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THE APPLICATION OF THE DIFFERENTIAL-CONSTRAINTS METHOD TO THE TWO-DIMENSIONAL EQUATIONS OF GAS DYNAMICS†

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The differential-constraints (DC) method is used to distinguish and construct individual classes of solutions of the two-dimensional equations of gas dynamics with plane and axial symmetry. The solutions (the DP-solutions), which satisfy one, two and three first-order differential constraints are classified. The solution of the Cauchy problem with data on the line of common position in these cases has three, two and one arbitrary functions respectively. In the case of solutions with three- and two-function arbitrariness all the differential constraints compatible with the system considered are indicated. The individual DP-solutions of the gas-dynamics equations are constructed using them. In the case of single-function arbitrariness the differential constraints compatible with the gas-dynamics equations are obtained. This class of solutions includes the well-known generalized Prandtl–Meyer waves. The construction of the DP-solutions of this class reduces to the integration of a system of ordinary differential equations, which is a new representation of this class of solutions. © 2000 Elsevier Science Ltd. All rights reserved.

We will use the differential-constraints (DC) method [1] to classify and construct particular solutions of the system of equations of two-dimensional gas dynamics. We will only dwell on well-known results and follow the scheme for using the method described in [2]; the new feature is its object of application.

Consider the system of equations describing the steady flow of gas in the plane ($\nu = 0$) and axisymmetric ($\nu = 1$) cases

$$u\tau_x + v\tau_y - \tau u_x - \tau v_y = \nu\tau v / y, \quad uu_x + vu_y + \tau p_x = 0 \tag{1}$$

$$uv_x + vv_y + \tau p_y = 0, \quad up_x + vp_y + A(u_x + v_y) = -\nu A v / y; \quad A = c^2/\tau$$

Here u and v are the components of the velocity vector along the x and y axes, p is the pressure, τ is the specific volume and c is the velocity of sound. We will write system of equations (1) in the form (system F)

$$F \equiv L\mathbf{u}_x + \Lambda L\mathbf{u}_y - Lf = 0$$

$$L = \begin{Vmatrix} A & 0 & 0 & \tau \\ 0 & u & v & \tau \\ 0 & -v & u & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \quad \Lambda = \begin{Vmatrix} v/u & 0 & 0 & 0 \\ 0 & v/u & 0 & 0 \\ 0 & 0 & uv/H & \tau(q^2 - A\tau)/H \\ 0 & 0 & A/H & uv/H \end{Vmatrix}$$

$$\mathbf{u} = (\tau, u, v, p)', \quad f = \left(\frac{\nu\tau v A}{yH}, 0, \frac{-\nu uv A}{yH}, \frac{\nu\tau uv}{yH}\right)'$$

$$H \equiv u^2 - \tau A, \quad q^2 = u^2 + v^2$$

In writing the vectors we will use boldface type only for the vector $\mathbf{u}$, and a prime on the vectors and matrices denotes transposition (unlike a prime on functions, which denotes differentiation).

We will consider the problem of distinguishing [1, 2], from the whole set of solutions of the specified system F, those solutions which satisfy $k = 1, 2, 3$ differential relations (system Ψ) algebraically independent of F

$$\Psi(\mathbf{u}_y, \mathbf{u}, x, y) = 0, \quad \Psi = \{\Psi_1, \Psi_2, \ldots, \Psi_k\}$$

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i.e. differential constraints, the form of which is not defined *a priori*. The joint system of equations $F = 0$, $\Psi = 0$ (system $F\Psi$) is over-defined. We will impose constraints—DP-conditions on the form of $\boldsymbol{\Psi}(\mathbf{u}_y, \mathbf{u}, x, y)$, the satisfaction of which is necessary and sufficient for the $F\Psi$ system corresponding to it (the DP-system) to have solutions (DP-solutions). These solutions of system F can often be found more easily than its other solutions.

Without going into detail, we note that, by differentiating with respect to the independent variables and eliminating the derivatives, one cannot obtain new first-order differential equations from the DP-system. To classify the DP-solutions we will consider the DP-systems with different subscripts k successively.

