}{2}(-1)^{\mu+1}\sigma^\mu\varepsilon_{-\mu}\delta(\vec{p}_i+\vec{p}_f-\vec{k}). \quad (3.17)$$

Here we have used

$$\vec{A} \cdot \vec{B} = \sum_{\mu} (-1)^{\mu} A_{\mu} B_{-\mu}, \tag{3.18}$$

with

$$\begin{aligned} A_1 &= -\frac{1}{\sqrt{2}} \left(A_x + i A_y \right), \\ A_0 &= A_z, \\ A_{-1} &= \frac{1}{\sqrt{2}} \left(A_x - i A_y \right). \end{aligned} \tag{3.19}$$

It can be easily proven that

$$\begin{aligned} \langle \bar{b} | \sigma^{-1} | a \rangle &= -\sqrt{2} \delta_{a,\bar{b}} \delta_{a,\frac{1}{2}}, \\ \langle \bar{b} | \sigma^{0} | a \rangle &= \sqrt{2} \delta_{a,-\bar{b}}, \\ \langle \bar{b} | \sigma^{1} | a \rangle &= -\sqrt{2} \delta_{a,\bar{b}} \delta_{a,-\frac{1}{2}}, \end{aligned} \tag{3.20}$$

where a and $\bar{b}$ are spin states of quark and antiquark respectively and σ^{μ} are defined as

$$\begin{aligned} \sigma^1 &= -\frac{1}{\sqrt{2}} \left(\sigma^x + i\, \sigma^y \right), \\ \sigma^0 &= \sigma^z, \\ \sigma^{-1} &= \frac{1}{\sqrt{2}} \left(\sigma^x - i\, \sigma^y \right). \end{aligned} \tag{3.21}$$

The operation of the $\vec{\sigma}$ could be understand as that it operates a quark state to an antiquark state, or that it projects a quark-antiquark pair onto a spin-1 state. We may write eq. (3.20) in the form

$$\langle 0,0|\sigma^{\mu}_{ij}|[\bar{\chi}_i \otimes \chi_j]_{JM}\rangle = (-1)^M\sqrt{2}\delta_{J,1}\delta_{M,-\mu}. \quad (3.22)$$

For the flavor a quark-antiquark pair which annihilates into a gluon must have zero isospin. So we may introduce a unit operator $\mathbf{1}^F_{ij}$ with the property

$$\langle 0,0|\hat{\mathbf{1}}^F_{ij}|T,T_z\rangle = \sqrt{2}\delta_{T,0}\delta_{T_z,0}. \quad (3.23)$$

The operation of the $\vec{\varepsilon}$ which operates on the unit vector of gluon can be in form

$$\langle 0,0|\varepsilon_{-\mu}|\mathbf{e}_{sm_s}\rangle = \delta_{-\mu,m_s}. \quad (3.24)$$

Finally one may write the ${}^3S_1$ operator in the form

$$V_{ij} = g\sum_{\mu}(-1)^{\mu}\sigma^{\mu}_{ij}\,\mathbf{1}^F_{ij}\varepsilon_{-\mu}\,A^{\alpha}\frac{\lambda^{\alpha}}{2}\delta(\vec{p}_i+\vec{p}_j-\vec{k}), \quad (3.25)$$

where g is the effective coupling strength, and the two body matrix elements given by

$$\langle 0,0|\sigma^{\mu}_{ij}|[\bar{\chi}_i \otimes \chi_j]_{JM}\rangle = (-1)^M\sqrt{2}\delta_{J,1}\delta_{M,-\mu}, \quad (3.26)$$

$$\langle 0,0|\hat{\mathbf{1}}^F|T,T_z\rangle = \sqrt{2}\delta_{T,0}\delta_{T_z,0}, \quad (3.27)$$

$$\langle 0,0|\varepsilon_{-\mu}|\mathbf{e}_{sm_s}\rangle = \delta_{-\mu,m_s}. \quad (3.28)$$

Chapter IV

The Transition Amplitude for $N^*(1440)$ Decay

The transition amplitude of the Roper resonance decay into nucleon and pion is evaluated in hybrid baryon picture. The method developed in the chapter is general, and could be applied to other decay channels without modification. The whole transition amplitude is derived by calculating the spatial part, spin-flavor part and color part separately.

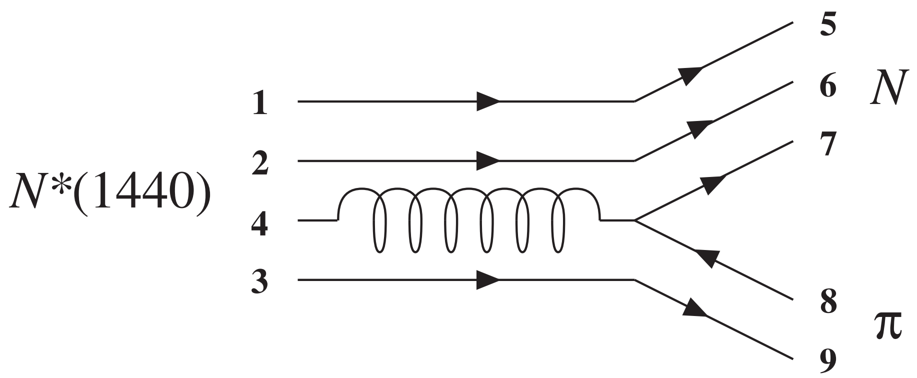


Figure 4.1: Feynman diagram for $N^*(1440) \rightarrow N\pi$

4.1 The Derivation of the Transition Amplitude

The transition amplitude for the decay of the Roper resonance $N^*(1440)$ into nucleon and π-meson is defined as

$$T_{N^*(1440)\to N\pi} = \int \prod_{i=1}^{9} d\vec{p}_i \Psi^\dagger_{N\pi} \mathcal{O} \Psi_{N^*(1440)}, \tag{4.1}$$

where $\Psi_{N\pi}$ and $\Psi_{N^*(1440)}$ are the final and initial wave functions, respectively. The momenta $\vec{p}_i$ are labelled as in Fig. 4.1. The operator $\mathcal{O}$ is defined as

$$\mathcal{O} = \delta(\vec{p}_1 - \vec{p}_5)\delta(\vec{p}_2 - \vec{p}_6)\delta(\vec{p}_3 - \vec{p}_9)V^\dagger_{78}(^3S_1), \tag{4.2}$$

with the $^3S_1$ interaction vertex

$$V_{ij}(^3S_1) = g\sum_\mu (-1)^{\mu+1}\sigma^\mu_{ij}\mathbf{1}^F_{ij}\varepsilon_{-\mu}A^\alpha \frac{\lambda^\alpha}{2}\delta(\vec{p}_i + \vec{p}_j - \vec{k}). \tag{4.3}$$

In the center-of-mass frame, the wave function of the Roper resonance includes the internal motion of three quarks and a TE gluon and a plane wave (delta function in momentum space) representing the global motion of the particle, taking the form

$$\begin{aligned}\Psi_{N^*(1440)} &= N_{N^*} \exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_1 - \vec{p}_2}{\sqrt{2}}\right)^2\right] \exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_1 + \vec{p}_2 - 2\vec{p}_3}{\sqrt{6}}\right)^2\right] \\ &\quad \exp\left[-\frac{c^2}{8}\left(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 - \vec{p}_4\right)^2\right] \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4) \\ &\quad \Psi^{\text{spin-flavor-color}}_{N^*(1440)}.\end{aligned} \tag{4.4}$$

The wave function of the final state is composed the internal motions of the nucleon and meson quark clusters and the relative motion between nucleon and meson. It takes the form

$$\begin{aligned} \Psi_{N\pi} &= N_N N_\pi \exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_5-\vec{p}_6}{\sqrt{2}}\right)^2\right]\exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_5+\vec{p}_6-2\vec{p}_7}{\sqrt{6}}\right)^2\right] \\ &\quad\cdot\exp\left[-\frac{1}{8}b^2\left(\vec{p}_8-\vec{p}_9\right)^2\right]\delta(\vec{p}_8+\vec{p}_9-\vec{p})\Psi_{N\pi}^{\text{spin-flavor-color}}. \end{aligned} \tag{4.5}$$

There is one delta function either in $\Psi_{N^*(1440)}$ or in $\Psi_{N\pi}$ which describes the center of mass motion. In fact only one of these functions,

$$\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4)$$

and

$$\delta(\vec{p}_5+\vec{p}_6+\vec{p}_7+\vec{p}_8+\vec{p}_9)$$

needs to appear in the eq.(4.1) with the overall momentum conservation maintained.

In order to evaluate the transition amplitude for a certain final state, we expand the total transition amplitude in partial waves

$$T_{N^*(1440)\to N\pi} = \sum_{lm_l} T_{lm_l}\mathcal{Y}^*_{lm_l}(\hat{p}), \tag{4.6}$$

where $\mathcal{Y}^*_{lm_l}$ is the spherical harmonics with l and m_l denoting the total orbital angular momentum and its projection of the final $N\pi$ state.

The partial wave transition amplitude T_{lm_l} for the $N^*(1440)$ decay into nucleon and π-meson takes the form

$$\begin{aligned} T_{lm_l} &= g \sum_{m_1, m_{123}} P_{lm_l} \left[\frac{A}{2} C\left(\frac{1}{2}\frac{1}{2}1; S''_z m_{123}(S''_z - m_{123})\right) \right. \\ &\qquad C\left(111; (S''_z - m_{123}) m_1 (S''_z - m_{123} - m_1)\right) Q_1 \\ &\qquad + \frac{B}{\sqrt{2}} C\left(\frac{1}{2}\frac{3}{2}1; S''_z m_{123}(S''_z - m_{123})\right) \\ &\qquad \left. C\left(111; (S''_z - m_{123}) m_1 (S''_z - m_{123} - m_1)\right) Q_2 \right], \end{aligned} \tag{4.7}$$

with

$$\begin{aligned} P_{lm_l} &= N_{N^*} N_N N_\pi \int \prod_{i=1}^{9} d\vec{p}_i d\hat{p} \mathcal{Y}_{lm_l}(\hat{p}) \cdot \exp\left[-\frac{a^2}{2} \left(\frac{\vec{p}_5 - \vec{p}_6}{\sqrt{2}} \right)^2 \right] \\ &\cdot \exp\left[-\frac{a^2}{2} \left(\frac{\vec{p}_5 + \vec{p}_6 - 2\vec{p}_7}{\sqrt{6}} \right)^2 \right] \exp\left[-\frac{b^2}{8} \left(\vec{p}_8 - \vec{p}_9 \right)^2 \right] \\ &\cdot \delta(\vec{p}_8 + \vec{p}_9 - \vec{p}) \exp\left[-\frac{a^2}{2} \left(\frac{\vec{p}_1 - \vec{p}_2}{\sqrt{2}} \right)^2 \right] \exp\left[-\frac{a^2}{2} \left(\frac{\vec{p}_1 + \vec{p}_2 - 2\vec{p}_3}{\sqrt{6}} \right)^2 \right] \\ &\cdot \exp\left[-\frac{c^2}{8} \left(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 - \vec{p}_4 \right)^2 \right] \delta\left(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4 \right) \delta\left(\vec{p}_1 - \vec{p}_5 \right) \\ &\cdot \delta\left(\vec{p}_2 - \vec{p}_6 \right) \delta\left(\vec{p}_3 - \vec{p}_9 \right) \delta\left(\vec{p}_4 - \vec{p}_7 - \vec{p}_8 \right) \\ &\mathcal{Y}_{1m_1}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 - \vec{p}_4), \end{aligned} \tag{4.8}$$

$$Q_1 = \left\langle \Psi_{N\pi}^{\text{spin}-\text{flavor}-\text{color}} | \mathcal{O}_{\text{spin}-\text{flavor}-\text{color}} | {}^2 N''_g \right\rangle, \tag{4.9}$$

and

$$Q_2 = \left\langle \Psi_{N\pi}^{\text{spin}-\text{flavor}-\text{color}} | \mathcal{O}_{\text{spin}-\text{flavor}-\text{color}} | {}^4 N''_g \right\rangle, \tag{4.10}$$

