QUANTUM DYNAMICAL PRINCIPLE OF CONSTRAINED DYNAMICS IN QUANTUM PHYSICS AND QUANTUM FIELD THEORY

Tukkamon Vijaktanawudhi

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หลักการเชิงพลวัตควอนตัมของพลศาสตร์เชิงบังคับ
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Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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ทักษ์กมนต์ วิจักษณ์ธนาวุฒิ : หลักการเชิงพลวัตควอนตัมของพลศาสตร์เชิงบังคับในฟิสิกส์ควอนตัมและทฤษฎีสนามควอนตัม (QUANTUM DYNAMICAL PRINCIPLE OF CONSTRAINED DYNAMICS IN QUANTUM PHYSICS AND QUANTUM FIELD THEORY) อาจารย์ที่ปรึกษา : ศาสตราจารย์ ดร. เอ็ดเวิร์ด มานูเกียน, 272 หน้า.

เริ่มจากโครงสร้างของลากรางเจียนในปัจจุบัน สำหรับพลวัตของอนุภาคมูลฐานในฟิสิกส์พลังงานสูง และแม้แต่การเพิ่มเติมลักษณะทั่วไปของลากรางเจียนการวิเคราะห์อนุกรมวิธานและการวิเคราะห์เอกภาพได้หาพลวัตบังคับในสูตรอนุพันธ์เชิงฟังก์ชันของทฤษฎีสนามควอนตัม ผ่านการประยุกต์ของหลักการเชิงพลวัตควอนตัม ดังที่ทุกทฤษฎีของอันตรกิริยาพื้นฐานในปัจจุบันนั้นคือทฤษฎีเกจดังนั้นจึงจำเป็นต้องมีเงื่อนไขบังคับเกจในทฤษฎีหลังจากการหามาอย่างละเอียด ของแฟกเตอร์ที่เรียกว่าแฟกเตอร์ Faddeev-Popov โดยสูตรข้างต้นเราแสดงให้เห็นว่าทฤษฎีการไม่แปรเปลี่ยนเกจ ไม่จำเป็นต้องแสดงในรูปของแฟกเตอร์ที่คุ้นเคยนี้ และการดัดแปรเพิ่มเติม มีการแสดงรายละเอียดในส่วนของเนื้อหา โดยปกติจะเป็นการแสดงว่า โดยทั่วไปเป็นจริงเมื่อพิจารณาเทอมเกจแตกหัก วิธีการหาผลลัพธ์ของทฤษฎีบททั่วไปสำหรับพลวัตบังคับคือการพิสูจน์ และกฎของการประยุกต์เป็นการพัฒนาในสูตรข้างต้นดังต่อไปนี้ ทฤษฎีสนามทั่วไปถูกพิจารณาเกี่ยวกับความหนาแน่นอันตรกิริยาของลากรางเจียน $\mathcal{L}_I(x;\lambda)$ เมื่อ λ คือ ค่าคงตัวคู่ควบสามัญ ดังนิพจน์ต่อไปนี้ $\partial\mathcal{L}_I(x;\lambda)/\partial\lambda$ อาจแสดงในรูปของฟังก์ชันกำลังสองในสนามไม่อิสระ แต่บางทีโดยทั่วไปเป็นฟังก์ชันใดๆก็ตามของสนามอิสระ ความจำเป็นเหล่านี้รวมทั้งกรณีพิเศษก็คือ ทฤษฎีเกจที่สามารถทำให้เป็นปกติได้ในปัจจุบัน เป็นการแสดงว่าในวิธีการเอกภาพนั้นคือแอมพลิจูดของการเปลี่ยนสถานะสุญญากาศถึงสุญญากาศ (การกำเนิดเชิงฟังก์ชัน) อาจแสดงได้แน่ชัดในรูปอนุพันธ์เชิงฟังก์ชัน ซึ่งโดยทั่วไปนำไปสู่การแก้ไขกฎการคำนวณ โดยรวมถึงแฟกเตอร์ Faddeev-Popov และการดัดแปรดังกล่าวได้มาอย่างชัดเจน มีแหล่งกำเนิดภายนอกในผลที่ได้และไม่ขึ้นกับความสมมาตรใดๆ และข้อสรุปความไม่แปรเปลี่ยนอย่างที่เกิดบ่อยๆในทฤษฎีเกจไม่ปรากฏในวิธีการอินทิเกรตตามวิถีประเด็นทางฟิสิกส์ของทฤษฎีบทและประเด็นทางฟิสิกส์ภายใต้การวิเคราะห์แบบทั่วไปในทฤษฎีสนามควอนตัมในสูตรอนุพันธ์เชิงฟังก์ชันเป็นที่ชัดเจนเราได้ทำการวิเคราะห์พลวัตบังคับในฟิสิกส์ควอนตัม และสองวิธีการที่ต่างกัน นำมาใช้อีกในสูตรอนุพันธ์เชิงฟังก์ชัน [1] กำหนดฮามิลโทเนียน $H(\mathbf{q},\mathbf{p})$ และเซตของฟังก์ชันตัวดำเนินการสับเปลี่ยนคู่ $G_j(\mathbf{q}(\tau),\mathbf{p}(\tau))$ เมื่อ $j=1,\ldots,k$ ฟังก์ชันการแปลงที่ได้คล้องจองกับฮามิลโทเนียนใดๆ $\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})$ ด้วย

เงื่อนไขบังคับ $\mathbf{Q}(\tau) - \mathbf{G}(\mathbf{q}(\tau), \mathbf{p}(\tau)) = 0$ ซึ่ง $\mathbf{P} = 0$ และ $\tilde{H}(\mathbf{q}, \mathbf{p}, \mathbf{G}(\mathbf{q}, \mathbf{p}), 0) = H(\mathbf{q}, \mathbf{p})$ [2] กำหนดฮามิลโทเนียน $\tilde{H}(\mathbf{q}, \mathbf{p})$ เราพิจารณาระบบใหม่โดยนิยามฟังก์ชันตัวดำเนินการบังคับ $G_j(\mathbf{q}(\tau), \mathbf{p}(\tau))$ เมื่อ $j = 1, \ldots, k$ ทำให้ $\mathbf{G}(\mathbf{q}(\tau), \mathbf{p}(\tau)) = \mathbf{0}$, $\hat{\mathbf{G}}(\mathbf{q}(\tau), \mathbf{p}(\tau)) = \mathbf{0}$ และฮามิลโทเนียนใหม่กำหนดได้โดย $H(\mathbf{q}^*, \mathbf{p}^*) = \tilde{H}(\mathbf{q}, \mathbf{p})\big|_{\mathbf{G}=\mathbf{0}, \hat{\mathbf{G}}=\mathbf{0}}$ ด้วย $(\mathbf{q}, \mathbf{p}) \to (\mathbf{q}^*, \mathbf{p}^*, \mathbf{G}, \hat{\mathbf{G}})$ ซึ่งเป็นการแปลงแบบบัญญัติ

สาขาวิชาฟิสิกส์ ลายมือชื่อนักศึกษา ____________________

ปีการศึกษา 2551 ลายมือชื่ออาจารย์ที่ปรึกษา ____________________

TUKKAMON VIJAKTANAWUDHI : QUANTUM DYNAMICAL PRINCIPLE OF CONSTRAINED DYNAMICS IN QUANTUM PHYSICS AND QUANTUM FIELD THEORY. THESIS ADVISOR : PROF. EDOUARD B. MANOUKIAN, Ph.D. 272 PP.

FUNCTIONAL DIFFERENTIAL FORMALISM OF QUANTUM FIELD THEORY/ DEPENDENT FIELDS/ ACTION PRINCIPLE/ QUANTIZATION RULES/ GAUGE THEORIES/ FADDEEV-POPOV FACTOR/ CONSTRAINTS/ THE QUANTUM DYNAMICAL PRINCIPLE AND FUNCTIONAL CALCULUS/ GAUGE INVARIANCE/ GAUGE BREAKING INTERACTIONS.

Guided by the structures of present Lagrangians for elementary particles' dynamics in High-Energy Physics and even their further *generalizations*, a systematic and a unified analysis is carried out of constrained dynamics in the functional *differential* formalism of quantum field theory via the application of the Quantum Dynamical Principle. As all of the present theories of the fundamental interactions are gauge theories a gauge constraint then necessarily arises in the theory. After a detailed derivation of the so-called Faddeev-Popov factor by the above formalism, we show that a gauge invariant theory does not necessarily imply the presence of this familiar factor and further *modifications*, derived in the text, may arise. In particular this is shown to be also generally true when gauge breaking terms are considered. Equipped with such results, a general Theorem for constrained dynamics is proved and rules of applications are developed in the above formalism as follows. General field theories are considered with interaction Lagrangian densities $\mathcal{L}_I(x;\lambda)$, with λ a generic coupling constant, such that the following expression $\partial\mathcal{L}_I(x;\lambda)/\partial\lambda$ may be expressed as quadratic functions in *dependent* fields but may, in general, be arbitrary functions of *independent* fields. These necessarily include, as special cases, present renormalizable gauge theories. It is shown, in a unified manner, that the vacuum-to-vacuum transition amplitude (the generating func-