It was shown in [2] that, in view of the hyperbolic form of the initial system of differential constraints in the given case, we must seek the representation

$$\Psi \equiv B_1 L\mathbf{u}_y + \psi(B_2 L\mathbf{u}_y, \mathbf{u}, y, x) = 0$$

Here the form of the differential constraints is specified by the choice of the matrices B_1 and B_2, in each row of which there is only one non-zero element, equal to unity; the matrix

$$B = \left\| \begin{matrix} B_1 \\ B_2 \end{matrix} \right\|$$

is orthogonal and in matrix B_1 there are k rows. The DP-conditions have the form [2]

$$(D_x\Psi + Z\Lambda B_1' D_y \Psi - ZD_y F)|_{(F,\Psi)} = 0 \tag{2}$$

$$Z\Lambda - Z\Lambda B_1' Z = 0 \tag{3}$$

where D_x and D_y are operators of total differentiation with respect to x and y, $Z = B_1 + \psi_h B_2$, $h = B_2 L\mathbf{u}_y$ is a vector with $m-k$ components $h_l (l = 1, 2, \ldots, m-k)$ and ψ_h – $[k \times (m-k)]$ is a matrix with components $(\psi_h)_{il} = \partial\psi_i/\partial h_l (i = 1, 2, \ldots, k; l = 1, 2, \ldots, m-k)$. The symbol $(F\Psi)$ denotes a manifold in the space of the variables $(\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}, y, x)$, defined by the equations $F = 0$, $\Psi = 0$ as algebraic relations in these variables

It was shown in [2] that the DP-system $F\Psi$, consisting of a hyperbolic system of m equations $F = 0$ and K differential constraints $\Psi = 0$, has $m-k$ families of characteristics, which occur in the number m of families of characteristics of the system $F = 0$. For a hyperbolic system, the differential constraints can be found somewhat more simply if the initial system is written in the characteristic form

$$F \equiv L_0\mathbf{u}_x + \Lambda_0 L_0 \mathbf{u}_y - L_0 f = 0, \quad \Lambda_0 = L_0 L^{-1}\Lambda L L_0^{-1}$$

Here L_0 is a matrix of the left eigenvectors of the matrix $L^{-1}\Lambda L$. The diagonal elements of the matrix Λ_0—the eigenvalues of the matrix $L^{-1}\Lambda L$—at each point (x, y) of the region in which the solution of system (1) is defined, specify the directions of its four characteristics

$$y_1' = \frac{uv - AQ}{H} \leftrightarrow y = y_1(x), \quad y_{2,3}' = \frac{v}{u} \leftrightarrow y = y_{2,3}(x)$$

$$y_4' = \frac{uv + AQ}{H} \leftrightarrow y = y_4(x), \quad Q = \frac{\sqrt{\tau(q^2 - \tau A)}}{A}$$

For subsonic velocities the matrix $L^{-1}\Lambda L$ does not have four real eigenvalues, and there are only two identical values corresponding to the trajectories of the gas particles. Hence, the characteristic form of writing $F = 0$ is not used here.

1. Suppose $B_1 = \| 1\ 0\ 0\ 0 \|$. In this case the differential constraint specifies the relation between the entropy distribution and the Bernoulli integral, in particular for a polytropic gas, between the quantities $S = p\tau^\gamma$ and $I = q^2/2 + \gamma p\tau/(\gamma - 1)$. For the case in question the following representation follows from the DP-conditions

$$\psi = y^\nu u \Phi(q^2, p, \tau), \quad q^2 \equiv u^2 + v^2$$

Here the function $\boldsymbol{\Phi}\ (q^2, p, \tau)$ satisfies the equation

$$\tau A\Phi_q / q + \tau\Phi_\tau - A\Phi_p + A_p\Phi = 0$$

In particular, for a polytropic gas ($A = \gamma p$) the general solution of the last equation is $\Phi = \tau^{-\gamma}(I, S)$ with an arbitrary function of two arguments ϕ. In the DP-solutions three characteristics pass through each points (x, y), where the characteristic $y = y_2(x)$ is simple.

