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Invariants and invariant description of second-order ODEs with three infinitesimal symmetries. I

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Abstract

Lie's group classification of ODEs shows that the second-order equations can possess one, two, three or eight infinitesimal symmetries. The equations with eight symmetries and only these equations can be linearized by a change of variables. Lie showed that the latter equations are at most cubic in the first derivative and gave a convenient invariant description of all linearizable equations. Our aim is to provide a similar description of the equations with three symmetries. There are four different types of such equations. We present here the candidates for all four types. We give an invariant test for existence of three symmetries for one of these candidates.

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1. Introduction

According to Lie's classification [1] in the complex domain, any ordinary differential equation of the second order

$$y'' = f(x, y, y') \tag{1}$$

admitting a three-dimensional Lie algebra belongs to one of four distinctly different types. Each of these four types is obtained by a change of variables from the following canonical representatives (see, e.g., [2, Section 8.4]):

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$$y'' + Cy^{-3} = 0, \tag{2}$$
$$y'' + Ce^{y'} = 0, \tag{3}$$
$$y'' + Cy'^{(k-2)/(k-1)} = 0, \tag{4}$$
$$y'' + 2\frac{y' + Cy'^{3/2} + y'^2}{x - y} = 0, \tag{5}$$

where $k \neq 0, 1/2, 1, 2$ in (4), and $C = \text{const}$.

Eqs. (2)–(5) admit non-similar three-dimensional Lie algebras L_3 spanned by the operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \tag{6}$$
$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + (y - x)\frac{\partial}{\partial y}, \tag{7}$$
$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}, \tag{8}$$

and

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \tag{9}$$

respectively (see, e.g., [2, Section 8.4]).

2. Candidates for equations with three symmetries

Let us subject each of Eqs. (2)–(5) to the arbitrary change of variables

$$t = \varphi(x,y), \qquad u = \psi(x,y), \tag{10}$$

where t is a new independent variable and u is a new dependent variable. Then we obtain from (2)–(5) the equations of the form

$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 = 0, \tag{11}$$
$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + (b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8)\exp\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right) = 0, \tag{12}$$
$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + (b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8)\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right)^{(k-2)/(k-1)} = 0, \tag{13}$$

and

$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + (b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8)\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right)^{3/2} = 0, \tag{14}$$

respectively, where $b_i = b_i(t,u)$, $i = 1,\ldots,12$. Eqs. (11)–(14) are the *candidates for the equations with three symmetries.*

All candidates can be encapsulated in the formula

$$u'' + b_1u'^3 + 3b_2u'^2 + 3b_3u' + b_4 + (b_5u'^3 + 3b_6u'^2 + 3b_7u' + b_8)f\left(\frac{b_9u' + b_{10}}{b_{11}u' + b_{12}}\right) = 0.$$

Namely, Eqs. (11)–(14) are obtained by letting

$$f(z) = 0, \qquad f(z) = e^z, \qquad f(z) = z^{(k-2)/(k-1)}, \qquad f(z) = z^{3/2}.$$

Using the usual formula for the transformation of derivatives under the change of variables (10), we obtain the following statement.

Theorem 1. *Any equation of the form*

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 + (b_5 u'^3 + 3b_6 u'^2 + 3b_7 u' + b_8) f\left(\frac{b_9 u' + b_{10}}{b_{11} u' + b_{12}}\right) = 0$$

is transformed by the change of variables (10) *into an equation of the same form*:

$$y'' + a_1 y'^3 + 3a_2 y'^2 + 3a_3 y' + a_4 + (a_5 y'^3 + 3a_6 y'^2 + 3a_7 y' + a_8) f\left(\frac{a_9 y' + a_{10}}{a_{11} y' + a_{12}}\right) = 0.$$

Here a_i and b_i are functions of x, y and t, u, respectively, and are connected by