where

$$
\begin{aligned}
|{}^2N_g''\rangle &= \sum_{J_{12}} \Bigg\{ \Bigg(\Bigg[\Bigg| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_{J_{12}} \otimes \frac{1}{2}^{(3)} \Bigg\rangle_{\frac{1}{2},m_{123}} \mathbf{e}_{1m_s} \Bigg]^{\text{spin}}_{\frac{1}{2},S_z''} \\
&\qquad \Bigg| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_{J_{12}} \otimes \frac{1}{2}^{(3)} \Bigg\rangle^{\text{flavor}}_{\frac{1}{2},T_z''} (-1)^{J_{12}} \frac{1}{\sqrt{8}} \sum_\alpha \psi^\rho_\alpha |g\rangle^{\text{color}}_\alpha \Bigg) \\
&\quad - \Bigg(\Bigg[\Bigg| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_{1-J_{12}} \otimes \frac{1}{2}^{(3)} \Bigg\rangle_{\frac{1}{2},m_{123}} \mathbf{e}_{1m_s} \Bigg]^{\text{spin}}_{\frac{1}{2},S_z''} \\
&\qquad \Bigg| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_{J_{12}} \otimes \frac{1}{2}^{(3)} \Bigg\rangle^{\text{flavor}}_{\frac{1}{2},T_z''} \frac{1}{\sqrt{8}} \sum_\alpha \psi^\lambda_\alpha |g\rangle^{\text{color}}_\alpha \Bigg) \Bigg\}, \qquad (4.11)
\end{aligned}
$$

and

$$
\begin{aligned}
|{}^4N_g''\rangle &= \Bigg(\Bigg| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_1 \otimes \frac{1}{2}^{(3)} \Bigg\rangle^{\text{spin}}_{\frac{3}{2},m_{123}} \mathbf{e}_{1m_s} \Bigg) \\
&\qquad \Bigg\{ \Bigg| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_1 \otimes \frac{1}{2}^{(3)} \Bigg\rangle^{\text{flavor}}_{\frac{1}{2},T_z''} \frac{1}{\sqrt{8}} \sum_\alpha \psi^\rho_\alpha |g\rangle^{\text{color}}_\alpha \\
&\quad - \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_0 \otimes \frac{1}{2}^{(3)} \Bigg\rangle^{\text{flavor}}_{\frac{1}{2},T_z''} \frac{1}{\sqrt{8}} \sum_\alpha \psi^\lambda_\alpha |g\rangle^{\text{color}}_\alpha \Bigg\}. \qquad (4.12)
\end{aligned}
$$

4.1.1 The Spatial Part of the Transition Amplitude

In this subsection we calculate the spatial part P_{lm_l} of the transition amplitude for the decay of the Roper resonance into nucleon and π-meson,

$$\begin{aligned}
P_{lm_l} &= N_{N^*}N_N N_\pi \int d\hat{p} \prod_{i=1}^{9} \vec{p}_i \delta\left(\vec{p}_1 - \vec{p}_5\right)\delta\left(\vec{p}_2 - \vec{p}_6\right)\delta\left(\vec{p}_3 - \vec{p}_9\right) \\
&\cdot\delta\left(\vec{p}_4 - \vec{p}_7 - \vec{p}_8\right)\delta\left(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4\right)\delta\left(\vec{p}_8 + \vec{p}_9 - \vec{p}\right) \\
&\cdot \exp\left[-\frac{a^2}{2}\left(\frac{\vec{p}_5 - \vec{p}_6}{\sqrt{2}}\right)^2\right]\exp\left[-\frac{a^2}{2}\left(\frac{\vec{p}_5 + \vec{p}_6 - 2\vec{p}_7}{\sqrt{6}}\right)^2\right] \\
&\cdot \exp\left[-\frac{a^2}{2}\left(\frac{\vec{p}_1 - \vec{p}_2}{\sqrt{2}}\right)^2\right]\exp\left[-\frac{a^2}{2}\left(\frac{\vec{p}_1 + \vec{p}_2 - 2\vec{p}_3}{\sqrt{6}}\right)^2\right] \\
&\cdot \exp\left[-\frac{b^2}{8}\left(\vec{p}_8 - \vec{p}_9\right)^2\right]\exp\left[-\frac{c^2}{8}\left(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 - \vec{p}_4\right)^2\right] \\
&\cdot\mathcal{Y}_{1m_1}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 - \vec{p}_4)\mathcal{Y}_{lm_l}(\hat{p}). \qquad (4.13)
\end{aligned}$$

Due to the δ-functions in the equation, the integration could be largely simplified. One can easily derive

$$\begin{aligned}
P_{lm_l} &= N_{N^*}N_N N_\pi \int d\hat{p}\vec{p}_4\vec{p}_5\vec{p}_6 \mathcal{Y}_{lm_l}(\hat{p})\mathcal{Y}_{1m_1}(2\vec{p}_4)\exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_5 - \vec{p}_6}{\sqrt{2}}\right)^2\right] \\
&\cdot \exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_5 + \vec{p}_6 - 2(-\vec{p}_4 - \vec{p}_5 - \vec{p}6)}{\sqrt{6}}\right)^2\right] \\
&\cdot \exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_5 - \vec{p}_6}{\sqrt{6}}\right)^2\right]\exp\left[-\frac{1}{2}a^2\left(\frac{\vec{p}_5 + \vec{p}_6 - 2(-\vec{p}_4 - \vec{p}_5 - \vec{p}_6)}{\sqrt{6}}\right)\right] \\
&\cdot \exp\left[-\frac{1}{8}b^2\left(\vec{p} + 2\vec{p}_4 + 2\vec{p}_5 + 2\vec{p}_6\right)^2\right]\exp\left[-\frac{1}{8}c^2(2\vec{p}_4)^2\right]
\end{aligned}$$

$$
\begin{aligned}
= \quad & N_{N^*} N_N N_\pi \int d\hat{p}\vec{p}_4\vec{p}_5\vec{p}_6 \mathcal{Y}_{lm_l}(\hat{p})\mathcal{Y}_{1m_1}(\vec{2}p_4) \\
& \exp\left[-\left(\frac{a^2}{3}+\frac{b^2}{2}+\frac{c^2}{2}\right)\vec{p}_4^{\,2} - \left(2a^2+\frac{b^2}{2}\right)\vec{p}_5^{\,2} - \left(2a^2+\frac{b^2}{2}\right)\vec{p}_6^{\,2}\right. \\
& \quad - \left(\frac{a^2}{3}+\frac{b^2}{8}\right)\vec{p}^{\,2} - \left(a^2+b^2\right)\left(\vec{p}_4\cdot\vec{p}_5+\vec{p}_4\cdot\vec{p}_6\right) - \frac{b^2}{2}\left(\vec{p}\cdot\vec{p}_4\right) \\
& \quad \left. - \left(a^2+\frac{b^2}{2}\right)\left(\vec{p}\cdot\vec{p}_5+\vec{p}\cdot\vec{p}_6\right) - \left(2a^2+b^2\right)\left(\vec{p}_5\cdot\vec{p}_6\right)\right] \qquad (4.14) \\
= \quad & N_{N^*} N_N N_\pi \exp\left[-\left(\frac{a^2}{3}+\frac{b^2}{8}\right)\vec{p}^{\,2}\right] \int d\hat{p}\vec{p}_4\vec{p}_5\vec{p}_6 \mathcal{Y}_{lm_l}(\hat{p})\mathcal{Y}_{1m_1}(2\vec{p}_4) \\
& \exp\left[-\left(\frac{a^2}{3}+\frac{b^2}{2}+\frac{c^2}{2}\right)\vec{p}_4^{\,2} - \left(2a^2+\frac{b^2}{2}\right)\vec{p}_5^{\,2} - \left(2a^2+\frac{b^2}{2}\right)\vec{p}_6^{\,2}\right. \\
& \quad - \left(a^2+b^2\right)\left(\vec{p}_4\cdot\vec{p}_5+\vec{p}_4\cdot\vec{p}_6\right) - \frac{b^2}{2}\left(\vec{p}\cdot\vec{p}_4\right) \\
& \quad \left. - \left(a^2+\frac{b^2}{2}\right)\left(\vec{p}\cdot\vec{p}_5+\vec{p}\cdot\vec{p}_6\right) - \left(2a^2+b^2\right)\left(\vec{p}_5\cdot\vec{p}_6\right)\right]. \qquad (4.15)
\end{aligned}
$$

The standard way to carry out the integration above is to first eliminate the cross terms. We make the following transformation

$$
\begin{aligned}
\vec{Q}_1 &= a_1\vec{p}_4 + a_2\vec{p}_5 + a_3\vec{p}_6 + a_4\vec{p}, \\
\vec{Q}_2 &= b_1\vec{p}_4 + b_2\vec{p}_5 + b_3\vec{p}_6 + b_4\vec{p}, \qquad (4.16) \\
\vec{Q}_3 &= c_1\vec{p}_4 + c_2\vec{p}_5 + c_3\vec{p}_6 + c_4\vec{p}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&-\vec{Q}_1^{\,2} - \vec{Q}_2^{\,2} - \vec{Q}_3^{\,2} - \alpha \vec{p}^{\,2} \\
&\quad = -\left(a_1^{\,2} + b_1^{\,2} + c_1^{\,2}\right) \vec{p}_4^{\,2} - \left(a_2^{\,2} + b_2^{\,2} + c_2^{\,2}\right) \vec{p}_5^{\,2} \\
&\qquad -\left(a_3^{\,2} + b_3^{\,2} + c_3^{\,2}\right) \vec{p}_6^{\,2} - \left(a_4^{\,2} + b_4^{\,2} + c_4^{\,2} - \alpha\right) \vec{p}^{\,2} \\
&\qquad -2\left(a_1 a_2 + b_1 b_2 + c_1 c_2\right) \vec{p}_4 \cdot \vec{p}_5 - 2\left(a_1 a_3 + b_1 b_3 + c_1 c_3\right) \vec{p}_4 \cdot \vec{p}_6 \\
&\qquad -2\left(a_1 a_4 + b_1 b_4 + c_1 c_4\right) \vec{p} \cdot \vec{p}_4 - 2\left(a_2 a_3 + b_2 b_3 + c_2 c_3\right) \vec{p}_5 \cdot \vec{p}_6 \\
&\qquad -2\left(a_2 a_4 + b_2 b_4 + c_2 c_4\right) \vec{p} \cdot \vec{p}_5 - 2\left(a_3 a_4 + b_3 b_4 + c_3 c_4\right) \vec{p} \cdot \vec{p}_6. \qquad (4.17)
\end{aligned}
$$

Let the coefficients of all the cross terms zero and also let $b_1 = 0$, $c_1 = 0$ and $c_2 = 0$, we obtain

$$
\begin{aligned}
a_1 &= \left(\frac{2a^2 + 3b^2 + 3c^2}{6}\right)^{\frac{1}{2}}, \\
a_2 &= \frac{a^2 + b^2}{2}\left(\frac{6}{2a^2 + 3b^2 + 3c^2}\right)^{\frac{1}{2}}, \\
a_3 &= \frac{a^2 + b^2}{2}\left(\frac{6}{2a^2 + 3b^2 + 3c^2}\right)^{\frac{1}{2}}, \\
a_4 &= \frac{b^2}{4}\left(\frac{6}{2a^2 + 3b^2 + 3c^2}\right)^{\frac{1}{2}}, \\
b_2 &= \left(\frac{5a^4 + 8a^2b^2 + 12a^2c^2 + 3b^2c^2}{4a^2 + 6b^2 + 6c^2}\right)^{\frac{1}{2}}, \\
b_3 &= \frac{a^4 + 2a^2b^2 + 6a^2c^2 + 3b^2c^2}{\left[\left(5a^4 + 8a^2b^2 + 12a^2c^2 + 3b^2c^2\right)\left(4a^2 + 6b^2 + 6c^2\right)\right]^{\frac{1}{2}}}, \\
b_4 &= \frac{4a^4 + 5a^2b^2 + 6a^2c^2 + 3b^2c^2}{2\left[\left(5a^4 + 8a^2b^2 + 12a^2c^2 + 3b^2c^2\right)\left(4a^2 + 6b^2 + 6c^2\right)\right]^{\frac{1}{2}}}, \\
c_3 &= \left(\frac{6a^6 + 6a^2b^2c^2 + 18a^4c^2 + 10a^4b^2}{5a^4 + 8a^2b^2 + 12a^2c^2 + 3b^2c^2}\right)^{\frac{1}{2}}, \\
c_4 &= \frac{4a^6 + 3a^2b^2c^2 + 5a^4b^2 + 6a^4c^4}{2\left[\left(5a^4 + 8a^2b^2 + 12a^2c^2 + 3b^2c^2\right)\left(6a^6 + 6a^2b^2c^2 + 18a^4c^2 + 10a^4b^2\right)\right]^{\frac{1}{2}}},
\end{aligned}
$$

$$-\alpha = \frac{8a^6+3b^4c^2+8a^4b^2+12a^4c^2+5a^2b^4+12a^2b^2c^2}{24a^4+24b^2c^2+40a^2b^2+72a^2c^2}.$$