tional) may be explicitly derived in functional differential form which, in general, leads to modifications to computational rules by including such factors as Faddeev-Popov ones and *modifications* thereof which are explicitly obtained. The derivation is given in the *presence* of external sources and does not rely on any symmetry and invariance arguments as is often done in gauge theories and no appeal is made to path integrals. The *physical relevance* of such a Theorem and of the underlying general analysis in quantum field theory in the functional differential formalism is clear. We have also carried out analyses of constrained dynamics in quantum physics and two different approaches were taken again in the functional *differential* formalism: [1] Given a Hamiltonian $H(\mathbf{q},\mathbf{p})$ and a set of pairwise commuting operator functions $G_j(\mathbf{q}(\tau),\mathbf{p}(\tau))$, $j=1,\ldots,k$, transformation functions are derived corresponding to any Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})$ with constraints $\mathbf{Q}(\tau)-\mathbf{G}(\mathbf{q}(\tau),\mathbf{p}(\tau))=0$, for which $\mathbf{P}=0$, and $\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{G}(\mathbf{q},\mathbf{p}),0)=H(\mathbf{q},\mathbf{p})$. [2] Given a Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p})$, we consider a new system by defining constraint operator functions $G_j(\mathbf{q}(\tau),\mathbf{p}(\tau))$, $j=1,\ldots,k$, and canonical conjugate momenta defined for them $\hat{G}_j(\mathbf{q}(\tau),\mathbf{p}(\tau))$, $j=1,\ldots,k$, such that $\mathbf{G}(\mathbf{q}(\tau),\mathbf{p}(\tau))=\mathbf{0}$, $\hat{\mathbf{G}}(\mathbf{q}(\tau),\mathbf{p}(\tau))=\mathbf{0}$ and the new Hamiltonian is given by $H(\mathbf{q}^*,\mathbf{p}^*)=\tilde{H}(\mathbf{q},\mathbf{p})\big|_{\mathbf{G}=\mathbf{0},\hat{\mathbf{G}}=\mathbf{0}}$ with $(\mathbf{q},\mathbf{p})\rightarrow(\mathbf{q}^*,\mathbf{p}^*,\mathbf{G},\hat{\mathbf{G}})$ defining a canonical transformation.

School of Physics	Student's Signature ____________
Academic Year 2008	Advisor's Signature ____________

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CHAPTER I

INTRODUCTION

Quantum field theory, successfully uniting quantum physics and relativity, provides the non-phenomenological theoretical approach in describing the dynamics of Elementary Particle Physics and is the basic tool for practical computations in High-Energy Physics. The theories introduced so far in history to describe the fundamental interactions in physics, include Quantum Electrodynamics (Dirac, 1927; Fermi, 1930; Schwinger, 1948, 1949a, 1949b, 1951a; Feynman, 1949a, 1949b, 1950; Tomonaga, 1948; Dyson, 1949a, 1949b), the Unified Weak-Electromagnetic Theory (Salam, 1968, 1980; Salam and Strathdee, 1972; Weinberg, 1967, 1974, 1980; Glashow, 1959, 1961, 1980), Quantum Chromodynamics and unified theories involving strong interactions (Bjorken, 1972; Pati and Salam, 1973; Georgi and Glashow, 1974; Buras et. al., 1978; Gross, 1999; Ioffe, 2001; Gross, Wilczek, Politzer, 2004) even theories attempting to include Einstein's theory of gravitation and modifications thereof (Zumino, 1975; Arnowitt et. al., 1975; Akulov et. al., 1975; Deser and Zumino, 1976; Wess and Zumino, 1977; Brink et. al., 1978, Salam and Strathdee, 1978; Deser, 1986; 't Hooft, 1986). The reason why it took years from the time of the development of quantum electrodynamics to its extension to the weak and strong interactions was the necessity of obtaining renormalizable theories (Salam, 1980; Weinberg, 1980; Glashow, 1980; 't Hooft and Veltman, 1999; see also Manoukian, 1983). At present there is no renormalizable theory of gravitation and this fundamental interaction remains left out of the realm of quantum physics as no theory is acceptable if it cannot be consistently renormalized for proper physical interpretation and for actual computations. The reason for going through the history of the fundamental interactions in physics is that *all of these theories are* not only gauge theories but *are* also *constrained dynamical systems*. [Theories with constraints are difficult to handle even at the classical level and very little may be found on it even in most authoritative books (cf. Goldstein, Poole and Safko, 2002) on Classical Dynamics.]. The breakthrough in constrained dynamics came through the classic work of Dirac (1950, 1951, 1958, 1967) restricted to Hamiltonian systems which, in particular, necessitated generalizing the expression of the well known Poisson bracket which had been used for years before. This method was successfully applied in a path-integral context (Feynman, 1948; Feynman and Hibbs, 1965) to gauge theories by Faddeev (1969), Faddeev and Popov (1967); with pertinent contributions due to De Witt (1964), Senjanovik (1976), Gribov (1978), Fradkin and Vilkovisky (1977) (see also Henneaux and Teitelboim, 1994; Garcia, Vergara and Urrutia, 1996; Galvão and Boechat, 1990; Batalin and Fradkin, 1986, 1987; Batalin, Fradkin and Fradkina, 1990; Batalin and Tyutin, 1993, Bizdadea and Saliu, 1996; Shimuzu, 1997; Bartlett and Rowe, 2003). Due to the non-uniqueness problem encountered in gauge field theories (e.g., Gribov, 1978; Zwanziger, 1981) several approaches have been taken in the literature to constrained dynamics. The first is the path integral approach just mentioned. The second is the canonical operator approach (cf. Utiyama and Sakamoto, 1977; Mohapatra, 1971a, 1971b, 1972; 't Hooft and Veltman, 1973; Weinberg, 1980) which turned out to be not too economical in details as it involves S-matrix techniques, using Wick's Theorem, field equations, commutation rules and several difficulties involving with Schwinger terms in ill defined commutators. The third approach used for the first time in the quantization problem of gauge theories (Manoukian, 1986, 1987a, 1987b); also the recent work of Manoukian and Siranan, 2005) is based on the *Quantum Dynamical principle* or *Quantum Action Principle*, pioneered by Schwinger, 1951b, 1951c,1953a, 1953b, 1954; see also Lam, 1965; Manoukian, 1985). This approach generates the so-called Faddeev-Popov ghost factor (Manoukian, 1986, 1987a) of gauge theories with no difficulty. The advantages of the quantum dynamical approach is that it avoids making appeal to path integrals, avoids using commutation rules; it avoids, in general, going through the complicated structure of the Hamiltonian in non-abelian gauge theories, it avoids using S-matrix-Wick's product techniques, it avoids guessing weight factors as generated from the Feynman rules of the canonical formalism and avoids altogether solving for field equations. The quantum dynamical principle gives the variation of the vacuum-to-vacuum transition amplitude with respect to charges or couplings, masses, frequencies, external sources and with respect to any parameter that the theory may depend on. Unlike the path integral approach which depends on continual integrals, as an infinite product over spacetime points, which are often ill defined, the quantum dynamical principle involves only differentiations with respect to external sources of a well defined quantity, and is obviously much easier to differentiate than to integrate.

Constrained dynamics in the quantum field theory of gauge fields lead, in general, to a modification of the so-called naïve Feynman rules. In a classic paper published in Acta Physica Polonica in 1963, Feynman, dealing with quantum gravity (Feynman, 1963) as a gauge theory, had already emphasized that naïve Feynman rules cannot be applied in the theory of gravitation and modifications are necessary to ensure the positivity of the underlying theory and consistent positive definite probabilities of fundamental processes may be obtained. Otherwise unwanted Ghosts would appear in the theory leading to unphysical singularities and unphysical repulsive negative probabilities.

The main purpose of the present thesis is to prove, develop and analyse within the functional *differential* treatment of quantum field theory (Schwinger, 1951a, 1951b, 1953, 1954, 1972; Manoukian, 1985, 1986, 1987, 2006; Manoukian and Siranan, 2005; Limboonsong and Manoukian, 2006), as based on the Quantum Dynamical Principle (QDP), also popularly known as Schwinger's Dynamical (Action) Principle, constrained dynamical systems. Our thesis is also involved with constraints in quantum physics, in general, as well as will be discussed below and is developed in Chapters III and IV. The Quantum Dynamical Principle in its very general form is given by the very useful formula spelled out below. Suppressing spinor and tensor indices and denoting a general field by $\chi(x)$, coupled to an external source $J(x)$ in the Lagrangian density, then for an operator $\mathscr{O}(x)$, the functional derivative of the matrix element $\langle 0_+ | \mathscr{O}(x) | 0_- \rangle$, with respect to the external source $J(x')$ is rigorously given by

$$(-\mathrm{i})\,\frac{\delta}{\delta J(x')}\,\langle 0_+\,|\mathscr{O}(x)|0_-\rangle \;=\; \langle 0_+\,|(\chi(x')\mathscr{O}(x))_+|0_-\rangle - \mathrm{i}\left\langle 0_+\left|\frac{\delta}{\delta J(x')}\mathscr{O}(x)\right|0_-\right\rangle, \tag{1.0.1}$$

where $\langle 0_+\,|\,0_-\rangle$ denotes the vacuum transition amplitude of the theory, $(\chi(x')\mathscr{O}(x))_+$ denotes time ordering with $\chi(x')$ appearing first on the left-hand side for $x'^0 > x^0$ and vice versa, and most importantly, the functional derivative $\delta\mathscr{O}(x)/\delta J(x')$ in the last term in Eq. (1.0.1) is taken with the *independent* fields and their canonical conjugate momenta kept *fixed*. A complete rigorous proof of Eq. (1.0.1) is now available (see Manoukian, Sukkhasena and Siranan, 2007).

