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Quantum dynamics of the Stern-Gerlach (S-G) effect

E.B. Manoukian[a] and A. Rotjanakusol

School of Physics, Suranaree University of Technology Nakorn Ratchasima 3000, Thailand

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Abstract. A quantum dynamical treatment of the S-G effect, to the leading order in $|e|/\sqrt{\hbar c} \equiv \sqrt{\alpha}$ for the electron, where α is the fine-structure constant, and for spin 1/2 charged particles (*e.g.*, the proton), in general, leads to a unitary expression for the probability density on the observation screen, where the magnetic field has a controllable longitudinal uniform component along the initial average direction of propagation of the particle, in addition to a non-uniform, almost longitudinal, magnetic field lying in the plane defined by the quantization axis, in question, of the spin and the initial average direction of propagation.

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1 Introduction

Much effort has been done in the literature (*e.g.*, [1–9]) on the theoretical description of the S-G [10–12] effect over the years. Unfortunately, there does not seem to exist an *analytical* dynamical treatment of the problem which is: (1) quantum mechanical, and takes into account (2) the field equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, (3) the quantum counterpart of the Lorentz force, (4) the two, rather than one, dimensional aspect of the beam hitting the observation screen, (5) the rather non-trivial correlations that occur between the dynamical variables, as will be shown, to exist, describing the intensity distribution on the screen. It is the purpose of this paper to carry out a theoretical analysis of the problem which takes into account *all* of the above five points just mentioned. An analytical dynamical treatment of the S-G effect to the leading order in $|e|/\sqrt{\hbar c} \equiv \sqrt{\alpha}$ for the electron, where α is the fine-structure constant, and for spin 1/2 charged particles (*e.g.* the proton), in general, leads to a unitary, *i.e.*, positive definite, expression for the probability intensity distribution on the observation screen, where the magnetic field has a controllable uniform component along the initial average direction of propagation of the particle, in addition to a non-uniform, almost longitudinal, magnetic field lying in the longitudinal plane defined by the quantization axis, in question, of the spin and the initial average direction of propagation. With an initially prepared Gaussian wavepacket, it leads to a sum of so-called bivariate normal distributions for the probability intensity distribution with non-zero correlation [13]. The uniform longitudinal controllable magnetic field, as will be shown, has a dual role in our analysis. Although longitudinal, it reduces effectively the quantum Lorentz force contribution by reducing, in turn, the correlation between the dynamical variables describing the probability density of observation, and also provides a positive definite expression for the latter.

It is perhaps surprising, but nevertheless well-known [2,11,12], that the S-G experiment, as such a basic experiment of quantum physics, has not been carried out for the electron. The reason for the obstacle in carrying out the experiment, is that the Lorentz force arising form a transversal magnetic field to the initial average direction of propagation, in the classic apparatus, causes an obvious deviation of the particle from its initial path thus leading to a blurring [2,14] of the expected splitting of a beam. Because of this, the feasibility of performing the experiment with longitudinal non-uniform magnetic fields was emphasized many years ago [14] and emphasized again and brought to the attention of the physics community recently [2]. Another aspect of a transversal magnetic field is that a non-uniform magnetic field perpendicular to the non-uniform component along the quantization axis of the spin, tends to cause, in general, a further splitting of the beam in a direction perpendicular to the quantization axis as well. The importance of the consideration of the S-G effect for the electron itself is evident. To this end, it worth recalling the statement made by Albert Einstein: "... We know, it would be sufficient to really understand the electron" as quoted in [12]. As a quantum measurement problem, the one involving the S-G effect is perhaps the simplest [4–6,12,13,15,16] and one of the easiest for interpretation. Finally, we will see that our analysis also applies to neutral particles.

[a] e-mail: edouard@ccs.sut.ac.th

2 Preliminaries

We consider the Pauli Hamiltonian

$$H = \frac{\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2}{2M} - \boldsymbol{\mu}\cdot\mathbf{B} \tag{2.1}$$

with

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \quad \boldsymbol{\nabla}\cdot\mathbf{A} = 0 \tag{2.2}$$

$$\boldsymbol{\mu} = \mu\boldsymbol{\sigma}, \quad \mu = \frac{q\hbar}{4Mc}g \tag{2.3}$$

and where the σ_i denote the Pauli matrices. For g-factor one has, *e.g.*, $g \simeq 2$ for the electron and $g \simeq 5.59$ for the proton. In terms of the dimensionless parameter

$$\alpha_q = \frac{|q|^2}{\hbar c}, \quad \varepsilon(q) = \mathrm{sgn}(q) \tag{2.4}$$

we may write the interaction Hamiltonian in (2.1) as

$$H_I = \varepsilon(q)\sqrt{\alpha_q} \times \left[-\sqrt{\frac{\hbar}{c}}\mathbf{A}\cdot\mathbf{p} + \varepsilon(q)\sqrt{\alpha_q}\frac{\hbar}{2Mc}\mathbf{A}^2 - \frac{1}{4M}\sqrt{\frac{\hbar^3}{c}}g\boldsymbol{\sigma}\cdot\mathbf{B}\right]. \tag{2.5}$$

For the electron, $q = -|e|$, α denotes the fine-structure constant.