Suppose $B_1 = \| 1\ 0\ 0\ 0 \|$. In view of the equality

$$uu_y + vv_y + \tau p_y = u(u_y - v_x)$$

the differential constraint can be taken in the form

$$u_y - v_x + y^\nu \psi = 0$$

Hence it can be seen that there is a relation between the entropy, the Bernoulli integral and the vorticity in the solutions which satisfy this differential constraint. By writing conditions (3) minutely in this case, we establish that ψ is independent of two components of the vector $\mathbf{h}$: $\| 1\ 0\ 0\ 1\ 0 \| L\mathbf{u}_y$ and $\| 1\ 0\ 0\ 1\ 0 \| L\mathbf{u}_y$, i.e.

$$\psi = \psi\,(h_1, u, v, p, \tau, y, x),\quad h_1 = A\tau_y + \tau p_y$$

Conditions (3) in this case take the form

$$[uD_x\Psi + v\, D_y\Psi - \psi_{h_1}(AD_yF_1 + \tau D_yF_1) - (uD_yF_2 + v\, D_yF_3)]\,|_{(F\Psi)} = 0 \tag{4}$$

In order for the over-defined system $F\Psi$ to be in an involution, it is necessary that the derivatives v_y and p_y should not be determined from (4), i.e. it should be an identity in these variables. Taking this requirement into account and also the relation $\tau_y = (h_1 - \tau p_y)/A$ (splitting (4) with respect to the variables v_y and p_y) we obtain

$$h_1\psi_{h_1}(\tau A - u^2)A_p + \tau u^2\psi_\tau + \tau uA\psi_u - u^2A\psi_p + u^2h_1 + (u^2 - \tau A)\psi = 0 \tag{5}$$

$$h_1\psi_{h_1}v + uv\,\psi_u - u^2\psi_v = v\,\psi \tag{6}$$

$$\begin{aligned} &v\,[y\tau A\psi + \nu u^4 - A_p(u^2y\psi + u^4)]h_1\psi_{h_1} + h_1u^2v\,(y\psi + \nu u^2) + \\ &+ v\,y\psi^2(u^2 - \tau A) + \tau v\,uA(y\psi + \nu u^2)\psi_u + v\,yu^2(u^2 - \tau A)\psi_y + \\ &+ \tau v\,u^2(y\psi + \nu u^2)\psi_\tau - v\,Au^2(y\psi + \nu u^2)\psi p + yu^2(u^2 - \tau A)\psi_x = 0 \end{aligned} \tag{7}$$

We construct the general solution of Eq. (6)

$$\psi = u\Phi(w_1, w_2, p, \tau, y, x),\quad w_1 \equiv h_1/u,\quad w_2 \equiv u^2 + v^2 \tag{8}$$

Substituting (8) into (5) and (7), we obtain

$$-w_1A_p\Phi_{w_1} + \Phi + w_1 + 2\tau A\Phi_{w_2} + \tau\Phi_\tau - A\Phi_p = 0 \tag{9}$$

$$\nu w_1\Phi_{w_1} - \nu\Phi + y\Phi_y + yu\Phi_x / v = 0 \tag{10}$$

Consider Eq. (10). If $\nu = 0$, it follows from relations (8) and Eq. (10) that $\Phi_y = \Phi_x = 0$. If $\nu = 1$, the general solution of Eq. (10) will be

$$\Phi = yg\,(\bar{w}_1, w_2, p, \tau),\quad \bar{w}_1 \equiv w_1/y$$

Hence, the general solution of Eq. (10) for both cases ($\nu = 0$ and $\nu = 1$) can be written in the form

$$\Phi = y^\nu g\,(\bar{w}_1, w_2, p, \tau),\quad \bar{w}_1 \equiv w_1/y^\nu \tag{11}$$

Substituting (11) into (9) we obtain

$$-\bar{w}_1A_pg_{\bar{w}_1} + 2\tau Ag_{w_2} + \tau g_\tau - Ag_p = -(g + \bar{w}_1) \tag{12}$$

Hence, the problem of determining the function ψ has been reduced to solving Eq. (12). In this case $\psi = uy^{\nu}g(\overline{w}_1, w_2, p, \tau)$.