$$\begin{aligned}
a_1 &= \Delta^{-1}[\varphi_y \psi_{yy} - \varphi_{yy}\psi_y + b_4 \varphi_y^3 + 3b_3 \varphi_y^2 \psi_y + 3b_2 \varphi_y \psi_y^2 + b_1 \psi_y^3],\\
a_2 &= \Delta^{-1}[b_4 \varphi_x \varphi_y^2 + b_3 \varphi_y(2\varphi_x \psi_y + \varphi_y \psi_x) + b_2 \psi_y(\varphi_x \psi_y + 2\varphi_y \psi_x)\\
&\quad + b_1 \psi_x \psi_y^2 + (\varphi_x \psi_{yy} - \varphi_{yy}\psi_x - 2\varphi_{xy}\psi_y + 2\varphi_y \psi_{xy})/3],\\
a_3 &= \Delta^{-1}[b_4 \varphi_x^2 \varphi_y + b_3 \varphi_x(\varphi_x \psi_y + 2\varphi_y \psi_x) + b_2 \psi_x(2\varphi_x \psi_y + \varphi_y \psi_x)\\
&\quad + b_1 \psi_x^2 \psi_y + (\varphi_y \psi_{xx} - \varphi_{xx}\psi_y - 2\varphi_{xy}\psi_x + 2\varphi_x \psi_{xy})/3],\\
a_4 &= \Delta^{-1}[b_4 \varphi_x^3 + 3b_3 \varphi_x^2 \psi_x + 3b_2 \varphi_x \psi_x^2 + b_1 \psi_x^3 - \varphi_{xx}\psi_x + \varphi_x \psi_{xx}],\\
a_5 &= \Delta^{-1}[b_8 \varphi_y^3 + 3b_7 \varphi_y^2 \psi_y + 3b_6 \varphi_y \psi_y^2 + b_5 \psi_y^3],\\
a_6 &= \Delta^{-1}[b_8 \varphi_x \varphi_y^2 + b_7 \varphi_y(2\varphi_x \psi_y + \varphi_y \psi_x) + b_6 \psi_y(\varphi_x \psi_y + 2\varphi_y \psi_x) + b_5 \psi_x \psi_y^2],\\
a_7 &= \Delta^{-1}[b_8 \varphi_x^2 \varphi_y + b_7 \varphi_x(\varphi_x \psi_y + 2\varphi_y \psi_x) + b_6 \psi_x(2\varphi_x \psi_y + \varphi_y \psi_x) + b_5 \psi_x^2 \psi_y],\\
a_8 &= \Delta^{-1}[b_8 \varphi_x^3 + 3b_7 \varphi_x^2 \psi_x + 3b_6 \varphi_x \psi_x^2 + b_5 \psi_x^3],\\
a_9 &= b_{10}\varphi_y + b_9 \psi_y,\\
a_{10} &= b_{10}\varphi_x + b_9 \psi_x,\\
a_{11} &= b_{12}\varphi_y + b_{11}\psi_y,\\
a_{12} &= b_{12}\varphi_x + b_{11}\psi_x,
\end{aligned}$$

where

$$\Delta = (\varphi_x \psi_y - \varphi_y \psi_x) \neq 0$$

is the Jacobian of the change of variables (10).

3. Equations equivalent to Eq. (2)

In this paper, we will dwell on the first candidate, i.e., on equations of the form (11). Other candidates will be considered elsewhere.

We know that all equations obtained from Eq. (2),

$$y'' + Ky^{-3} = 0 \quad (K = \text{const.} \neq 0), \tag{15}$$

by the change of variables (10) are contained in the family of the equations of the form (11):

$$u'' + b_1 u'^3 + 3b_2 u'^2 + 3b_3 u' + b_4 = 0.$$

We also know from Theorem 1 that any Eq. (11) is transformed by the change of variables (10) into an equation of the same form:

$$y'' + a_1 y'^3 + 3a_2 y'^2 + 3a_3 y' + a_4 = 0, \tag{16}$$

and that the coefficients of Eqs. (11) and (16) are related by the following equations:

$$\begin{aligned} a_1 &= \Delta^{-1}[\varphi_y\psi_{yy} - \varphi_{yy}\psi_y + b_4\varphi_y^3 + 3b_3\varphi_y^2\psi_y + 3b_2\varphi_y\psi_y^2 + b_1\psi_y^3], \\ a_2 &= \Delta^{-1}[b_4\varphi_x\varphi_y^2 + b_3\varphi_y(2\varphi_x\psi_y + \varphi_y\psi_x) + b_2\psi_y(\varphi_x\psi_y + 2\varphi_y\psi_x) \\ &\quad + b_1\psi_x\psi_y^2 + (\varphi_x\psi_{yy} - \varphi_{yy}\psi_x - 2\varphi_{xy}\psi_y + 2\varphi_y\psi_{xy})/3], \\ a_3 &= \Delta^{-1}[b_4\varphi_x^2\varphi_y + b_3\varphi_x(\varphi_x\psi_y + 2\varphi_y\psi_x) + b_2\psi_x(2\varphi_x\psi_y + \varphi_y\psi_x) \\ &\quad + b_1\psi_x^2\psi_y + (\varphi_y\psi_{xx} - \varphi_{xx}\psi_y - 2\varphi_{xy}\psi_x + 2\varphi_x\psi_{xy})/3], \\ a_4 &= \Delta^{-1}[b_4\varphi_x^3 + 3b_3\varphi_x^2\psi_x + 3b_2\varphi_x\psi_x^2 + b_1\psi_x^3 - \varphi_{xx}\psi_x + \varphi_x\psi_{xx}]. \end{aligned} \tag{17}$$

