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A Particular Class of Partially Invariant Solutions of the Navier–Stokes Equations

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Abstract. One class of partially invariant solutions of the Navier–Stokes equations is studied here. This class of solutions is constructed on the basis of the four-dimensional algebra L_4 with the generators

$$X_1 = \phi_1 \partial_x + \phi_1' \partial_u - x\phi_1'' \partial_p, \qquad X_2 = \phi_2 \partial_x + \phi_2' \partial_u - x\phi_2'' \partial_p,$$
$$Y_1 = \psi_1 \partial_y + \psi_1' \partial_v - y\psi_1'' \partial_p, \qquad Y_2 = \psi_2 \partial_y + \psi_2' \partial_v - y\psi_2'' \partial_p.$$

Systematic investigation of the case, where the Monge–Ampere equation (10) is hyperbolic ($Lf_z + k + l \geq 0$) is given. It is shown that this class of solutions is a particular case of the solutions with linear velocity profile with respect to one or two space variables.

Key words: Group classification, group stratification, invariant and partially invariant solutions, Navier–Stokes equations

1. Introduction

An unsteady motion of incompressible viscous fluid is governed by the Navier–Stokes equations

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \Delta \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0, \tag{1}$$

where $\mathbf{u} = (u, v, w)$ is the velocity field, p is the fluid pressure, ∇ is the gradient operator in the three-dimensional space $\mathbf{x} = (x, y, z)$ and Δ is the Laplacian. The Navier–Stokes equations contain complete information about the structure of flows under usual temperature and pressure. Despite progress in numerical methods and techniques, there is considerable interest in finding exact solutions of the Navier–Stokes equations. Each exact solution has value, first, as the exact description of the real process in the framework of a given model; secondly, as a model to compare various numerical methods; and thirdly, as theoretical tool to improve the models used.

One method of constructing exact solutions is group analysis [1]. A historical review of the development of group analysis can be found in [2]. Many results obtained by group analysis are collected in [3]. The method is based on symmetries of given equations. Note that many of invariant solutions of the Navier–Stokes equations have been known for a long time: these solutions were obtained by assuming a form of the representation of the solution. Group analysis gives a method for obtaining a representation of a solution. The first group classification of the Navier–Stokes equations in the three-dimensional case was done in [4]. The first classification of the two-dimensional Navier–Stokes equations was studied in [5]. It was shown that the Lie algebra admitted by the Navier–Stokes equations is infinite-dimensional

Classification of infinite-dimensional subalgebras of this algebra was studied in [6]. There is still no complete classification of subalgebras of this algebra. For each subalgebra of the admitted algebra one can try to find an invariant or partially invariant solution. Several papers [7–13] are devoted to invariant solutions of the Navier–Stokes equations. Short reviews devoted to invariant solutions of the Navier–Stokes equations can be found in [7, 14–16]. Another class of solutions proposed by group analysis is the class of partially invariant solutions [1, 17]. The theory of partially invariant solutions is still developing [18, 19]. While partially invariant solutions of the Navier–Stokes equations have been less studied [7], there has been substantial progress in studying such classes of solutions of the inviscid gas dynamics equations [1, 20–26].

It should be noted here that there are also other approaches for constructing exact solutions of the Navier–Stokes equations. We mention two of them: nonclassical symmetry reductions and direct methods [16, 27], and linear profile of velocity [28, 29].

This manuscript is devoted to the class of partially invariant solutions, which generalizes the class considered in [30].

2. One Class of Partially Invariant Solutions

The class of solutions studied in [30] is a class of partially invariant solutions with respect to the group H with the generators

$$X = \partial_x, \qquad Y = \partial_y, \qquad U = t\partial_x + \partial_u, \qquad V = t\partial_y + \partial_v.$$

There exist no invariant solutions that correspond to this group. In fact, the universal invariant of this group is t, z, w, p. Hence, the rank of the Jacobi matrix of the universal invariant with respect to the dependent variables q equals two. Therefore, $\delta \geq 2$ and one can only construct partially invariant solutions with respect to this group. According to the classification [18], a partially invariant solution with minimum defect $\delta = 2$ is a regular partially invariant solution of $H(2, 2)$. In this case a representation of the partially invariant solution is

$$w = 2f(z,t), \qquad p = h(z,t), \qquad u = u(x,y,z,t), \qquad v = v(x,y,z,t).$$

For the gas dynamics equations such a class of solutions was studied in [19]. V. V. Pukhnachov (oral communication) noted that for the Navier–Stokes equations this representation can be generalized by including two arbitrary functions $k = k(t)$ and $l = l(t)$:

$$w = 2f(z,t), \qquad p = h(z,t) - k(t)x^2 - l(t)y^2, \qquad u = u(x,y,z,t), \qquad v = v(x,y,z,t). \quad (2)$$

The arbitrariness of the functions $k(t)$ and $l(t)$ gives additional possibilities for satisfying boundary conditions. Representation (2) can also be explained from the group point of view. In fact, let us consider the four-dimensional group H^4, which is generated by the operators

$$X_1 = \phi_1\partial_x + \phi_1'\partial_u - x\phi_1''\partial_p, \qquad X_2 = \phi_2\partial_x + \phi_2'\partial_u - x\phi_2''\partial_p,$$

$$Y_1 = \psi_1\partial_y + \psi_1'\partial_v - y\psi_1''\partial_p, \qquad Y_2 = \psi_2\partial_y + \psi_2'\partial_v - y\psi_2''\partial_p.$$

Here the functions $\phi_i = \phi_i(t),\ \psi_i = \psi_i(t),\ (i = 1, 2)$ satisfy the natural conditions for the algebra to be a four-dimensional algebra:

$$\phi_1\phi_2' - \phi_1'\phi_2 \neq 0, \qquad \psi_1\psi_2' - \psi_1'\psi_2 \neq 0,$$
$$\phi_1\phi_2'' - \phi_1''\phi_2 = 0, \qquad \psi_1\psi_2'' - \psi_1''\psi_2 = 0.$$

A regular partially invariant solution with respect to the Lie group H^4 has representation (2), where $k = \phi_i''/(2\phi_i),\ l = \psi_i''/(2\psi_i)$. The solutions studied in [34] and one of the solutions in [27] are particular cases of (2).

3. Compatibility Conditions

As is well known, the main difficulty in the study of partially invariant solutions is compatibility analysis of reduced systems. The compatibility analysis can be reduced to a consecutive performance of algebraic operations of symbolic nature [35, 36]. These operations are related with prolongation of a system, substitution of composite expressions (transition onto manifold), and finding ranks of matrices. Typically, the compatibility study of systems of partial differential equations requires a large amount of analytical calculations, and it is necessary to use a computer system for these calculations. Here the system REDUCE [37] was used.

For the case $k = 0,\ l = 0$ the analysis of compatibility was done in [30]. As was mentioned, the arbitrariness of the functions $k(t)$ and $l(t)$ gives additional possibilities; however, compatibility analysis of the overdetermined system obtained after substituting representation (2) into the Navier–Stokes equations (1), becomes more difficult. Here the compatibility analysis of this overdetermined system is given.

Introducing the functions $\hat{u}(x, y, z, t),\ \hat{v}(x, y, z, t)$ by the formulae:

$$u = \hat{u} - x\frac{\partial f}{\partial z}, \qquad v = \hat{v} - y\frac{\partial f}{\partial z},$$

the second equation of (1) becomes

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0.$$

The general solution of the last equation can be given through the analog of the stream function $\psi = \psi(x, y, z, t)$:

$$\hat{u} = \frac{\partial \psi}{\partial y}, \qquad \hat{v} = -\frac{\partial \psi}{\partial x}.$$

The first two scalar equations of (1) take the form

$$\begin{aligned}
&\psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy} + 2f\psi_{yz} - x\left(f_{zt} + f_z\psi_{xy} + 2ff_{zz} - f_z^2\right) - yf_z\psi_{yy}\\
&\quad = \triangle\psi_y - xf_{zzz} + 2xk,\\
&-\psi_{xt} - \psi_y\psi_{xx} + \psi_x\psi_{xy} - 2f\psi_{xz} - y\left(f_{zt} - f_z\psi_{xy} + 2ff_{zz} - f_z^2\right) + xf_z\psi_{xx}\\
&\quad = -\triangle\psi_x - yf_{zzz} + 2yl.
\end{aligned} \tag{3}$$

The third equation serves for determining the function $h(z, t)$ (if the function $f(z, t)$ is known):

$$h_z + 2f_t - 2f_{zz} + 4ff_z = 0.$$

Compatibility conditions are derived with respect to the following equivalence transformations: representation (2) is invariant with respect to rotations in the (x, y)–plane and shifts in (x, y, z) and t.