The spatial part of the transition amplitude P_{lm_l} is expressed in the new coordinates $\vec{Q}_1$, $\vec{Q}_2$, $\vec{Q}_3$ as

$$\begin{aligned} P_{lm_l} &= N_{N^*}N_N N_\pi |C|^3 \int d\hat{p}d\vec{Q}_1 d\vec{Q}_2 d\vec{Q}_3 \mathcal{Y}_{lm_l}(\hat{p}) \\ &\quad \mathcal{Y}_{1m_1}(\beta_1\vec{Q}_1+\beta_2\vec{Q}_2+\beta_3\vec{Q}_3+\beta_4\vec{p})\exp\left(-\alpha\vec{p}^{\,2}\right) \\ &\quad \exp\left(-\vec{Q}_1^{\,2}-\vec{Q}_2^{\,2}-\vec{Q}_3^{\,2}\right), \end{aligned} \tag{4.18}$$

where $|C|$ is the Jacobian, coming from the coordinate transformation in eq. (4.16). All the coefficients in the above equation are listed below as:

$$|C| = \frac{\sqrt{6}}{(3a^6+5a^4b^2+9a^4c^2+3a^2b^2c^2)^{1/2}}, \tag{4.19}$$

$$\beta_1 = \frac{2\sqrt{6}}{(2a^2+3b^2+3c^2)^{1/2}}, \tag{4.20}$$

$$\beta_2 = \frac{6\sqrt{2}(a^2+b^2)}{(5a^4+8a^2b^2+12a^2c^2+3b^2c^2)^{1/2}(2a^2+3b^2+3c^2)^{1/2}}, \tag{4.21}$$

$$\beta_3 = \frac{6\sqrt{2}(a^4+a^2b^2)}{(5a^4+8a^2b^2+12a^2c^2+3b^2c^2)^{1/2}(3a^6+3a^2b^2c^2+9a^4c^2+5a^4b^2)}, \tag{4.22}$$

$$\begin{aligned} \beta_4 &= 12\Big(-12a^{12}-13a^{10}b^2+47a^8b^4+60a^6b^6-18a^{10}c^2+123a^8b^2c^2 \\ &\quad +294a^6b^4c^2+81a^4b^6c^2+234a^6b^2c^4+153a^4b^4c^4+27a^2b^6c^4\Big) \\ &\quad \times\frac{1}{(4a^2+6b^2+6c^2)(10a^4+16a^2b^2+24a^2c^2+6b^2c^2)} \\ &\quad \times\frac{1}{(6a^6+6a^2b^2c^2+18a^4c^2+10a^4b^2)}, \end{aligned} \tag{4.23}$$

$$\alpha = \frac{-(8a^6+3b^4c^2+8a^4b^2+12a^4c^2+5a^2b^4+12a^2b^2c^2)}{(24a^4+24b^2c^2+40a^2b^2+72a^2c^2)}. \tag{4.24}$$

Integrating over the coordinates $\vec{Q}_1$, $\vec{Q}_1$, $\vec{Q}_1$ and $\hat{p}$ in eq. (4.18), we derive

$$\begin{aligned}
P_{lm_l} &= N_{N^*}N_N N_\pi \left|C\right|^3 (4\pi)^3 \int d\hat{p}\mathcal{Y}_{1m_1}(\hat{p})\mathcal{Y}^*_{l(-m_l)}(\hat{p})(-1)^{m_l} \\
&\quad \int dQ_1 dQ_2 dQ_3 Q_1^2 Q_2^2 Q_3^2 \beta_4 p \exp\left(-Q_1^2 - Q_2^2 - Q_3^2 - \alpha p^2\right) \\
&= N_{N^*}N_N N_\pi \left|C\right|^3 (4\pi)^3 (-1)^{m_l}\delta_{1,l}\delta_{m_1,-m_l}\beta_4 p \exp(-\alpha p^2) \\
&\quad \int Q_1^2 e^{-Q_1^2} dQ_1 \int Q_2^2 e^{-Q_2^2} dQ_2 \int Q_3^2 e^{-Q_3^2} dQ_3 \\
&= N_{N^*}N_N N_\pi \left|C\right|^3 (4\pi)^3 (-1)^{m_l}\delta_{1,l}\delta_{m_1,-m_l}\beta_4 p \exp(-\alpha p^2) \left(\frac{\Gamma(3/2)}{2^{3/2}}\right) \\
&= N_{N^*}N_N N_\pi \left|C\right|^3 (4\pi)^3 (-1)^{m_l}\delta_{1,l}\delta_{m_1,-m_l}\beta_4 p \exp(-\alpha p^2) \left[\left(\frac{\sqrt{\pi}}{2}^3\right)\left(\frac{1}{2^{3/2}}\right)\right] \\
&= \frac{\pi^{9/2}}{2\sqrt{2}} N_{N^*}N_N N_\pi \left|C\right|^3 (-1)^{m_l}\delta_{1,l}\delta_{m_1,-m_l}\beta_4 p \exp(-\alpha p^2). \qquad (4.25)
\end{aligned}$$

Here we have used

$$\begin{aligned}
&\mathcal{Y}_{1m_1}(\beta_1\vec{Q}_1 + \beta_2\vec{Q}_2 + \beta_3\vec{Q}_3 + \beta_4\vec{p}) \\
&\qquad = \beta_1 Q_1 \mathcal{Y}_{1m_1}(\hat{Q}_1) + \beta_2 Q_2 \mathcal{Y}_{1m_1}\beta(\hat{Q}_2) + \beta_3 Q_3 \mathcal{Y}_{1m_1}(\hat{Q}_3) + \beta_4 p \mathcal{Y}_{1m_1}(\hat{p}),
\end{aligned}$$

$$d\vec{Q} = Q^2 dQ d\hat{Q},$$

$$\int d\hat{p} d\hat{Q}_1 \mathcal{Y}_{1m_1}(\hat{Q}_1)\mathcal{Y}_{lm_l}(\hat{p}) = 0,$$

and

$$\mathcal{Y}_{1m_1}(\hat{p})\mathcal{Y}_{lm_l}(\hat{p}) = (-1)^{m_l}\mathcal{Y}_{1m_1}(\hat{p})\mathcal{Y}^*_{l(-m_l)}(\hat{p}).$$

4.1.2 The Spin-Flavor-Color Part of the Transition Amplitude

Now we turn to evaluate the spin-flavor-color part of the transition amplitude

$$Q_1 = \langle \Psi_{N\pi}^{\text{spin}-\text{flavor}-\text{color}} | \mathcal{O}_{\text{spin}-\text{flavor}} \mathcal{O}_{\text{color}} | {}^2 N_g'' \rangle, \tag{4.26}$$

and

$$Q_2 = \langle \Psi_{N\pi}^{\text{spin}-\text{flavor}-\text{color}} | \mathcal{O}_{\text{spin}-\text{flavor}} \mathcal{O}_{\text{color}} | {}^4 N_g'' \rangle. \tag{4.27}$$

The operators for the spin-flavor part and color part are

$$\mathcal{O}_{\text{spin}-\text{flavor}} = (-1)^{\mu+1} \sigma_{78}^{\mu} \varepsilon_{-\mu} \mathbf{1}_{78}^{F}, \tag{4.28}$$

$$\mathcal{O}_{\text{color}} = \frac{1}{2} \lambda_{78}^{\alpha} \mathbf{A}^{\alpha}, \tag{4.29}$$

respectively. The spin-flavor-color wave function of the Roper resonance, nucleon and π-meson have been constructed in chapter II. Eq.(4.26) is more complicated than eq.(4.27), so we evaluate the latter first as follows:

$$
\begin{aligned}
&\langle \Psi_{N\pi}^{\mathrm{spin-flavor-color}} | \mathcal{O}_{\mathrm{spin-flavor}} \mathcal{O}_{\mathrm{color}} | {}^4 N_g'' \rangle \\
&= \left\langle \Psi_N^{\mathrm{sfc}} \Psi_\pi^{\mathrm{sfc}} | \mathcal{O}_{\mathrm{spin-flavor}} \mathcal{O}_{\mathrm{color}} | {}^4 N_g'' \right\rangle \\
&= \frac{1}{\sqrt{2}} \sum_{J_{56}} \left[\left\langle \left(\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)} \right)_{J_{56}} \otimes \frac{1}{2}^{(7)} \right|_{\frac{1}{2}} \otimes \left\langle \left(\frac{1}{2}^{(8)} \otimes \frac{1}{2}^{(9)} \right) \right|_{0} \right]_{S,S_z}^{\mathrm{spin}} \\
&\quad \left[\left\langle \left(\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)} \right)_{J_{56}} \otimes \frac{1}{2}^{(7)} \right|_{\frac{1}{2}} \otimes \left\langle \left(\frac{1}{2}^{(8)} \otimes \frac{1}{2}^{(9)} \right) \right|_{1} \right]_{T,T_z}^{\mathrm{flavor}} \mathcal{O}_{\mathrm{spin-flavor}} \\
&\quad \left[\left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)} \right)_{1} \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{3}{2},m_{123}} \mathbf{e}_{1m_s} \right]_{\frac{1}{2},S_z''}^{\mathrm{spin}} \\
&\quad \left\{ \langle N|^{\mathrm{color}} \langle \pi|^{\mathrm{color}} \mathcal{O}_{\mathrm{color}} | \frac{1}{\sqrt{8}} \sum_{\alpha} \psi_\alpha^\rho | g \rangle_\alpha^{\mathrm{color}} \left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)} \right)_{1} \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{1}{2},T_z''}^{\mathrm{flavor}} \right. \\
&\quad \left. - \langle N|^{\mathrm{color}} \langle \pi|^{\mathrm{color}} \mathcal{O}_{\mathrm{color}} | \frac{1}{\sqrt{8}} \sum_{\alpha} \psi_\alpha^\lambda | g \rangle_\alpha^{\mathrm{color}} \left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)} \right)_{0} \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{1}{2},T_z''}^{\mathrm{flavor}} \right\} \\
&= \frac{\sqrt{2}}{3} \sum_{J_{56}} \left[\left\langle \left(\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)} \right)_{J_{56}} \otimes \frac{1}{2}^{(7)} \right|_{\frac{1}{2}} \otimes \left\langle \left(\frac{1}{2}^{(8)} \otimes \frac{1}{2}^{(9)} \right) \right|_{0} \right]_{S,S_z}^{\mathrm{spin}} \\
&\quad \left[\left\langle \left(\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)} \right)_{J_{56}} \otimes \frac{1}{2}^{(7)} \right|_{\frac{1}{2}} \otimes \left\langle \left(\frac{1}{2}^{(8)} \otimes \frac{1}{2}^{(9)} \right) \right|_{1} \right]_{T,T_z}^{\mathrm{flavor}} \mathcal{O}_{\mathrm{spin-flavor}} \\
&\quad \left[\left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)} \right)_{1} \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{3}{2},m_{123}} \mathbf{e}_{1m_s} \right]_{\frac{1}{2},S_z''}^{\mathrm{spin}} \left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)} \right)_{1} \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{1}{2},T_z''}^{\mathrm{flavor}},
\end{aligned}
\tag{4.30}
$$

where the two color parts of the transition amplitude are evaluated as follows:

$$
\begin{aligned}
&\langle N|^{\mathrm{color}} \langle \pi|^{\mathrm{color}} \mathcal{O}_{\mathrm{color}} | \frac{1}{\sqrt{8}} \sum_{\alpha} \psi_\alpha^\rho | g \rangle_\alpha^{\mathrm{color}} \\
&= \langle qqq | \langle q\bar{q} | \mathcal{O}_{\mathrm{color}} | \frac{1}{\sqrt{8}} \sum_{\alpha} \psi_\alpha^\rho | g \rangle_\alpha^{\mathrm{color}} \rangle \\
&= \left(\frac{1}{\sqrt{6}} \epsilon_{ijk} \langle q_5|_i \langle q_6|_j \langle q_7|_k \right) \left(\frac{1}{\sqrt{3}} \langle q_9|_l \langle \bar{q}_8|_l \right) \frac{1}{2} \lambda_{78}^{\alpha} \mathbf{A}^{\alpha} \\
&\quad \cdot \left(\frac{1}{4\sqrt{2}} \sum_{i',j',k',l',\beta} |q_3\rangle_{i'} \lambda_{i'j'}^{\beta} \cdot \epsilon_{j'k'l'} |q_1\rangle_{k'} |q_2\rangle_{l'} |g\rangle^{\beta} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{48}\epsilon_{ijk}\sum_{i',j',k',l',\beta}\epsilon_{j'k'l'}\langle q_5|_i|q_1\rangle_{k'}\ \langle q_6|_j|q_2\rangle_{l'}\ \langle q_9|_l|q_3\rangle_{i'}\ \langle\bar{q}_8|_l\langle q_7|_k\lambda^{\alpha}_{78}\ \mathbf{A}^{\alpha}|g\rangle^{\beta}\lambda^{\beta}_{i'j'} \\
&= \frac{1}{48}\epsilon_{ijk}\sum_{i',j',k',l',\beta}\epsilon_{j'k'l'}\delta_{ik'}\delta_{jl'}\delta_{li'}\lambda^{\alpha}_{lk}\delta_{\alpha\beta}\lambda^{\beta}_{i'j'} \\
&= \frac{1}{48}\epsilon_{ijk}\sum_{j'}\epsilon_{j'ij}\lambda^{\alpha}_{lk}\lambda^{\alpha}_{lj'} \\
&= \frac{1}{48}\sum_{j'}\epsilon_{ijk}\epsilon_{ijj'}\lambda^{\alpha}_{lk}\lambda^{\alpha}_{lj'} \\
&= \frac{1}{24}\mathrm{Tr}\left[\lambda^{\alpha}\cdot\lambda^{\alpha}\right] \ = \ \frac{16}{24} \ = \ \frac{2}{3},
\end{aligned}
\tag{4.31}
$$