For the initial wavepacket ($t = 0$), in the $\mathbf{x}$-description, we take the Gaussian type

$$\Psi_0(\mathbf{x}) = \frac{1}{(2\pi)^{3/4}\gamma^{3/2}}\exp\left(\frac{\mathrm{i}}{\hbar}\mathbf{p}_0\cdot\mathbf{x}\right)\exp\left(-\frac{\mathbf{x}^2}{4\gamma^2}\right) \tag{2.6}$$

where we have denoted the variance by γ^2 in order not to confuse it with the Pauli matrices, and

$$\mathbf{p}_0 = (0, p_0, 0). \tag{2.7}$$

Here the x_2-axis denotes the initial average direction of propagation of the particle. In the absence of a magnetic field, the state function in (2.6) evolves in time to

$$\Psi_0(\mathbf{x}, t) = \frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathbf{p}_0\cdot\mathbf{x}}\mathrm{e}^{\frac{-\mathrm{i}\mathbf{p}_0^2 t}{2M\hbar}}}{(2\pi)^{3/4}\gamma^{3/2}\left(1 + \frac{\mathrm{i}\hbar t}{2M\gamma^2}\right)^{3/2}}\exp\left(-\frac{\left(\mathbf{x} - \frac{\mathbf{p}_0}{M}t\right)^2}{4\gamma^2\left(1 + \frac{\mathrm{i}\hbar t}{2M\gamma^2}\right)}\right) \tag{2.8}$$

and

$$|\Psi_0(\mathbf{x}, t)|^2 = \frac{1}{(2\pi)^{3/2}\gamma^3(t)}\exp\left(-\frac{\left(\mathbf{x} - \frac{\mathbf{p}_0}{M}t\right)^2}{2\gamma^2(t)}\right) \tag{2.9}$$

where

$$\gamma(t) = \gamma\left(1 + \frac{h^2 t^2}{4M^2\gamma^4}\right)^{1/2}. \tag{2.10}$$

Here we have chosen a common γ-width in all directions to simplify the grouping together of the various terms in the final expression for the probability density in carrying out the x_2-integration. This is not a serious restriction.

For the magnetic field we choose the simple form

$$\mathbf{B} = (0,\ b - \beta x_2,\ \beta x_3) \tag{2.11}$$

with

$$\mathbf{A} = (b x_3,\ \beta x_1 x_3,\ \beta x_1 x_2) \tag{2.12}$$

satisfying (2.2), where b and β are some constants. We note that the component $(b - \beta x_2)$ is along the initial average direction of propagation specified by $\mathbf{p}_0$. The observation screen is defined by the (x_1, x_3)-plan for $t > 0$, with x_3 denoting the traditional quantization axis of the spin. We will see that the uniform part $(0, b, 0)$ of the magnetic field, although longitudinal, may be appropriately set up to effectively reduce the quantum mechanical counterpart of the Lorentz-force contribution by reducing, in turn, the correlation that occurs between the x_1 and x_3 variables on the screen, an also provide a positive definite expression for the probability density distribution in question. Concerning the non-uniform part $(0, -\beta x_2, \beta x_3)$ of the magnetic field, we note that since $|x_2|$, a macroscopic distance, is much larger than $|x_3|$ (providing a measure of the splitting of the beam), this non-uniform magnetic field is almost longitudinal along the direction of propagation.

In the above set up, as a working hypothesis, we treat the particles as if they are throughout in the magnetic field. Otherwise, an analytical treatment is not so manageable. We hope to turn to a such refinement in a subsequent report.

The dynamics is most elegantly described in terms of the density operator, which at $t = 0$, is given by

$$\rho = \omega_+\begin{pmatrix}1\\0\end{pmatrix}|\Psi\rangle\langle\Psi|\begin{pmatrix}1 & 0\end{pmatrix} + \omega_-\begin{pmatrix}0\\1\end{pmatrix}|\Psi\rangle\langle\Psi|\begin{pmatrix}0 & 1\end{pmatrix} \tag{2.13}$$

where

$$\omega_+ + \omega_- = 1 \tag{2.14}$$

for

$$\omega_+ = \omega_- = \frac{1}{2} \tag{2.15}$$

one would be dealing with an unpolarized beam. For $t > 0$, the density operator is given by

$$\rho(t) = \omega_+\mathrm{e}^{-\frac{\mathrm{i}t}{\hbar}H}\begin{pmatrix}1\\0\end{pmatrix}|\Psi\rangle\langle\Psi|\begin{pmatrix}1 & 0\end{pmatrix}\mathrm{e}^{\frac{\mathrm{i}t}{\hbar}H} + \omega_-\mathrm{e}^{-\frac{\mathrm{i}t}{\hbar}H}\begin{pmatrix}0\\1\end{pmatrix}|\Psi\rangle\langle\Psi|\begin{pmatrix}0 & 1\end{pmatrix}\mathrm{e}^{\frac{\mathrm{i}t}{\hbar}H}. \tag{2.16}$$