We will use the following information about invariants of Eqs. (11). Lie [1] showed that any second-order equation obtained from the linear equation $y'' = 0$ by the change of variables (10) belongs to the family of Eqs. (11) and obtained the necessary and sufficient conditions for Eqs. (11) to be equivalent to the linear equation. Lie's linearization test can be expressed by means of the equations $L_1 = 0$, $L_2 = 0$ (see, e.g., [3]). These equations are invariant with respect to the change of variables (10). Therefore the quantities L_1 and L_2 are called *relative invariants* for Eq. (11). They involve the coefficients of Eq. (11) and their derivatives of up to second order and can be readily calculated by means of the infinitesimal method [4]. We will write them, using notation from [5], in the following form:

$$\begin{aligned} L_1 &= -\frac{\partial \Pi_{11}}{\partial u} + \frac{\partial \Pi_{12}}{\partial t} - b_4\Pi_{22} - b_2\Pi_{11} + 2b_3\Pi_{12}, \\ L_2 &= -\frac{\partial \Pi_{12}}{\partial u} + \frac{\partial \Pi_{22}}{\partial t} - b_1\Pi_{11} - b_3\Pi_{22} + 2b_2\Pi_{12}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} \Pi_{11} &= 2(b_3^2 - b_2b_4) + b_{3t} - b_{4u}, \\ \Pi_{22} &= 2(b_2^2 - 3b_1b_3) + b_{1t} - b_{2u}, \\ \Pi_{12} &= b_2b_3 - b_1b_4 + b_{2t} - b_{3u}. \end{aligned} \tag{19}$$

The change of variables (10) converts the quantities (18) into the following relative invariants for Eq. (11):

$$\widetilde{L}_1 = \Delta(L_1\varphi_x + L_2\psi_x), \qquad \widetilde{L}_2 = \Delta(L_1\varphi_y + L_2\psi_y). \tag{20}$$

For Eq. (15), the relative invariants (20) are written

$$\widetilde{L}_1 = 12Ky^{-5}, \qquad \widetilde{L}_2 = 0. \tag{21}$$

Hence, the following statement is valid.

Lemma 1. *For all Eq.* (11) *obtained from Eq.* (15) *by a change of variables, at least one of the relative invariants* L_1, L_2 *does not vanish, and the corresponding change of the variables* (10) *obeys the equation*

$$L_1\varphi_y + L_2\psi_y = 0. \tag{22}$$

We will use the following relative invariants of higher order given in [5–7]:

$$v_5 = L_2(L_1L_{2t} - L_2L_{1t}) + L_1(L_2L_{1u} - L_1L_{2u}) - b_1L_1^3 + 3b_2L_1^2L_2 - 3b_3L_1L_2^2 + b_4L_2^3 \tag{23}$$

$$w_1 = L_1^{-4}[-L_1^3(\Pi_{12}L_1 - \Pi_{11}L_2) + R_1(L_1^2)_t - L_1^2R_{1t} + L_1R_1(b_3L_1 - b_4L_2)], \tag{24}$$

and

$$I_2 = 3R_1L_1^{-1} + L_{2t} - L_{1u}, \tag{25}$$

where

$$R_1 = L_1L_{2t} - L_2L_{1t} + b_2L_1^2 - 2b_3L_1L_2 + b_4L_2^2.$$

If the relative invariant $I_2 \neq 0$, there is the set of absolute invariants

$$J_{2m} = I_{2m} I_2^{-m} \quad (m \geqslant 1),$$

where

$$I_{2m+2} = L_1 \frac{\partial I_{2m}}{\partial u} - L_2 \frac{\partial I_{2m}}{\partial t} + 2m I_{2m}(L_{2t} - L_{1u}).$$

The similar relative invariants for Eq. (16) are denoted by $\widetilde{v}_5$, $\widetilde{w}_1$, $\widetilde{I}_2$ and $\widetilde{J}_4$. For Eq. (15), invoking Eqs. (21), we obtain:

$$\widetilde{v}_5 = 0, \qquad \widetilde{w}_1 = 0, \qquad \widetilde{I}_2 = 60Ky^{-6}, \qquad \widetilde{J}_4 = 4/5. \tag{26}$$

Hence, we have the following necessary conditions for Eqs. (11) obtained from Eq. (15) by a change of variables:

$$v_5 = 0, \qquad w_1 = 0, \qquad I_2 \neq 0, \qquad J_4 = 4/5. \tag{27}$$

We will obtain now the necessary and sufficient conditions.

