Available online at www.sciencedirect.com

SCIENCE DIRECT®

Mathematics and Computers in Simulation 69 (2005) 282–289

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

A new approach related with group analysis and hodograph type transformation for constructing exact solutions

A. Hematulin [a,*], S.V. Meleshko [b]

[a] *Nakhon Ratchasima Rajabhat University, School of Mathematics and Applied Statistics, Nakhon Ratchasima 30000, Thailand*
[b] *Suranaree University of Technology, School of Mathematics, Nakhon Ratchasima 30000, Thailand*

Available online 26 April 2005

Abstract

The method suggested in the manuscript uses the idea of the hodograph transformation method, which exchanges the independent and dependent variables. Here a change of the independent variables into dependent variables is applied to first derivatives. For the derivatives one obtains a system of differential equations. Group analysis is applied to this system. New invariant solutions, which are not invariant for the original equations, are obtained. The approach is illustrated by the semi-linear wave equation. For example, for Pion Meson equation one obtains a solution, which is reduced to quadrature.

Keywords: Admitted Lie group; Invariant solution; Semi-linear wave equation; Pion Meson equation

1. Introduction

The manuscript is concerned with the methodology of group analysis [1] for finding exact solutions of partial differential equations. Group analysis provides three types of solutions: invariant [2], partially invariant [1] and solutions related with conditional (weak) symmetries [3]. Applications of group analysis are discussed in numerous articles. Many of these results are collected in [4]. An approach for obtaining exact solutions by using invariant solutions of system which is related with original is suggested in the

* Corresponding author.
E-mail addresses: api9904@yahoo.com (A. Hematulin), sergey@math.sut.ac.th (S.V. Meleshko).

0378-4754/$30.00 © 2005 IMACS. Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.matcom.2005.01.014

article. The idea of the method is also related with the method of differential constraints [5]. An application of the suggested approach is illustrated by the semi-linear wave equation.

2. The wave equation

The semi-linear wave equation[1]

$$u_{tt} - u_{xx} = h(u) \tag{1}$$

is used for modeling many nonlinear wave phenomena. Since in the case $h'(u) = 0$ Eq. (1) is reduced to the classical wave equation $u_{tt} - u_{xx} = 0$, the restriction $h'(u) \neq 0$ is assumed in the manuscript. There are complete classifications of contact [6] and Lie–Bäcklund [7–9] symmetries of Eq. (1). The kernel of admitted Lie algebras is three-dimensional and consists of the generators

$$X_1 = \partial_t, \qquad X_2 = \partial_x, \qquad X_3 = x\partial_t + t\partial_x. \tag{2}$$

The group of equivalence transformations is infinite and corresponds to the generators

$$X_1{}^e = \partial_u, \qquad X_2{}^e = u\partial_u + h\partial_h, \qquad X_3{}^e = (\phi + \psi)\partial_x + (\phi - \psi)\partial_t + 2h(\phi' + \psi')\partial_h,$$

where $\phi = \phi(x + t)$ and $\psi = \psi(x - t)$ are arbitrary functions.

Eq. (1) can be reduced to the equivalent system of first order quasilinear equations[2]

$$u_t - u_x = v, \qquad v_t + v_x = h(u). \tag{3}$$

By the first equation of system (3), any infinitesimal symmetry of system (3) provides a contact symmetry of Eq. (1). Since contact symmetries of Eq. (1) are prolongations of the point symmetries [6], there is a simple correspondence between the group (2) admitted by Eq. (1) and the group admitted by system (3). Therefore the Lie algebra admitted by system (3) is spanned by the generators

$$X_1 = \partial_t, \qquad X_2 = \partial_x, \qquad \tilde{X}_3 = x\partial_t + t\partial_x + v\partial_v. \tag{4}$$

An optimal system of subalgebras of the Lie algebra (4) consists of the subalgebras

$$\{X_1, X_2, \tilde{X}_3\}, \{X_1, X_2\}, \{X_1, \tilde{X}_3\}, \{\tilde{X}_3\}, \{X_1\}.$$

All invariant solutions (up to transformations of the admitted group) are exhausted by these two representations

$$u = f(x), \qquad v = g(x) \tag{5}$$

and

$$u = f(x^2 - t^2), \qquad v = (x + t)g(x^2 - t^2). \tag{6}$$

Let $u = u(x, t)$ and $v = v(x, t)$ be a solution of system (3). If $u(x, t)$ and $v(x, t)$ are functionally dependent, then this solution is reduced to a well-known class of invariant solutions: travelling waves. Therefore one needs to study only functionally independent solutions of system (3).