If $\mathscr{L}_I(x)$ is the interaction Lagrangian density of the theory, then, in the functional *differential* treatment of the theory, the vacuum-to-vacuum transition amplitude $\langle 0_+\,|\,0_-\rangle$, in very special cases, in given by (see, e.g., Manoukian, 1986)

$$\langle 0_+\,|\,0_-\rangle \;=\; \exp\left[\mathrm{i}\int(\mathrm{d}x)\;\mathscr{L}_I{}'(x)\right]\langle 0_+\,|\,0_-\rangle_0\,, \tag{1.0.2}$$

where $\langle 0_+\,|\,0_-\rangle_0$ is the vacuum-to-vacuum transition amplitude in the absence of the interaction term $\mathscr{L}_I(x)$. Also $\mathscr{L}_I{}'(x)$ is the interaction Lagrangian density with the fields $\chi(x)$ replaced by the functional differential operator $(-\mathrm{i})\delta/\delta J(x)$. In gauge theories, and all present elementary particle dynamical theories in quantum field theory are gauge field theories involving constraints. These constraints lead to modification of the naïve rules obtained from Eq. (1.0.2) by involving an *additional multiplicative functional* factor in Eq. (1.0.2) as functions of functional differential operators $(-\mathrm{i})\delta/\delta J(x)$. The determination of such factors is quite difficult and this thesis is a rigorous study to determine *explicitly* such factors in theories with constraints.

In developing such rules, we were guided by the explicit structure of present Lagrangian densities in elementary particle physics. We have even generalized such structures and obtained some very general rules. It is instructive to write down some of the Lagrangians densities used in particle physics. For example quantum electrodynamics is described by the Lagrangian density given below.

$$\mathscr{L}_{QED} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[\frac{\partial_\mu\bar{\psi}}{\mathrm{i}}\,\gamma^\mu\psi - \bar{\psi}\gamma^\mu\frac{\partial_\mu\psi}{\mathrm{i}}\right] - m_0\bar{\psi}\psi + e_0\bar{\psi}\gamma_\mu\psi A^\mu \;, \quad (1.0.3)$$

in and obvious notation. Mathematically, QED has the structure of an abelian gauge theory, with the symmetry group U(1) as gauge group. The gauge field which mediates the interaction between the charged spin -1/2 fields is the electromagnetic field. While the strong interaction describing the dynamics of quarks and gluons, referred to as Quantum Chromo_Dynamics (QCD) is defined by

$$\mathscr{L}_{QCD} \;= \frac{1}{2}\left[\frac{\partial_\mu\bar{\psi}_i}{\mathrm{i}}\,\partial^\mu\delta_{ij}\,\psi_j - \bar{\psi}_i\gamma^\mu\delta_{ij}\frac{\partial_\mu\psi_j}{\mathrm{i}}\right] - m_0\delta_{ij}\bar{\psi}_i\psi_j - \frac{1}{4}\,G^a_{\mu\nu}G^{\mu\nu}_a \;, \quad (1.0.4)$$

where $G^a_{\mu\nu}$ are the gluon field strength tensor defined by

$$G^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 f^{abc}A^b_\mu A^c_\nu \;, \quad (1.0.5)$$

f^{abc} are the structure constants satisfying the relation:

$$f^{abc} \;=\; \mathrm{i}(t^a)^{bc} \;, \quad (1.0.6)$$

and for the matrices t^a we have

$$[t^a, t^b] \;=\; \mathrm{i}f^{abc}t^c \;. \quad (1.0.7)$$

The matrices t^a are the generators of the underlying algebra. QCD is a non-abelian gauge theory as the generators t^a do *not* commute. Quarks are massive spin -1/2 fermions which carry a color charge whose gauging is the content of QCD. Quarks ae represented by Dirac fields in the fundamental representation 3 of the gauge group SU(3). They also carry electric charge (either -1/3 or 2/3) and participate in weak interactions as part of weak isospin doublets. They carry global quantum numbers including the baryon number, which is 1/3 for each quark, hypercharge and one of the flavor quantum numbers. Gluons are spin -1 bosons which also carry color charges, since they lie in the adjoint representation 8 of SU(3). They have no electric charge, do not participate in the weak interactions, and have no flavor. They lie in the singlet representation 1 of all these symmetry groups. The electro-weak theory is also a non-abelian gauge theory with Lagrangian density having the a general structure similar to the one in Eq. (1.0.4) but also involve the so-called Higgs boson which is a scalar field and causes no further difficulties as a constrained dynamics is concerned. The gravitational interaction most popularly given by Einstein's Lagrangian density is also a non-abelian gauge theory and requires very special tools and will be discussed in the concluding chapter of the thesis.

The outline of the thesis is as follows. In Chapter III, we develop the following construction of transformation functions for constrained dynamics in quantum physics. We are given a Hamiltonian $H(\mathbf{q}, \mathbf{p})$ as a function of independent pairs of canonical conjugate variables $\{q_i, p_i, i = 1, \ldots, n\} \equiv \{\mathbf{q}, \mathbf{p}\}$, that is, it is defined in a phase space of dimensionality equal to $2n$. We are also given a set of pairwise commuting operator functions $\{G_j(\mathbf{q}(t), \mathbf{p}(t)), j = 1, \ldots, k\}$ of these variables. These allow us to describe the dynamics of any Hamiltonian $\tilde{H}(\mathbf{q}, \mathbf{p}, \mathbf{Q}, \mathbf{P})$ in, *a priori*, $(2n + 2k)$ dimensional phase space in which constraints are imposed given by

$$Q_j(t) - G_j(\mathbf{q}(t), \mathbf{p}(t)) \;=\; 0 \quad , \quad j = 1, \ldots, k \; , \tag{1.0.8}$$

with $\mathbf{Q} = (Q_1, \ldots, Q_k)$, for which $\mathbf{P} = \mathbf{0}$, such that

$$\tilde{H}(\mathbf{q}, \mathbf{p}, \mathbf{G}(\mathbf{q}, \mathbf{p}), \mathbf{0}) \;=\; H(\mathbf{q}, \mathbf{p}) \; . \tag{1.0.9}$$

In Chapter 4, we consider constrained dynamics in quantum physics in the following manner. Given a Hamiltonian $\tilde{H}(\mathbf{q}, \mathbf{p})$ as a function of independent variables $\mathbf{q} = (q_1, \ldots, q_n)$ and their canonical conjugate momenta $\mathbf{p} = (p_1, \ldots, p_n)$, we consider a new system by defining constraint operator functions

$$\mathbf{G}(\mathbf{q}(t),\mathbf{p}(t)) \;=\; \big\{G_1(\mathbf{q}(t),\mathbf{p}(t)),\ldots,G_k(\mathbf{q}(t),\mathbf{p}(t))\big\}\,, \tag{1.0.10}$$

as of pairwise commuting operator functions $G_j(\mathbf{q}(t),\mathbf{p}(t))$, which together we introduce canonical conjugate momenta for them

$$\hat{\mathbf{G}}(\mathbf{q}(t),\mathbf{p}(t)) \;=\; \big\{\hat{G}_1(\mathbf{q}(t),\mathbf{p}(t)),\ldots,\hat{G}_k(\mathbf{q}(t),\mathbf{p}(t))\big\}\,, \tag{1.0.11}$$

such that

$$\begin{aligned}\mathbf{G}(\mathbf{q}(t),\mathbf{p}(t)) &= \mathbf{0}\,,\\ \hat{\mathbf{G}}(\mathbf{q}(t),\mathbf{p}(t)) &= \mathbf{0}\,,\end{aligned}$$

and

$$H(\mathbf{q}^*,\mathbf{p}^*) \;=\; \tilde{H}(\mathbf{q},\mathbf{p})\big|_{\mathbf{G}=\mathbf{0},\hat{\mathbf{G}}=\mathbf{0}}\,, \tag{1.0.12}$$

defines the new Hamiltonian of the system with constraints and with $(\mathbf{q},\mathbf{p}) \rightarrow (\mathbf{q}^*,\mathbf{p}^*,\mathbf{G},\hat{\mathbf{G}})$ defining a canonical transformation $H(\mathbf{q}^*,\mathbf{p}^*)$ describes dynamics in a $2(n-k)$ dimensional phase space. As the analyses involve extensive applications of functionals in quantum physics, we first, in Chapter II, apply such methods to a simpler problem of determining the number of eigenvalues of a given potential. Chapters V, VI and VII are entirely devoted to quantum field theory. In Chapter 5, functional calculus is developed for dependent fields with applications to Maxwell's Lagrangian with, *a priori*, non-conserved external current $\partial_\mu J^\mu \neq 0$, so that variations with respect to all the components of J^μ may be carried out independently - a required mathematical fact. Chapter VI deals systematically with the modification of the so-called Faddeev-Popov factor with explicit examples given for gauge invariant theories as well as to theories which break gauge invariance. Chapter VII is of central importance in the entire thesis as it establishes a Theorem for quadratic actions in *dependent* fields which are arbitrary functions in the independent fields. Applications of this general Theorem is also given. In quantum field theory, we consider units such that $\hbar = 1$, $c = 1$ as is often taken. The final chapter (Chapter VIII) deals with our conclusion and a summary of the main results obtained in the thesis.