One has to find the conditions for the coefficients $b_1(t,u)$, $b_2(t,u)$, $b_3(t,u)$ and $b_4(t,u)$ that guarantee the existence of the functions $\varphi(x,y)$ and $\psi(x,y)$ such that the change of variables (10) transforms the coefficients of Eq. (11) into

$$a_1 = 0, \qquad a_2 = 0, \qquad a_3 = 0, \qquad a_4 = Ky^{-3},$$

where a_1, a_2, a_3, a_4 are defined by formulae (17). Thus, we have to investigate the consistency of the following over-determined system:

$$\varphi_y\psi_{yy} - \varphi_{yy}\psi_y + b_4\varphi_y^3 + 3b_3\varphi_y^2\psi_y + 3b_2\varphi_y\psi_y^2 + b_1\psi_y^3 = 0, \tag{28}$$

$$\begin{aligned} &3b_4\varphi_x\varphi_y^2 + 6b_3\varphi_x\varphi_y\psi_y + 3b_3\varphi_y^2\psi_x + 3b_2\varphi_x\psi_y^2 + 6b_2\varphi_y\psi_x\psi_y \\ &\quad + 3b_1\psi_x\psi_y^2 - 2\varphi_{xy}\psi_y + \varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2\varphi_y\psi_{xy} = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} &3b_4\varphi_x^2\varphi_y + 3b_3\varphi_x^2\psi_y + 6b_3\varphi_x\varphi_y\psi_x + 6b_2\varphi_x\psi_x\psi_y + 3b_2\varphi_y\psi_x^2 \\ &\quad + 3b_1\psi_x^2\psi_y - 2\varphi_{xy}\psi_x - \varphi_{xx}\psi_y + 2\varphi_x\psi_{xy} + \varphi_y\psi_{xx} = 0, \end{aligned} \tag{30}$$

$$y^3(b_4\varphi_x^3 + 3b_3\varphi_x^2\psi_x + 3b_2\varphi_x\psi_x^2 + b_1\psi_x^3 - \varphi_{xx}\psi_x + \varphi_x\psi_{xx}) - K\Delta = 0. \tag{31}$$

Remark 1. For Eq. (11) equivalent to Eq. (15) one of the values, either L_1 or L_2, is not equal to zero. Notice that if $L_1 = 0$ and $L_2 \neq 0$, then the change

$$t = y, \quad u = -x$$

leads to the change

$$\widetilde{L}_1 = L_2, \qquad \widetilde{L}_2 = L_1, \qquad a_1 = b_4, \qquad a_2 = -b_4, \qquad a_3 = b_2, \qquad a_4 = -b_1.$$

Further without loss of generality it is assumed that $L_1 \neq 0$.

Lemma 2. *Let Eq.* (11) *which is equivalent to Eq.* (15) *has* $L_1 \neq 0$. *Then* $\varphi_y = 0$ *if and only if* $L_2 = 0$.

Proof. The statement follows from Lemma 1. □

Theorem 2. *Eq.* (11) *with* $L_2 = 0$ *is equivalent to Eq.* (15) *if and only if its coefficients satisfy the following equations*:

$$b_1 = 0, \qquad b_{3u} - 2b_{2t} = 0, \tag{32}$$

$$L_{1tt} - \frac{6}{5L_1}(L_{1t})^2 + \frac{3}{5}b_3L_{1t} - 15b_4b_2L_1 + b_4I_2 - 5b_{4u}L_1 + 6b_{3t}L_1 + \frac{54}{5}b_3^2L_1 - \frac{25}{3I_2}L_1^3 = 0, \tag{33}$$

$$I_{2t} - \frac{6}{5L_1} I_2 L_{1t} - \frac{6}{5} b_3 I_2 = 0, \tag{34}$$

$$I_{2u} - 6b_2 I_2 + \frac{6}{5L_1} I_2^2 = 0. \tag{35}$$

Let Eqs. (32)–(35) *be satisfied. Then Eq.* (11) *is mapped to Eq.* (15) *by the change of variables* (10) *of the form*

$$t = \varphi(x), \qquad u = \psi(x,y), \tag{36}$$

where $\varphi(x)$ *is determined by the equation*

$$\varphi_x^2 = \frac{12KI_2}{5L_1^2 y^4} \tag{37}$$

and $\psi(x,y)$ *by the following integrable system:*

$$\psi_y = \frac{5L_1}{I_2 y}, \tag{38}$$

$$\psi_{xx} = \frac{1}{25\varphi_x I_2 L_1^3 y^4}[5\varphi_x \psi_x^2 I_2 L_1^2 y^4 (2I_2 - 15b_2 L_1) + 5\varphi_x K L_1 (25L_1^3 - 12b_4 I_2^2) - 24\psi_x K I_2^2 (L_{1t} + 6L_1 b_3)]. \tag{39}$$

Remark 2. The left-hand sides of Eqs. (32)–(35) are relative invariants with respect to the transformation (36). The equations $v_5 = 0$, $w_1 = 0$ (see (27)) are Eqs. (32), the equation $J_4 = 4/5$ is Eq. (35). In these equations, the variable I_2 is given by $I_2 = 3b_2 L_1 - L_{1u}$.