The probability density is then (for $t > 0$)

$$\langle\mathbf{x}|\rho(t)|\mathbf{x}\rangle \tag{2.17}$$

and for the probability density, in question, on the screen one may then most conveniently write it as

$$\begin{aligned} f(x_1,x_3;t) &= \int_{-\infty}^{\infty} \mathrm{d}x_2 \langle \mathbf{x}|\rho(t)|\mathbf{x}\rangle \\ &= \omega_+ \int_{-\infty}^{\infty} \mathrm{d}x_2 \left| \langle \mathbf{x}| \mathrm{e}^{-\frac{\mathrm{i}t}{\hbar}H} \begin{pmatrix} 1 \\ 0 \end{pmatrix} |\Psi\rangle \right|^2 \\ &\quad + \omega_- \int_{-\infty}^{\infty} \mathrm{d}x_2 \left| \langle \mathbf{x}| \mathrm{e}^{-\frac{\mathrm{i}t}{\hbar}H} \begin{pmatrix} 0 \\ 1 \end{pmatrix} |\Psi\rangle \right|^2 \end{aligned} \tag{2.18}$$

where we note the unitarity of $\exp(-\mathrm{i}tH/\hbar)$, and the orthogonality of $\begin{pmatrix} 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \end{pmatrix}$.

3 The intensity distribution

With $\exp(-\mathrm{i}tH/\hbar)$ as the time-evolution operator, the following expectation values of the Heisenberg operators in the state (2.6), relevant to the observation screen, to the leading order in $\sqrt{\alpha_q}$, are readily obtained:

$$\langle x_1(t)\rangle = 0 \tag{3.1}$$

$$\langle x_3(t)\rangle = \frac{\mu\beta}{2M}\sigma_3 t^2 \tag{3.2}$$

and the important non-trivial correlation occurring between the dynamical variables $x_1(t)$, $x_3(t)$:

$$\langle (x_1(t)-\langle x_1(t)\rangle)(x_3(t)-\langle x_3(t)\rangle)\rangle = -\frac{qbt\gamma^2}{Mc} + \frac{q\beta p_0 t^2\gamma^2}{2M^2c} + \frac{q\beta p_0 t^4\hbar^2}{24M^2\gamma^2 c} \tag{3.3}$$

with

$$\left(\left\langle (x_1(t)-\langle x_1(t)\rangle)^2\right\rangle\right)^{1/2} = \left(\left\langle (x_3(t)-\langle x_3(t)\rangle)^2\right\rangle\right)^{1/2} = \gamma(t). \tag{3.4}$$

To study the probability intensity distribution, we note the following commutation relations:

$$\begin{aligned} \left[\frac{\mathbf{p}^2}{2M}, H_\mathrm{I}\right] &= \frac{\mathrm{i}\hbar q\beta}{M^2c}x_3p_1p_2 + \frac{\mathrm{i}\hbar q}{M^2c}(\beta x_2+b)p_1p_3 \\ &\quad + \frac{2\mathrm{i}\hbar q\beta}{M^2c}x_1p_2p_3 - \frac{\mathrm{i}\hbar\mu\beta}{M}\sigma_2p_2 \\ &\quad + \frac{\mathrm{i}\hbar\mu\beta}{M}\sigma_3p_3 \end{aligned} \tag{3.5}$$

$$\left[\frac{\mathbf{p}^2}{2M}, \left[\frac{\mathbf{p}^2}{2M}, H_\mathrm{I}\right]\right] = \frac{4\hbar^2 q\beta}{M^3c}p_1p_2p_3 \tag{3.6}$$

with all the other commutators with $\mathbf{p}^2/2M$ vanish or are of higher order. We use a variation of the Baker-Campbell-Hausdorff formula: if

$$[B,[A,B]] = 0 \tag{3.7}$$

$$[B,[A,[A,B]]] = 0 \tag{3.8}$$

$$[A,[A,[A,B]]] = 0 \tag{3.9}$$

for two operators A, B, then

$$\mathrm{e}^{A+B} = \mathrm{e}^{\left(\frac{1}{2}[A,B]+\frac{1}{6}[A,[A,B]]\right)}\mathrm{e}^B\mathrm{e}^A. \tag{3.10}$$