3.1. Preliminary Analysis

Let us consider some solutions of (1), which we call simple.

The first solution is a solution of the form

$$\psi(x, y, z, t) = \frac{1}{2}(x^2\gamma(z, t) + y^2c(z, t)) + x\lambda(z, t) + yb(z, t) + xy\alpha(z, t). \tag{4}$$

This representation is a particular case of the solutions with linear profile of velocity[1]

$$u = x(\alpha - f_z) + yc + b, \qquad v = -x\gamma - y(\alpha + f_z) - \lambda.$$

Substituting the representation (4) into (3) and splitting with respect to x and y, one obtains the compatibility conditions:

$$\begin{aligned} &Lf_z + k + l = -c\gamma + \alpha^2, \qquad L\alpha = \alpha f_z + k - l, \\ &L\gamma = f_z\gamma, \qquad L\lambda = \lambda\alpha - b\gamma \qquad Lc = f_zc, \qquad Lb = \lambda c - \alpha b, \end{aligned} \tag{5}$$

where L is the linear operator

$$LF \equiv F_t + 2fF_z - F_{zz} - f_zF.$$

The second type of solutions has the representation

$$\psi(x, y, z, t) = x^2a(y, z, t) + xb(y, z, t) + g(y, z, t). \tag{6}$$

Because the Navier–Stokes equations are symmetric with respect to rotations, the case $\psi_{yyy} = 0$ is similar to the case $\psi_{xxx} = 0$. As in the previous case, after substituting the representation of the solution into (3) and splitting with respect to x, one obtains compatibility conditions. Two of these conditions are $a_y = 0,\ b_{yy} = 0$. Hence, the function $b(y, z, t)$ is linear with respect to y: $b(y, z, t) = y\alpha(z, t) + \lambda(z, t)$. If $a \neq 0$, then $g_{yyy} = 0$, but this case corresponds to (4), which was considered earlier. Hence, $a = 0$. The remaining compatibility conditions are

$$\begin{aligned} &Lf_z + k + l = \alpha^2, \qquad L\alpha = \alpha f_z + k - l, \\ &L\lambda = \alpha\lambda,\ L\varphi - \varphi_{yy} - (y(\alpha + f_z) + \lambda)\varphi_y + \alpha\varphi = 0, \end{aligned} \tag{7}$$

where $\varphi = g_y$. This solution has a linear profile of velocity with respect to x

$$u = x(\alpha - f_z) + \varphi, \qquad v = -y(\alpha + f_z) - \lambda.$$

[1] Solutions with linear profile of velocity with respect to one, two or three space variables were studied in [28, 29].

Let us consider the representation

$$\psi(x, y, z, t) = a(x, z, t) + b(y, z, t) + xy\alpha(z, t). \tag{8}$$

After substitution of this representation into the Navier–Stokes equations one obtains

$$a_{xxx}b_{yy} = 0, \qquad a_{xx}b_{yyy} = 0.$$

Without loss of generality, this case can be considered as a particular case of representation (6).

We exclude the above considered solutions from the further study of the compatibility conditions of system (3).

Remark. A solution of the form

$$\psi(x, y, z, t) = x^2\varphi(z, t) + x\lambda(z, t) + y^2c(z, t) + yb(z, t) + xy\alpha(z, t) + Q(x + yq(z, t), z, t) \tag{9}$$

is a particular case of (4) if the function $Q = Q(\xi, z, t)$ is a quadratic function with respect to the first argument. This case corresponds to a linear profile of velocity, which was studied before. If $Q_{\xi\xi\xi} \neq 0$, then the compatibility conditions require that q is a constant. By rotating in the (x, y)–plane this case can be transformed to (6).

3.2. Monge–Ampere Equation

Adding the first equation of (3) differentiated with respect to x to the second equation differentiated with respect to y, one obtains

$$\psi_{xy}^2 - \psi_{xx}\psi_{yy} = Lf_z + k + l. \tag{10}$$

The right side of this equation only depends on z and t; therefore it can be regarded as the Monge–Ampere equation with a constant (depending on the parameters z and t) right side. A method for solving the Monge–Ampere equation depends on the sign of the right side.

The next theorem is one of the main results of this paper.

Theorem. *Any solution of system (3) satisfies the Monge–Ampere equation (10). If the right side of the Monge–Ampere equation is non-negative, $Lf_z + k + l \geq 0$, then the solution of the overdetermined system (3) is either a solution of system (5) or system (7).*

Before proving the theorem a few comments are in order.

Particular solutions of the Navier–Stokes equations of type (2) with both positive and negative right sides are known. For example, solutions with linear profile of velocity (4) with respect to x and y can be of both types, depending on the value of $\alpha^2 - c\gamma$. For solutions that are linear with respect to one independent variable x (9), and essentially nonlinear with respect to y, the right side of the Monge–Ampere equation is positive. In case (6) the type of the Monge–Ampere equation is hyperbolic.

Here we also present two known solutions [27, 34].

As the first example one can consider a slight generalization of the solution [34][2]

$$u = -\Omega(y - g_1(z,t)), \qquad u = \Omega(x - g_2(z,t)), \qquad w = w_0,$$

where w_0 is constant and Ω denotes constant angular velocity. Compatibility conditions for this solution are

$$g_{1t} + w_0 g_{1z} - g_{1zz} + \Omega g_2 = 0, \qquad g_{2t} + w_0 g_{2z} - g_{2zz} - \Omega g_1 = 0.$$

This solution can be represented as type (2) if $k = -\Omega^2/2$, $l = -\Omega^2/2$, $h = h(t)$, $2f = w_0$. In this case

$$Lf_z + k + l = -\Omega^2 \le 0. \tag{11}$$

The second example is the class of steady solutions studied in [27]:

$$f = f(z), \qquad h = 2(f'(z) - f^2(z)), \qquad u = x\widetilde{u}(z), \qquad v = -y(\widetilde{u}(z) + 2(f'(z))^2,$$

and constants k and l. The functions $f(z)$ and $\widetilde{u}(z)$ satisfy the equations

$$\widetilde{u}'' - 2f\widetilde{u}' - \widetilde{u}^2 + 2l = 0, \qquad f''' - 2ff'' + 2(f')^2 + 2\widetilde{u}f' + \widetilde{u}^2 = k + l.$$

For this solution

$$Lf_z + k + l = (f' + \widetilde{u})^2 \ge 0. \tag{12}$$

The next section is devoted to the proof of the theorem.

3.3. Hyperbolic Case

Further we consider the hyperbolic case, where the right side of the Monge–Ampere equation (10) is non-negative. By virtue of this assumption it is denoted

$$\alpha^2(z,t) \equiv Lf_z + k + l.$$

It is well known [38] that in this case the Monge–Ampere equation can be integrated[3]

$$g_y = 2\alpha x + G(g_x, z, t). \tag{13}$$

where $g(x,y,z,t) = \psi(x,y,z,t) + xy\alpha(z,t)$, and $G = G(z,t,\xi)$ is an arbitrary function. Substituting this representation into the first equation (3), with the help of the second equation, one can exclude the third-order derivatives:

$$S \equiv b_4 g_{xx}^2 + b_5 g_{xz}^2 + b_1 g_{xx} + b_2 g_{xz} - b_3 = 0, \tag{14}$$

[2] In [34] the functions g and f do not depend on time t. But this is not significant, because without loss of generality one can include in these functions dependence on time.