Theorem 3. *Eq.* (11) *with* $L_2 \neq 0$ *is equivalent to Eq.* (15) *if and only if its coefficients satisfy the following equations:*

$$5L_1 I_{2t} - 6I_2(L_{1t} - b_4 L_2 + b_3 L_1) = 0, \tag{40}$$

$$L_1^2 L_{2u} - b_4 L_2^3 + 3b_3 L_1 L_2^2 - 3b_2 L_1^2 L_2 + b_1 L_1^3 + L_{1t} L_2^2 - L_{1u} L_1 L_2 - L_{2t} L_1 L_2 = 0, \tag{41}$$

$$5L_1^2 I_{2u} - 6I_2(4b_4 L_2^2 - 9b_3 L_1 L_2 + 5b_2 L_1^2 - 4L_{1t} L_2 + 5L_{2t} L_1 - I_2 L_1) = 0, \tag{42}$$

$$\begin{aligned} &15I_2 L_1 L_{1tt} - 63b_4^2 I_2 L_2^2 + 126b_4 b_3 I_2 L_1 L_2 - 225b_4 b_2 I_2 L_1^2 + 81b_4 L_{1t} I_2 L_2 - 90b_4 L_{2t} I_2 L_1 + 15b_4 I_2^2 L_1 \\ &\quad + 162b_3^2 I_2 L_1^2 + 9b_3 L_{1t} I_2 L_1 - 15b_{4t} I_2 L_1 L_2 - 75b_{4u} I_2 L_1^2 + 90b_{3t} I_2 L_1^2 - 18L_{1t}^2 I_2 - 125L_1^4 = 0, \end{aligned} \tag{43}$$

$$\begin{aligned} &L_1^2 L_{2tt} + b_4^2 L_2^3 - 3b_4 b_3 L_1 L_2^2 + 3b_4 b_2 L_1^2 L_2 - b_4 b_1 L_1^3 - 3b_4 L_{1t} L_2^2 + 3b_4 L_{2t} L_1 L_2 + 3b_3 L_{1t} L_1 L_2 - 3b_3 L_{2t} L_1^2 \\ &\quad + b_{4t} L_1 L_2^2 + b_{4u} L_1^2 L_2 - 3b_{3t} L_1^2 L_2 - b_{3u} L_1^3 + 2b_{2t} L_1^3 - L_{1tt} L_1 L_2 + 2L_{1t}^2 L_2 - 2L_{1t} L_{2t} L_1) = 0. \end{aligned} \tag{44}$$

Let Eqs. (40)–(44) *be satisfied. Then Eq.* (11) *is mapped to Eq.* (15) *by the change of variables* (10) *with* $\varphi_y \neq 0$. *The functions* $\varphi(x,y)$ *and* $\psi(x,y)$ *are determined by the following integrable system:*

$$L_1 \varphi_y + L_2 \psi_y = 0, \tag{45}$$

$$5y^4(\varphi_x L_1 + \psi_x L_2)^2 - 12KI_2 = 0, \tag{46}$$

$$\psi_y = \frac{5L_1}{I_2 y}, \tag{47}$$

$$\begin{aligned} 5L_1^2 \psi_{xx} &= \psi_x^2(-15b_4 L_2^2 + 30b_3 L_1 L_2 - 15b_2 L_1^2 + 10L_{1t} L_2 - 10L_{2t} L_1 + 2I_2 L_1) \\ &\quad + 24\psi_x \Delta^{-1} y^{-5} K(6b_4 L_2 - 6b_3 L_1 - L_{1t}) + Ky^{-4} I_2^{-1}(-12b_4 I_2^2 + 25L_1^3). \end{aligned} \tag{48}$$

Remark 3. The left-hand sides of Eqs. (40)–(44) are relative invariants with respect to the general change of variables (10). The equations $v_5 = 0$, $w_1 = 0$ (see (27)) are Eqs. (41) and (44), the equation $J_4 = 4/5$ is Eq. (42).

Remark 4. The conditions of Theorem 2 are particular cases of the conditions of Theorem 3 provided that $L_2 = 0$.

4. Proof of Theorem 2

We use the method similar to that employed in [8,9]. Routine calculations were made by means of the system for symbolic calculations *Reduce* [10].