and

$$
\begin{aligned}
&\langle N|^{\mathrm{color}}\langle\pi|^{\mathrm{color}}\mathcal{O}_{\mathrm{color}}|\frac{1}{\sqrt{8}}\sum_{\alpha}\psi^{\lambda}_{\alpha}|g\rangle^{\mathrm{color}}_{\alpha} \\
&\quad= \langle qqq|\langle q\bar{q}|\mathcal{O}_{\mathrm{color}}|\frac{1}{\sqrt{8}}\sum_{\alpha}\psi^{\lambda}_{\alpha}|g\rangle^{\mathrm{color}}_{\alpha} \\
&\quad= \left(\frac{1}{\sqrt{6}}\epsilon_{ijk}\langle q_5|_i\langle q_6|_j\langle q_7|_k\right)\left(\frac{1}{\sqrt{3}}\langle q_9|_l\langle\bar{q}_8|_l\right)\frac{1}{2}\lambda^{\alpha}_{78}\mathbf{A}^{\alpha} \\
&\qquad\cdot\left(\frac{1}{4\sqrt{2}}\sum_{i',j',k',l',\beta}\left[|q_1\rangle_{i'}|q_2\rangle_{j'}\lambda^{\beta}_{i'j'}\epsilon_{j'k'l'}|q_3\rangle_{k'} + |q_1\rangle_{i'}|q_2\rangle_{j'}\lambda^{\beta}_{j'l'}\epsilon_{i'k'l'}|q_3\rangle_{k'}\right]\right) \\
&\quad= \frac{1}{48}\epsilon_{ijk}\sum_{i',j',k',i',\beta}\Big[\epsilon_{j'k'l'}\ \langle q_5|_i|q_1\rangle_{i'}\ \langle q_6|_j|q_2\rangle_{j'}\ \langle q_9|_l|q_3\rangle_{k'}\ \langle q_7|_k\langle\bar{q}_8|_l\lambda^{\alpha}_{78}\mathbf{A}^{\alpha}|g\rangle^{\beta}\lambda^{\beta}_{i'l'} \\
&\qquad\qquad + \epsilon_{i'k'l'}\ \langle q_5|_i|q_1\rangle_{i'}\ \langle q_6|_j|q_2\rangle_{j'}\ \langle q_9|_l|q_3\rangle_{k'}\ \langle q_7|_k\langle\bar{q}_8|_l\lambda^{\alpha}_{78}\mathbf{A}^{\alpha}|g\rangle^{\beta}\lambda^{\beta}_{j'l'}\Big] \\
&\quad= \frac{1}{48}\epsilon_{ijk}\sum_{i',j',k',l',\beta}\left[\epsilon_{j'k'l'}\delta_{ii'}\delta_{jj'}\delta_{lk'}\lambda^{\alpha}_{kl}\delta_{\alpha\beta}\lambda^{\beta}_{i'l'} + \epsilon_{i'k'l'}\delta_{ii'}\delta_{jj'}\delta_{lk'}\lambda^{\alpha}_{kl}\delta_{\alpha\beta}\lambda^{\beta}_{j'l'}\right] \\
&\quad= \frac{1}{48}\sum_{l'}\left[-\epsilon_{jik}\epsilon_{jll'}\lambda^{\alpha}_{kl}\lambda^{\alpha}_{il'} + \epsilon_{ijk}\epsilon_{ill'}\lambda^{\alpha}_{kl}\lambda^{\alpha}_{jl'}\right] \\
&\quad= \frac{1}{48}\sum_{l'}\left[-\left(\delta_{il}\delta_{kl'}-\delta_{il'}\delta_{kl}\right)\lambda^{\alpha}_{kl}\lambda^{\alpha}_{il'} + \left(\delta_{jl}\delta_{kl'}-\delta_{jl'}\delta_{kl}\right)\lambda^{\alpha}_{kl}\lambda^{\alpha}_{jl'}\right] \\
&\quad= \frac{1}{48}\left[-\left(\lambda^{\alpha}_{kl}\lambda^{\alpha}_{lk}-\lambda^{\alpha}_{kk}\lambda^{\alpha}_{ii}\right) + \left(\lambda^{\alpha}_{kl}\lambda^{\alpha}_{lk}-\lambda^{\alpha}_{kk}\lambda^{\alpha}_{jj}\right)\right] \\
&\quad= \frac{1}{48}\left[(\mathrm{Tr}[\lambda^{\alpha}])^2-(\mathrm{Tr}[\lambda^{\alpha}])^2\right] \ = \ 0.
\end{aligned}
\tag{4.32}
$$

An easy way to evaluate the spin-flavor part of the transition amplitude is to recouple the quark spins and flavors using the formulas

$$
\begin{aligned}
&|(j_1 \otimes j_2)_{J_{12}} \otimes (j_3 \otimes j_4)_{J_{34}}\rangle_{J,M} \\
&\quad = \sum_{J_{13},J_{24}} \langle (j_1 j_3)_{J_{13}}, (j_2 j_4)_{J_{24}}; JM | (j_1 j_2)_{J_{12}}, (j_3 j_4)_{J_{34}}; JM\rangle \\
&\qquad |(j_1 \otimes j_3)_{J_{13}} \otimes (j_2 \otimes j_4)_{J_{24}}\rangle_{J,M},
\end{aligned}
$$

with

$$
\begin{aligned}
&\langle (j_1 j_3)_{J_{13}}, (j_2 j_4)_{J_{24}}; JM | (j_1 j_2)_{J_{12}}, (j_3 j_4)_{J_{34}}; JM\rangle \\
&\quad = \sqrt{(2J_{12}+1)(2J_{34}+1)(2J_{13}+1)(2J_{24}+1)} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix}. \qquad (4.33)
\end{aligned}
$$

Eq.(4.30) may be rewritten as

$$
\begin{aligned}
&\langle \Psi_{N\pi}^{\text{spin}-\text{flavor}-\text{color}} | \mathcal{O}_{\text{spin}-\text{flavor}} \mathcal{O}_{\text{color}} | {}^4N_g'' \rangle \\
&= \frac{\sqrt{2}}{3} \sum_{J_{56}} \sum_{i,j} \left\langle \left(J_{56}\frac{1}{2}\right)_i, \left(\frac{1}{2}\frac{1}{2}\right)_j ; TT_z \middle| \left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_1 ; TT_z \right\rangle \\
&\cdot \left[\left\langle \left(\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)} \right)_{J_{56}} \otimes \frac{1}{2}^{(9)} \right|_i \otimes \left\langle \left(\frac{1}{2}^{(7)} \otimes \frac{1}{2}^{(8)} \right) \right|_j \right]_{T,T_z}^{\text{flavor}} \\
&\sum_{\alpha,\beta} \left\langle \left(J_{56}\frac{1}{2}\right)_\alpha, \left(\frac{1}{2}\frac{1}{2}\right)_\beta ; SS_z \middle| \left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_0 ; SS_z \right\rangle \\
&\cdot \left[\left\langle \left(\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)} \right)_{J_{56}} \otimes \frac{1}{2}^{(9)} \right|_\alpha \otimes \left\langle \left(\frac{1}{2}^{(7)} \otimes \frac{1}{2}^{(8)} \right) \right|_\beta \right]_{S,S_z}^{\text{spin}} \\
&(-1)^{\mu+1} \sigma_{78}^{\mu} \varepsilon_{-\mu} \mathbf{1}_{78} \left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)} \right)_1 \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{1}{2},T_z''}^{\text{flavor}} \\
&\cdot \left[\left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)} \right)_1 \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{3}{2},m_{123}} \mathbf{e}_{1m_s} \right]_{\frac{1}{2},S_z''}^{\text{spin}}
\end{aligned} \tag{4.34}
$$

$$
\begin{aligned}
&= \frac{\sqrt{2}}{3} \sum_{J_{56}} \sum_{i,j} \sum_{\alpha,\beta} \left\langle \left(J_{56}\frac{1}{2}\right)_i, \left(\frac{1}{2}\frac{1}{2}\right)_j ; TT_z \middle| \left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_1 ; TT_z \right\rangle \\
&\left\langle \left(J_{56}\frac{1}{2}\right)_\alpha, \left(\frac{1}{2}\frac{1}{2}\right)_\beta ; SS_z \middle| \left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_0 ; SS_z \right\rangle \\
&\sum_{m_i} \langle im_i, j(T_z - m_i) | TT_z \rangle \left\langle \left(\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)} \right)_{J_{56}} \otimes \frac{1}{2}^{(9)} \right|_{i,m_i}^{\text{flavor}} \\
&\left\langle \left(\frac{1}{2}^{(7)} \otimes \frac{1}{2}^{(8)} \right) \right|_{j,(T_z-m_i)}^{\text{flavor}} \sum_{m_\alpha} \langle \alpha m_\alpha, \beta(S_z - m_\alpha) | SS_z \rangle \\
&\left\langle \left(\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)} \right)_{J_{56}} \otimes \frac{1}{2}^{(9)} \right|_{\alpha,m_\alpha}^{\text{spin}} \left\langle \left(\frac{1}{2}^{(7)} \otimes \frac{1}{2}^{(8)} \right) \right|_{\beta,(S_z-m_\alpha)}^{\text{spin}} \\
&(-1)^{\mu+1} \sigma_{78}^{\mu} \varepsilon_{-\mu} \mathbf{1}_{78}^{F} \left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)} \right)_1 \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{1}{2},T_z''}^{\text{flavor}} \\
&\left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)} \right)_1 \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{3}{2},m_{123}}^{\text{spin}} \mathbf{e}_{1(S_z''-m_{123})}
\end{aligned} \tag{4.35}
$$

$$
\begin{aligned}
= \ & \frac{\sqrt{2}}{3}\sum_{J_{56}}\sum_{i,j}\sum_{m_i,m_\alpha}\sum_{\mu}\left\langle\left(J_{56}\frac{1}{2}\right)_i,\left(\frac{1}{2}\frac{1}{2}\right)_j;TT_z\right|\left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_1;TT_z\Big\rangle \\
& \left\langle\left(J_{56}\frac{1}{2}\right)_\alpha,\left(\frac{1}{2}\frac{1}{2}\right)_\beta;SS_z\right|\left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_0;SS_z\Big\rangle \\
& \langle im_i,j(T_z-m_i)|TT_z\rangle\,\langle\alpha m_\alpha,\beta(S_z-m_\alpha)|SS_z\rangle \\
& \left\langle\left(\frac{1}{2}^{(5)}\otimes\frac{1}{2}^{(6)}\right)_{J_{56}}\otimes\frac{1}{2}^{(9)}\right|_{im_i}\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_1\otimes\frac{1}{2}^{(3)}\right\rangle_{\frac{1}{2},T_z''} \\
& \left\langle\left(\frac{1}{2}^{(7)}\otimes\frac{1}{2}^{(8)}\right)\right|_{j,(T_z-m_i)}\mathbf{1}^F_{78}\Big|0,0\Big\rangle_{0,0} \\
& \left\langle\left(\frac{1}{2}^{(5)}\otimes\frac{1}{2}^{(6)}\right)_{J_{56}}\otimes\frac{1}{2}^{(9)}\right|_{\alpha,m_\alpha}\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_1\otimes\frac{1}{2}^{(3)}\right\rangle_{\frac{3}{2},m_{123}} \\
& (-1)^{\mu+1}\left\langle\left(\frac{1}{2}^{(7)}\otimes\frac{1}{2}^{(8)}\right)\right|_{\beta,(S_z-m_\alpha)}\sigma^\mu_{78}\Big|0,0\Big\rangle\,\langle 0,0|\varepsilon_{-\mu}|\mathbf{e}_{1(S_z''-m_{123})}\rangle. \qquad (4.36)
\end{aligned}
$$