[3] There are some studies of an elliptic case of the Monge–Ampere equation, for example [39, 40].

where

$$b_1 = 4\alpha G_\xi G_{\xi\xi}, \qquad b_2 = 2G_{\xi z}, \qquad b_4 = \left(G_\xi^2 + 1\right) G_{\xi\xi}, \qquad b_5 = G_{\xi\xi},$$
$$b_3 = x(\widehat{\alpha} - 2(k-l)) + y\widehat{\alpha}G_\xi + (f_z - \alpha)\left(\xi G_\xi - G\right) + G_t + 2fG_z - G_{zz} - 4\alpha^2 G_{\xi\xi},$$
$$\widehat{\alpha} \equiv L\alpha - \alpha f_z + k - l$$

By direct calculations the expression $D_y S - G_\xi D_x S - 2g_{xx}G_{\xi\xi}S = 0$ can be rewritten as a polynomial of second order with respect to the derivatives g_{xx}, g_{xz}:

$$\alpha\left(1 + G_\xi^2\right) G_{\xi\xi\xi} g_{xx}^2 + \alpha G_{\xi\xi\xi} g_{xz}^2 + f_1 g_{xx} + f_2 g_{xz} + f_3 = 0. \tag{15}$$

Here D_x and D_y are the total derivatives with respect to x and y, respectively,

$$f_1 = (x(\widehat{\alpha} - 2(k-l)) + y\widehat{\alpha}G_\xi)G_{\xi\xi} + \widehat{f}_1, \quad f_2 = 2(\alpha_z G_{\xi\xi} + \alpha G_{\xi\xi z}), \quad f_3 = -y\alpha\widehat{\alpha}G_{\xi\xi} + \widehat{f}_3$$

with some functions $\widehat{f}_i$, $(i = 1, 3)$, which are not explicitly dependent on x and y. Because the expressions of the functions $\widehat{f}_1$ and $\widehat{f}_3$ are very cumbersome we omit their representations here. For the treatment of complicated mathematical expressions we used the system REDUCE [37].

Note that if $G_{\xi\xi} = 0$, then this is a particular case of the representation (8) or (9). In fact, assume that $G_{\xi\xi} = 0$ or $G = qg_x + \beta$ for some functions $q = q(z, t)$, $\beta = \beta(z, t)$. By (13) the function $g(x, y, z, t)$ has to satisfy the equation

$$g_y - qg_x = 2\alpha x + \beta. \tag{16}$$

If $q = 0$, the general solution of (16) is

$$g = 2\alpha xy + y\beta + \varphi(x, z, t),$$

which is a particular case of (8). If $q \neq 0$, the general solution of (16) is

$$g = q^{-1}(\alpha x^2 + \beta x) + \varphi(x - qy, z, t),$$

which is a particular case of (9).

3.4. The Nonlinear Case ($G_{\xi\xi} \neq 0$)

Let $G_{\xi\xi} \neq 0$, then Equation (15) with the help of (14) can be rewritten as the quasilinear equation

$$a_1 g_{xx} + a_2 g_{xz} + a_3 = 0, \tag{17}$$

with the coefficients $a_i = b_i G_{\xi\xi\xi} - f_i G_{\xi\xi}$, $(i = 1, 2, 3)$. The last equation and Equation (14) can be regarded as a system of linear algebraic equations with respect to x and y. The determinant of this system is equal to $G_{\xi\xi}\alpha\widehat{\alpha}(\widehat{\alpha} - 2(k-l))$.

If $\alpha = 0$, then by virtue of the definition of $\widehat{\alpha}$ we get $\widehat{\alpha} = (k - l)$, and the following prolongation

$$D_y H - G_\xi D_x H - g_{xx}G_{\xi\xi}H = -6(k-l)g_{xx}G_\xi G_{\xi\xi} = 0,$$

where $H = D_yS - G_\xi D_xS - 2g_{xx}G_{\xi\xi}S$. Because $g_{xx}G_\xi G_{\xi\xi} \neq 0$, then $k - l = 0$. This means, that $\widehat{\alpha} = 0$. In this case

$$H + 2g_{xx}G_{\xi\xi}S = -2g_{xx}\left((g_{xz}G_{\xi\xi} + G_{\xi z})^2 + g_{xx}^2G_{\xi\xi}^2(1 + G_\xi^2)\right) = 0.$$

The last equation contradicts the condition $g_{xx}G_{\xi\xi} \neq 0$. Therefore, $\alpha \neq 0$.

3.4.1. *The Case* $\widehat{\alpha}(\widehat{\alpha} - 2(k - l)) \neq 0$

If $\widehat{\alpha}(\widehat{\alpha} - 2(k - l)) \neq 0$, then $\alpha\widehat{\alpha}(\widehat{\alpha} - 2(k - l))G_{\xi\xi} \neq 0$ and, hence, Equations (14) and (17) can be solved with respect to x and y:

$$x = \Phi_1(g_{xx}, g_{xz}, g_x, z, t), \qquad y = \Phi_2(g_{xx}, g_{xz}, g_x, z, t). \tag{18}$$

Differentiating the last equations with respect to x and y, substituting the expressions of $g_y, g_{xy}, g_{xyz}, g_{xxy}$ into them and taking linear combinations, one obtains

$$D_y\Phi_1 - G_\xi D_x\Phi_1 = g_{xx}^2\Phi_{1,1}G_{\xi\xi} + \Phi_{1,2}(2\alpha_z + g_{xx}g_{xz}G_{\xi\xi} + g_{xx}G_{\xi z}) + 2\Phi_{1,3}\alpha + G_\xi = 0, \tag{19}$$

$$\begin{aligned} H(g_{xx}, g_{xz}, g_x, z, t) &\equiv D_y\Phi_2 - G_\xi D_x\Phi_2 \\ &= g_{xx}^2\Phi_{2,1}G_{\xi\xi} + \Phi_{2,2}(2\alpha_z + g_{xx}g_{xz}G_{\xi\xi} + g_{xx}G_{\xi z}) + 2\Phi_{2,3}\alpha - 1 = 0, \end{aligned} \tag{20}$$

$$\begin{aligned} H_3(g_{xx}, g_{xz}, g_x, z, t) &\equiv -\Phi_{2,1}D_x\Phi_1 + \Phi_{1,1}D_x\Phi_2 - \Phi_{2,2}D_z\Phi_1 + \Phi_{1,2}D_z\Phi_2 \\ &= g_{xx}(\Phi_{1,1}\Phi_{2,3} - \Phi_{1,3}\Phi_{2,1}) + g_{xz}(\Phi_{1,2}\Phi_{2,3} - \Phi_{1,3}\Phi_{2,2}) + \Phi_{1,2}\Phi_{2,4} - \Phi_{1,4}\Phi_{2,2} + \Phi_{2,1} = 0 \end{aligned}$$

where $\Phi_{i,1} = \frac{\partial\Phi_i}{\partial g_{xx}}$, $\Phi_{i,2} = \frac{\partial\Phi_i}{\partial g_{xz}}$, $\Phi_{i,3} = \frac{\partial\Phi_i}{\partial z}$. Note that after substituting the expressions of the functions Φ_i, $(i = 1, 2)$ into the last equations, Equation (19) is a consequence of Equation (20) and the function $H(g_{xx}, g_{xz}, g_x, z, t)$ is a polynomial of fourth degree with respect to g_{xx} and second degree with respect to g_{xz}

$$H = h_2g_{xz}^2 + h_1g_{xz} + h_0,$$

where

$$h_2 = 3g_{xx}^2G_{\xi\xi}^4 + 4\alpha g_{xx}G_{\xi\xi\xi}G_{\xi\xi}^2 + 2\alpha^2\left(G_{\xi\xi\xi\xi}G_{\xi\xi} - G_{\xi\xi\xi}^2\right)$$

The coefficient of the polynomial H with respect to g_{xx}^4 is $3G_{\xi\xi}^4(1 + G_\xi^2) \neq 0$ and does not depend on g_{xz}. Hence, the equation $H(g_{xx}, g_{xz}, g_x, z, t) = 0$ can be rewritten as $H_1 \equiv g_{xx} - \chi(g_{xz}, g_x, z, t) = 0$. In the same way, after differentiating the last equation with respect to x and y, substituting the expressions of $g_y, g_{xy}, g_{xyz}, g_{xxy}$ into them, one obtains

$$D_yH - G_\xi D_xH = g_{xx}^2H_1G_{\xi\xi} + H_2(2\alpha_z + g_{xx}g_{xz}G_{\xi\xi} + g_{xx}G_{\xi z}) + 2H_3\alpha = 0.$$