1 There is also another representation of this equation, $z_{xy} = F(z)$.

2 It should be noted here that the method used in [10] cannot be applied to Eq. (1).

If $u(x, t)$ and $v(x, t)$ are functionally independent, then any autonomous homogeneous system of quasilinear equations can be linearized by a hodograph transformation. The hodograph transformation applied to system (3) does not simplify it. There is another way of using the variables u and v as the independent variables. For solutions with functionally independent u and v the derivatives u_x and v_x can be expressed as

$$u_x = U(u, v), \qquad v_x = V(u, v).$$

Substituting them into system (3), and using the integrability conditions $(u_x)_t = (u_t)_x$, $(v_x)_t = (v_t)_x$, one obtains the system

$$-vU_u + (2V - h)U_v + V = 0, \qquad -(2U + v)V_u - hV_u + h'U = 0. \tag{7}$$

This manuscript is devoted to the group classification of system (7) and constructing its invariant solutions.

3. Group classification of system (7)

Group classification of system (7) is regarded with respect to an arbitrary function $h(u)$. The first step in the group classification is the step of obtaining a group of equivalence transformations. An equivalence transformation is a nondegenerate change of the dependent and independent variables, and arbitrary elements, which transforms a system of differential equations of a given class to a system of equations of the same class. These transformations allow using the simplest representation of the given equations. The next step in the group classification is searching an admitted group of transformations, which is admitted for all arbitrary elements. This group is called a kernel of admitted groups. Note that an admitted group depends on specialization of arbitrary elements. A specialization of the arbitrary elements can extend the admitted group.

3.1. Group of equivalent transformations

For the calculation of equivalence transformations we follow to the approach developed in [11,12]. In this approach all coefficients of the generator

$$X^e = \xi^u \partial_u + \xi^v \partial_v + \zeta^U \partial_U + \zeta^V \partial_V + \zeta^h \partial_h$$

are assumed dependent on all involved variables (u, v, U, V, h). The coefficients of the prolonged generator

$$\tilde{X}^e = X + \zeta^{U_u} \partial_{U_u} + \zeta^{U_v} \partial_{U_v} + \zeta^{V_u} \partial_{V_u} + \zeta^{V_v} \partial_{V_v} + \zeta^{h_u} \partial_{h_u} \cdots$$

are defined by the formulae

$$\zeta^{U_u} = D_u^e \zeta^U - U_u D_u^e \xi^u - U_v D_u^e \xi^v, \qquad \zeta^{U_v} = D_v^e \zeta^U - U_u D_v^e \xi^u - U_v D_v^e \xi^v,$$
$$\zeta^{V_u} = D_u^e \zeta^V - V_u D_u^e \xi^u - V_v D_u^e \xi^v, \qquad \zeta^{V_v} = D_v^e \zeta^V - V_u D_v^e \xi^u - V_v D_v^e \xi^v$$

and

$$\zeta^{h_U} = \zeta^h_U - h_u \xi^u_U, \qquad \zeta^{h_V} = \zeta^h_V - h_u \xi^u_V, \qquad \zeta^{h_u} = \zeta^h_u - h_u \xi^u_u, \qquad \zeta^{h_v} = \zeta^h_v - h_u \xi^u_v.$$

Here

$$D^e_u = \partial_u + U_u\partial_U + V_u\partial_V + h_u\partial_h, \qquad D^e_v = \partial_v + U_v\partial_U + V_v\partial_V,$$

and the property that the function $h(u)$ only depends on u is used. The coefficients ζ^{h_U}, ζ^{h_V}, ζ^{h_v} are also needed since the equations

$$h_U = 0, \qquad h_V = 0, \qquad h_v = 0 \tag{8}$$

have to be invariant with respect to the equivalence group. All necessary calculations were carried on a computer using the symbolic manipulation program REDUCE [13]. The calculations showed that the group of equivalence transformations of system (7) and (8) corresponds to a Lie algebra with the generators

$$X^e_1 = -u\partial_u + V\partial_V + h\partial_h, \qquad X^e_2 = \partial_u, \qquad X^e_3 = 2u\partial_u + v\partial_v + U\partial_U,$$
$$X^e_4 = -v\partial_v + (U+v)\partial_U + (h-2V)\partial_V.$$

The transformations corresponding to the generators X^e_2 and X^e_3 do not change the function $h(u)$. Thus they belong to the kernel of admitted Lie groups. The generator X^e_4 shows the importance of the assumption that all coefficients of the generator of an equivalence group must be considered as dependent on all variables, including arbitrary elements. Without this assumption the generator X^e_4 would be lost.