We let

$$A = -\frac{\mathrm{i}t}{\hbar}\frac{\mathbf{p}^2}{2M} \tag{3.11}$$

$$B = -\frac{\mathrm{i}t}{\hbar}H_\mathrm{I} \tag{3.12}$$

and note that (3.7–3.9) hold true to the accuracy retained. Equation (3.10) then gives

$$\begin{aligned} \exp\left(-\frac{\mathrm{i}t}{\hbar}H\right) &= \exp\left(-\frac{t^2}{2\hbar^2}\left[\frac{\mathbf{p}^2}{2M}, H_\mathrm{I}\right]\right) \\ &\quad \times \exp\left(\frac{2\mathrm{i}t^3 q\beta}{3\hbar M^3 c}p_1p_2p_3\right) \\ &\quad \times \exp\left(-\frac{\mathrm{i}t}{\hbar}H_\mathrm{I}\right)\exp\left(-\frac{\mathrm{i}t}{\hbar}H_0\right) \end{aligned} \tag{3.13}$$

where $H_0 = \mathbf{p}^2/2M$ is the free Hamiltonian, and $(\exp(-\mathrm{i}tH_0/\hbar)\Psi)(\mathbf{x}) \equiv \Psi_0(\mathbf{x},t)$ is explicitly given in (2.8).

To carry out the time-evolution operation given in (3.13) on Ψ we use, in the process, the identity

$$\mathrm{e}^{\mathrm{i}a\frac{p}{\hbar}}f(x) = f(x+a). \tag{3.14}$$

The operation defined on the right-hand side of (3.13) on Ψ may be then carried out. The analysis is very tedious but straightforward. Up to a normalization factor, and the phase factor $\exp(\mathrm{i}t\mathbf{p}_0\cdot\mathbf{x}/\hbar)$, $(\exp(-\mathrm{i}tH/\hbar)\Psi)(\mathbf{x})$, is given by the expression:

$$F_1(x_1, x_2-\frac{p_0}{M}t, x_3; t)F_2\left(x_2-\frac{p_0}{M}t; t\right)F_3(x_1,x_3;t) \tag{3.15}$$

where we have conveniently isolated the terms dependent on the variable x_2, in the two factors F_1, F_2, as we have to integrate over it as indicated in (2.18). With

$$x_2' = x_2 - \frac{p_0}{M}t \tag{3.16}$$

$$F = \frac{1}{4\gamma^2\left(1+\frac{\mathrm{i}\hbar t}{2M\gamma^2}\right)} \equiv F(t) \tag{3.17}$$

we have

$$\begin{aligned} F_1(x_1,x_2',x_3) &= \exp\left(-x_2'^2F\right)\exp\left(-\frac{4tq\beta}{Mc}x_1x_2'x_3F\right) \\ &\quad \times \exp\left(\frac{8\mathrm{i}\hbar q\beta t^2}{M^2c}x_1x_2'x_3F^2\right)\exp\left(\frac{16q\beta\hbar^2t^3}{3M^3c}x_1x_2'x_3F^3\right) \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} F_2(x_2';t) &= \exp\left(\frac{\mathrm{i}t\mu}{\hbar}\sigma_2\left(b-\beta\frac{p_0}{M}t\right)\right)\exp\left(-\frac{\mathrm{i}t\mu\beta\sigma_2}{\hbar}x_2'\right) \\ &\quad \times \exp\left(\frac{\mathrm{i}p_0\mu\beta t^2}{2\hbar M}\sigma_2\right)\exp\left(-\frac{\mu\beta t^2}{M}\sigma_2x_2'F\right) \end{aligned} \tag{3.19}$$

$$F_3(x_1,x_3;t) = \exp\left(\frac{\mathrm{i}tq\beta p_0}{\hbar Mc}x_1x_3\right)\exp\left(\frac{\mathrm{i}t\mu\beta}{\hbar}\sigma_3x_3\right)$$
$$\times\exp\left(\frac{p_0q\beta t^2}{M^2c}x_1x_3F\right)\exp\left(-\frac{2tqb}{Mc}x_1x_3F\right)$$
$$\times\exp\left(-\frac{8\mathrm{i}p_0t^3q\beta\hbar}{3M^3c}x_1x_3F^2\right)\exp\left(\frac{\mu\beta t^2}{M}\sigma_3x_3F\right)$$
$$\times\exp\left(\frac{2\mathrm{i}\hbar qt^2}{M^2c}\left(\beta\frac{p_0}{M}t+b\right)x_1x_3F^2\right)\exp\left(-\left(x_1^2+x_3^2\right)F\right). \quad (3.20)$$