Since $H(g_{xx}, g_{xz}, g_x, z, t) = 0$, the left side of the last equation can be rewritten as a polynomial of degree three with respect to g_{xx}:

$$H_2(g_{xx}, g_{xz}, g_x, z, t) = 0. \tag{21}$$

If the Jacobian $\frac{\partial(H_1,H_2)}{\partial(g_{xx},g_{xz})}$ is not equal to zero, then from the equations

$$H_1(g_{xx}, g_{xz}, g_x, z, t) = 0, \qquad H_2(g_{xx}, g_{xz}, g_x, z, t) = 0.$$

one can define

$$g_{xx} = \Psi_1(g_x, z, t), \qquad g_{xz} = \Psi_2(g_x, z, t).$$

Substitution of these derivatives into (18) gives the contradictory equalities

$$x = \widehat{\Phi}_1(g_x, z, t), \qquad y = \widehat{\Phi}_2(g_x, z, t). \tag{22}$$

If the Jacobian $\frac{\partial(H_1,H_2)}{\partial(g_{xx},g_{xz})} = \frac{\partial H_2}{\partial g_{xx}}\frac{\partial \chi}{\partial g_{xz}} - \frac{\partial H_2}{\partial g_{xz}} = 0$, this means that the function $\widehat{H_2} = H_2(\chi(g_{xz}, g_x, z, t), g_{xz}, g_x, z, t)$ does not depend on g_{xz}. Furthermore $\widehat{H_2}_{g_x} = 0$, because otherwise one can define g_x as a function of z and t only, which contradicts the condition $g_{xx} \neq 0$. Therefore $H_2 = F(H_1)$. In our case,

$$H_2 = \widehat{a}_3 H_1^3 + \widehat{a}_2 H_1^2 + \widehat{a}_1 H_1 + \widehat{a}_0.$$

Thus, the coefficients $\widehat{a}_i$ must be constants and $\widehat{a}_0 = 0$. Note that

$$\widehat{a}_2 = \widehat{b}_1 \chi + \widehat{b}_2, \qquad \widehat{a}_1 = \widehat{b}_3 \chi^2 + \widehat{b}_4 \chi + \widehat{b}_5 g_{xz}^2 + \widehat{b}_6 g_{xz} + \widehat{b}_7,$$

where $\widehat{b}_i$ are functions of the variables g_x, z, t and

$$\widehat{b}_3 = \widehat{b}_1 = 3(1 + G_\xi^2)\widehat{b}_5,\ \widehat{b}_5 = 3G_{\xi\xi}G_{\xi\xi\xi\xi} - 5G_{\xi\xi\xi}^2.$$

If $\widehat{b}_1 \neq 0$, then from the equation $\widehat{a}_2 = \text{const}$ we have $\chi = -\widehat{b}_1^{-1}(\widehat{b}_2 - \widehat{a}_2)$, which does not depend on g_{xz}. In this case the equation $\widehat{a}_1 = \text{const}$ is a polynomial of degree two with respect to g_{xz} with coefficient $\widehat{b}_5 \neq 0$. This means that one can obtain contradictory equations of the type (22). Therefore, $\widehat{b}_1 = 0$ or

$$3G_{\xi\xi}G_{\xi\xi\xi\xi} - 5G_{\xi\xi\xi}^2 = 0.$$

This equation can be integrated twice with respect to ξ:

$$G_{\xi\xi} = \lambda(G_\xi + q)^3,$$

Two more integrations with respect to ξ give:

$$\lambda(G + \xi q + \gamma)^2 + 2\xi + \beta = 0.$$

Here the functions $\lambda = \lambda(z, t),\ q = q(z, t),\ \gamma = \gamma(z, t),\ \beta = \beta(z, t)$ are arbitrary and $\lambda \neq 0$. Note that in this case $\widehat{a}_3 = 0,\ b_2 = b_3 = b_4 = b_5 = b_6 = 0,\ \widehat{a}_3 = b_7,$

$$\widehat{a}_0 = \varphi_1(g_x, z, t)\chi + \varphi_0(g_x, z, t), \tag{23}$$

and

$$h_2 = 3\lambda^2(G_\xi + q)^5(g_{xx}(G_\xi + q) + 2\alpha)^2.$$

Assume that the function $\chi(g_{xz}, g_x, z, t)$ does not depend on g_{xz} : $\chi = \chi(g_x, z, t)$. Because of the prohibition for obtaining equations of the type (22), the coefficients h_i $(i = 1, 2)$ of the polynomial H have to be equal to zero. Since $G_{\xi\xi} \neq 0$, we have

$$g_{xx}(G_\xi + q) = -2\alpha.$$

The left side of this expression is the total derivative with respect to x of $G(g_x, z, t) + q(z, t)g_x$. Thus,

$$G(g_x, z, t) + q(z, t)g_x + 2\alpha(z, t)x = \phi(y, z, t). \tag{24}$$

Because $g_{xy} = G_\xi g_{xx} + 2\alpha$, then

$$\phi_y = g_{xy}(G_\xi + q) = (G_\xi + q)G_\xi g_{xx} + 2\alpha(G_\xi + q) = 2\alpha q.$$

After integrating the last equation with respect to y, there is $\phi(y, z, t) = 2y\alpha(z, t)q(z, t) + h(z, t)$. Substituting the function $\phi(y, z, t)$ and g_y into (24), one obtains

$$g_y + qg_x = 2y\alpha q + h.$$

The general solution of this equation is

$$g(x, y, z, t) = yh(z, t) + y^2\alpha(z, t)q(z, t) + \Phi(x - yq(z, t), z, t)$$

or

$$\psi(x, y, z, t) = -xy\alpha(z, t) + yh(z, t) + y^2\alpha(z, t)q(z, t) + \Phi(x - yq(z, t), z, t).$$

This is a particular case of (9). Therefore, we only need to study the case $\frac{\partial\chi}{\partial g_{xz}} \neq 0$.

Assume that $\frac{\partial\chi}{\partial g_{xz}} \neq 0$. From the expression for the function $\widehat{a}_0 = 0$ (23) we conclude that

$$\varphi_1(g_x, z, t) = 0, \qquad \varphi_0(g_x, z, t) = 0.$$

Splitting these equations with respect to g_x, one obtains

$$q_z = 0, \qquad \alpha q_t + q(k - l) = 0,$$
$$2\alpha^2\lambda(\lambda_t + 2f\lambda_z - \lambda_{zz} + f_z\lambda - \alpha\lambda) - (\alpha_z\lambda + \alpha\lambda_z)^2 + \lambda^2\alpha(\widehat{\alpha} - 4(k - l)) + 4\lambda_z^2\alpha^2 = 0.$$

The same analysis of the equation $H_3(g_{xx}, g_{xz}, g_x, z, t) = 0$ as for the equation $H_2 = 0$ leads to a contradiction. Therefore, the case $\widehat{\alpha}(\widehat{\alpha} - 2(k - l)) = 0$ is studied.

3.4.2. *The Case* $\widehat{\alpha} = 0$
Let us consider $\widehat{\alpha} = 0$ or

$$\frac{\partial \alpha}{\partial t} + 2f\frac{\partial \alpha}{\partial z} - \frac{\partial^2 \alpha}{\partial z^2} - 2\alpha\frac{\partial f}{\partial z} + k - l = 0.$$

In this case the coefficients $a_i, b_i, f_i,\ (i = 1, 2, 3),\ b_4, b_5$ do not explicitly depend on y.