By using the operations in eq.(3.26), eq.(3.27) and eq.(3.28), we get

$$
\begin{aligned}
& \langle\Psi_{N\pi}^{\rm spin-flavor-color}|\mathcal{O}_{\rm spin-flavor}\mathcal{O}_{\rm color}|^4N_g''\rangle \\
= \ & \frac{\sqrt{2}}{3}\sum_{J_{56}}\sum_{i,j}\sum_{m_i,m_\alpha}\sum_{\mu}\left\langle\left(J_{56}\frac{1}{2}\right)_i,\left(\frac{1}{2}\frac{1}{2}\right)_j;TT_z\right|\left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_1;TT_z\Big\rangle \\
& \left\langle\left(J_{56}\frac{1}{2}\right)_\alpha,\left(\frac{1}{2}\frac{1}{2}\right)_\beta;SS_z\right|\left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_0;SS_z\Big\rangle \\
& \langle im_i,j(T_z-m_i)|TT_z\rangle\,\langle\alpha m_\alpha,\beta(S_z-m_\alpha)|SS_z\rangle \\
& \delta_{J_{56},1}\delta_{i,\frac{1}{2}}\delta_{m_i,T_z''}\sqrt{2}\delta_{j,0}\delta_{T_z-m_i,0}\delta_{\alpha,\frac{3}{2}}\delta_{m_\alpha,m_{123}} \\
& (-1)^{\mu+1}\sqrt{2}(-1)^{(S_z-m_\alpha)}\delta_{\beta,1}\delta_{(S_z-m_\alpha),-\mu}\delta_{(S_z''-m_{123}),-\mu} \\
= \ & -\frac{2\sqrt{2}}{3}\left\langle\left(1\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_0;TT_z\right|\left(1\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_1;TT_z\Big\rangle \\
& \left\langle\left(1\frac{1}{2}\right)_{\frac{3}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_1;SS_z\right|\left(1\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_0;SS_z\Big\rangle \\
& \left\langle\frac{1}{2}T_z'',00\right|TT_z\Big\rangle\left\langle\frac{3}{2}m_{123},1(S_z-m_{123})\right|SS_z\Big\rangle\,\delta_{T_z,T_z''}\delta_{S_z,S_z''}. \qquad (4.37)
\end{aligned}
$$

Next, we calculate eq.(4.26)

$$
\begin{aligned}
&\langle\Psi_{N\pi}^{\mathrm{spin-flavor-color}}|\mathcal{O}_{\mathrm{spin-flavor}}\mathcal{O}_{\mathrm{color}}|^2N_g''\rangle\\
&= \left\langle\Psi_N^{\mathrm{sfc}}\Psi_\pi^{\mathrm{sfc}}|\mathcal{O}_{\mathrm{spin-flavor}}\mathcal{O}_{\mathrm{color}}|^2N_g''\right\rangle\\
&= \frac{1}{\sqrt{2}}\sum_{J_{56}}\left[\left\langle\left(\frac{1}{2}^{(5)}\otimes\frac{1}{2}^{(6)}\right)_{J_{56}}\otimes\frac{1}{2}^{(7)}\right|_{\frac{1}{2}}\otimes\left\langle\left(\frac{1}{2}^{(8)}\otimes\frac{1}{2}^{(9)}\right)\right|_0\right]^{\mathrm{spin}}_{S,S_z}\\
&\quad\left[\left\langle\left(\frac{1}{2}^{(5)}\otimes\frac{1}{2}^{(6)}\right)_{J_{56}}\otimes\frac{1}{2}^{(7)}\right|_{\frac{1}{2}}\otimes\left\langle\left(\frac{1}{2}^{(8)}\otimes\frac{1}{2}^{(9)}\right)\right|_1\right]^{\mathrm{flavor}}_{T,T_z}\mathcal{O}_{\mathrm{spin-flavor}}\\
&\quad\sum_{J_{12}}\left\{\left(\left[\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\right\rangle_{\frac{1}{2},m_{123}}\mathbf{e}_{1m_s}\right]^{\mathrm{spin}}_{\frac{1}{2},S_z''}\right.\right.\\
&\quad\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\right\rangle^{\mathrm{flavor}}_{\frac{1}{2},T_z''}\\
&\quad\left.(-1)^{J_{12}}\,\langle N|^{\mathrm{color}}\langle\pi|^{\mathrm{color}}\mathcal{O}_{\mathrm{color}}|\frac{1}{\sqrt{8}}\sum_\alpha\psi_\alpha^\rho|g\rangle_\alpha^{\mathrm{color}}\right)\\
&\quad-\left(\left[\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{1-J_{12}}\otimes\frac{1}{2}^{(3)}\right\rangle_{\frac{1}{2},m_{123}}\mathbf{e}_{1m_s}\right]^{\mathrm{spin}}_{\frac{1}{2},S_z''}\right.\\
&\quad\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\right\rangle^{\mathrm{flavor}}_{\frac{1}{2},T_z''}\\
&\quad\left.\left.\langle N|^{\mathrm{color}}\langle\pi|^{\mathrm{color}}\mathcal{O}_{\mathrm{color}}|\frac{1}{\sqrt{8}}\sum_\alpha\psi_\alpha^\lambda|g\rangle_\alpha^{\mathrm{color}}\right)\right\}\\
&= \frac{\sqrt{2}}{3}\sum_{J_{56}}\left[\left\langle\left(\frac{1}{2}^{(5)}\otimes\frac{1}{2}^{(6)}\right)_{J_{56}}\otimes\frac{1}{2}^{(7)}\right|_{\frac{1}{2}}\otimes\left\langle\left(\frac{1}{2}^{(8)}\otimes\frac{1}{2}^{(9)}\right)\right|_0\right]^{\mathrm{spin}}_{S,S_z}\\
&\quad\left[\left\langle\left(\frac{1}{2}^{(5)}\otimes\frac{1}{2}^{(6)}\right)_{J_{56}}\otimes\frac{1}{2}^{(7)}\right|_{\frac{1}{2}}\otimes\left\langle\left(\frac{1}{2}^{(8)}\otimes\frac{1}{2}^{(9)}\right)\right|_1\right]^{\mathrm{flavor}}_{T,T_z}\mathcal{O}_{\mathrm{spin-flavor}}\\
&\quad\sum_{J_{12}}\left(\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\right\rangle^{\mathrm{spin}}_{\frac{1}{2},m_{123}}\mathbf{e}_{1(S_z''-m_{123})}\right.\\
&\quad\left.\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\right\rangle^{\mathrm{flavor}}_{\frac{1}{2},T_z}(-1)^{J_{12}}\right),
\end{aligned}
\tag{4.38}
$$

with

$$\langle N|^{\text{color}}\langle\pi|^{\text{color}}\mathcal{O}_{\text{color}}|\frac{1}{\sqrt{8}}\sum_{\alpha}\psi^{\rho}_{\alpha}|g\rangle^{\text{color}}_{\alpha} = \frac{2}{3},$$
$$\langle N|^{\text{color}}\langle\pi|^{\text{color}}\mathcal{O}_{\text{color}}|\frac{1}{\sqrt{8}}\sum_{\alpha}\psi^{\lambda}_{\alpha}|g\rangle^{\text{color}}_{\alpha} = 0.$$

Eq.(4.38) can be rewritten as

$$\begin{aligned}
&\langle\Psi^{\text{spin}-\text{flavor}-\text{color}}_{N\pi}|\mathcal{O}_{\text{spin}-\text{flavor}}\mathcal{O}_{\text{color}}|^{2}N''_{g}\rangle \\
&= \frac{\sqrt{2}}{3}\sum_{J_{56}}\sum_{i,j}\left\langle\left(J_{56}\frac{1}{2}\right)_{i},\left(\frac{1}{2}\frac{1}{2}\right)_{j};TT_{z}\right|\left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_{1};TT_{z}\right\rangle \\
&\left[\left\langle\left(\frac{1}{2}^{(5)}\otimes\frac{1}{2}^{(6)}\right)_{J_{56}}\otimes\frac{1}{2}^{(9)}\right|_{i}\otimes\left\langle\left(\frac{1}{2}^{(7)}\otimes\frac{1}{2}^{(8)}\right)\right|_{j}\right]^{\text{flavor}}_{T,T_{z}} \\
&\sum_{\alpha,\beta}\left\langle\left(J_{56}\frac{1}{2}\right)_{\alpha},\left(\frac{1}{2}\frac{1}{2}\right)_{\beta};SS_{z}\right|\left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_{0};SS_{z}\right\rangle \\
&\left[\left\langle\left(\frac{1}{2}^{(5)}\otimes\frac{1}{2}^{(6)}\right)_{J_{56}}\otimes\frac{1}{2}^{(9)}\right|_{\alpha}\otimes\left\langle\left(\frac{1}{2}^{(7)}\otimes\frac{1}{2}^{(8)}\right)\right|_{\beta}\right]^{\text{spin}}_{S,S_{z}} \\
&(-1)^{\mu+1}\sigma^{\mu}_{78}\epsilon_{-\mu}\mathbf{1}^{F}_{78}\sum_{J_{12}}\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\right\rangle^{\text{flavor}}_{\frac{1}{2},T''_{z}} \\
&\left|\left(\frac{1}{2}^{(1)}\otimes\frac{1}{2}^{(2)}\right)_{J_{12}}\otimes\frac{1}{2}^{(3)}\right\rangle^{\text{spin}}_{\frac{1}{2},m_{123}}\mathbf{e}_{1(S''_{z}-m_{123})}(-1)^{J_{12}}
\end{aligned}$$

$$
\begin{aligned}
= \; & \frac{\sqrt{2}}{3} \sum_{J_{12},J_{56}} \sum_{i,j} \sum_{\alpha,\beta} \left\langle \left(J_{56}\frac{1}{2}\right)_i, \left(\frac{1}{2}\frac{1}{2}\right)_j; TT_z \middle| \left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_1; TT_z \right\rangle \\
& \left\langle \left(J_{56}\frac{1}{2}\right)_\alpha, \left(\frac{1}{2}\frac{1}{2}\right)_\beta; SS_z \middle| \left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_0; SS_z \right\rangle \\
& \sum_{m_i,m_\alpha} \langle im_i, j(T_z - m_i)|TT_z\rangle \langle \alpha m_\alpha, \beta(S_z - m_\alpha)|SS_z\rangle \\
& \left\langle \left(\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right)_{J_{56}} \otimes \frac{1}{2}^{(3)} \right|_{i,m_i}^{\text{flavor}} \left\langle \left(\frac{1}{2}^{(7)} \otimes \frac{1}{2}^{(8)}\right) \right|_{j,(T_z-m_i)}^{\text{flavor}} \\
& \left\langle \left(\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right)_{J_{56}} \otimes \frac{1}{2}^{(3)} \right|_{\alpha,m_\alpha}^{\text{spin}} \left\langle \left(\frac{1}{2}^{(7)} \otimes \frac{1}{2}^{(8)}\right) \right|_{\beta,(T_z-m_\alpha)}^{\text{spin}} \\
& (-1)^{\mu+1}\sigma_{78}^{\mu}\epsilon_{-\mu}\mathbf{1}_{78}^{F} \sum_{J_{12}} \left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_{J_{12}} \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{1}{2},T_z''}^{\text{flavor}} \\
& \left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_{J_{12}} \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{1}{2},m_{123}}^{\text{spin}} \mathbf{e}_{1(S_z''-m_{123})}(-1)^{J_{12}} \\
= \; & \frac{\sqrt{2}}{3} \sum_{J_{12},J_{56}} \sum_{i,j} \sum_{\alpha,\beta} \sum_{m_i,m_\alpha} \sum_{\mu} \left\langle \left(J_{56}\frac{1}{2}\right)_i, \left(\frac{1}{2}\frac{1}{2}\right)_j; TT_z \middle| \left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_1; TT_z \right\rangle \\
& \left\langle \left(J_{56}\frac{1}{2}\right)_\alpha, \left(\frac{1}{2}\frac{1}{2}\right)_\beta; SS_z \middle| \left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_0; SS_z \right\rangle \\
& \langle im_i, j(T_z - m_i)|TT_z\rangle \langle \alpha m_\alpha, \beta(S_z - m_\alpha)|SS_z\rangle \\
& \left\langle \left(\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right)_{J_{56}} \otimes \frac{1}{2}^{(9)} \right|_{im_i} \left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_{J_{12}} \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{1}{2},T_z''} \\
& \left\langle \left(\frac{1}{2}^{(7)} \otimes \frac{1}{2}^{(8)}\right) \right|_{j,(T_z-m_i)} \mathbf{1}_{78}^{F} \left| 0,0 \right\rangle_{0,0} \\
& \left\langle \left(\frac{1}{2}^{(5)} \otimes \frac{1}{2}^{(6)}\right)_{J_{56}} \otimes \frac{1}{2}^{(9)} \right|_{\alpha,m_\alpha} \left| \left(\frac{1}{2}^{(1)} \otimes \frac{1}{2}^{(2)}\right)_{J_{12}} \otimes \frac{1}{2}^{(3)} \right\rangle_{\frac{1}{2},m_{123}} \\
& (-1)^{\mu+1+J_{12}} \left\langle \left(\frac{1}{2}^{(7)} \otimes \frac{1}{2}^{(8)}\right) \right|_{\beta,(S_z-m_\alpha)} \sigma_{78}^{\mu} \left| 0,0 \right\rangle \langle|\varepsilon_{-\mu}|\mathbf{e}_{1(S_z''-m_{123})}\rangle.
\end{aligned}
$$