3.2. Admitted Lie group of transformations

A generator of the admitted Lie group has the form

$$X = \xi^u\partial_u + \xi^v\partial_v + \zeta^U\partial_U + \zeta^V\partial_V,$$

where the coefficients of the generator X are functions of the variables (u, v, U, V). Calculations yield the following results.

The kernel of principal Lie algebras is empty. The generator

$$Y_h = -v\partial_v + (U+v)\partial_U + (h-2V)\partial_V,$$

plays a role of an operator from the kernel: it is admitted for any function $h(u)$, but it also depends on $h(u)$. This symmetry is induced by the transformations corresponding to the generator $\tilde{X}_3$.

An extension of the kernel of principal Lie algebras occurs by specializing the function $h(u)$. It occurs for the functions (up to equivalence transformations): $h(u) = u^\beta$ and $h(u) = e^u$.

If $h(u) = u^\beta$ there is the additional symmetry

$$Y_1 = 2u\partial_u + (\beta+1)v\partial_v + (\beta+1)U\partial_U + 2\beta V\partial_V.$$

If $h(u) = e^u$ (the Liouville equation) there is the additional symmetry

$$Y_2 = 2\partial_u + v\partial_v + U\partial_U + 2V\partial_V.$$

4. Invariant solutions

Here the general case of the function $h(u)$ is studied. In this case system (3) only admits the generator Y_h.

Invariants of the generator Y_h are

$$u,\ v\left(U+\frac{v}{2}\right),\ \left(V-\frac{h}{2}\right)v^{-2}.$$

Thus, an invariant solution has the representation

$$U=-\frac{v}{2}+v^{-1}q(u),\qquad V=\frac{h}{2}+v^2p(u). \tag{9}$$

After substituting the representation of an invariant solution one obtains that the functions $p(u)$ and $q(u)$ have to satisfy the equations

$$4qp'+4hp-h'=0,\qquad 2q'+4pq-h=0. \tag{10}$$

From the second equation of (10) one can define

$$p=\frac{h-2q'}{4q}. \tag{11}$$

After substituting it into the first equation of (10) one has

$$2qq''-2(q')^2+3hq'-2qh'-h^2=0. \tag{12}$$

Since the symmetry Y_h is induced by the transformations corresponding to the generator $\tilde{X}_3$, one may think that the invariant solution (9) also corresponds to the solution of system (3), which is invariant with respect to $\tilde{X}_3$. This is not right. Let us show that the class of invariant solutions of system (7) defined by (9) is not an invariant solution of the original equation (1). Assume that a solution of (9) and (10) gives an invariant solution (6). For the invariant solution (6) the functions $(t+x)^{-1}v$ and u are functionally dependent:

$$((t+x)^{-1}v)_tu_x-((t+x)^{-1}v)_xu_t=(t+x)^{-1}v\left((t+x)^{-1}v-2pq-\frac{h}{2}\right)=0.$$

Hence, $g=2pq+h/2$ and

$$v_x=2pq+\frac{h}{2}+(x+t)\left(2pq+\frac{h}{2}\right)'u_x.$$

Since $u_x=U$, $v_x=V$ and (9), one obtains

$$\frac{h}{2}+v^2p=2pq+\frac{h}{2}+(x+t)\left(2pq+\frac{h}{2}\right)'\left(-\frac{v}{2}+v^{-1}q\right).$$

After substituting (11), (12), and $x+t=(2pq+h/2)^{-1}v$ into the last equation, it becomes

$$(2q'-h)(2q-v^2)=0.$$

Since $v=(x+t)g(x^2-t^2)$ and $q=q(f(x^2-t^2))$ the last equation leads to $q'=h/2$. In this case $p=0$ and (10) gives the condition $h'=0$, which contradicts the assumption about $h(u)$.