Assume first that $k \neq l$, then one can define the value of x from (14) and substitute it into (17), which becomes a third-order polynomial with respect to g_{xx}

$$H_1 = h_3 g_{xx}^3 + h_2 g_{xx}^2 + h_1 g_{xx} + h_0,$$

where[4]

$$h_3 = G_{\xi\xi}^2 \left(1 + G_\xi^2\right) \neq 0$$

This means that one can define $g_{xx} = \chi(g_{xz}, g_x, z, t)$ from this equation. Note that the coefficient in H_1, which is related with the maximal order (second) with respect to g_{xz}, is equal to

$$g_{xx} G_{\xi\xi}^2 + \alpha G_{\xi\xi\xi}. \tag{25}$$

By the equation $H_1(g_{xx}, g_{xz}, g_x, z, t) = 0$, the left side of the expression

$$H_2 \equiv D_y H_1 - G_\xi D_x H_1 = 0.$$

is a polynomial of second degree with respect to g_{xx}:

$$H_2 = a_2 g_{xx}^2 + a_1 g_{xx} + a_0 = 0.$$

Before further consideration, we note that if from the equations

$$H_1(g_{xx}, g_{xz}, g_x, z, t) = 0, \qquad H_2(g_{xx}, g_{xz}, g_x, z, t) = 0.$$

one can define

$$g_{xx} = \Psi_1(g_x, z, t), \qquad g_{xz} = \Psi_2(g_x, z, t),$$

then after substitution of these derivatives into (14) one has the equality

$$x = \Phi(g_x, z, t). \tag{26}$$

Differentiating the last equality with respect to y we have $g_{xy}\Phi_\xi = 0$. If $\Phi_\xi = 0$, then (26) is a contradictory equality between the independent variables. The case $g_{xy} = 0$ was considered earlier.

[4] The analysis is similar to the previous case. For the polynomials and their coefficients we use the same symbols as in the previous case. However, the functions H, H_2, H_3, etc. are now different.

First assume that the function $\chi(g_{xz}, g_x, z, t)$ does not depend on g_{xz}. In this case all coefficients of the equation $H_1 = 0$ with respect to g_{xz} have to be equal to zero. Hence, from (25) we obtain

$$\chi = -\alpha G_{\xi\xi}^{-2} G_{\xi\xi\xi}. \tag{27}$$

Hence, $D_x S = H_3(g_{xz}, g_x, z, t)$, which give the equation $H_3 = 0$. Since $g_{xx} \neq 0$, then $G_{\xi\xi\xi} \neq 0$. Because it is prohibited to define g_{xz} from the equation $H_1 = 0$, $H_2 = 0$, $H_3 = 0$, all coefficients of these polynomials with respect to g_{xz} have to be equal to zero. In particular, from the coefficient related with the highest (second) degree of the equation $H_3 = 0$ we have

$$G_{\xi\xi\xi\xi} = \frac{3G_{\xi\xi\xi}^2}{2G_{\xi\xi}}.$$

Since $G_{\xi\xi\xi} \neq 0$, the general solution of the last equation is

$$G = -\lambda^{-1} \ln(\lambda g_x + \beta) + \mu g_x + \gamma,$$

where $\beta, \lambda, \mu, \gamma$ are arbitrary functions of the independent variables z, t. In this case,

$$g_{xx} = 2\alpha(\lambda g_x + \beta). \tag{28}$$

The general solution of Equation (28) is

$$g = -\frac{\beta}{2\alpha\lambda^2}(1 + 2\alpha\lambda x) + \varphi_1 e^{2\alpha\lambda x} + \varphi_2,$$

where $\varphi_1 = \varphi_1(y, z, t)$, $\varphi_2 = \varphi_2(y, z, t)$. All coefficients of the polynomials H_1 and H_3 with respect to g_{xz}, which have to be equal to zero, are polynomials with respect to g_x. This allows splitting them with respect to g_x, otherwise g_x can be defined as a function only of z and t. Further examination of all these coefficients leads to the equality $\mu = 0$. By virtue of $\mu = 0$ and substituting g into the equation

$$g_y = G(g_x, z, t) + 2\alpha x,$$

one obtains $\varphi_{1,y} = 0$, $\varphi_{2,yy} = 0$. This means that $g_{yy} = 0$ or $\psi_{yy} = 0$. This case was studied earlier.

Assume that $\chi_{g_{xz}} \neq 0$. The study of this case is similar to the previous case where $\widehat{\alpha}(\widehat{\alpha} - 2(k - l)) \neq 0$. Because the Jacobian $\frac{\partial(H_1, H_2)}{\partial(g_{xx}, g_{xz})}$ has to be equal to zero, then $H_2 = F(g_{xx} - \chi(g_{xz}, g_x, z, t))$. In our case

$$H_2 = \widehat{a}_2 H_1^2 + \widehat{a}_1 H_1 + \widehat{a}_0,$$

The coefficients $\widehat{a}_i$ must be constant and $\widehat{a}_0 = 0$. Note that

$$\widehat{a}_1 = \widehat{b}_1 \chi + \widehat{b}_2,\ \widehat{a}_0 = \widehat{b}_3 \chi^2 + \widehat{b}_4 \chi + \widehat{b}_5 g_{xz}^2 + \widehat{b}_6 g_{xz} + \widehat{b}_7,$$

where $\widehat{b}_i$ are functions of the variables g_x, z, t and

$$\widehat{a}_2 = \widehat{b}_1 = \widehat{b}_3 = \left(1 + G_\xi^2\right) \widehat{b}_5, \qquad \widehat{b}_5 = \left(2G_{\xi\xi} G_{\xi\xi\xi\xi} - 3G_{\xi\xi\xi}^2\right).$$

If $\widehat{b}_1 \neq 0$, then $\chi = -\widehat{b}_1^{-1}(\widehat{b}_2 - \widehat{a}_2)$ does not depend on g_{xz}. This case has already been studied. If $G_{\xi\xi\xi} = 0$, then $\widehat{a}_2 = \widehat{b}_1 = \widehat{b}_3 = \widehat{b}_5 = \widehat{b}_6 = 0$. This requires $\widehat{b}_4 = \widehat{b}_7 = 0$. Analysis of these coefficients by splitting them with respect to g_x leads to the condition that $a_2 = a_3 = 0$ in Equation (17) and that a_1 is linear with respect to g_x: $a_1 = a(z,t)g_x + \phi(x,z,t)$, where $a \neq 0$. This gives the contradiction $g_{xx}g_{xy} = 0$. Thus, $\widehat{b}_1 = 0$ and $G_{\xi\xi\xi} \neq 0$ or

$$G = -\frac{1}{\lambda}\ln(\lambda\xi + \beta) + \mu\xi + \gamma.$$

In this case

$$\widehat{a}_0 = \widehat{b}_4\chi + \widehat{b}_6 g_{xz} + \widehat{b}_7 = 0.$$

If the coefficient $\widehat{b}_4 = 0$, then as it is done earlier, analysis of the coefficients $\widehat{b}_4 = \widehat{b}_6 = \widehat{b}_7 = 0$ by splitting them with respect to g_x leads to the condition that Equation (17) be written in the form

$$a_1(g_{xx} - 2\alpha(\lambda g_x + \beta)) = 0,$$

where $a_1 = a(z,t)g_x + \phi(x,z,t)$ with $a \neq 0$. These cases have been already studied.

If $\widehat{b}_4 \neq 0$, then

$$\chi = -\widehat{b}_4^{-1}(\widehat{b}_6 g_{xz} + \widehat{b}_7).$$

Returning to the equation $H_1 = 0$, which becomes a cubic polynomial with respect to g_{xz} and analyzing the coefficients of this polynomial, which have to be equal to zero, leads to a contradiction. This completes the study of the case $k \neq l$.