In the special case of a polytropic case, the function

$$g = \frac{\overline{w}_1}{\gamma - 1} + \overline{w}_1^{1/\gamma}\phi\left(\frac{\overline{w}_1}{p}, I, S\right) \tag{13}$$

with arbitrary function ϕ is the general solution of Eq. (12).

The DP-system has the same characteristics as in case 1.

The DP-solutions with one differential constraint have an arbitrariness in three functions. In particular, all solutions with a constant Bernoulli integral (I = const), all isentropic solutions (S = const) and all irrotational solutions $u_y - \upsilon_x = 0$ occur in these two classes of DP-solutions. But in these classes, obviously, there are also more general solutions of the gas-dynamics equations with these variable quantities. System (1) has no other DP-solutions with one first-order differential constraint. The cases $B_1 = \| 0, 0, 1, 0 \|$ and $B_1 = \| 0, 0, 0, 1 \|$ lead to contradictory governing systems.

3. Suppose

$$B_1 = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}$$

These differential constraints define the dependence of the entropy and vortex distribution on the stream function. For the components of the vector $\psi = (\psi_1, \psi_2)$ we have the following representation, obtained from the DP-conditions

$$\psi_1 = y^{\nu}u\Phi_1(w_2, p, \tau), \quad \psi_2 = y^{\nu}u\Phi_2(w_2, p, \tau)$$

with functions Φ_1 and Φ_2 which satisfy the equations

$$2\tau A\Phi_{1w_2} + \tau\Phi_{1\tau} - A\Phi_{1p} + A_p\Phi_1 = 0$$

$$2\tau A\Phi_{2w_2} + \tau\Phi_{2\tau} - A\Phi_{2p} + \Phi_2 = \Phi_1$$

For a polytropic gas, the last equations have the general solution

$$\Phi_1 = (\gamma - 1)\tau^{-\gamma}\phi_1(I, S), \quad \Phi_2 = -\tau^{-\gamma}\phi_1(I, S) + \tau^{-1}\phi_2(I, S)$$

where ϕ_1 and ϕ_2 are arbitrary functions of their arguments, and they give the general solution of the last system of equations. The DP-system has two families of characteristics $y = y_1(x)$ and $y = y_4(x)$.

4. Suppose

$$B_1 = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

The DP-conditions are only satisfied when $\nu = 0$. In this case

$$\psi_1 = u^2\Phi, \quad \psi_2 = -\frac{u^3}{\upsilon\tau}\Phi$$

where either $\Phi = 0$ or $\Phi = \upsilon/[u(y-g)]$ with a function $g = g(\alpha, \beta)$, which satisfies Hopf's equation

$$g_\alpha + gg_\beta = 0; \quad \alpha \equiv \upsilon / u, \quad \beta \equiv x + y\alpha$$

In this well-known class of gas flows, the vorticity, the velocity modulus, the density and the pressure are constant along the streamlines, which are concentric circles. In the DP-system $y = y_2(x)$ and $y = y_3(x)$ are a multiple characteristic.

System (1) has no other DP-solutions with two first-order differential constraints.

The DP-solutions with two differential constraints possess two-function arbitrariness. Compared with the general solution of system (1) their arbitrariness is much narrower, being simultaneously restricted by two differential constraints. In particular, it can be seen from Section 3 that flows with both constant entropy and Bernoulli integral belong to this class. It was shown in [2] that more general DP-solutions with one differential constraint or more degenerate solutions with three differential constraints, which we will consider later, may be adjacent to them via the characteristics and via the discontinuity lines.