Now we have to apply the operator $(F_1F_2F_3)$ in (3.15) to $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, and perform the operations defined in (2.18). To this end, we use the identities:

$$\sigma_3\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},\quad \sigma_3\begin{pmatrix}0\\1\end{pmatrix}=-\begin{pmatrix}0\\1\end{pmatrix} \quad (3.21)$$

$$\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\\mathrm{i}\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\-\mathrm{i}\end{pmatrix} \quad (3.22)$$

$$\begin{pmatrix}0\\1\end{pmatrix}=-\frac{\mathrm{i}}{2}\begin{pmatrix}1\\\mathrm{i}\end{pmatrix}+\frac{\mathrm{i}}{2}\begin{pmatrix}1\\-\mathrm{i}\end{pmatrix} \quad (3.23)$$

$$\sigma_2\begin{pmatrix}1\\\mathrm{i}\end{pmatrix}=\begin{pmatrix}1\\\mathrm{i}\end{pmatrix},\quad \sigma_2\begin{pmatrix}1\\-\mathrm{i}\end{pmatrix}=-\begin{pmatrix}1\\-\mathrm{i}\end{pmatrix} \quad (3.24)$$

and the orthogonality of $\begin{pmatrix}1 & \mathrm{i}\end{pmatrix}$, $\begin{pmatrix}1 & -\mathrm{i}\end{pmatrix}$. Also we note that

$$F(t)+F^*(t)=\frac{1}{2\gamma^2(t)} \quad (3.25)$$
$$\mathrm{i}F(t)+(\mathrm{i}F(t))^*=\frac{1}{4\gamma^2(t)}\frac{\hbar t}{M\gamma^2} \quad (3.26)$$
$$\mathrm{i}F^2(t)+\left(\mathrm{i}F^2(t)\right)^*=\frac{1}{8\gamma^4(t)}\frac{\hbar t}{M\gamma^2}. \quad (3.27)$$

From (3.15–3.27) we obtain, up to a normalization factor, the following expression for the x_2-integrand in (2.18):

$$\exp\left(-\frac{1}{2\gamma^2(t)}\left[x_2'^2+a(t)x_1x_2'x_3\right]\right)\left[\exp\left(\frac{t^2}{2\gamma^2(t)}\frac{\mu\beta}{M}x_2'\right)+\exp\left(-\frac{t^2}{2\gamma^2(t)}\frac{\mu\beta}{M}x_2'\right)\right]f(x_1,x_3;t) \quad (3.28)$$

where $a(t)$, of order $\sqrt{\alpha_q}$, is a function of t only, and up to a mutiplicative time-dependent constant,

$$f(x_1,x_3;t)\equiv\omega_+f_+(x_1,x_3;t)+\omega_-f_-(x_1,x_3;t)$$
$$=\omega_+\exp\left(-\frac{1}{2\gamma^2(t)}\left[x_1^2+x_3^2-\frac{t^2}{M}\mu\beta x_3-\frac{x_iA_{ij}x_j}{\gamma^2(t)}\right]\right)$$
$$+\omega_-\exp\left(-\frac{1}{2\gamma^2(t)}\left[x_1^2+x_3^2+\frac{t^2}{M}\mu\beta x_3-\frac{x_iA_{ij}x_j}{\gamma^2(t)}\right]\right). \quad (3.29)$$

A summation over the repeated indices $i,j=1,3$ in (3.29) is understood,

$$A_{13}=A_{31}=-\frac{qbt\gamma^2}{Mc}+\frac{q\beta p_0t^2\gamma^2}{2M^2c}+\frac{q\beta p_0t^4\hbar^2}{24M^4\gamma^2c},$$
$$A_{11}=A_{33}=0. \quad (3.30)$$

The later expression in (3.30) is identical to the correlation of the dynamical variables $x_1(t)$, $x_3(t)$ in (3.3).

In reference to the x_2-integral in (2.18), we have, with

$$b(t)=\frac{\mu\beta t^2}{M} \quad (3.31)$$

for the shifted x_2'-integral,

$$\int_{-\infty}^{\infty}\mathrm{d}x_2'\exp\left(-\frac{1}{2\gamma^2(t)}(x_2'^2+[a(t)x_1x_3\pm b(t)]x_2')\right)=\sqrt{2\pi}\gamma(t)\exp\left(\frac{1}{8\gamma^2(t)}[a(t)x_1x_3\pm b(t)]^2\right) \quad (3.32)$$

where $[a(t)x_1x_3\pm b(t)]^2$ is necessarily of a higher order correction in $\sqrt{\alpha_q}$.

Accordingly, for the probability density $f(x_1,x_3;t)$ we obtain the preliminary expression given in (3.29). Upon setting

$$\langle g\rangle_t^{\pm}=\int\mathrm{d}x_1\mathrm{d}x_3g(x_1,x_3)f_{\pm}(x_1,x_3;t) \quad (3.33)$$
$$\langle g\rangle_0^{\pm}=\langle g\rangle^{\pm} \quad (3.34)$$

where $\gamma(t)$ as defined in (2.10), we note that any significant correction to the one derived in (3.29) consistent with following constraints, as dictated by the explicit expectation values in (3.1–3.4), normalizability and positivity, are easily obtained:

C.1 – $\langle x_1\rangle_t^{\pm}=0+$ higher orders

C.2 – $\langle x_3\rangle_t^{\pm}=\frac{\mu\beta t^2}{2M}\langle\sigma_3\rangle^{\pm}+$ higher orders