According to Lemma 2, $L_2 = 0$ implies $\varphi_y = 0$. Since $\Delta \neq 0$, one obtains $\varphi_x \psi_y \neq 0$. Eqs. (28)–(31) yield

$$b_1 = 0, \qquad \psi_{yy} = -3\psi_y^2 b_2, \qquad \psi_{xy} = (2\varphi_x)^{-1}(\psi_y \varphi_{xx} - 3\varphi_x^2 \psi_y b_3 - 6\varphi_x \psi_x \psi_y b_2),$$
$$\psi_{xx} = \varphi_x^{-1}\psi_x \varphi_{xx} - \varphi_x^2 b_4 - 3\varphi_x \psi_x b_3 - 3\psi_x^2 b_2 + y^{-3}\psi_y K.$$

Equating the mixed derivatives $(\psi_{xy})_x = (\psi_{xx})_y$ and $(\psi_{xy})_y = (\psi_{yy})_x$ one has

$$\varphi_x^4(4b_{4u} - 6b_{3t} + 12b_4 b_2 - 9b_3^2) + 6\varphi_x^3\psi_x(b_{3u} - 2b_{2t}) + 12\varphi_x^2 y^{-4}K + 2\varphi_x\varphi_{xxx} - 3\varphi_{xx}^2 = 0 \tag{49}$$

and

$$b_{3u} = 2b_{2t}. \tag{50}$$

The derivative φ_{xxx} is found from Eq. (49). The equation $(\varphi_{xxx})_y = 0$ gives

$$\varphi_x^2\psi_y L_1 = 12Ky^{-5}. \tag{51}$$

Differentiating this equation with respect to x and y, one obtains

$$\varphi_x^2(3b_3L_1 - 2L_{1t}) + 2\varphi_x\psi_x(3b_2L_1 - L_{1u}) - 5\varphi_{xx}L_1 = 0, \tag{52}$$
$$\psi_y y(3b_2L_1 - L_{1u}) - 5L_1 = 0. \tag{53}$$

Since $L_1 \neq 0$, one has $(3b_2L_1 - L_{1u}) \neq 0$. Using Eqs. (51) and (53), one finds

$$\psi_y = \frac{5L_1}{y(3b_2L_1 - L_{1u})}, \qquad \varphi_x^2 = \frac{12K(3b_2L_1 - L_{1u})}{5L_1^2 y^4}.$$

Substituting them into (52), one obtains

$$10\varphi_x\psi_x L_1^2 y^4(3b_2L_1 - L_{1u}) + 12(3b_2L_1 - L_{1u})K(3b_3L_1 - 2L_{1t}) - 25\varphi_{xx}L_1^3 y^4 = 0. \tag{54}$$

Since $\varphi_y = 0$, the equation $(\varphi_x^2)_y = 0$ gives

$$5L_1L_{1uu} = 3(-12b_2^2L_1^2 + 3b_2L_{1u}L_1 + 5b_{2u}L_1^2 + 2L_{1u}^2). \tag{55}$$

Using Eq. (54) one can find the derivative φ_{xx}:

$$\varphi_{xx} = 2(3b_2L_1 - L_{1u})(5\varphi_x\psi_x L_1^2 y^4 + 6K(3b_3L_1 - 2L_{1t}))/(25L_1^3 y^4).$$

By considering the equations

$$(\varphi_x^2)_{xx} - 2(\varphi_{xx}^2 + \varphi_x\varphi_{xxx}) = 0, \tag{56}$$
$$(\varphi_x^2)_x - 2\varphi_x\varphi_{xx} = 0, \tag{57}$$

one can obtain conditions for the coefficients b_1, b_2, b_3, b_4. For example, Eq. (57) gives

$$5L_1L_{1tu} = 3(-6b_3b_2L_1^2 + 2b_3L_{1u}L_1 - b_2L_{1t}L_1 + 5b_{2t}L_1^2 + 2L_{1t}L_{1u}).$$

Invoking that $(\varphi_{xx})_y \equiv 0$ and using the equation $2\varphi_x\varphi_{xx} - (\varphi_x^2)_x \equiv 0$ one obtains

$$L_{1tt} = 3L_{1t}(2L_{1t} - b_3L_1)/(5L_1) + 15b_4b_2L_1 - b_4I_2 + 5b_{4u}L_1 - 6b_{3t}L_1 - 54b_3^2L_1/5 + 25L_1^3/(3I_2),$$

where the relative invariant I_2 (25) becomes

$$I_2 = 3b_2L_1 - L_{1u}.$$

Eqs. (56) and $(\psi_y)_{xx} - (\psi_{xx})_y = 0$ are satisfied. Summing up the above results, we complete the proof of Theorem 2.