CHAPTER II

NUMBER OF EIGENVALUES OF A GIVEN POTENTIAL: EXPLICIT FUNCTIONAL EXPRESSIONS

2.1 Introduction

Over the years, upper bounds have been derived for the number of eigenvalues, falling within specific ranges, for given potentials. The first bound was due to Bargmann (1952) who worked with spherically symmetric potentials and, in the process, obtained a bound depending on the orbital angular momentum. This was then extended by Schwinger (1961) for more general potentials, not mecessarily spherically symmetric, and a similar result was obtained by Birman (1966). Related upper bounds have been also derived by others, cf. Ghirardi and Rimini (1965). The most significant application of the Schwinger bound for the number of eigenvalues of a given potential, or more precisely of the negative of the sum of the negative eigenvalues, was carried out in the problem of the stability of matter, (Lieb and Thirring, 1975; Manoukian and Sirininlakul, 2005) and, in particular, in deriving a lower bound to the expectation value of the kinetic energy operator. The purpose of this chapter is to derive an explicit functional expression for the number of eigenvalues as well as for their sum. Our strategy of attack is the following. We first obtain expressions for the quantities we are seeking in terms of the spectral measure of the underlying Hamiltonian H in the problem. We relate these expressions to corresponding integrals involving Green functions. We then recast the derived results, by using in the process the quantum dynamical (action) principle (Manoukian, 1985; Schwinger, 1951, 1953, 1960, 1962) in terms of trace functionals of the transformation function $\langle \mathbf{x}T \,|\, \mathbf{x}0 \rangle$ and we finally carry out a Fourier decomposition (Schwinger, 1951, 1953) of the latter.

2.2 Explicit Functional Expressions for $N(\xi)$ and $N[\xi]$

For a given Hamiltonian H, its spectral decomposition may be written as

$$H = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}P_H(\lambda) \ . \tag{2.2.1}$$

The number of eigenvalues $< \xi$, counting degeneracy, may be simply written in the form

$$N(\xi) = \int \mathrm{d}^\nu \mathbf{x} \int_{-\infty}^{\infty} \Theta(\xi - \lambda) \, \mathrm{d} \langle \mathbf{x} | P_H(\lambda) | \mathbf{x} \rangle \ , \tag{2.2.2}$$

where ν denotes the dimensionality of space, and Θ is the step function. ξ may be taken to fall between eigenvalues. We may introduce an integral representation for Θ, and from the residue theorem, to rewrite $(\epsilon \to +0)$

$$\Theta(\xi - \lambda) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon} \, \mathrm{e}^{\mathrm{i}(\xi - \lambda)T/\hbar} \ , \tag{2.2.3}$$

$$N(\xi) = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^\nu \mathbf{x} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon} \, \mathrm{e}^{\mathrm{i}\xi T/\hbar} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\lambda T/\hbar} \, \mathrm{d} \langle \mathbf{x} | P_H(\lambda) | \mathbf{x} \rangle \ . \tag{2.2.4}$$

On the other hand, the Green (transformation) function $\langle \mathbf{x}T \,|\, \mathbf{x}'0 \rangle$ is given by

$$\langle \mathbf{x}T \,|\, \mathbf{x}'0 \rangle = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\lambda T/\hbar} \, \mathrm{d} \langle \mathbf{x} | P_H(\lambda) | \mathbf{x}' \rangle \ , \tag{2.2.5}$$

from the time evolution of the problem. Accordingly, (2.2.4) becomes

$$N(\xi) = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^\nu \mathbf{x} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon} \, \mathrm{e}^{\mathrm{i}\xi T/\hbar} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\lambda T/\hbar} \langle \mathbf{x} \, T \,|\, \mathbf{x} \, 0 \rangle \ . \tag{2.2.6}$$

For the *sum* of eigenvalues $N[\xi]$ having values $< \xi$, we have to multiply the integrand in Eq. (2.2.2) by λ, to obtain

$$N[\xi] = \int \mathrm{d}^\nu \mathbf{x} \int_{-\infty}^{\infty} \lambda \Theta(\xi - \lambda)\, \mathrm{d}\, \langle \mathbf{x} | P_H(\lambda) | \mathbf{x} \rangle \,. \tag{2.2.7}$$

From Eqs. (2.2.5), (2.2.6) and (2.2.7), we then have

$$N[\xi] = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^\nu \mathbf{x} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\, \mathrm{e}^{\mathrm{i}\xi T/\hbar}\, \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}T} \langle \mathbf{x}\, T | \mathbf{x}\, 0 \rangle \,. \tag{2.2.8}$$

Given a Hamiltonian $H(\mathbf{x}, \mathbf{p})$, we may couple $\mathbf{x}$ and $\mathbf{p}$ linearly to external c-number sources $\mathbf{F}(\tau)$, $\mathbf{S}(\tau)$ and define the new Hamiltonian:

$$\tilde{H}(\tau) = \lambda H - \mathbf{x} \cdot \mathbf{F}(\tau) + \mathbf{p} \cdot \mathbf{S}(\tau) \,. \tag{2.2.9}$$

We may now use the quantum dynamical (action) principle (Schwinger, 1951, 1953, 1960, 1962; Manoukian, 1985), expression

$$\delta \langle \mathbf{x}\, T | \mathbf{x}\, 0 \rangle = -\frac{\mathrm{i}}{\hbar} \int_0^T \mathrm{d}\tau \left\langle \mathbf{x}\, T \middle| \delta \tilde{H} \middle| \mathbf{x}\, 0 \right\rangle . \tag{2.2.10}$$

Hence for the functional derivative of $\langle \mathbf{x}\, T | \mathbf{x}\, 0 \rangle$ with respect to $\mathbf{F}(\tau)$ we obtain

$$\begin{aligned} (-\mathrm{i}\hbar) \frac{\delta}{\delta \mathbf{F}(\tau)} \langle \mathbf{x}\, T | \mathbf{x}\, 0 \rangle &= (-\mathrm{i}\hbar) \left(-\frac{\mathrm{i}}{\hbar} \right) \int_0^T \mathrm{d}\tau \left\langle \mathbf{x}\, T \middle| \frac{\delta \tilde{H}}{\delta \mathbf{F}(\tau)} \middle| \mathbf{x}\, 0 \right\rangle \\ &= - \langle \mathbf{x}\, T | {-\mathbf{x}} | \mathbf{x}\, 0 \rangle \\ &= \langle \mathbf{x}\, T | \mathbf{x} | \mathbf{x}\, 0 \rangle \,. \end{aligned} \tag{2.2.11}$$

On the other hand, for the functional derivative of $\langle \mathbf{x}\, T | \mathbf{x}\, 0 \rangle$ with respect to $\mathbf{S}(\tau)$, we obtain

$$(\mathrm{i}\hbar)\frac{\delta}{\delta\mathbf{S}(\tau)}\langle\mathbf{x}\,T\,|\,\mathbf{x}\,0\rangle \;=\; (\mathrm{i}\hbar)\left(-\frac{\mathrm{i}}{\hbar}\right)\int_0^T \mathrm{d}\tau\left\langle\mathbf{x}\,T\left|\frac{\delta\tilde{H}}{\delta\mathbf{S}(\tau)}\right|\mathbf{x}\,0\right\rangle$$

$$= \langle\mathbf{x}\,T\,|\mathbf{p}|\mathbf{x}\,0\rangle\,. \tag{2.2.12}$$

The action principle gives

$$\frac{\partial}{\partial\lambda}\langle\mathbf{x}\,T\,|\,\mathbf{x}\,0\rangle = -\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\,\langle\mathbf{x}\,T\,|H(\mathbf{x},\mathbf{p})|\mathbf{x}\,0\rangle_\lambda$$

$$= -\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\; H'(\tau)\,\langle\mathbf{x}\,T\,|\,\mathbf{x}\,0\rangle\,, \tag{2.2.13}$$

where

$$H'(\tau) \;=\; H\left(-\mathrm{i}\hbar\,\frac{\delta}{\delta\mathbf{F}(\tau)},\mathrm{i}\hbar\,\frac{\delta}{\delta\mathbf{S}(\tau)}\right). \tag{2.2.14}$$

That is,

$$\frac{\partial}{\partial\lambda}\langle\mathbf{x}\,T\,|\,\mathbf{x}\,0\rangle = -\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau H\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)},\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)\langle\mathbf{x}\,T\,|\,\mathbf{x}\,0\rangle\,, \tag{2.2.15}$$

or

$$\int\delta\,\langle\mathbf{x}\,T\,|\,\mathbf{x}\,0\rangle = -\frac{\mathrm{i}}{\hbar}\int\delta\lambda\int_0^T \mathrm{d}\tau\; H\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)},\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)\langle\mathbf{x}\,T\,|\,\mathbf{x}\,0\rangle\,, \tag{2.2.16}$$

which upon integration gives

$$\langle\mathbf{x}\,T\,|\,\mathbf{x}\,0\rangle = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\; H\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)},\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)\right]\langle\mathbf{x}\,T\,|\,\mathbf{x}\,0\rangle_0\,, \tag{2.2.17}$$

where

$$\langle\mathbf{x}\,T\,|\,\mathbf{x}\,0\rangle \;=\; \langle\mathbf{x}\,T\,|\,\mathbf{x}\,0\rangle_{\lambda=1}\;, \tag{2.2.18}$$

and

$$\langle \mathbf{x}\, T\,|\,\mathbf{x}\, 0\rangle_0 \;=\; \langle \mathbf{x}\, T\,|\,\mathbf{x}\, 0\rangle\,\big|_{\lambda=0}\;. \tag{2.2.19}$$

Consider the simple Hamiltonian

$$\hat{H} = -\mathbf{x}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau)\;. \tag{2.2.20}$$