C.3 – $\sqrt{\langle x_1^2\rangle_t^{\pm}}=\gamma(t)+$ higher orders

C.4 – $\left(\langle x_3^2\rangle_t^{\pm}-\left(\langle x_3^2\rangle_t^{\pm}\right)^2\right)^{1/2}=\gamma(t)+$ higher orders

C.5 – $\left\langle\left(x_1-\langle x_1\rangle_t^{\pm}\right)\left(x_3-\langle x_3\rangle_t^{\pm}\right)\right\rangle_t^{\pm}=A_{13}+$higher orders

C.6 – $\int\mathrm{d}x_1\mathrm{d}x_3f(x_1,x_3;t)=1$

C.7 – $f(x_1,x_3;t)$ is real and positive

where A_{13} is given in (3.3, 3.30), and higher orders stand relative to the parameter $\sqrt{\alpha_q}$.

To satisfy, in the process, constraint C.2 (see also (3.2)), we multiply the right-hand side of (3.29) by an overall normalizing factor $\exp\left(-(\mu\beta t^2/2M)^2/2\gamma^2(t)\right)$

giving

$$f(x_1,x_3;t) \propto$$
$$\omega_+ \exp\left(-\frac{1}{2\gamma^2(t)}\left[x_1^2+\left(x_3-\frac{\mu\beta t^2}{2M}\right)^2-\frac{x_iA_{ij}x_j}{\gamma^2(t)}\right]\right)$$
$$+\,\omega_- \exp\left(-\frac{1}{2\gamma^2(t)}\left[x_1^2+\left(x_3+\frac{\mu\beta t^2}{2M}\right)^2-\frac{x_iA_{ij}x_j}{\gamma^2(t)}\right]\right). \tag{3.35}$$

Consistency with the constraints C.1–C.6 necessarily gives

$$f(x_1,x_3;t) = \frac{\sqrt{\det \underset{\sim}{C}}}{2\pi}\left[\begin{array}{l}\omega_+ \exp\left(-\frac{1}{2}(x_i-x_{i0})C^{ij}(x_j-x_{j0})\right)\\ +\omega_- \exp\left(-\frac{1}{2}(x_i+x_{i0})C^{ij}(x_j+x_{j0})\right)\end{array}\right] \tag{3.36}$$

where $[\underset{\sim}{C}]=[C^{ij}]$, $C^{11}=C^{33}=1/\gamma^2(t)$, $i,j=1,3$,

$$C^{13}=C^{31} = \frac{1}{\gamma^4(t)}\left(\frac{qbt\gamma^2}{Mc}-\frac{q\beta p_0t^2\gamma^2}{2M^2c}-\frac{q\beta p_0t^4\hbar^2}{24M^4\gamma^2c}\right) \tag{3.37}$$

$$x_{i0}=\frac{\mu\beta}{2M}t^2\delta_{i3} \tag{3.38}$$

and $\omega_+=\omega_-=1/2$ for an unpolarized beam.

The probability density in (3.36) is a sum of bivariate normal distributions (*e.g.*, [11]) and

$$\left[\Sigma^{ij}\right]=\left[[\underset{\sim}{C}^{-1}]^{ij}\right] \tag{3.39}$$

is the so-called covariance matrix describing the correlation between x_1 and x_3 on the screen for $i\neq j$. $\underset{\sim}{\Sigma}$ is a measure of dispersion in all directions in the (x_1,x_3)-plane. The multiplicative factor $\sqrt{\det\underset{\sim}{C}}/2\pi$ is the standard normalization factor.

Finally, the constrain C.7 implies that $\det\underset{\sim}{C}>0$, *i.e.*, it leads to a positivity requirement. This in turn implies that we should have

$$\frac{|q|t}{Mc}\left|b-\frac{\beta p_0}{2M}t-\frac{\beta p_0t^3\hbar^2}{24M^3\gamma^4}\right|<1+\frac{\hbar^2t^2}{4M^2\gamma^4}. \tag{3.40}$$

In reference to this inequality consider first the case with $b=0$, *i.e.*, the constraint

$$C<1+\frac{\hbar^2t^2}{4M^2\gamma^4} \tag{3.41}$$

with

$$C=\frac{|q|\beta p_0t^2}{2M^2c}\left(1+\frac{\hbar^2t^2}{12M^2\gamma^4}\right). \tag{3.42}$$

By setting,

$$\Delta z=\frac{|\mu|\beta t^2}{2M} \tag{3.43}$$

$$\frac{p_0}{M}t=L \tag{3.44}$$

with the latter denoting the macroscopic distance from the particle's initial center of the wavepacket to the observation screen, we may rewrite C as

$$C=\frac{4L}{|g|}\frac{M}{\hbar}\frac{\Delta z}{t}\left(1+\frac{\hbar^2t^2}{12M^2\gamma^4}\right). \tag{3.45}$$