By using the operations in eq.(3.26) and eq.(3.27), we get

$$
\begin{aligned}
&\langle \Psi_{N\pi}^{\text{spin}-\text{flavor}-\text{color}} | \mathcal{O}_{\text{spin}-\text{flavor}} \mathcal{O}_{\text{color}} |^2 N_g'' \rangle \\
&= \frac{\sqrt{2}}{3} \sum_{J_{12},J_{56}} \sum_{i,j} \sum_{\alpha,\beta} \sum_{m_i,m_\alpha} \sum_{\mu} \left\langle \left(J_{56}\frac{1}{2}\right)_i, \left(\frac{1}{2}\frac{1}{2}\right)_j; TT_z \middle| \left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_1; TT_z \right\rangle \\
&\quad \left\langle \left(J_{56}\frac{1}{2}\right)_\alpha, \left(\frac{1}{2}\frac{1}{2}\right)_\beta; SS_z \middle| \left(J_{56}\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_0; SS_z \right\rangle \\
&\quad \langle im_i, j(T_z - m_i) | TT_z \rangle \, \langle \alpha m_\alpha, \beta(S_z - m_\alpha) | SS_z \rangle \\
&\quad \delta_{J_{56},J_{12}} \delta_{i,\frac{1}{2}} \delta_{m_i,T_z''} \sqrt{2} \delta_{j,0} \delta_{T_z - m_i,0} \delta_{\alpha,\frac{1}{2}} \delta_{m_\alpha,m_{123}} \\
&\quad (-1)^{\mu+1+J_{12}} \sqrt{2} (-1)^{(S_z - m_\alpha)} \delta_{\beta,1} \delta_{(S_z - m_\alpha),-\mu} \delta_{(S_z'' - m_{123}),-\mu} \\
&= -\frac{2\sqrt{2}}{3} \Bigg[\left\langle \left(0\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_0; TT_z \middle| \left(0\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_1; TT_z \right\rangle \\
&\quad \left\langle \left(0\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_1; SS_z \middle| \left(0\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_0; SS_z \right\rangle \\
&\quad \left\langle \frac{1}{2} T_z'', 00 \middle| TT_z \right\rangle \left\langle \frac{1}{2} m_{123}, 1(S_z - m_{123}) \middle| SS_z \right\rangle \delta_{T_z,T_z''} \delta_{S_z,S_z''} \\
&\quad + \left\langle \left(1\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_0; TT_z \middle| \left(1\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_1; TT_z \right\rangle \\
&\quad \left\langle \left(1\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_1; SS_z \middle| \left(1\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_0; SS_z \right\rangle \\
&\quad \left\langle \frac{1}{2} T_z'', 00 \middle| TT_z \right\rangle \left\langle \frac{1}{2} m_{123}, 1(S_z - m_{123}) \middle| SS_z \right\rangle \delta_{T_z,T_z''} \delta_{S_z,S_z''} \Bigg].
\end{aligned}
\tag{4.39}
$$

4.2 Decay Width of N*(1440) → Nπ

We are now ready to evaluate the decay width of the reaction $N^*(1440) \rightarrow N\pi$. With the the transition amplitude of the reaction is expressed in the partial wave expansion

$$
T = \sum_{lm_l} T_{lm_l} Y_{lm_l}^*, \tag{4.40}
$$

the partial wave amplitudes T_{lm_l} are derived in the previous sections as

$$\begin{aligned}
T_{lm_l} &= g \sum_{m_1,m_{123}} P_{lm_l}\Bigg[\frac{A}{2}C\left(\frac{1}{2}\frac{1}{2}1; S''_z m_{123}(S''_z - m_{123})\right) \\
&\quad C\left(111; (S''_z - m_{123})m_1(S''_z - m_{123} - m_1)\right) Q_1 \\
&\quad + \frac{B}{\sqrt{2}}C\left(\frac{1}{2}\frac{3}{2}1; S''_z m_{123}(S''_z - m_{123})\right) \\
&\quad C\left(111; (S''_z - m_{123})m_1(S''_z - m_{123} - m_1)\right) Q_2\Bigg] \\
&= g\delta_{S_z,S''_z}\delta_{T_z,T''_z}P_{1m_l}\left[\frac{A}{2}Q'_1 + \frac{B}{\sqrt{2}}Q'_2\right], \qquad (4.41)
\end{aligned}$$

with

$$P_{1m_l} = (-1)^{m_l}\beta p \exp(-\alpha p^2), \qquad (4.42)$$

$$\begin{aligned}
Q'_1 &= -\frac{2\sqrt{2}}{3}\sum_{m_{123}}\Bigg[\left\langle\left(0\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_0; TT_z\middle|\left(0\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_1; TT_z\right\rangle \\
&\quad \left\langle\left(0\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_1; SS_z\middle|\left(0\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_0; SS_z\right\rangle \\
&\quad \left\langle\frac{1}{2}T''_z, 00\middle|TT_z\right\rangle\left\langle\frac{1}{2}m_{123}, 1(S_z - m_{123})\middle|SS_z\right\rangle \\
&\quad + \left\langle\left(1\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_0; TT_z\middle|\left(1\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_1; TT_z\right\rangle \\
&\quad \left\langle\left(1\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_1; SS_z\middle|\left(1\frac{1}{2}\right)_{\frac{1}{2}},\left(\frac{1}{2}\frac{1}{2}\right)_0; SS_z\right\rangle \\
&\quad \left\langle\frac{1}{2}T''_z, 00\middle|TT_z\right\rangle\left\langle\frac{1}{2}m_{123}, 1(S_z - m_{123})\middle|SS_z\right\rangle\Bigg] \\
&\quad C\left(\frac{1}{2}\frac{1}{2}1; S''_z m_{123}(S''_z - m_{123})\right) \\
&\quad C\left(111; (S''_z - m_{123})(-m_l)(S''_z - m_{123} + m_l)\right), \qquad (4.43)
\end{aligned}$$

$$\begin{aligned} Q_2' &= -\frac{2\sqrt{2}}{3}\sum_{m_{123}} \left\langle \left(1\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_0; TT_z \middle| \left(1\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_1; TT_z \right\rangle \\ &\quad \left\langle \left(1\frac{1}{2}\right)_{\frac{3}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_1; SS_z \middle| \left(1\frac{1}{2}\right)_{\frac{1}{2}}, \left(\frac{1}{2}\frac{1}{2}\right)_0; SS_z \right\rangle \\ &\quad \left\langle \frac{1}{2}T_z'', 00 \middle| TT_z \right\rangle \left\langle \frac{3}{2}m_{123}, 1(S_z - m_{123}) \middle| SS_z \right\rangle \\ &\quad C\left(\frac{1}{2}\frac{3}{2}1; S_z'' m_{123}(S_z'' - m_{123})\right) \\ &\quad C\left(111; (S_z'' - m_{123})(-m_l)(S_z'' - m_{123} + m_l)\right), \end{aligned} \tag{4.44}$$

where

$$\beta = \frac{\pi^{9/2}}{2\sqrt{2}} N_{N^*} N_N N_\pi |C|^3 \beta_4. \tag{4.45}$$

Listed below are the values of Q_1' and Q_2' for the different initial and final states with ($T = 1/2$, $T_z = \pm 1/2$) and ($S = 1/2$, $S_z = \pm 1/2$):

Table 4.1: Values of Q_1' and Q_2' for different initial and final states.

m_l	T_z	S_z	Q_1'	Q_2'
-1	1/2	1/2	-0.064	0.091
	1/2	-1/2	-0.032	0.091
	-1/2	1/2	-0.064	0.091
	-1/2	-1/2	-0.032	0.091
0	1/2	1/2	0.032	0
	1/2	-1/2	-0.032	0
	-1/2	1/2	0.032	0
	-1/2	-1/2	-0.032	0
1	1/2	1/2	0.032	-0.091
	1/2	-1/2	0.064	-0.091
	-1/2	1/2	0.032	-0.091
	-1/2	-1/2	0.064	-0.091

The decay width of the process $N^*(1440) \rightarrow N\pi$ can be evaluated in the formula

$$\Gamma_{N^*(1440)\to N\pi} = 2\pi \frac{E_N E_\pi p}{M_{N^*}} \int d\Omega \, |T_{N^*(1440)\to N\pi}|^2, \tag{4.46}$$

where $T_{N^*(1440)\to N\pi}$ is the total transition amplitude and p the magnitude of the final momentum of π or N. Integrating over the solid angle Ω of the final particle N or π and averaging over the initial states, one derives the unpolarized decay width in the partial wave transition amplitudes

$$\begin{aligned}
\Gamma_{N^*(1440)\to N\pi} &= 2\pi \frac{E_N E_\pi p}{M_{N^*}} \frac{1}{2} \sum_{S_z} \sum_{l m_l} |\, T_{l m_l}|^2 \\
&= 2\pi \frac{E_N E_\pi p}{M_{N^*}} \frac{1}{2} \sum_{S_z} \sum_{m_l} |\, T_{1 m_l}|^2 \\
&= 2\pi \frac{E_N E_\pi}{M_{N^*}} g^2 \beta^2 p^3 \exp\left(-2\alpha p^2\right) \frac{1}{2} \sum_{S_z} \sum_{m_l} \left| (-1)^{m_l} \left(\frac{A}{2} Q'_1 + \frac{B}{\sqrt{2}} Q'_2 \right) \right|^2,
\end{aligned} \tag{4.47}$$

where the factor $\frac{1}{2}$ comes from the average over the initial states. It is found that the Roper resonance decays into $N\pi$ through only the $l = 1$ channel in the hybrid picture.

The ratio of the decay widths for the difference values of the free parameters A and B is derived as follows:

$$\begin{aligned}
&\Gamma(A = 1, B = 0) : \Gamma(A = 0, B = 1) : \Gamma(A/B = 1) : \Gamma(A/B = -1) \\
&\qquad = 0.307 : 1.656 : 0.388 : 1.574.
\end{aligned}$$

Note that it is independent of the effective coupling constant g and the length parameters a, b and c.

It is too early to seriously compare the theoretical predictions with the experimental data because there are more than one free parameters needed to be nailed down. The model has the effective coupling constant g of the quark-gluon interaction and the mixture parameter A or B free, even if the length parameters a, b and c could be borrowed from other works in low-energy quark models.