The Heisenberg equations are

$$\dot{\mathbf{x}}(\tau) \;=\; \mathbf{S}(\tau)\;, \tag{2.2.21}$$

$$\dot{\mathbf{p}}(\tau) \;=\; \mathbf{F}(\tau)\;. \tag{2.2.22}$$

These equations may be integrated to

$$\int_\tau^t \mathrm{d}\mathbf{x}(\tau) \;=\; \int_\tau^t \mathrm{d}\tau\, \mathbf{S}(\tau)\;, \tag{2.2.23}$$

$$\mathbf{x}(t) - \mathbf{x}(\tau) \;=\; \int_{t'}^t \mathrm{d}\tau'\, \Theta(\tau'-\tau)\, \mathbf{S}(\tau')\;, \tag{2.2.24}$$

$$\mathbf{x}(\tau) \;=\; \mathbf{x}(t) - \int_{t'}^t \mathrm{d}\tau'\, \Theta(\tau'-\tau)\, \mathbf{S}(\tau')\;, \tag{2.2.25}$$

and

$$\int_{t'}^\tau \mathrm{d}\mathbf{p}(\tau) \;=\; \int_{t'}^\tau \mathrm{d}\tau\, \mathbf{F}(\tau)\;, \tag{2.2.26}$$

$$\mathbf{p}(\tau) - \mathbf{p}(t') \;=\; \int_{t'}^t \mathrm{d}\tau'\, \Theta(\tau-\tau')\, \mathbf{F}(\tau')\;, \tag{2.2.27}$$

$$\mathbf{p}(\tau) \;=\; \mathbf{p}(t') + \int_{t'}^t \mathrm{d}\tau'\, \Theta(\tau-\tau')\, \mathbf{F}(\tau')\;. \tag{2.2.28}$$

Upon taking the matrix element of the above solutions between $\langle \mathbf{x}\, T|$ and $|\mathbf{x}\, 0\rangle$ for $\lambda = 0$, we obtain

$$\langle \mathbf{x}\, T|\mathbf{x}(\tau)|\mathbf{x}\, 0\rangle_0 = \left[\mathbf{x}(T) - \int_0^T \mathrm{d}\tau'\, \Theta(\tau'-\tau)\, \mathbf{S}(\tau')\right] \langle \mathbf{x}\, T|\, \mathbf{x}\, 0\rangle_0 \,, \quad (2.2.29)$$

$$\langle \mathbf{x}\, T|\mathbf{p}(\tau)|\mathbf{x}\, 0\rangle_0 = \left[\mathbf{p}(0) + \int_0^T \mathrm{d}\tau'\, \Theta(\tau-\tau')\, \mathbf{F}(\tau')\right] \langle \mathbf{x}\, T|\, \mathbf{x}\, 0\rangle_0 \,, \quad (2.2.30)$$

where $\mathbf{x}$ and $\mathbf{p}$ within the square brackets on the right-hand sides of the above two equations are c-numbers, and we have used the relations

$$_0\langle \mathbf{x}\, T|\, \mathbf{x}(T) = \mathbf{x}\ _0\langle \mathbf{x}\, T| \ , \quad (2.2.31)$$

$$\mathbf{p}(0)\, |\mathbf{p}\, 0\rangle_0 = \mathbf{p}\, |\mathbf{p}\, 0\rangle_0 \ , \quad (2.2.32)$$

for $\lambda = 0$ at coincident times. Eqs. (2.2.29) and (2.2.30) may be rewritten as

$$-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{F}(\tau)} \langle \mathbf{x}\, T|\, \mathbf{p}\, 0\rangle_0 = \left[\mathbf{x} - \int_0^T \mathrm{d}\tau'\, \Theta(\tau'-\tau)\, \mathbf{S}(\tau')\right] \langle \mathbf{x}\, T|\, \mathbf{p}\, 0\rangle_0 \ , \quad (2.2.33)$$

$$\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{S}(\tau)} \langle \mathbf{x}\, T|\, \mathbf{p}\, 0\rangle_0 = \left[\mathbf{p} + \int_0^T \mathrm{d}\tau'\, \Theta(\tau-\tau')\, \mathbf{F}(\tau')\right] \langle \mathbf{x}\, T|\, \mathbf{p}\, 0\rangle_0 \ . \quad (2.2.34)$$

These equations may be integrated to yield

$$-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{F}(\tau)} \langle \mathbf{x}\, T|\, \mathbf{p}\, 0\rangle_0 = \left[\mathbf{x} - \int_0^T \mathrm{d}\tau'\, \Theta(\tau'-\tau)\, \mathbf{S}(\tau')\right] \langle \mathbf{x}\, T|\, \mathbf{p}\, 0\rangle_0 \ , \quad (2.2.35)$$

$$\int \delta\, \langle \mathbf{x}\, T|\, \mathbf{p}\, 0\rangle_0 = \frac{\mathrm{i}}{\hbar} \int \delta \mathbf{F}(\tau) \left[\mathbf{x} - \int_0^T \mathrm{d}\tau'\, \Theta(\tau'-\tau)\, \mathbf{S}(\tau')\right] \langle \mathbf{x}\, T|\, \mathbf{p}\, 0\rangle_0 \ , \quad (2.2.36)$$

$$\langle \mathbf{x}\, T \,|\, \mathbf{p}\; 0 \rangle_0 = A \;\exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{x}\cdot\int_0^T \mathbf{F}(\tau)\,\mathrm{d}\tau\right]$$

$$\times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\,\mathbf{F}(\tau)\cdot\Theta(\tau'-\tau)\,\mathbf{S}(\tau)\right], \tag{2.2.37}$$

and

$$\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\,\langle \mathbf{x}\, T \,|\, \mathbf{p}\; 0 \rangle_0 = \left[\mathbf{p} + \int_0^T \mathrm{d}\tau'\,\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right]\langle \mathbf{x}\, T \,|\, \mathbf{p}\; 0 \rangle_0 \;, \tag{2.2.38}$$

$$\int \delta\,\langle \mathbf{x}\, T \,|\, \mathbf{p}\; 0 \rangle_0 = -\,\frac{\mathrm{i}}{\hbar}\int \delta\mathbf{S}(\tau)\left[\mathbf{p} + \int_0^T \mathrm{d}\tau'\,\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right]\langle \mathbf{x}\, T \,|\, \mathbf{p}\; 0 \rangle_0 \;, \tag{2.2.39}$$

$$\langle \mathbf{x}\, T \,|\, \mathbf{p}\; 0 \rangle_0 = B \;\exp\left[-\frac{\mathrm{i}}{\hbar}\,\mathbf{p}\cdot\int_0^T \mathbf{S}(\tau)\,\mathrm{d}\tau\right]$$

$$\times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\,\mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right]. \tag{2.2.40}$$

To find A and B in Eqs. (2.2.37) and (2.2.40), respectively, we use the identity

$$\text{Eq.}(2.2.34) \;=\; \text{Eq.}(2.2.37)\;. \tag{2.2.41}$$

That is,

$$A\;\exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{x}\cdot\int_0^T \mathbf{F}(\tau)\,\mathrm{d}\tau\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\int_0^T \mathrm{d}\tau'\,\mathbf{F}(\tau)\cdot\Theta(\tau'-\tau)\,\mathbf{S}(\tau')\right]$$

$$=\; B\;\exp\left[-\frac{\mathrm{i}}{\hbar}\,\mathbf{p}\cdot\int_0^T \mathbf{S}(\tau)\,\mathrm{d}\tau\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\int_0^T \mathrm{d}\tau'\,\mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right]. \tag{2.2.42}$$

The above equality gives the expressions

$$A = \exp\left[-\frac{\mathrm{i}}{\hbar}\,\mathbf{p}\cdot\int_0^T \mathbf{S}(\tau)\,\mathrm{d}\tau\right] \quad , \quad B = \exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{x}\cdot\int_0^T \mathbf{F}(\tau)\,\mathrm{d}\tau\right] . \tag{2.2.43}$$

For the boundary condition $\mathbf{F} = 0$, $\mathbf{S} = 0$, $\hat{H} \to 0$ and we have

$$\tilde{H} = H \; . \tag{2.2.44}$$

For $T = 0$, that is, for $t = t'$, we obtain

$$\begin{aligned} \langle \mathbf{x}\, t \,|\, \mathbf{p}\, t\rangle &= \big\langle \mathbf{x} \big| U(t) U^\dagger(t) \big| \mathbf{p} \big\rangle \\ &= \langle \mathbf{x} \,|\, \mathbf{p} \rangle \\ &= \exp\left(\frac{\mathrm{i}}{\hbar}\mathbf{x}\cdot\mathbf{p}\right) . \end{aligned} \tag{2.2.45}$$

Hence

$$\begin{aligned} \langle \mathbf{x}\, T \,|\, \mathbf{p}\, 0\rangle_0 = {}& \exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{x}\cdot\int_0^T \mathrm{d}\tau\,\mathbf{F}(\tau)\right] \exp\left[-\frac{\mathrm{i}}{\hbar}\,\mathbf{p}\cdot\int_0^T \mathrm{d}\tau\,\mathbf{S}(\tau)\right] \exp\left(\frac{\mathrm{i}}{\hbar}\,\mathbf{x}\cdot\mathbf{p}\right) \\ & \times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\,\mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right] . \end{aligned} \tag{2.2.46}$$