5. Suppose

$$B_1 = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

In this case we will not give the expanded form of the DP-conditions in view of their length. It follows from the second part of the DP-conditions that

$$\psi_3 = \pm[\tau(u^2 + v^2 - A\tau)/A]^{1/2} p_y + \omega(\mathbf{u}, y, x)$$

It has not so far been possible to investigate the problem of their general solution. For $\nu = 0$ and homogeneous constraints ($\psi_1 = \psi_2 = \omega = 0$) it can be seen that the DP-conditions are satisfied. The DP-solution in this case is a Prandtl–Meyer wave [6]. Other separate solutions of the DP-conditions can also be obtained; for example, in the case of a polytropic gas with $\nu = 0$ we have

$$\psi_1 = (\gamma - 1)u\tau^{-\gamma} f(I, S), \quad \psi_2 = u[(\gamma - 1)I/(\gamma S\tau) - \tau^{-\gamma}]f(I, S), \quad \omega = 0$$

where f is an arbitrary function of its arguments. The differential constraints are inhomogeneous. In the general case, the characteristic of the gas-dynamics equations on these solutions are curvilinear. The DP-system has one family of characteristics: either $y = y_1(x)$ or $y = y_4(x)$ depending on the sign of $(\psi_3)_{py}$.

6. Suppose

$$B_1 = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

It follows from the second part of the DP-conditions that

$$(\psi_1)_h = (\psi_2)_h = 0, \quad (\psi_3)_h = \pm[\tau(u^2 + v^2 - f\tau)/f]^{-1/2}, \quad h = -vu_y + uv_y$$

In this case the differential constraints are identical with the differential constraints of the previous case (the subcase of a functional dependence of the entropy on the velocity components is of no interest from the point of view of the analysis of the differential constraints and is not considered here). When $\nu = 0$ the homogeneous differential constraints satisfy the DP-conditions. The DP-system has a single family of characteristics $y = y_1(x)$.

7. Suppose

$$B_1 = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

It follows from the second part of the DP-conditions that

$$(\psi_1)_h = (\psi_3)_h = 0, \quad h = \tau p_y + A\tau_y$$

Hence, we have established that for $\nu = 0$ and homogeneous differential constraints the DP-conditions are satisfied. The DP-system has one family of characteristics $y = y_2(x)$.

8. Suppose

$$B_1 = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

It follows from the second part of the DP-conditions that

$$(\psi_2)_h = (\psi_3)_h = 0, \quad h = uu_y + \upsilon v_y + \tau p_y$$

and when $\nu = 0$ the homogeneous differential constraints satisfy the DP-conditions. If $(\psi_1)_h \neq 0$, then in this fundamental case, apart from the notation of the functional relationship, the differential constraints are identical with those in Section 7. The case $(\psi_1)_h = 0$ is a separate one. Other solutions are not found. The DP-system has a single family of characteristics $y = y_2(x)$.

We will return again to the DP-system in Section 5, which is obtained from the solution of the DP-conditions indicated at the end of Section 5. Suppose that in the system $F\Psi$, of the two possible families of characteristics there is a family $y = y_4(x)$. Along the characteristic of the DP-system there is the usual relation along the fourth family of characteristics of system F

$$l^4\left(f_s - \frac{du}{dx}\right) = 0 \tag{14}$$

and three relations

$$B_1 L_0\left(f_s - \frac{du}{dx}\right) + B_1(\Lambda_0 - y'_{x,4}E_4)B_1^*\Psi = 0 \tag{15}$$

a consequence of the three differential constraints of system $F\Psi$ [2] (l^4 is the fourth eigenvector of the matrix L_0). These four relations, together with the relation

$$y'_{x,4} = (uv + AQ)/H \tag{16}$$

are a system of five ordinary differential equations in five unknowns $u(x)$, $\upsilon(x)$, $p(x)$, $\tau(x)$ and $y(x)$.

We will write a more detailed form of relations (14) and (15)

$$\frac{du}{dx} = \frac{A}{q^2 H}[Q(\upsilon\omega - u\psi_2) - \tau(u\omega + \upsilon\psi_2)], \quad \frac{d\upsilon}{dx} = -\frac{A}{uq^2 H}(u\omega + \upsilon\psi_2)(Qu + \upsilon\tau)$$

$$\frac{dp}{dx} = \frac{A\omega}{H}, \quad \frac{d\tau}{dx} = -\frac{u\tau\omega + \psi_1(Qu + \upsilon\tau)}{uH}$$

This system enables us to solve the Cauchy problem and to determine the vector **u** along the characteristic y_4, if it is specified at a certain initial point (y_0, x_0). The functions ψ_1, ψ_2 and ω are found as the solution of the DP-system. For the case of inhomogeneous differential constraints a particular solution ψ_1, ψ_2 ω is given in Section 5.