For the electron with $\Delta z\simeq 10^{-3}$ m, $t\simeq 10^{-6}$ s, $L\simeq 1$ m, $\gamma<10^{-6}$ m

$$C\simeq 1.73\times 10^7\left(1+\frac{1.12\times 10^{-21}}{\gamma^4}\right)$$

which is a very large number and the positivity constraint in (3.41) cannot be satisfied. On the other hand, the uniform magnetic field $(0,b,0)$ may *a priori* be set at

$$b=\frac{\beta}{2}L \tag{3.46}$$

defined simply in terms of the non-uniform magnetic field gradient $\partial B_2/\partial x_2=-\beta=-\partial B_3/\partial x_3$ (see (2.11)) chosen, and the distance to the observation screen L, independently of any of the details of the spin 1/2 charged particle considered and of the (initial) spread γ. [The uniform magnetic field component b may be, of course, chosen so that $C^{13}=0$, but this would mean to choose a different uniform magnetic field for every different charged particle, and a different spread γ, and would not be physically as interesting.] The matrix elements in (3.37) then simply become

$$C^{13}=C^{31}=-\varepsilon(q)\frac{1}{3|g|}\frac{\Delta z}{\gamma}\frac{L}{\gamma}\frac{\hbar t}{M}\frac{1}{\gamma^4(t)} \tag{3.47}$$

and the positivity constraint

$$\frac{1}{3|g|}\frac{\Delta z}{\gamma}\frac{L}{\gamma}\frac{\hbar t}{M\gamma^2}<1+\frac{\hbar^2t^2}{4M^2\gamma^4} \tag{3.48}$$

is readily satisfied. For example, for the electron with $\Delta z=10^{-3}$ m, $L=0.7$ m, $\gamma=0.55\times 10^{-3}$ m, $t=4\times 10^{-6}$ s, corresponding to an initial average speed of 1.75×10^5 m/s, a magnetic field gradient $\beta=12.280$ T/m, and a uniform longitudinal magnetic field $b=4.298$ T, the left-hand side of (3.48) is $\simeq 0.59$. In Figure 1, the probability density $f(x_1,x_3;t)$ for $\mathbf{B}=0$, $t=4\times 10^{-6}$ s is plotted for $\gamma=0.55\times 10^{-3}$ m and the corresponding density for $\mathbf{B}\neq 0$, for the above just given parameters, is plotted in Figure 2, for an initially unpolarized beam, showing a clear splitting of the beam along the quantization axis. [The magnetic field b may be chosen to be even smaller. For example, for slower electrons $t=5.93\times 10^{-6}$ s, $L=0.5$ m, $b=1.4$ T consistent with (3.48).] The asymmetry with elongations in the second and the fourth quadrants in Figure 2 are easy to understand. For the electron $\varepsilon(q)=-1$, $C^{13}=C^{31}>0$ and the probability density gets, respectively, positive amplifying contributions for $x_3>x_{30}$, $x_1<0$ and $x_3<-x_{30}$, $x_1>0$. This graph corresponds to a negatively charged particle. The formal physical argument for this asymmetry is that it arises as a consequence of the direction of the

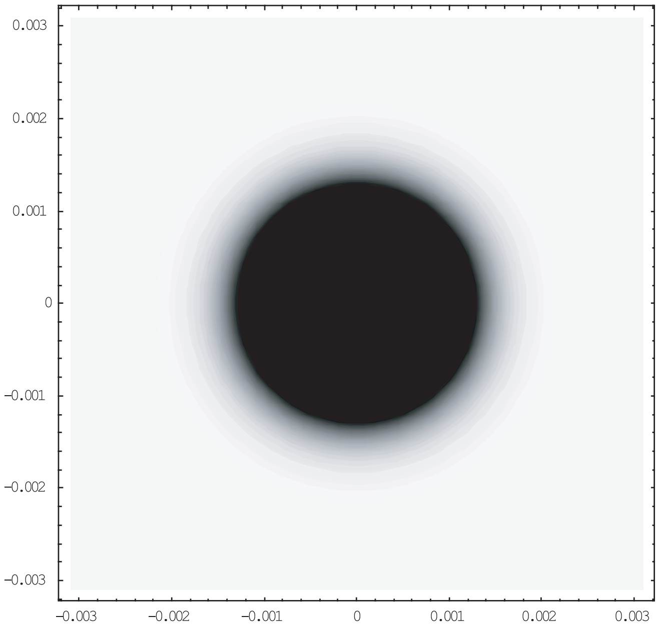

Fig. 1. Plot of the density $f(x_1, x_3; t)$ for $\mathbf{B} = 0$, $\gamma = 0.55 \times 10^{-3}$ m, $t = 4 \times 10^{-6}$ s.