To obtain the expression for $\langle \mathbf{x}\, T \,|\, \mathbf{x}\, 0\rangle_0$, we multiply Eq. (2.2.46) by $\langle \mathbf{p}\, 0 \,|\, \mathbf{x}\, 0\rangle = \exp(-\mathrm{i}\,\mathbf{x}\cdot\mathbf{p}/\hbar)$ and integrate over $\mathbf{p}$, with measure $\mathrm{d}^\nu\mathbf{p}/(2\pi\hbar)^\nu$, to obtain

$$
\begin{aligned}
\langle \mathbf{x}\, T \,|\, \mathbf{x}\, 0\rangle_0 &= \int \frac{\mathrm{d}^\nu \mathbf{p}}{(2\pi\hbar)^\nu} \langle \mathbf{x}\, T \,|\, \mathbf{p}\, 0\rangle_0 \langle \mathbf{p}\, 0 \,|\, \mathbf{x}\, 0\rangle \\
&= \int \frac{\mathrm{d}^\nu \mathbf{p}}{(2\pi\hbar)^\nu} \; \exp\left[\frac{\mathrm{i}}{\hbar}\, \mathbf{x}\cdot \int_0^T \mathrm{d}\tau\, \mathbf{F}(\tau)\right] \exp\left[-\frac{\mathrm{i}}{\hbar}\, \mathbf{p}\cdot \int_0^T \mathrm{d}\tau\, \mathbf{S}(\tau)\right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\, \mathbf{F}(\tau')\right] \\
&\qquad \times \exp\left(\frac{\mathrm{i}}{\hbar}\, \mathbf{x}\cdot\mathbf{p}\right) \exp\left(-\frac{\mathrm{i}}{\hbar}\, \mathbf{x}\cdot\mathbf{p}\right) \\
&= \int \frac{\mathrm{d}^\nu \mathbf{p}}{(2\pi\hbar)^\nu} \; \exp\left[\frac{\mathrm{i}}{\hbar}\, \mathbf{x}\cdot \int_0^T \mathrm{d}\tau\, \mathbf{F}(\tau)\right] \exp\left[\mathrm{i}\left(\mathbf{x}-\mathbf{x}-\int_0^T \mathrm{d}\tau \mathbf{S}(\tau)\right)\frac{\mathbf{p}}{\hbar}\right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\, \mathbf{F}(\tau')\right] \\
&= \delta^\nu\left(\mathbf{x}-\mathbf{x}-\int_0^T \mathrm{d}\tau\, \mathbf{S}(\tau)\right) \exp\left[\frac{\mathrm{i}}{\hbar}\, \mathbf{x}\cdot \int_0^T \mathrm{d}\tau\, \mathbf{F}(\tau)\right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\, \mathbf{F}(\tau')\right]. \qquad (2.2.47)
\end{aligned}
$$

That is,

$$
\begin{aligned}
\langle \mathbf{x}\, T \,|\, \mathbf{x}\, 0\rangle_0 &= \delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{S}(\tau)\right) \exp\left[\frac{\mathrm{i}}{\hbar}\, \mathbf{x}\cdot \int_0^T \mathrm{d}\tau\, \mathbf{F}(\tau)\right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\, \mathbf{F}(\tau')\right]. \qquad (2.2.48)
\end{aligned}
$$

Substituting Eqs. (2.2.17) and (2.2.48) into Eqs. (2.2.6) and (2.2.8) gives

$$
\begin{aligned}
N(\xi) &= \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^\nu \mathbf{x} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\ \mathrm{e}^{\mathrm{i}\xi T/\hbar} \, \langle \mathbf{x}\, T \,|\, \mathbf{x}\, 0 \rangle \\
&= \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^\nu \mathbf{x} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\ \mathrm{e}^{\mathrm{i}\xi T/\hbar} \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_0^T \mathrm{d}\tau\ H\left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{F}(\tau)}, \mathrm{i}\hbar \frac{\delta}{\delta \mathbf{S}(\tau)} \right) \right] \langle \mathbf{x}\, T \,|\, \mathbf{x}\, 0 \rangle_0 \\
&= \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^\nu \mathbf{x} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\ \mathrm{e}^{\mathrm{i}\xi T/\hbar} \exp\left[-\frac{\mathrm{i}}{\hbar} \int_0^T \mathrm{d}\tau\ H\left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{F}(\tau)}, \mathrm{i}\hbar \frac{\delta}{\delta \mathbf{S}(\tau)} \right) \right] \\
&\qquad \times \delta^\nu \left(\int_0^T \mathrm{d}\tau\ \mathbf{S}(\tau) \right) \exp\left[\frac{\mathrm{i}}{\hbar}\, \mathbf{x} \cdot \int_0^T \mathrm{d}\tau\ \mathbf{F}(\tau) \right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\ \mathbf{S}(\tau) \cdot \Theta(\tau - \tau')\, \mathbf{F}(\tau') \right] \\
&= \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\ \mathrm{e}^{\mathrm{i}\xi T/\hbar} \exp\left[-\frac{\mathrm{i}}{\hbar} \int_0^T \mathrm{d}\tau\ H\left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{F}(\tau)}, \mathrm{i}\hbar \frac{\delta}{\delta \mathbf{S}(\tau)} \right) \right] \\
&\qquad \times \delta^\nu \left(\int_0^T \mathrm{d}\tau\ \mathbf{S}(\tau) \right) \int \mathrm{d}^\nu \mathbf{x}\ \exp\left[\frac{\mathrm{i}}{\hbar}\, \mathbf{x} \cdot \int_0^T \mathrm{d}\tau\ \mathbf{F}(\tau) \right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\ \mathbf{S}(\tau) \cdot \Theta(\tau - \tau')\, \mathbf{F}(\tau') \right] \\
&= \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\ \mathrm{e}^{\mathrm{i}\xi T/\hbar} \exp\left[-\frac{\mathrm{i}}{\hbar} \int_0^T \mathrm{d}\tau\ H\left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{F}(\tau)}, \mathrm{i}\hbar \frac{\delta}{\delta \mathbf{S}(\tau)} \right) \right] \\
&\qquad \times \delta^\nu \left(\int_0^T \mathrm{d}\tau\ \mathbf{S}(\tau) \right) \int \mathrm{d}^\nu \frac{\mathbf{x}}{\hbar} (\hbar)\ \exp\left[\frac{\mathrm{i}}{\hbar}\ \left(\frac{\mathbf{x}}{\hbar} \right) \cdot \int_0^T \mathrm{d}\tau\ \mathbf{F}(\tau) \right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\ \mathbf{S}(\tau) \cdot \Theta(\tau - \tau')\, \mathbf{F}(\tau') \right]
\end{aligned}
$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\, \mathrm{e}^{\mathrm{i}\xi T/\hbar} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\, H\left(-\mathrm{i}\hbar\frac{\delta}{\delta \mathbf{F}(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta \mathbf{S}(\tau)}\right)\right]$$

$$\times \delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{S}(\tau)\right)(2\pi\hbar)^\nu\, \delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{F}(\tau)\right)$$

$$\times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right],$$

$$N(\xi) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\, \mathrm{e}^{\mathrm{i}\xi T/\hbar} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\, H\left(-\mathrm{i}\hbar\frac{\delta}{\delta \mathbf{F}(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta \mathbf{S}(\tau)}\right)\right]$$

$$\times(2\pi\hbar)^\nu\, \delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{S}(\tau)\right)\, \delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{F}(\tau)\right)$$

$$\times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right]. \tag{2.2.49}$$

Similarly, we have

$$N[\xi] = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^\nu\mathbf{x} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\, \mathrm{e}^{\mathrm{i}\xi T/\hbar}\, \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}T}\, \langle \mathbf{x}\, T\,|\,\mathbf{x}\, 0\rangle$$

$$= \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^\nu\mathbf{x} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\, \mathrm{e}^{\mathrm{i}\xi T/\hbar}\, \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}T}$$

$$\times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\, H\left(-\mathrm{i}\hbar\frac{\delta}{\delta \mathbf{F}(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta \mathbf{S}(\tau)}\right)\right] \langle \mathbf{x}\, T\,|\,\mathbf{x}\, 0\rangle_0$$

$$= \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^\nu\mathbf{x} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\, \mathrm{e}^{\mathrm{i}\xi T/\hbar}\, \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}T}$$

$$\times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\, H\left(-\mathrm{i}\hbar\frac{\delta}{\delta \mathbf{F}(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta \mathbf{S}(\tau)}\right)\right]$$

$$\times \delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{S}(\tau)\right) \exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{x}\cdot\int_0^T \mathrm{d}\tau\, \mathbf{F}(\tau)\right]$$

$$\times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right]$$

$$
\begin{aligned}
&= \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\, \mathrm{e}^{\mathrm{i}\xi T/\hbar}\, \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\, H\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)\right] \\
&\quad \times\delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{S}(\tau)\right) \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right] \\
&\quad \times \int \mathrm{d}^\nu\mathbf{x}\, \exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{x}\cdot\int_0^T \mathrm{d}\tau\, \mathbf{F}(\tau)\right] \\
&= \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\, \mathrm{e}^{\mathrm{i}\xi T/\hbar}\, \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\, H\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)\right] \\
&\quad \times\delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{S}(\tau)\right) \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right] \\
&\quad \times(2\pi\hbar)^\nu\, \delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{F}(\tau)\right),
\end{aligned}
$$

$$
\begin{aligned}
N[\xi] &= \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\, \mathrm{e}^{\mathrm{i}\xi T/\hbar}\, \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\, H\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)\right] \\
&\quad \times(2\pi\hbar)^\nu\, \delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{S}(\tau)\right)\, \delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{F}(\tau)\right) \\
&\quad \times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right]. \qquad (2.2.50)
\end{aligned}
$$

Therefore we have obtained the following expression for $N(\xi)$, $N[\xi]$:

$$
N(\xi) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\, \mathrm{e}^{\mathrm{i}\xi T/\hbar}\, K(T)\,, \qquad (2.2.51)
$$

$$
N[\xi] = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon}\, \mathrm{e}^{\mathrm{i}\xi T/\hbar}\, \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}T}\, K(T)\,, \qquad (2.2.52)
$$

where

$$
\begin{aligned}
K(T) \;=\; & \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\, H\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)\right] \\
& \times(2\pi\hbar)^\nu\, \delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{S}(\tau)\right) \delta^\nu\left(\int_0^T \mathrm{d}\tau\, \mathbf{F}(\tau)\right) \\
& \times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right]\Bigg| , \qquad (2.2.53)
\end{aligned}
$$

and the bar | corresponds to taking the limits $\mathbf{S}, \mathbf{F} \to 0$, *after* the functional differentiations are carried out.