Suppose we are given the values of the vector **u** along some line $\Gamma: y = y(x)$, which is not a characteristic of the fourth family. Using these as Cauchy data at points of the lint Γ, in the region adjacent to it, we locally construct a solution of system $F\Psi$ [2], covering it with characteristics of the family y_4, the origin of which is on Γ.

On the solution of system $F\Psi$, between the four components of the vector **u** on an arbitrary line Γ there are three relations, which differ from (15) in the fact that the quantity y' in them is equal to the slope of the tangent to Γ. In particular, along the streamline $y' = \upsilon/u$. If one of the components of the vector **u** is specified in the form of a function of x, these three equations, together with the equation for $y(x)$, enable us to construct Cauchy data along the chosen line, which do not contradict system $F\Psi$. If the vector **u** is taken as variable along Γ, the line is obtained as a curve. Near the concave wall the characteristics begin to intersect in the flow region. An advantage of the numerical solution-test, constructed using high-accuracy (Runge–Kutta) procedures is the fact that the point of intersection of the characteristics—the origin of a shock in flow, like the other features of the solutions, is reproduced relatively poorly by the general numerical methods of solving boundary-value problems for partial differential equations.

Remark. The DP-solutions of this class in general have curvilinear characteristics, along which the vector **u** is variable. But in special cases **u** may be constant on characteristics, and they are rectilinear. In this case the DP-solutions are generalized Prandtl–Meyer waves. In the general case, the previously known technique for constructing generalized Prandtl–Meyer waves [7, 8], cannot be used to construct DP-solutions of this class. Relations (14)–(16), together with the Cauchy data on the line Γ, without contradicting differential constraints, define the representation

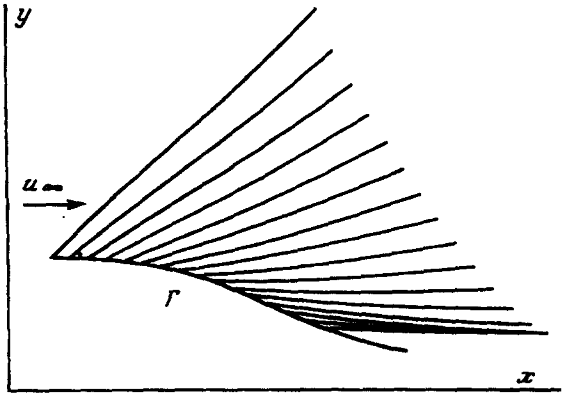

Fig. 1.

of the DP-solutions of this class in the form of a system of ordinary differential equations. If in the axisymmetric case we can construct a solution of the DP-conditions, a similar representation is also obtained in the axisymmetric case and enables solutions to be constructed by one method in both cases.

Figure 1 shows the pattern of the characteristics obtained in one of the numerical calculations of the DP-solution considered. The initial line Γ is a streamline. The calculation was carried out using the scheme proposed here for constructing a numerical solution. It is still an open question when the solution will adjoin the constructed solution via the shock in flow and the characteristic emerging from the origin of the shock in flow and incident on the all around which the flow occurs.

It was shown in [2] that, using separate DP-solutions of different classes (here, of the numbers indicated in the different Sections) by a continuous adjoinment through the characteristic of the system F or with a discontinuity in the values of **u** on the discontinuity line and satisfaction of the conservation laws one can construct generalized solutions of the system F. Many cases of possible adjoinment of these solutions are possible here.

It is extremely useful to use computer algebra in applications of the differential-constraints method [2, 9–11]. In this investigation we used the REDUCE system of computer algebra [12] to carry out analytic calculations at all stages right up to the conversion of the arithmetic expressions obtained into programming language operators for the numerical solution of the problem.

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