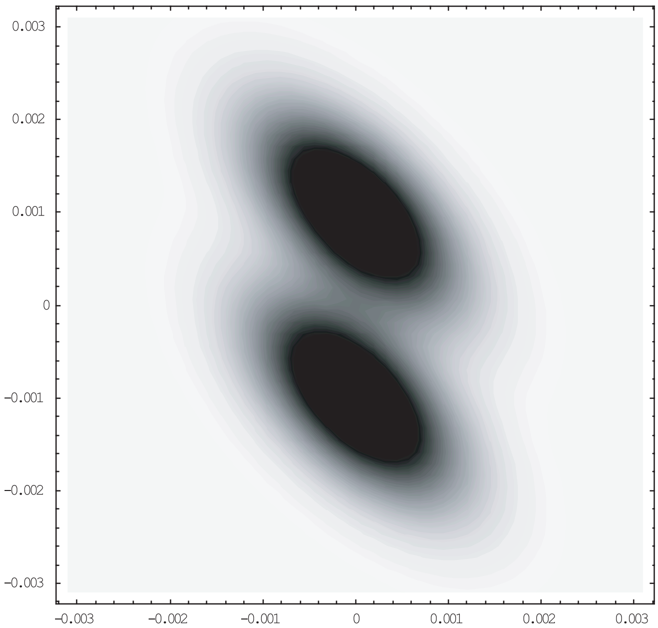

Fig. 2. Plot of the density $f(x_1, x_3; t)$ for the electron, based on (3.36, 3.46, 3.47) with $\Delta z = 10^{-3}$ m, $\gamma = 0.55 \times 10^{-3}$ m, $t = 4 \times 10^{-6}$ s, $L = 0.7$ m, corresponding to an initial average speed of 1.75×10^5 m/s, a magnetic field gradient $\beta = 12.280$ T/m, and a uniform longitudinal magnetic field $b = 4.298$ T.

Lorentz force, as determined by the so-called right-hand rule, on a charged particle as applied to the transverse part of the non-uniform magnetic field. For a positive charge, the elongations in opposite directions occur in the corresponding first and third quadrants.

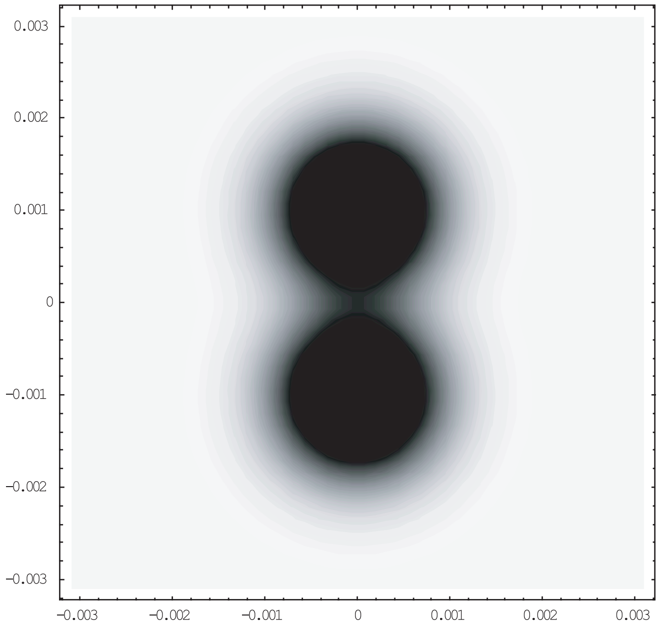

Fig. 3. Plot of the density $f(x_1, x_3; t)$ for uncharged particles, based on (3.49) for, $|x_0| = 1 \times 10^{-3}$ m, $\gamma = 0.55 \times 10^{-3}$ m, $t = 4 \times 10^{-6}$ s.

We note that the correlation in (3.3) and $C^{13} = C^{31}$ in (3.37) vanish for neutral particles. The analysis carried above (with $C^{13} = C^{31}$ set equal to zero), is equally valid for neutral spin 1/2 particles with magnetic moment $\boldsymbol{\mu} = \mu\boldsymbol{\sigma}$, as carried to the leading order in $M|\mu|/|g|\left(\hbar^3 c\right)^{1/2}$, and finally leads to the expression

$$f(x_1, x_3; t) = \frac{1}{2\pi\gamma^2(t)} \left[\begin{array}{c} \omega_+ \exp\left(-\frac{1}{2}\frac{x_1^2 + (x_3 - x_0)^2}{\gamma^2(t)}\right) \\ +\omega_- \exp\left(-\frac{1}{2}\frac{x_1^2 + (x_3 + x_0)^2}{\gamma^2(t)}\right) \end{array} \right] \quad (3.49)$$

where $x_0 = \mu\beta t^2/2M$. For an unpolarized beam, this is plotted in Figure 3 for $t = 4 \times 10^{-6}$ s, $|x_0| = 1 \times 10^{-3}$ m, $\gamma = 0.55 \times 10^{-3}$ m, showing the difference of the densities for the charged and uncharged cases in the presence of an appropriately chosen longitudinal uniform magnetic field.

It is expected that an experiment, as described in the bulk of this work, may be indeed carried out with relatively small magnetic fields and we hope that it will encourage experimentalists to do so.

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