2.3 Functional Fourier Analysis of $N(\xi)$ and $N[\xi]$

Since in Eqs. (2.2.6) and (2.2.8), we are considering the trace operation, we may carry out Fourier decompositions as follows:

$$
\mathbf{F}(\tau) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \mathbf{F}_n\, \mathrm{e}^{-\mathrm{i}2\pi n\tau/T} , \qquad (2.3.1)
$$

$$
\mathbf{S}(\tau) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \mathbf{S}_n\, \mathrm{e}^{\mathrm{i}2\pi n\tau/T} , \qquad (2.3.2)
$$

$$
\frac{\delta}{\delta\mathbf{F}(\tau)} = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi n\tau/T}\, \frac{\partial}{\partial\mathbf{F}_n} , \qquad (2.3.3)
$$

$$
\frac{\delta}{\delta\mathbf{S}(\tau)} = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi n\tau/T}\, \frac{\partial}{\partial\mathbf{S}_n} , \qquad (2.3.4)
$$

where $\partial F_n^i/\partial F_m^j = \delta^{ij}\,\delta_{nm}$ and so on. These spectral decompositions of the auxiliary c-fields correspond to protections on subspaces, labeled by n, with which eigenvectors of the Hamiltonian in question will be associated.

To simplify $K(T)$, consider the term $\delta^\nu\left(\int_0^T \mathrm{d}\tau\ \mathbf{S}(\tau)\right)$ in Eq. (2.2.53)

For $n = 0$;

$$\frac{1}{T}\int_0^T \mathrm{d}\tau\ \mathbf{S}_0 \quad = \quad \mathbf{S}_0\ . \tag{2.3.5}$$

For $n \neq 0$;

$$\begin{aligned}
\frac{1}{T}\sum_{n\neq 0}\int_0^T \mathrm{d}\tau\ \mathrm{e}^{\mathrm{i}2\pi n\tau/T}\ \mathbf{S}_n \quad &= \quad \frac{1}{T}\sum_{n\neq 0}\ \frac{T\ \mathrm{e}^{\mathrm{i}2\pi n\tau/T}}{\mathrm{i}2\pi n}\ \Bigg|_0^T \mathbf{S}_n \\
&= \quad \frac{1}{T}\sum_{n\neq 0}\ \frac{T}{\mathrm{i}2\pi n}\ (\mathrm{e}^{\mathrm{i}2\pi n} - 1)\ \mathbf{S}_n \\
&= \quad \frac{1}{T}\sum_{n\neq 0}\ \frac{T}{\mathrm{i}2\pi n}\ (1 - 1)\ \mathbf{S}_n \\
&= \quad 0\ .
\end{aligned} \tag{2.3.6}$$

For all n, we may combine Eqs. (2.3.5) and (2.3.6), to obtain simply

$$\delta^\nu\left(\int_0^T \mathrm{d}\tau\ \mathbf{S}(\tau)\right) \quad = \quad \delta^\nu(\mathbf{S}_0)\ . \tag{2.3.7}$$

Similarly, we may write

$$\delta^\nu\left(\int_0^T \mathrm{d}\tau\ \mathbf{F}(\tau)\right) \quad = \quad \delta^\nu(\mathbf{F}_0)\ . \tag{2.3.8}$$

For the last term of the right-hand side of Eq. (2.2.53) we have the following equations:

$$\begin{aligned}
&\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\ \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\ \mathbf{F}(\tau')\right] \\
= \quad &\exp\left[-\frac{\mathrm{i}}{\hbar}\ \frac{1}{T^2}\sum_{n,m}\ \mathbf{S}_n\cdot\mathbf{F}_m \int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'\ \mathrm{e}^{\mathrm{i}2\pi n\tau/T}\Theta(\tau-\tau')\ \mathrm{e}^{-\mathrm{i}2\pi m\tau'/T}\right]
\end{aligned}$$

$$= \exp\left[-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\sum_{n,m}\,\mathbf{S}_n\cdot\mathbf{F}_m\int_0^T \mathrm{d}\tau\left(\int_0^\tau \mathrm{d}\tau'\,\mathrm{e}^{-\mathrm{i}2\pi m\tau'/T}\right)\,\mathrm{e}^{\mathrm{i}2\pi n\tau/T}\right]. \quad (2.3.9)$$

For $n, m = 0$;

$$\begin{aligned}
\exp\left[-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\mathbf{S}_0\cdot\mathbf{F}_0\int_0^T \mathrm{d}\tau\int_0^\tau \mathrm{d}\tau'\right] &= \exp\left[-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\mathbf{S}_0\cdot\mathbf{F}_0\int_0^T \tau\,\mathrm{d}\tau\right] \\
&= \exp\left[-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\mathbf{S}_0\cdot\mathbf{F}_0\,\frac{\tau^2}{2}\,\Big|_{\tau=0}^{T}\right] \\
&= \exp\left[-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\mathbf{S}_0\cdot\mathbf{F}_0\,\frac{T^2}{2}\right] \\
&= \exp\left[-\frac{\mathrm{i}}{2\hbar}\,\mathbf{S}_0\cdot\mathbf{F}_0\right]. \quad (2.3.10)
\end{aligned}$$

Therefore,

$$\delta^\nu(\mathbf{F}_0)\delta^\nu(\mathbf{S}_0)\exp\left[-\frac{\mathrm{i}}{2\hbar}\,\mathbf{S}_0\cdot\mathbf{F}_0\right] = \delta^\nu(\mathbf{F}_0)\delta^\nu(\mathbf{S}_0)\,. \quad (2.3.11)$$

For $n, m \neq 0$, we obtain

$$\begin{aligned}
&\exp\left[-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\mathbf{S}_n\cdot\mathbf{F}_m\int_0^T \mathrm{d}\tau\left(\int_0^\tau \mathrm{d}\tau'\,\mathrm{e}^{-\mathrm{i}2\pi m\tau'/T}\right)\,\mathrm{e}^{\mathrm{i}2\pi n\tau/T}\right] \\
&= \exp\left[-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\mathbf{S}_n\cdot\mathbf{F}_m\int_0^T \mathrm{d}\tau\left(\frac{T}{-\mathrm{i}2\pi m}\,\mathrm{e}^{-\mathrm{i}2\pi m\tau'/T}\,\Big|_{\tau'=0}^{\tau}\right)\,\mathrm{e}^{\mathrm{i}2\pi n\tau/T}\right] \\
&= \exp\left[-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\mathbf{S}_n\cdot\mathbf{F}_m\int_0^T \mathrm{d}\tau\left(\frac{T}{-\mathrm{i}2\pi m}\right)\left(\mathrm{e}^{-\mathrm{i}2\pi m\tau/T}-1\right)\,\mathrm{e}^{\mathrm{i}2\pi n\tau/T}\right] \\
&= \exp\left[-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\mathbf{S}_n\cdot\mathbf{F}_m\int_0^T \mathrm{d}\tau\left(\frac{T}{-\mathrm{i}2\pi m}\right)\left(\mathrm{e}^{\mathrm{i}2\pi(n-m)\tau/T}-\mathrm{e}^{\mathrm{i}2\pi n\tau/T}\right)\right] \\
&= \exp\left[-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\delta_{nm}\,\mathbf{S}_n\cdot\mathbf{F}_m\int_0^T \mathrm{d}\tau\left(\frac{T}{-\mathrm{i}2\pi m}\right)\left(1-\mathrm{e}^{\mathrm{i}2\pi n\tau/T}\right)+0\right] \\
&= \exp\left[-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\delta_{nm}\,\mathbf{S}_n\cdot\mathbf{F}_m\left(\frac{T}{-\mathrm{i}2\pi m}\right)\left(\int_0^T \mathrm{d}\tau-\int_0^T \mathrm{d}\tau\,\mathrm{e}^{\mathrm{i}2\pi n\tau/T}\right)\right]
\end{aligned}$$

$$= \exp\left\{-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\delta_{nm}\,\mathbf{S}_n\cdot\mathbf{F}_m\left(\frac{T}{-\mathrm{i}2\pi m}\right)\left[\tau\,\Big|_{\tau=0}^{T}-\left(\frac{T}{\mathrm{i}2\pi n}\right)\,\mathrm{e}^{\mathrm{i}2\pi n\tau/T}\,\Big|_{\tau=0}^{T}\right]\right\}$$

$$= \exp\left\{-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\delta_{nm}\,\mathbf{S}_n\cdot\mathbf{F}_m\left(\frac{T}{-\mathrm{i}2\pi m}\right)\left[T-\left(\frac{T}{\mathrm{i}2\pi n}\right)\left(\mathrm{e}^{\mathrm{i}2\pi n}-1\right)\right]\right\}$$

$$= \exp\left\{-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\delta_{nm}\,\mathbf{S}_n\cdot\mathbf{F}_m\left(\frac{T}{-\mathrm{i}2\pi m}\right)\left[T-\left(\frac{T}{\mathrm{i}2\pi n}\right)(1-1)\right]\right\}$$

$$= \exp\left[-\frac{\mathrm{i}}{\hbar}\,\frac{1}{T^2}\,\delta_{nm}\,\mathbf{S}_n\cdot\mathbf{F}_m\left(\frac{T^2}{-\mathrm{i}2\pi m}\right)\right]$$

$$= \exp\left[\frac{1}{\hbar}\frac{\mathbf{S}_n\cdot\mathbf{F}_n}{2\pi n}\right]. \tag{2.3.12}$$