5. Proof of Theorem 3

We deal now with the case $L_2 \neq 0$, and hence $\varphi_y \neq 0$. Using Eqs. (28)–(31), one finds the derivatives ψ_{yy}, ψ_{xy}, ψ_{xx}, φ_{xx}:

$$\psi_{yy} = \varphi_y^{-1}(\varphi_{yy}\psi_y - \psi_y^3 b_1) - 3\psi_y^2 b_2 - 3\psi_y b_3\varphi_y - b_4\varphi_y^2,$$
$$2\psi_{xy} = (\varphi_y^2)^{-1}(2\varphi_{xy}\psi_y\varphi_y - \Delta\varphi_{yy} + b_1\psi_y^2(\varphi_x\psi_y - 3\psi_x\varphi_y)) - 3b_3(\varphi_x\psi_y + \psi_x\varphi_y) - 2\varphi_x b_4\varphi_y - 6\psi_x\psi_y b_2,$$
$$\psi_{xx} = \varphi_y^{-3}(-2\varphi_{xy}\varphi_x\psi_y\varphi_y + 2\varphi_{xy}\psi_x\varphi_y^2 + \varphi_{xx}\psi_y\varphi_y^2 + \Delta\varphi_x\varphi_{yy} - b_1\psi_y(\varphi_x^2\psi_y^2 - 3\varphi_x\psi_x\psi_y\varphi_y + 3\psi_x^2\varphi_y^2)) - \varphi_x^2 b_4 - 3\varphi_x\psi_x b_3 - 3\psi_x^2 b_2,$$
$$\varphi_{xx} = \varphi_x\varphi_y^{-2}(2\varphi_{xy}\varphi_y - \varphi_x\varphi_{yy} + \varphi_x\psi_y^2 b_1 - 2\psi_x\psi_y b_1\varphi_y) + \psi_x^2 b_1 + \varphi_y K y^{-3}.$$

Furthermore, the equations $(\psi_{xy})_x = (\psi_{xx})_y$ and $(\psi_{xy})_y = (\psi_{yy})_x$ determine the derivatives φ_{xyy} and φ_{yyy}, respectively,

$$2\varphi_{xyy} = \varphi_y^{-2}(4\varphi_{xy}\varphi_{yy}\varphi_y - \varphi_x\varphi_{yy}^2 + \varphi_x\psi_y^4 b_1^2 - 4\psi_x\psi_y^3 b_1^2\varphi_y) + 2\varphi_x\psi_y^2(b_{1t} - 3b_3b_1) + 4\varphi_x\psi_y\varphi_y(2b_{2t} - b_{3u} - 2b_4b_1) + \varphi_x\varphi_y^2(6b_{3t} - 4b_{4u} - 12b_4b_2 + 9b_3^2) + 2\psi_x\psi_y^2(b_{1u} - 6b_2b_1) + 4\psi_x\psi_y\varphi_y(b_{1t} - 3b_3b_1) + 2\psi_x\varphi_y^2(2b_{2t} - b_{3u} - 2b_4b_1),$$
$$2\varphi_{yyy} = \varphi_y^{-1}(3\varphi_{yy}^2 - 3\psi_y^4 b_1^2) + 2\psi_y^3(-6b_2b_1 + b_{1u}) + 6\psi_y^2\varphi_y(-3b_3b_1 + b_{1t}) + 6_y\varphi_y^2(2b_{2t} - 2b_4b_1 - b_{3u}) + \varphi_y^3(6b_{3t} - 12b_4b_2 + 9b_3^2 - 4b_{4u}).$$

The equations $(\varphi_{xyy})_y = (\varphi_{yyy})_x$ and $(\varphi_{xyy})_y = (\varphi_{xx})_{yy}$ yield:

$$y^5\Delta(L_2(\varphi_x\psi_y + \psi_x\varphi_y) + 2L_1\varphi_x\varphi_y) - 12\varphi_y K = 0. \quad (58)$$

Eq. (22) and the condition $\Delta \neq 0$ yield that $L_2\psi_y \neq 0$ and $L_1\varphi_x + L_2\psi_x \neq 0$. By virtue of Eq. (22), Eq. (58) becomes

$$y^5\psi_y(\varphi_x L_1 + \psi_x L_2)^2 - 12KL_1 = 0. \quad (59)$$

Differentiating (22) with respect to x and y, and substituting the derivatives ψ_{xx}, ψ_{xy}, ψ_{yy}, and φ_{xx}, one obtains

$$-\varphi_{yy}L_1^2(\varphi_x L_1 + \psi_x L_2) + \psi_y^2\varphi_x(2b_4L_2^3 - 3b_3L_1L_2^2 + b_1L_1^3 - 2L_{1t}L_2^2 + 2L_{2t}L_1L_2) + \psi_y^2\psi_x(3b_3L_2^3 - 6b_2L_1L_2^2 + 3b_1L_1^2L_2 - 2L_{1u}L_2^2 + 2L_{2u}L_1L_2) = 0, \quad (60)$$
$$-b_4L_2^3 + 3b_3L_1L_2^2 - 3b_2L_1^2L_2 + b_1L_1^3 + L_{1t}L_2^2 - L_{1u}L_1L_2 - L_{2t}L_1L_2 + L_{2u}L_1^2 = 0. \quad (61)$$