On this stage, however, we may estimate the strength of the effective coupling constant g by using the parameters employed in other works. Here we would take $a = 3.1\ \mathrm{GeV}^{-1}$ and $b = 4.1\ \mathrm{GeV}^{-1}$ as determined in Maruyama et al. (1987) and Gutsche et al. (1989), $A = -B = 1/\sqrt{2}$ as used in Li et al. (1992) and Li (1991). As for the length parameter c, we just let it equal to the parameter a since we have nowhere to refer to. Using the nucleon, pion and the Roper resonance masses respectively as $M_{N^*} = 1440$ MeV, $M_N = 939$ MeV and $M_\pi = 135$ MeV, we obtain

$$p = \sqrt{\left(\frac{M_{N^*}^2 + M_N^2 - {M_\pi}^2}{2M_{N^*}}\right)^2 - {M_N}^2} = 397.9\ \ \mathrm{MeV}, \quad E_N = 1019.83\ \ \mathrm{MeV},$$

$$E_\pi = 420.17\ \ \mathrm{MeV}, \quad \beta = 0.545 \times 10^{-3}\ \ \mathrm{MeV}^{-1}, \quad \text{and} \quad \alpha = 3.67 \times 10^{-6}\ \ \mathrm{MeV}^{-2}.$$

Putting all the pieces above together, we get

$$\Gamma_{N^*(1440)\to N\pi} = g^2[\mathrm{MeV}]^{-1} \cdot 85.91[\mathrm{MeV}]^2. \tag{4.48}$$

By comparing with the experimental data $\Gamma_{N^*(1440)\to N\pi} \approx 200$ MeV, we get

$$g^2/4\pi \approx 0.18 \text{ MeV}^{-1}. \tag{4.49}$$

To compare with the electromagnetic coupling constant $\alpha = 1/137$ and the effective strength $GM_N^2 = 1.01 \times 10^{-5}$ of the weak interaction and the coupling constants of the nucleon-nucleon-meson systems in models of hadron level, we may multiply the effective coupling constant g^2 by a typical mass scale ($m_q \sim 100$ MeV) in nonrelativistic quark models to derive the dimensionless effective coupling constant in our model

$$\frac{g^2}{4\pi} m_q \approx 18. \tag{4.50}$$

The value of the effective coupling constant is reasonable considering the coupling constants $g^2_{NN\pi}/(4\pi) \approx 14$ and $g^2_{NN\omega}/(4\pi) \approx 10 \sim 20$ in models of hadron level.

Chapter V

Discussions and Conclusions

In the thesis we have evaluated the transition amplitude of the decay process of the Roper resonance to $N\pi$ in a nonrelativistic quark-gluon model. The Roper resonance is treated as a hybrid, that is, composed of three valence quarks and a gluon of the TE (transverse electric) mode (a TE mode is the lowest eigenmode). The wave function of the Roper resonance has been constructed to properly establish the gluonic degree of freedom, which has been a fascinating challenge in nowadays non-perturbative QCD physics.

We have estimated the effective coupling constant g^2 in the best knowledge of the mixture parameters A and B and the length parameters a and b, by comparing our theoretical prediction for the decay width of the process $N^*(1440) \to N\pi$ with the experimental data. It is found that the coupling constant is very reasonable. We may conclude at this stage that the model is promising.

The work here is not sufficient to judge if the Roper resonance could be reasonably interpreted as a bound state of three valence quarks and a TE gluon since only the $N\pi$ channel has been studied. However, the thesis work has paved the way for the whole project to study all the decay channels of the Roper resonance such as to $N\pi\pi$, $N\rho$, $\Delta\pi$ and $N\gamma$. The method developed here can be applied, without modification, to other decay channels. Once all those decay channels are evaluated, one could adjust all the free parameters (the effective coupling constant g of the quark-gluon interaction and the A or B which gives information of the combination of the $|^2N_g\rangle$ and $|^4N_g\rangle$ parts of the Roper resonance) to experimental data and find out if the theoretical predictions are consistent with data.

It has been indicated in the study of photoproduction of baryons (Li, 1991) that not only the Roper resonance but also other lower-lying baryons like N and Δ may have a component of gluon. To confirm or rule out the argument, a systematical study for the strong process of those baryons are essential. The presence of the gluonic degrees of freedom may solve the long-standing puzzle of the Roper resonance, and hopefully provide an explanation of the observation that the spin content of nucleon is not carried dominantly by valence quarks. It might be argued that nucleon may also include glunic degrees of freedom since q^3 and q^3G states could be strongly mixed in physical baryon resonances because of the quark-gluon coupling. The gluonic components of nucleon do not change the isospin and flavor structure, and therefore the ratio of the magnetic moments will be the same as in the conventional q^3 picture, namely, $\mu_p/\mu_n = -3/2$.

References

References

Barnes, T. and Close, F. E. (1983). Photoproduction of hermaphrodite baryons. **Phys. Lett. B** 128: 277–280.

Burkert, V. (1994). **Perspectives in the structure of hadronic systems**. New York: Plenum Press.

Capstick, S. and Page, P. R. (1999). Constructing hybrid baryons with flux tubes. **Phys. Rev. D** 60: R111501–R111504.

Chow, C. K., Pirjol, D., and Yan, T. M. (1999). Hybrid baryons in large $N_c QCD$. **Phys. Rev. D** 59: 056002.

Close, F. E. (1981). **An introduction to quarks and partons**. Academic Press,Inc.

Cutkosky, R. E., Forsyth, C. P., Hendrick, R. E., and Kelly, R. L. (1979). Pion-nucleon partial-wave amplitudes. **Phys. Rev. D** 20(11): 2839–2853.

Cutkosky, R. E. and Wang, S. (1990). Poles of πN P_{11} partial-wave amplitude. **Phys. Rev. D** 42(1): 235–237.

DeGrand, T. A. and Rebbi, C. (1978). Radial excitation of hadronic bags. **Phys. Rev. D** 17: 2358–2363.

Ericson, T. and Weise, W. (1988). **Pions and Nuclei**. Oxford: Clarendon Press.

Golowich, E., Hagg, E., and Karl, G. (1983). Are there baryons which contains constituent gluons? **Phys. Rev. D** 28: 160–169.

Groom, D. E. (2000). Particle data group. **Eur. Phys. Jour. C** 15(1): 700–702.

Gutsche, T., Maruyama, M., and Faessler, A. (1989). Proton-antiproton annihilation into three mesons in flight in the $^3P_0$ model with a planar quark diagram topology. **Nucl. Phys. A** 503: 737–765.

Høgaasen, H. and Richard, J. M. (1983). Nucleon resonances and the quark model. **Phys. Lett. B** 124: 520–522.

Isgur, N. and Karl, G. (1978). P-wave baryons in the quark model. **Phys. Rev. D** 18(11): 4187.

Kisslinger, L. S. (1998). Gluonic hadrons. **Nucl. Phys. A** 629: 30–36.

Kisslinger, L. S. and Li, Z. (1995). Hyrid baryons via QCD sum rules. **Phys. Rev. D** 51: R5986–R5989.

Kisslinger, L. S. and Li, Z. (1999). Sigma signal for hybrid baryon decay. **Phys. Lett. B** 445: 271–273.

Li, Z. (1991). Photoproduction signatures of hybrid baryon:an application of the quark model with gluoinc degrees of freedom. **Phys. Rev. D** 44(9): 2841–2850.

Li, Z., Burkert, V., and Li, Z. (1992). Electroproduction of the roper resonance as a hybrid state. **Phys. Rev. D** 46(1): 70–74.

Liu, K. F. and Wong, C. W. (1983). Cluster model of baryons. **Phys. Rev. D** 28(1): 170–180.

Maruyama, M., Furu, S., and Faessler, A. (1987). $p\bar{p}$ annihilation into three mesons in the $^3P_0$ model. **Nucl. Phys. A** 473: 649–666.

Moorhouse, R. G. and Roper, L. D. (1974). The development of pion-nucleon scattering analysis [On-line]. Available:

http://arts.bev.net/roperldavid/roperres.htm.

Page, P. R. (1999). (Hybrid) Baryons: symmetries and masses [On-line]. Available: http://xxx.lanl.gov/ps/nucl-th/9904044.html.

Page, P. R. (2000). Hybrid baryon signatures [On-line]. Avaliable: http://xxx.lanl.gov/ps/nucl-th/0004053.html.

Perkins, D. H. (1986). **Introduction to high energy physics**. Addison-Wesley Publishing Company,Inc.

Schütz, C., Haidenbauer, J., Speth, J., and Durso, J. W. (1998). Extended coupled channels model for πN scattering and the structure of $N^*(1440)$ and $N^*(1535)$. **Phys. Rev. C** 57(3): 1464–1477.

Yan, Y. (1994). $\bar{N}N$ **Bound states in a microscopic quark model**. Ph.D. Thesis, Universit$\ddot{a}$t T$\ddot{u}$bingen.

Appendices

Appendix A

Baryons Wave function

The quark model description of baryons is more complicated than for meson. Since quarks have baryon number $B = \frac{1}{3}$ the simplest way to construct baryons from the basic quark triplet is to form qqq states. The quark content of these states are unambiguous but in order to explain the observed baryon spectrum we need to consider the symmetry of the quark wave functions. The overall wave functions

$$\Psi = \Psi(\text{spatial})\phi(\text{flavor})\chi(\text{spin})\psi(\text{color})$$

must be antisymmetric. Each quark flavor comes in three colors, red, green and blue (rgb), which form a fundamental triplet of the SU(3) color group, $\mathrm{SU}(3)_c$, which, unlike SU(3) flavor symmetry, is assumed to be exact. The $\mathrm{SU}(3)_c$ singlet wave function for baryons

$$\psi = \frac{1}{\sqrt{6}}\left(rgb + gbr + brg - grb - bgr - rbg\right) \tag{A.1}$$

is antisymmetric in the exchange of any two quark colors. Its inclusion in the overall wave function Ψ guarantees antisymmetry provided $\Psi(\text{spatial})\phi(\text{flavor})\chi(\text{spin})$ is symmetric.

Let us focus on the lowest-lying baryon multiplets, the $J^P = \frac{1}{2}^+$ octet and

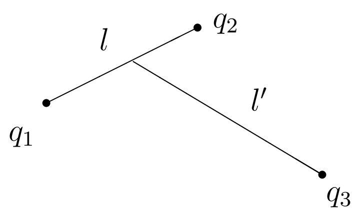


Figure A.1: Relative orbital angular momenta in a three-quark system

the $\frac{3}{2}^+$ decuplet. The relative orbital angular momenta l and l' in these three-quark state (Figure.A.1) are assumed to be zero and therefore Ψ(spatial) is symmetric.

Appendix B

Theoretical Expectation for Hybrid Baryons

B.1 General Expectation

Augmenting the quark q and antiquark $\bar{q}$ by gluons g leads to additional states in the spectrum relative to the expectations of the naive $q\bar{q}$ and qqq quark model. Physically allowed (color singlet) states in the baryon spectrum may be constructed from $|qqqg\rangle$ "hybrid" basis states, in addition to the familiar $|qqq\rangle$ quark model states:

$$|qqq\rangle\Big|_{\text{color}} = \mathbf{1} \otimes \mathbf{8} \otimes \mathbf{8} \otimes \mathbf{10} \tag{B.1}$$

$$|qqqg\rangle\Big|_{\text{color}} = (\mathbf{1} \otimes \mathbf{8} \otimes \mathbf{8} \otimes \mathbf{10}) \otimes \mathbf{8} = \mathbf{1^2} \otimes \mathbf{8^5} \otimes \ldots \tag{B.2}$$

The lowest hybrid baryon basis state is color octet and spatially symmetric in the qqq part of $|qqqg\rangle$, making it a **70** of SU(6). Since this qqq subsystem is combined with the angular momentum of the gluon, we predict that $|qqqg\rangle$ multiplets do not span the same $|\text{flavor}, \text{J}_{\text{tot}}\rangle$ states as an SU(6) $|qqq\rangle$ multiplet. Thus we should find evidence for "incomplete" or "overcomplete" SU(6) baryon multiplets due to the presence of hybrids. More detailed predictions for the multiplet content typically require the application of a specific model, although there has recently been work on the derivation of general properties of hybrid baryon states and their decays in the large quark mass and large N_c limit (Chow, Pirjol, and Yan, 1999).

B.2 Bag Model

This model places relativistic quarks and gluon in a spherical cavity and allows them to interact through QCD forces such as one gluon exchange (OGE), the color Compton effect and so forth.

The calculation for light bag model hybrids of Barnes and Close (Barnes and Close, 1983) was the first published. This reference found the spectrum shown in Fig.B.1 ; in order of increasing mass the states are

$$(1/2^+N)^2; (3/2^+N)^2; (1/2^+\Delta); (3/2^+\Delta); (5/2^+N).$$

Note that the lightest hybrid baryon is predicted to be an extra $1/2^+N^*$ P_{11} state at about 1.6 GeV, which might possibly be identified with the Roper resonance. A subsequent calculation by Golowich, Hagg and Karl (Golowich, Hagg, and Karl, 1983) basically confirmed these results, but used a parameter set that gave a mass of about 1.5 GeV. for this lightest hybrid, so identification with the Roper was given more support.