Assume that $k = l$. Note that if $a_1 = 0$ in Equation (17), then Equation (14) is reduced to

$$(g_{xz}G_{\xi\xi} + G_{\xi z})^2 + G_{\xi\xi}^2(g_{xx}G_\xi + 2\alpha)^2 + (g_{xx}G_{\xi\xi})^2 = 0.$$

Hence, $a_1 \neq 0$ and from Equation (17) one can define $g_{xx} = -a_1^{-1}(a_2 g_{xz} + a_0)$. Substituting g_{xx} into (14) gives a polynomial of second degree with respect to g_{xz}:

$$S = a_1^{-2}G_{\xi\xi}(a_1^2 + a_2^2\left(1 + G_\xi^2\right))\, g_{xz}^2 + \widehat{b}_1 g_{xz} + \widehat{b}_0 = 0.$$

This means that Equations (14) and (17) can be solved with respect to g_{xx} and g_{xz}:

$$g_{xx} = \widehat{\Phi}_1(g_x, z, t), \;\; g_{xz} = \widehat{\Phi}_2(g_x, z, t). \tag{29}$$

Because $g_{xx} \neq 0$, then the first equation of (29) can be integrated

$$\widehat{\Phi}(g_x, z, t) = x + q(y, z, t)$$

or

$$g_x = \Phi(x + q(y, z, t), z, t).$$

Here the function $q = q(y, z, t)$ is an arbitrary function of integration. The general solution of the last equation is expressed by the formula

$$g(x, y, z, t) = \Phi_1(x + q(y, z, t), z, t) + \Phi_2(y, z, t)$$

Note that

$$G(g_x, z, t) = \widehat{G}(x + q(y, z, t), z, t)$$

and the equation $g_y - (2\alpha x + G) = 0$ is rewritten as

$$q_y \Phi_{1,x'}(x', z, t) + \Phi_{2,y}(y, z, t) = 2\alpha x' + \widehat{G}(x', z, t) - 2\alpha q,$$

where $x' = x + q(y, z, t)$. Differentiating the last equation with respect to y one obtains

$$q_{yy} \Phi_{1,x'}(x', z, t) + \Phi_{2,yy}(y, z, t) = -2\alpha q_y. \tag{30}$$

Differentiating one more with respect to x' gives

$$q_{yy} \Phi_{1,x'x'} = 0.$$

If $\Phi_{1,x'x'} = 0$, this is a particular case of the representation (6). If $q_{yy} = 0$ or $q = yk_1(z, t) + k_2(z, t)$, then integrating equation (30), we have

$$\Phi_2 = -\alpha k_1 y^2 + y\psi_1(z, t) + \psi_2(z, t).$$

This is a particular case of the representation (9).

The case $\widehat{\alpha} = 2(k - l)$ is studied in a similar way as the previous case $\widehat{\alpha} = 0$. Note that in this case $\alpha(k - l) \neq 0$. A detail analysis leads either to contradictions or to the studied cases.

4. Group Classification of System (7)

System (7) is split into three parts: the system of the first two equations

$$Lf_z + k + l = \alpha^2, \qquad L\alpha = \alpha f_z + k - l \tag{31}$$

is determined and can be studied independently; the equation

$$L\lambda = \alpha\lambda$$

is for determining the function $\lambda(z, t)$; and the equation

$$L\varphi - \varphi_{yy} - (y(\alpha + f_z) + \lambda)\varphi_y + \alpha\varphi = 0$$

is for the function $\varphi(y, z, t)$. In this section group classification of system (31) is studied.

4.1. Equivalence Transformations

The first stage of group classification requires determining a group of equivalence transformations of Equation (31). An equivalence transformation [1] is a nondegenerate change of the dependent and independent variables and arbitrary elements, which transforms any system of differential equations of a given class to a system of equations of the same class. It allows using the simplest representation of the given equations.

Since the arbitrary elements are $k = k(t)$, $l = l(t)$, then for calculating group of equivalence transformations we have to append to Equation (31) the following equations

$$k_z = 0, \qquad k_f = 0, \qquad k_\alpha = 0,$$
$$l_z = 0, \qquad l_f = 0, \qquad l_\alpha = 0$$

All coefficients of the infinitesimal generator of the equivalence group

$$X^e = \zeta^t \partial_t + \zeta^z \partial_z + \zeta^f \partial_f + \zeta^\alpha \partial_\alpha + \zeta^k \partial_k + \zeta^l \partial_l$$

are assumed to be dependent on the variables t, z, f, α, k, l.

Calculations show that the group of equivalence transformations of system (31) corresponds to the Lie algebra with the generators

$$X_1^e = \partial_t, \qquad X_2^e = 2\xi(t)\partial_z + \xi'(t)\partial_f, \qquad X_3^e = -2t\partial_t - z\partial_z + f\partial_f + 2\alpha\partial_\alpha + 4k\partial_k + 4l\partial_l.$$

4.2. Admitted Group

To find an admitted group we are looking for the generator

$$X = \zeta^t \partial_t + \zeta^z \partial_z + \zeta^f \partial_f + \zeta^\alpha \partial_\alpha$$

with the coefficients depending on t, z, f, α. Calculations lead to the following result.

The equations that determine the extensions are

$$c_1(tk' + 2k) + c_2 k' = 0, \qquad c_1(tl' + 2l) + c_2 l' = 0,$$

where c_1 and c_2 are constant. Analysis of these equations is similar to the analysis of the group classification of the gas dynamics equations [1]. Let us consider the vectors $\mathbf{v}_1(t) = (tk' + 2k, k')$ and $\mathbf{v}_2(t) = (tl' + 2l, l')$. If they generate a two-dimensional space (where t is changed), then $c_1 = 0$, $c_2 = 0$. This corresponds to the kernel of principal Lie algebras. The kernel is infinite and defined by the generators

$$X_1 = 2\xi(t)\partial_z + \xi'(t)\partial_f.$$

An extension of the kernel can be made by specializing the functions $k = k(t)$, $l = l(t)$.

Let the vectors $\mathbf{v}_1(t)$, $\mathbf{v}_2(t)$ generate a one-dimensional space

$$\mathbf{v}_1(t) = s_1(k_1, k_2), \qquad \mathbf{v}_2(t) = s_2(k_1, k_2),$$

with some scalars $s_1 = s_1(t)$, $s_2 = s_2(t)$. Note that in this case $s_1^2 + s_2^2 \neq 0$ and $k_1^2 + k_2^2 \neq 0$.

If $k_2 = 0$, then $k(t)$, $l(t)$ are constants and $k \neq l$ (otherwise the space is zero-dimensional). Hence, $c_1 = 0$ and the kernel is extended by the generator

$$X_2 = \partial_t.$$

Table 1. Group classification of system.

	Functions	Extension
1.	$k = q_1 t^{-2},\ l = q_2 t^{-2}$	X_3
2.	$k = \text{const},\ l = \text{const}$	X_2
3.	$k = l = 0$	X_2, X_3

If $k_2 \neq 0$, then

$$\left(t - k_2^{-1}k_1\right) k' + 2k = 0, \qquad \left(t - k_2^{-1}k_1\right) l' + 2l = 0.$$

By virtue of an equivalent transformation (shift with respect to t), one can assume that $k_1 = 0$. The general solution of the last equations is

$$k = q_1 t^{-2}, \qquad l = q_2 t^{-2}, \qquad \left(q_1^2 + q_2^2 \neq 0\right).$$

In this case $c_2 = 0$ and the extension of the kernel is

$$X_3 = 2t\partial_t + z\partial_z - f\partial_f - 2\alpha\partial_\alpha.$$

Assume that the vectors $\mathbf{v}_1(t),\ \mathbf{v}_2(t)$ generate a zero-dimensional space. This gives that $k(t) = l(t) =$ const. If this constant is not equal to zero, the kernel is extended by the generator X_2. If $k(t) = l(t) = 0$, the kernel is extended by the generators X_2, X_3.

The result of the group classification is given in Table 1.

Remark. A detailed analysis of invariant solutions of the case $k = l = 0$ is done in [30].

4.3. Group Stratification and Invariant Solutions

The group admitted by Equation (31) is infinite. Classification of an infinite group is more difficult. This obstacle can be overcome by studying group stratification of an infinite group [1]. Group stratification allows splitting the initial system into automorphic and resolving systems. Any solution of the automorphic system is obtained from one fixed solution by a transformation belonging to the group.

The infinite group with the operator X_1 has the prolonged operator

$$X_1 = 2\xi(t)\partial_z + \xi'(t)(\partial_f - 2f_z\partial_{f_t} - 2\alpha_z\partial_{\alpha_t} - 2\beta_z\partial_{\beta_t}) + \xi''(t)\partial_{f_t},$$

where $\beta = f_z$. The universal invariant of the first order of the operators, which are obtained as coefficients of ξ, ξ', ξ'' is

$$J = (t, \beta, \alpha, \alpha_z, \beta_z, \beta_t + 2f\beta_z, \alpha_t + 2f\alpha_z).$$

Hence, the automorphic system AG of rank 2 can be written in the form

$$\alpha = \alpha(t, \beta), \quad \alpha_z = \varphi(t, \beta), \beta_z = \gamma(t, \beta), \quad \beta_t + 2f\beta_z = \varsigma_1(t, \beta), \quad \alpha_t + 2f\alpha_z = \varsigma_2(t, \beta), \quad (32)$$

Table 2. Group classification of system.