Since the main problem is to find a solution of system (3), it is shown here how to find it through an invariant solution (9). The construction of this solution can be done as follows. First one finds the initial values for the functions $u(x, t)$ and $v(x, t)$, for example, along a particular characteristic line $x - t = \text{const}$. These values can be found by the quadrature of[3]

$$\frac{du}{dt} = 2v^{-1}q,$$

where $v = \exp\left(\int (h/2q)\, du\right)$. In fact, along these characteristics

$$\frac{du}{dt} = u_t + u_x = v + 2U = 2v^{-1}q, \qquad \frac{dv}{dt} = v_t + v_x = h.$$

or

$$\frac{dv}{du} = v\frac{h}{2q}.$$

After that in order to obtain a solution one needs to integrate along another set of characteristics $x + t = \text{const}$. Along these set of characteristics

$$\frac{du}{dt} = u_t - u_x = v, \qquad \frac{dv}{dt} = v_t - v_x = h - 2V = -2v^2 p,$$

or

$$\frac{dv}{du} = -2vp = -v\frac{2q' - h}{2q}.$$

Thus the function $u(x, t)$ is found by integrating the equation

$$\frac{du}{dt} = v,$$

where $v = q^{-1} \exp\left(\int \frac{h}{2q}\, du\right)$.

5. Pion Meson equation

The equation describing a one-dimensional motion of a Pion Meson particle in an atom is the following:[4]

$$u_{tt} - u_{xx} + m^2 u + \lambda u^3 = 0, \tag{13}$$

where m is the mass of the Pion Meson, and the cubic term in (13) describes the Pion self-interaction with the effective coupling constant λ. For this equation

$$h = -m^2 u - \lambda u^3.$$

[3] It is assumed that the function $q(u)$, which is a solution of Eq. (12), is known.
[4] See, for example [14–16].

One class of solutions of Eq. (12) is

$$q = -\frac{\lambda u^4}{4} - \frac{m^2u^2}{2} + a_0,$$

where a_0 is an arbitrary constant. Note that $h = q'$. In this case $\exp\left(\int (h/2q)\,\mathrm{d}u\right) = a_1\sqrt{q}$, where a_1 is the constant of integration. Hence, along the characteristic line $x - t = \text{const}$.

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{2}{a_1}\sqrt{q},$$

and along another set of characteristics $x + t = \text{const.}$,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{a_1}{\sqrt{q}}.$$

6. Conclusion

One more method of using technique of invariant solutions for constructing exact solutions of partial differential equations is presented in the article. The method allows constructing new solutions.

Acknowledgments

The author (AH) thanks the Thailand Research Fund for financial support.

References

[1] L.V. Ovsyannikov, Group Analysis of Differential Equations, Nauka, Moscow, 1978 (English translation, Academic Press, New York, 1982).
[2] S. Lie, Collected Works, B.G. Teubner/H. Aschehoug, Leipzig/Oslo, 1922.
[3] G.W. Bluman, J.D. Cole, The general similarity solution of the heat equation, J. Math. Mech. 18 (1969) 1025–1042.
[4] N.H. Ibragimov (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, vol. 1, CRC Press, Boca Raton, FL, 1994;
N.H. Ibragimov (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, vol. 2, CRC Press, Boca Raton, FL, 1995;
N.H. Ibragimov (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, vol. 3, CRC Press, Boca Raton, FL, 1996.
[5] N.N. Yanenko, Compatibility theory and methods of integration of systems of nonlinear partial differential equations, in: Proceedings of the Fourth All-union Mathematics Congress, 1964, p. 613.
[6] S. Lie, Diskussion der differentialgleichung $z_{xy} = f(z)$, Arch. Math. 6 (1) (1881) 112–124.
[7] A.V. Zhiber, A.B. Shabat, Klein–Gordon equations with a nontrivial group, Dokl. Akad. Nauk. SSSR 247 (1979) 1103–1107.
[8] A.V. Zhiber, N.H. Ibragimov, A.B. Shabat, Equations of the Houville type, Dokl. Akad. Nauk. SSSR 249 (1979) 26–29.
[9] N.H. Ibragimov, Transformation Groups Applied to Mathematical Physics, Nauka, Moscow, 1983.
[10] N.H. Ibragimov, S.V. Khabirov, Contact transformation group classification of nonlinear wave equations, Nonlinear Dyn. 22 (2000) 61–71.

[11] S.V. Meleshko, Group classification of the equations of two-dimensional motions of a gas, J. Appl. Math. Mech. 58 (1995) 629–635.
[12] S.V. Meleshko, Generalization of the equivalence transformations, J. Nonlinear Math. Phys. 3 (1996) 170–174.
[13] A.C. Hearn, REDUCE: User's and Contributed Packages Manual, Version 3.7, Rand Corp., California, 1999.
[14] A. Das, T. Ferbel, Introduction to Nuclear and Particle Physics, Wiley, New York, 1994.
[15] M.E. Peskin, D.V. Schvoedev, An Introduction to Quantum Field Theory, Addison-Wesley, New York, 1994.
[16] S.K. Kenneth, Introductory Nuclear Physics, Wiley, New York, 1987.