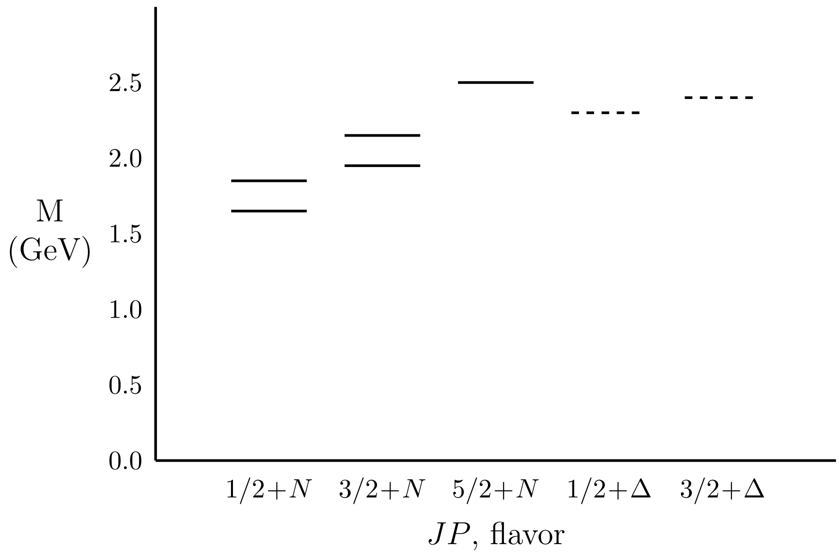


Figure B.1: The spectrum of light nonstrange hybrid baryons found by Barnes and Close (Barnes and Close, 1983) in the bag model

B.3 QCD Sum Rules

This approach, which has been applied to hybrid baryons is several papers of the previous decade, finds the masses and other parameters of the lowest-lying states in terms of numerically known vacuum expectation values (VEVs), called "condensates". Since the sum rules related known VEVs to a sum of resonance contributions, there are systematic uncertainties in separating the individual resonance and "continuum" parts. Identification of excited states such as hybrids is rather difficult in this approach, since higher-mass contributions to the sum rules are suppressed exponentially. This exercise can be carried out for hybrids, for example by calculating matrix elements of several operators and diagonalizing the result. In practice the calculations also use $qqqg$ operators, which one would expect to have larger hybrid couplings. Up to now only hybrids in the nucleon/Roper sector $1/2^+N$ have been studied by using QCD sum rules.

The first published hybrid baryon QCD sum rule calculations was in the Soviet journal of nuclear physics by A.P. Martynenko. He estimated the lightest $1/2^+N$ hybrid baryon mass to be near 2.1 GeV. A subsequent study by Kisslinger and Li (Kisslinger and Li, 1995) report algebra errors in matrix elements calculated in the Martynenko paper, and published a revised mass estimate of about 1.5 GeV, again vaery suggestive of the Roper. A more recent review by Kisslinger (Kisslinger, 1998) concluded that the Roper is largely a hybrid (meaning dominantly $|qqqg\rangle$), the nucleon has little evidence for a hybrid component, and also considers how one might calculate strong couplings. Some of this program of decay calculations was carried out by Kisslinger and Li (Kisslinger and Li, 1999), who conclude that the lightest hybrid should have a rather small branching fraction ratio $N(\pi\pi)s/N\pi$, consistent with observation for the Roper.

B.4 Flux Tube Model

The flux tube model assumes that glue forms a dynamically excitable tube between quarks and antiquarks, and that the lightest hybrids are states in which this flux tube is spatially excited. The determination of excited states in the baryon sector ($qqq+$ spatially excited flux tube) is a rather complicated problem which has only recently been treated. Flux tube model predictions for the lightest hybrid baryons were report by Capstick and Page in 1999 (Capstick and Page, 1999). They find that the lightest hybrid baryons is the nnn flavor sector are two each of $1/2^+N$ and $3/2^+N$, all at a mass of 1.87 GeV.

Thus the lowest flux-tube hybrid baryon level is predicted to include Roper quantum numbers, as was found in the bag model, though twofold degenerate and at a higher mass. In addition a degenerate $3/2^+N$ pair is expected. There are other differences in the multiplet content; the flux-tube hybrid baryon multiplet contains the states

$$(1/2^+N)^2\ ;\ (3/2^+N)^2\ ;\ (1/2^+\Delta)\ ;\ (3/2^+\Delta)\ ;\ (5/2^+\Delta),$$

so the flux tube model finds a high-mass $5/2^+\Delta$; in the bag model the high-mass state was a $5/2^+N$. The flux-tube Δ hybrids are predicted to be degenerate, with a mass of 2.09 GeV.

The implications of this work for searches for hybrid baryons, including various experimental search strategies such as overpopulation, strong decays, EM coupling and production amplitudes, were recently reviewed by Page (Page, 1999, 2000). In particular Page suggests seraches for hybrids in the final states $N\eta$, $N\rho$ and $N\omega$.

Appendix C

The Dirac Equation

The free spin-$\frac{1}{2}$ particle satisfies the free Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \tag{C.1}$$

where we have introduced four Dirac γ-matrices

$$\gamma^\mu \equiv (\gamma^0, \vec{\gamma}). \tag{C.2}$$

μ is running from 1 to 4 and the γ^μ are the 4×4 matrices with the properties

$$(\gamma^0)^2 = 1$$

$$(\gamma^i)^2 = -1 \text{ where } i = 1, 2, 3$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 0 \text{ for } \mu \neq \nu$$

We can summarize the above set of equations to be

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{C.3}$$

where $g^{\mu\nu}$ is the element of the Minkowski matric g defined as

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{C.4}$$

The explicit representation of the γ^μ matrices is not unique. In this work we use one of the two most popular forms listed, see below:

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{C.5}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{C.6}$$

where σ^i are the Pauli matrices taking the form

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{C.7}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{C.8}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.9}$$

Here below are some useful properties of the γ^μ

$$\gamma^{0\dagger} = \gamma^0, \tag{C.10}$$

$$\gamma^{i\dagger} = -\gamma^i; \;\; i = 1, 2, 3. \tag{C.11}$$

From the γ^μ matrices, we may define

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{C.12}$$

which has the explicit form

$$\gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{C.13}$$

The solutions of the Dirac equation take the general form

$$\psi = Au(p)e^{-ip\cdot x} + Bv(p)e^{ip\cdot x}, \tag{C.14}$$

with $u(p)$ and $v(p)$, the Dirac spinors, satisfy the equations

$$\begin{aligned} (\gamma^\mu p_\mu - m)u(p) = 0, \\ (\gamma^\mu p_\mu + m)v(p) = 0, \end{aligned} \tag{C.15}$$

which are indeed the Dirac equation in momentum space. By solving the above equations, one may derive the explicit form of the Dirac spinors as follows:

$$u_i(\vec{p}_i, \vec{\sigma}_i) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \chi_i \\ \frac{\vec{\sigma}_i \cdot \vec{p}_i}{E_i + m} \chi_i \end{pmatrix}, \tag{C.16}$$

and

$$v_i(\vec{p}_i, \vec{\sigma}_i) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \chi_i \\ \frac{\vec{\sigma}_i \cdot \vec{p}_i}{E_i + m} \chi_i \end{pmatrix}, \tag{C.17}$$

where $E = \sqrt{\vec{p}^{\,2} + m^2}$, and χ_i are 2-spinors with $i = 1, 2$. Usually, χ_i are chosen to be the eigenfunctions of the σ^3 or the helicity operators $h = \vec{p} \cdot \vec{\sigma} / |\vec{p}|$. The normalization constant $\sqrt{\frac{E+m}{2m}}$ is chosen so that the Dirac spinors are normalized according to

$$\bar{u}_i(p) u_j(p) = \delta_{ij}, \tag{C.18}$$

$$\bar{v}_i(p) v_j(p) = -\delta_{ij}, \tag{C.19}$$

where

$$\bar{u} \equiv u^\dagger \gamma^0, \tag{C.20}$$

$$\bar{v} \equiv v^\dagger \gamma^0. \tag{C.21}$$

The completeness relations for the Dirac spinors are

$$\sum_{i=1,2} u_i(p)\bar{u}_i(p) = \frac{\gamma \cdot p + m}{2m}, \tag{C.22}$$

$$\sum_{i=1,2} v_i(p)\bar{v}_i(p) = \frac{\gamma \cdot p - m}{2m}. \tag{C.23}$$

Appendix D

The Levi-Civita Tensor

The Levi-Civita symbol ϵ_{ijk} is a tensor of rank three and is defined by

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any two labels are the same} \\ 1, & \text{if } i, j, k \text{ is an even permutation of 1,2,3} \\ -1, & \text{if } i, j, k \text{ is an odd permutation of 1,2,3} \end{cases} \tag{D.1}$$

The Levi-Civita symbol ϵ_{ijk} is antisymmetric on each pair of indices.

The determinant of a matrix $\mathbf{A}$ with element a_{ij} can be written in term of ϵ_{ijk} as

$$\det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} a_{1i} a_{2j} a_{3k} = \epsilon_{ijk} a_{1i} a_{2j} a_{3k}. \tag{D.2}$$

Note the compact notation where the summation over the spatial direction is dropped. It is this one that is in use.

Note that the Levi-Civita symbol can therefore be expressed as the determinant, or mixed triple product, of any of the unit vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ of a normalized and orthogonal frame of reference.

$$\epsilon_{ijk} \;=\; \det(\hat{e}_i, \hat{e}_j, \hat{e}_k) \;=\; \hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k). \tag{D.3}$$

Now we can define by analogy to the definition of the determinant an additional type of product, the vector product or simply cross product

$$\vec{a} \times \vec{b} = \det \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \epsilon_{ijk} \hat{e}_i a_j b_k, \tag{D.4}$$

or for each coordinate

$$(\vec{a} \times \vec{b})_i = \epsilon_{ijk} a_j b_k. \tag{D.5}$$

The product of two Levi-Civita symbols can be expressed as a function of the Kronecker's symbol δ_{ij}

$$\begin{aligned} \epsilon_{ijk}\epsilon_{lmn} \;=\; & +\delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} \\ & -\delta_{im}\delta_{jl}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl}. \end{aligned} \tag{D.6}$$

Appendix E

Linear Transformation

In this appendix we show the calculation of $|C|$ in eq.(4.18) which is the determinant of the linear transformation matrix C, which is

$$\begin{pmatrix} \vec{Q}_1 \\ \vec{Q}_2 \\ \vec{Q}_3 \end{pmatrix} = C \begin{pmatrix} \vec{p}_4 \\ \vec{p}_5 \\ \vec{p}_6 \\ \vec{p} \end{pmatrix}, \tag{E.1}$$

with

$$C = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}. \tag{E.2}$$

Note that there is no cross term in the new expression of the P_{lm_l}.

$$\begin{aligned} \vec{p}_4 &= \gamma_{11}\vec{Q}_1 + \gamma_{12}\vec{Q}_2 + \gamma_{13}\vec{Q}_3 + \gamma_{14}\vec{p}, \\ \vec{p}_5 &= \gamma_{21}\vec{Q}_1 + \gamma_{22}\vec{Q}_2 + \gamma_{23}\vec{Q}_3 + \gamma_{24}\vec{p}, \\ \vec{p}_6 &= \gamma_{31}\vec{Q}_1 + \gamma_{32}\vec{Q}_2 + \gamma_{33}\vec{Q}_3 + \gamma_{34}\vec{p}, \end{aligned}$$

we can write those equations in matrix form

$$\begin{pmatrix} \vec{p}_4 \\ \vec{p}_5 \\ \vec{p}_6 \end{pmatrix} = \gamma \begin{pmatrix} \vec{Q}_1 \\ \vec{Q}_2 \\ \vec{Q}_3 \end{pmatrix}, \tag{E.3}$$

where

$$\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}. \tag{E.4}$$

But in our case use the transformation as eq.(4.16) and can be written in matrix form as

$$\begin{pmatrix} \vec{Q}_1 \\ \vec{Q}_2 \\ \vec{Q}_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} \vec{p}_4 \\ \vec{p}_5 \\ \vec{p}_6 \end{pmatrix}. \tag{E.5}$$

Thus,

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \gamma^{-1}. \tag{E.6}$$

From eq. (4.18)

$$C = \det \gamma = \frac{1}{\det \gamma^{-\mathbf{1}}}, \tag{E.7}$$

$$\det \gamma^{-\mathbf{1}} = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix} \tag{E.8}$$

$$= \left(\frac{3a^6 + 3a^2b^2c^2 + 9a^4c^2 + 5a^4b^2}{6} \right)^{\frac{1}{2}}. \tag{E.9}$$

Then,

$$C = \left(\frac{6}{3a^6 + 3a^2b^2c^2 + 9a^4c^2 + 5a^4b^2} \right)^{1/2} \tag{E.10}$$

Curriculum Vitae

Miss Nopmanee Supanam was born on January 3rd, 1979 in Bangkok. She graduated from Mahidol University, with a B.Sc. Degree in Physics in 1999. After that she joined the School of Physics, Institute of Science, Suranaree University of Technology for a Master's Degree.