	Functions	Extension
1.	$k = q_1 t^{-2},\ l = q_2 t^{-2}$	Y_2
2.	$k = \text{const},\ l = \text{const}$	Y_1
3.	$k = l = 0$	Y_1, Y_2

where $\alpha(t, \beta),\ \varphi(t, \beta),\ \gamma(t, \beta),\ \varsigma_1(t, \beta)$ and $\varsigma_2(t, \beta)$ are unknown functions. Compatibility conditions for the last system and the initial system (31) are

$$\varphi = \gamma\alpha_\beta, \qquad \varsigma_1 = \gamma\gamma_\beta + \alpha^2 + \beta^2, \tag{33}$$

$$\alpha_t + (\alpha^2 + \beta^2 - k - l)\alpha_\beta - \gamma^2\alpha_{\beta\beta} - 2\alpha\beta - k + l = 0,$$

$$\gamma_t + (\alpha^2 + \beta^2 - k - l)\gamma_\beta - \gamma^2\gamma_{\beta\beta} - 2\alpha\gamma = 0. \tag{34}$$

Thus, the group stratification of system (31) with respect to the infinite group with the operator X_1 is the union of the automorphic system (32) with the functions (33) and the resolving system, which consists of Equations (33).

The group of equivalence transformations of Equation (33) corresponds to the Lie algebra with the

$$Y_1^e = \partial_t, \qquad Y_2^e = -2t\partial_t + 2\beta\partial_\beta + 2\alpha\partial_\alpha + 3\gamma\partial_\gamma + 4k\partial_k + 4l\partial_l.$$

The kernel of the admitted group is empty. The group classification with respect to the arbitrary elements $k = k(t)$ and $l = l(t)$ is summarized in Table 2, where

$$Y_1 = \partial_t, \qquad Y_2 = 2t\partial_t - 2\beta\partial_\beta - 2\alpha\partial_\alpha - 3\gamma\partial_\gamma.$$

System (32) and (33) are equivalent to the initial system (31) provided $f_{zz} \neq 0$. Let us consider the degenerate case $f_{zz} = 0$. In this case the function $f(t, z)$ is $f = zq(t) + q_1(t)$, where the functions $q = q(t)$ and $q_1 = q_1(t)$ are arbitrary. Substituting this representation into system (31) one obtains that the function α depends only on t, and

$$(q - \alpha)' - (q - \alpha)^2 = -2k,$$

$$(q + \alpha)' - (q + \alpha)^2 = -2l.$$

These equations can be considered either as equations for the functions $\alpha = \alpha(t)$ and $q = q(t)$ with known functions $k = k(t)$ and $l = l(t)$, or the functions $\alpha = \alpha(t)$ and $q = q(t)$ are given, and the functions $k = k(t)$ and $l = l(t)$ are defined by these equations.

Let us consider invariant solutions of the resolving system with $f_{zz} \neq 0$ (or $\gamma \neq 0$). Because the case $k = l = 0$ has been studied in [30], then we only need to study two cases: a) $k = \text{const},\ l = \text{const}$ $(k^2 + l^2 \neq 0)$; b) $k = q_1 t^{-2},\ l = q_2 t^{-2},\ (q^2 + q_1^2 \neq 0)$.

The case $k = \text{const},\ l = \text{const}$. Further study is also valid for $k = l = 0$.

The admitted algebra of the resolving system consists of the generator $Y_1 = \partial_t$. An invariant solution has the representation

$$\alpha = \alpha(\beta), \qquad \gamma = \gamma(\beta). \tag{35}$$

The functions $\alpha(\beta)$ and $\gamma(\beta)$ have to satisfy the equations

$$(\beta^2+\alpha^2-k-l)\alpha'-\gamma^2\alpha''-2\alpha\beta-k+l=0,$$
$$(\beta^2+\alpha^2-k-l)\gamma'-\gamma^2\gamma''-2\alpha\gamma\alpha'=0.$$

Since the case $\gamma=0$ corresponds to $f_{zz}=0$, then $\gamma\neq 0$. In order to find a solution of the initial system (31) one has to solve the automorphic system. One of equations of the automorphic system is $\beta_z=\gamma(\beta)$. By virtue of $\gamma\neq 0$ and $\beta=f_z$, the function $f=f(t,z)$ has the representation $f=H(z+q(t))+s(t)$ with some functions $q=q(t)$ and $s=s(t)$. The solution of system (31), which corresponds to the invariant solution (35) has the representation

$$\alpha=\alpha(z+q(t)),\qquad f=H(z+q(t))+s(t). \tag{36}$$

Substituting this representation into system (31), one has

$$(2H+q'+2s)H''-H'''-(H')^2+k+l=\alpha^2,\quad (2H+q'+2s)\alpha'-\alpha''-2\alpha H'-k+l=0.$$

From the first equation, by considering $\widehat{z}=z+q(t)$ and $\widehat{t}=t$ as the new independent variables and differentiating the first equation with respect to $\widehat{t}$, one obtains $H''(q''+2s')=0$. Because $f_{zz}=H''\neq 0$, then $q'+2s=s_0=\text{const}$ and the last system becomes

$$(H'-\alpha)''-(2H+s_0)(H'-\alpha)'+(H'-\alpha)^2=2k,$$
$$(H'+\alpha)''-(2H+s_0)(H'+\alpha)'+(H'+\alpha)^2=2l.$$

The case $k=t^{-2}q_1,\ l=t^{-2}q_1$. The admitted group of the resolving system consists of the generator $Y_2=2t\partial_t-2\beta\partial_\beta-2\alpha\partial_\alpha-3\gamma\partial_\gamma$. An invariant solution has the representation

$$\alpha=t^{-1}\Lambda(t\beta),\qquad \gamma=t^{-3/2}\Gamma(t\beta). \tag{37}$$

Similar as in the previous case one obtains the solution of system (31), which corresponds to the invariant solution (37). This solution has the representation

$$\alpha=t^{-1}\Lambda(\xi),\qquad f=t^{-1/2}H(\xi)+s(t), \tag{38}$$

where $\xi=t^{-1/2}(z+q(t))$ and $q=q(t)$ is an arbitrary function. Substituting this representation into system (31), one has $t^{-1/2}(q'+2s)=s_0=\text{const}$ and the functions $\Lambda(\xi)$, $H(\xi)$ satisfy the equations

$$(H'-\Lambda)''+\left(\frac{\xi}{2}-2H-s_0\right)(H'-\Lambda)'+(H'-\Lambda)^2+(H'-\Lambda)=2q_1,$$
$$(H'+\Lambda)''+\left(\frac{\xi}{2}-2H-s_0\right)(H'+\Lambda)'+(H'+\Lambda)^2+(H'+\Lambda)=2q_2.$$

5. Group Classification of System (5)

System (5) is split into two parts: the system of the equations

$$Lf_z+k+l=-c\gamma+\alpha^2,\qquad L\alpha=\alpha f_z+k-l,\qquad L\gamma=f_z\gamma,\qquad Lc=f_zc \tag{39}$$

Table 3. Group classification of system (5).

	Functions	Extension
$k \neq l$		
1.	$k = q_1 t^{-2},\ l = q_2 t^{-2}\ (q_1 \neq q_2)$	X_5
2.	$k = \text{const},\ l = \text{const}$	X_6
$k = l$		
3.		X_3, X_4
4.	$k = l = qt^{-2}$	X_3, X_4, X_5
5.	$k = l = \text{const} \neq 0$	X_3, X_4, X_6
6.	$k = l = 0$	X_3, X_4, X_5, X_6

is closed and can be studied independently; the equations

$$L\lambda = \lambda\alpha - b\gamma, \qquad Lb = \lambda c - \alpha b \tag{40}$$

are for determining the functions $\lambda(z, t)$ and $b(z, t)$.