Eq. (60) yields:

$$\varphi_{yy} = L_1^{-2}(\varphi_x L_1 + \psi_x L_2)^{-1}(\varphi_x\psi_y^2(2b_4L_2^3 - 3b_3L_1L_2^2 + b_1L_1^3 - 2L_{1t}L_2^2 + 2L_{2t}L_1L_2) + \psi_x\psi_y^2L_2(3b_3L_2^2 - 6b_2L_1L_2 + 3b_1L_1^2 - 2L_{1u}L_2 + 2L_{2u}L_1)).$$

Furthermore, Eq. (61) determines L_{2u}:

$$L_{2u} = L_1^{-2}(b_4L_2^3 - 3b_3L_1L_2^2 + 3b_2L_1^2L_2 - b_1L_1^3 - L_{1t}L_2^2 + L_{1u}L_1L_2 + L_{2t}L_1L_2). \quad (62)$$

Using Eq. (59), one obtains the equation $(\varphi_{xy})_x - (\varphi_{xx})_y = 0$. Notice that the equations $(\varphi_y)_{yy} - \varphi_{yyy} = 0$ and $(\varphi_y)_{xy} - \varphi_{xyy} = 0$ are also satisfied. Now we find φ_x^2 from Eq. (59) and substitute it into the equation $\varphi_{xyy} - (\varphi_{yy})_x = 0$. This leads to the following expression for L_{2tt}:

$$L_{2tt} = L_1^{-2}(-b_4^2L_2^3 + 3b_4b_3L_1L_2^2 - 3b_4b_2L_1^2L_2 + b_4b_1L_1^3 + 3b_4L_{1t}L_2^2 - 3b_4L_{2t}L_1L_2 - 3b_3L_{1t}L_1L_2 + 3b_3L_{2t}L_1^2 - b_{4t}L_1L_2^2 - b_{4u}L_1^2L_2 + 3b_{3t}L_1^2L_2 + b_{3u}L_1^3 - 2b_{2t}L_1^3 + L_{1tt}L_1L_2 - 2L_{1t}^2L_2 + 2L_{1t}L_{2t}L_1). \quad (63)$$

Differentiating (59) with respect to x and y, one finds φ_{xy} and ψ_y, respectively,

$$\begin{aligned} \psi_y &= 5L_1(I_2y)^{-1}, \\ \varphi_{xy} &= 12K(5L_1^2y^5(\varphi_xL_1 + \psi_xL_2))^{-1}[6L_{1t}L_2 - 6b_4L_2^2 + 6b_3L_1L_2 - 5L_{2t}L_1] \\ &\quad + \psi_x(I_2L_1^2y)^{-1}[10b_4L_2^3 - 15b_3L_1L_2^2 + 5b_1L_1^3 10L_{1t}L_2^2 + 10L_{2t}L_1L_2 - I_2L_1L_2]. \end{aligned}$$

The equations $(\psi_y)_y = \psi_{yy}$ and $(\varphi_x^2)_x = 2\varphi_x\varphi_{xx}$ yield:

$$5L_1^2I_{2u} = 6I_2(4b_4L_2^2 - 9b_3L_1L_2 + 5b_2L_1^2 - 4L_{1t}L_2 + 5L_{2t}L_1 - I_2L_1), \tag{64}$$

$$5L_1I_{2t} = 6I_2(L_{1t} - b_4L_2 + b_3L_1). \tag{65}$$

Now the equations $(\psi_y)_x = \psi_{xy}$ and $(\varphi_{xy})_y = (\varphi_{yy})_x$ are satisfied.

The equation $(\varphi_{xy})_x = (\varphi_{xx})_y$ yields:

$$\begin{aligned} L_{1tt} &= (63b_4^2I_2L_2^2 - 126b_4b_3I_2L_1L_2 + 225b_4b_2I_2L_1^2 - 81b_4L_{1t}I_2L_2 + 90b_4L_{2t}I_2L_1 - 15b_4I_2^2L_1 - 162b_3^2I_2L_1^2 \\ &\quad - 9b_3L_{1t}I_2L_1 + 15b_{4t}I_2L_1L_2 + 75b_{4u}I_2L_1^2 - 90b_{3t}I_2L_1^2 + 18L_{1t}^2I_2 + 125L_1^4)/(15I_2L_1). \end{aligned} \tag{66}$$

Notice that the equations $(\psi_y)_x = \psi_{xy}$ and $(\psi_y)_y = \psi_{yy}$ are satisfied.

Summing up the above results, we complete the proof of Theorem 3.

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