Calculations showed that the group of equivalence transformations of system (39) corresponds to the Lie algebra with the generators

$$X_1^e = \partial_t, \qquad X_2^e = 2\xi(t)\partial_z + \xi'(t)\partial_f, \qquad X_3^e = \gamma\partial_\gamma - c\partial_c,$$
$$X_4^e = -2t\partial_t - z\partial_z + f\partial_f + 2\alpha\partial_\alpha + 2\gamma\partial_\gamma + 2c\partial_c + 4k\partial_k + 4l\partial_l.$$

The equations that determine the admitted Lie group are

$$c_1(tk' + 2k) + c_2 k' = 0, \qquad c_1(tl' + 2l) + c_2 l' = 0, c_3(k - l) = 0, \qquad c_4(k - l) = 0,$$

where c_1, c_2, c_3 and c_4 are constants. The same analysis as in the previous section gives the kernel, which consists of the generator

$$X_1 = 2\xi(t)\partial_z + \xi'(t)\partial_f, \qquad X_2 = \gamma\partial_\gamma - c\partial_c.$$

An extension of the kernel occurs by specializing the functions $k(t)$ and $l(t)$. The result of group classification of system (5) is presented in Table 3, where

$$X_3 = \gamma\partial_\alpha + 2\alpha\partial_c, \qquad X_4 = c\partial_\alpha + 2\alpha\partial_\gamma, \qquad X_5 = 2t\partial_t + z\partial_z - f\partial_f - 2\alpha\partial_\alpha - 4\gamma\partial_\gamma, \qquad X_6 = \partial_t.$$

5.1. Group Stratification and Invariant Solutions

The group admitted by Equation (39) is infinite. The infinite group with the operator X_1 has the prolonged operator

$$X_1 = 2\xi(t)\partial_z + \xi'(t)(\partial_f - 2f_z\partial_{f_t} - 2\alpha_z\partial_{\alpha_t} - 2\beta_z\partial_{\beta_t} - 2\gamma_z\partial_{\gamma_t} - 2c_z\partial_{c_t}) + \xi''(t)\partial_{f_t},$$

where $\beta = f_z$. The universal invariant of first order is

$$J = (t, \beta, \alpha, \alpha_z, \beta_z, \gamma_z, c_z, \beta_t + 2f\beta_z, \alpha_t + 2f\alpha_z, \gamma_t + 2f\gamma_z, c_t + 2fc_z).$$

Hence, the automorphic system AG of rank 2 can be written in the form

$$\begin{aligned}
&\alpha = \alpha(t,\beta), \qquad \gamma = \gamma(t,\beta), \qquad c = c(t,\beta), \qquad \alpha_z = \varphi_1(t,\beta), \qquad \beta_z = \varphi_2(t,\beta),\\
&\gamma_z = \varphi_3(t,\beta), \qquad c_z = \varphi_4(t,\beta), \qquad \beta_t + 2f\beta_z = \varphi_5(t,\beta),\\
&\alpha_t + 2f\alpha_z = \varphi_6(t,\beta), \qquad \gamma_t + 2f\gamma_z = \varphi_7(t,\beta), \qquad c_t + 2fc_z = \varphi_8(t,\beta),
\end{aligned} \tag{41}$$

where $\alpha(t,\beta)$, $\gamma(t,\beta)$, $c(t,\beta)$, $\varphi_i(t,\beta)$ $(i = 1, 2, \dots, 8)$ are unknown functions. The compatibility conditions for the last system and the initial system (39) are

$$\begin{aligned}
&\varphi_1 = \varphi_2\alpha_\beta, \qquad \varphi_3 = \varphi_2\gamma_\beta, \qquad \varphi_4 = \varphi_2 c_\beta,\\
&\varphi_5 = \varphi_2\varphi_{2\beta} + \beta^2 + \alpha^2 - c\gamma - k - l,\\
&\varphi_6 = \alpha_t + \alpha_\beta\varphi_5, \qquad \varphi_7 = \gamma_t + \gamma_\beta\varphi_5, \qquad \varphi_8 = c_t + c_\beta\varphi_5,
\end{aligned} \tag{42}$$

$$\begin{aligned}
&\alpha_t + (\alpha^2 + \beta^2 - c\gamma - k - l)\alpha_\beta - \varphi_2^2\alpha_{\beta\beta} - 2\alpha\beta - k + l = 0,\\
&\varphi_{2t} + (\alpha^2 + \beta^2 - c\gamma - k - l)\varphi_{2\beta} - \varphi_2^2\varphi_{2\beta\beta} - 2\alpha\varphi_2\alpha_\beta + \gamma\varphi_2 c_\beta + c\varphi_2\gamma_\beta = 0,\\
&\gamma_t + (\alpha^2 + \beta^2 - c\gamma - k - l)\gamma_\beta - \varphi_2^2\gamma_{\beta\beta} - 2\gamma\beta = 0,\\
&c_t + (\alpha^2 + \beta^2 - c\gamma - k - l)c_\beta - \varphi_2^2 c_{\beta\beta} - 2c\beta = 0.
\end{aligned} \tag{43}$$

Thus, the group stratification of system (39) with respect to the infinite group with the operator X_1 is the union of the automorphic system (41) with the functions (33) and the resolving system, which consists of Equation (43).

The group of equivalence transformations of Equations (43) corresponds to the Lie algebra with generators

$$\begin{aligned}
&Y_1^e = \partial_t, \qquad Y_2^e = -2t\partial_t + 2\beta\partial_\beta + 2\alpha\partial_\alpha + 2\gamma\partial_\gamma + 2c\partial_c + 3\varphi_2\partial_{\varphi_2} + 4k\partial_k + 4l\partial_l,\\
&Y_3^e = \gamma\partial_\gamma - c\partial_c.
\end{aligned}$$

The kernel of the admitted group is one-dimensional and consists of the group, corresponding to the generator

$$Y_1 = \gamma\partial_\gamma - c\partial_c.$$

The group classification with respect to the arbitrary elements $k = k(t)$ and $l = l(t)$ is summarized in Table 4, where

$$\begin{aligned}
&Y_2 = -2t\partial_t + 2\beta\partial_\beta + 2\alpha\partial_\alpha + 3\varphi_2\partial_{\varphi_2} + 2\gamma\partial_\gamma + 2c\partial_c,\\
&Y_3 = \partial_t, \qquad Y_4 = \gamma\partial_\alpha + 2\alpha\partial_c, \qquad Y_5 = c\partial_\alpha + 2\alpha\partial_\gamma.
\end{aligned}$$

5.2. Conclusion and Discussion

In this article we have systematically investigated a class of partially invariant solutions of the Navier–Stokes equations, where the Monge–Ampere equation (10) is hyperbolic ($Lf_z + k + l \geq 0$). It was shown that this class of solutions is a particular case of a solution either of system (5) or system (7). Note that the representation (2) is very rich and includes some solutions that were studied earlier. The presence of two arbitrary functions $k(t)$ and $l(t)$ gives additional possibilities for satisfying boundary

Table 4. Group classification of system (43).

	Functions	Extension
$k \neq l$		
1.	$k = q_1 t^{-2},\ l = q_2 t^{-2}\ (q_1 \neq q_2)$	Y_2
2.	$k = \text{const},\ l = \text{const}$	Y_3
$k = l$		
3.		Y_3, Y_4
4.	$k = l = qt^{-2}$	Y_1, Y_3, Y_4
5.	$k = l = \text{const} \neq 0$	Y_2, Y_3, Y_4
6.	$k = l = 0$	Y_1, Y_2, Y_3, Y_4

conditions. The problem of describing all solutions of the given representation (2) where the Monge–Ampere equation (10) is elliptic ($Lf_z + k + l < 0$) still remains open, although there are examples of solutions of such type of the Navier–Stokes equations(constructed here and known before).

In this paper the group classifications of systems (7) and (5) was discussed. These systems have infinite admitted groups. Infinite-dimensionality is an obstacle for classification of such groups. To overcome this difficulties, group stratification of these groups was done. Group stratification allows splitting the initial system into automorphic and resolving systems. Any solution of the automorphic system is obtained from one fixed solution by a transformation belonging to the group. Therefore the problem of constructing solutions is reduced to finding solutions of the resolving systems. Group classification of resolving systems was done. The admitted groups are finite-dimensional. All invariant solutions of system (7) were presented.

Note that we did not present here a comprehensive study of invariant solutions of the group admitted by (7). This study is a subject for the construction of new solutions of the Navier–Stokes equations.

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