All told, we derive

$$\delta^\nu(\mathbf{F}_0)\delta^\nu(\mathbf{S}_0)\exp\Big[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\int_0^T \mathrm{d}\tau'\,\mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\Big]$$
$$= \delta^\nu(\mathbf{F}_0)\delta^\nu(\mathbf{S}_0)\exp\left[\frac{1}{\hbar}\sum_{n\neq 0}\frac{\mathbf{S}_n\cdot\mathbf{F}_n}{2\pi n}\right]. \tag{2.3.13}$$

Substituting Eqs. (2.3.7), (2.3.8) and (2.3.12) into Eq. (2.2.53) for the expression of $K(T)$ the corresponding simplifies to

$$K(T) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}\tau\; H\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)},\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)\right]$$
$$\times\,(2\pi\hbar)^\nu\;\delta^\nu(\mathbf{S}_0)\;\delta^\nu(\mathbf{F}_0)\;\exp\left(\frac{1}{\hbar}\sum_{n\neq 0}\frac{\mathbf{S}_n\cdot\mathbf{F}_n}{2\pi n}\right)\Bigg|. \tag{2.3.14}$$

In evaluating the latter, we may use Eqs. (2.3.3) and (2.3.4) in H with the latter corresponding to the Hamiltonian of the system as appearing in Eq. (2.2.9) without external sources. The expressions in Eqs. (2.2.51) and (2.2.52) together with Eq. (2.3.14) are the main results of this chapter. The expression for $K(T)$ in Eq. (2.3.14) is directly related to the spectral resolution of the time evolution operator expressed in terms of

c-functional methods involving no quantum operators.

2.4 Illustration of the Rules

To verify the consistency of the formulation, consider the harmonic oscillator problem with $\nu = 1$, $H = p^2/2m + m\omega^2 x^2/2$. Then from Eqs. (2.3.3) and (2.3.4), we have

$$\int_0^T \mathrm{d}\tau\, H\left(-\mathrm{i}\hbar\frac{\delta}{\delta F(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta S(\tau)}\right)$$

$$= \int_0^T \mathrm{d}\tau \left[\frac{1}{2m}\left(\mathrm{i}\hbar\frac{\delta}{\delta S(\tau)}\right)^2 + \frac{m\omega^2}{2}\left(-\mathrm{i}\hbar\frac{\delta}{\delta F(\tau)}\right)^2\right]$$

$$= \int_0^T \mathrm{d}\tau \left[\frac{-\hbar^2}{2m}\left(\frac{\delta}{\delta S(\tau)}\right)^2 - \frac{m\omega^2\hbar^2}{2}\left(\frac{\delta}{\delta F(\tau)}\right)^2\right]$$

$$= \int_0^T \mathrm{d}\tau \left[\frac{-\hbar^2}{2m}\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi n\tau/T}\frac{\partial}{\partial S_n}\, \mathrm{e}^{\mathrm{i}2\pi n\tau/T}\,\frac{\partial}{\partial S_{-n}}\right.$$

$$\left. -\frac{m\omega^2\hbar^2}{2}\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi n\tau/T}\frac{\partial}{\partial F_n}\, \mathrm{e}^{-\mathrm{i}2\pi n\tau/T}\frac{\partial}{\partial F_{-n}}\right]$$

$$= \int_0^T \mathrm{d}\tau \left[\frac{-\hbar^2}{2m}\sum_{n=-\infty}^{\infty}\frac{\partial}{\partial S_n}\,\frac{\partial}{\partial S_{-n}} - \frac{m\omega^2\hbar^2}{2}\sum_{n=-\infty}^{\infty}\frac{\partial}{\partial F_n}\,\frac{\partial}{\partial F_{-n}}\right]$$

$$= \int_0^T \mathrm{d}\tau \left[-\hbar^2\sum_{n=-\infty}^{\infty}\left(\frac{1}{2m}\frac{\partial}{\partial S_n}\,\frac{\partial}{\partial S_{-n}} + \frac{m\omega^2}{2}\,\frac{\partial}{\partial F_n}\,\frac{\partial}{\partial F_{-n}}\right)\right]$$

$$= -\hbar^2\tau\,\Big|_{\tau=0}^{T}\,\sum_{n=-\infty}^{\infty}\left[\frac{1}{2m}\frac{\partial}{\partial S_n}\,\frac{\partial}{\partial S_{-n}} + \frac{m\omega^2}{2}\,\frac{\partial}{\partial F_n}\,\frac{\partial}{\partial F_{-n}}\right]$$

$$= -\hbar^2 T\sum_{n=-\infty}^{\infty}\left[\frac{1}{2m}\frac{\partial}{\partial S_n}\,\frac{\partial}{\partial S_{-n}} + \frac{m\omega^2}{2}\,\frac{\partial}{\partial F_n}\,\frac{\partial}{\partial F_{-n}}\right]. \tag{2.4.1}$$

Substituting Eq. (2.4.1) into Eqs. (2.3.14), we then obtain

$$K(T) \;=\; \exp\left[\mathrm{i}\hbar T \sum_{n=-\infty}^{\infty}\left(\frac{1}{2m}\,\frac{\partial}{\partial S_n}\,\frac{\partial}{\partial S_{-n}} + \frac{m\omega^2}{2}\,\frac{\partial}{\partial F_n}\,\frac{\partial}{\partial F_{-n}}\right)\right]$$

$$\times(2\pi\hbar)^{\nu}\,\delta^{\nu}(S_0)\,\delta^{\nu}(F_0)\,\exp\left(\frac{1}{\hbar}\sum_{n\neq 0}\frac{S_n\,F_n}{2\pi n}\right)\Bigg|\,, \qquad (2.4.2)$$

where $|$ means setting the external sources equal to zero after the functional differentiations are carried out.

The steps for carrying out the differentiations in Eq. (2.3.14) with respect to F_n are as follows:

For $n = 0$:

$$\exp\left\{\mathrm{i}\hbar T\left[\frac{1}{2m}\left(\frac{\partial}{\partial S_0}\right)^2 + \frac{m\omega^2}{2}\left(\frac{\partial}{\partial F_0}\right)^2\right]\right\}(2\pi\hbar)\,\delta(S_0)\,\delta(F_0)\,\Bigg|$$

$$=(2\pi\hbar)\exp\left[\frac{\mathrm{i}\hbar T}{2m}\left(\frac{\partial}{\partial S_0}\right)^2\right]\delta(S_0)\exp\left[\frac{\mathrm{i}\hbar T m\omega^2}{2}\left(\frac{\partial}{\partial F_0}\right)^2\right]\delta(F_0)\,\Bigg|$$

$$=(2\pi\hbar)\exp\left(\frac{\mathrm{i}\hbar T}{2m}\,\frac{\partial^2}{\partial S_0^2}\right)\left[\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda_1}{(2\pi)}\,\mathrm{e}^{\mathrm{i}\lambda_1 S_0}\right]$$

$$\times\exp\left(\frac{\mathrm{i}\hbar T m\omega^2}{2}\,\frac{\partial^2}{\partial F_0^2}\right)\left[\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda_2}{(2\pi)}\,\mathrm{e}^{\mathrm{i}\lambda_2 F_0}\right]\,\Bigg|$$

$$=(2\pi\hbar)\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda_1}{(2\pi)}\,\mathrm{e}^{\mathrm{i}\lambda_1 S_0}\,\mathrm{e}^{\frac{\mathrm{i}\hbar T}{2m}(-\lambda_1^2)}\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda_2}{(2\pi)}\,\mathrm{e}^{\mathrm{i}\lambda_2 F_0}\,\mathrm{e}^{\frac{\mathrm{i}\hbar T m\omega^2}{2}(-\lambda_2^2)}\,\Bigg|_{S_0\to 0,F_0\to 0}$$

$$=(2\pi\hbar)\frac{1}{(2\pi)}\frac{1}{(2\pi)}\int_{-\infty}^{\infty}\mathrm{d}\lambda_1\exp\left(-\frac{\mathrm{i}\hbar T\lambda_1^2}{2m}\right)\int_{-\infty}^{\infty}\mathrm{d}\lambda_2\exp\left(-\frac{\mathrm{i}\hbar m\omega^2 T\lambda_2^2}{2}\right)$$

$$
=\frac{\hbar}{2\pi}\left[\sqrt{2\pi}\,\sqrt{\frac{m}{\mathrm{i}\hbar T}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda_1}{\sqrt{2\pi}\sqrt{\frac{m}{\mathrm{i}\hbar T}}}\ \exp\left(-\frac{\mathrm{i}\hbar T\lambda_1^2}{2m}\right)\right]
$$

$$
\times\left[\sqrt{2\pi}\,\sqrt{\frac{1}{\mathrm{i}\hbar m\omega^2 T}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda_2}{\sqrt{2\pi}\sqrt{\frac{1}{\mathrm{i}\hbar m\omega^2 T}}}\ \exp\left(-\frac{\mathrm{i}\hbar m\omega^2 T\lambda_2^2}{2}\right)\right]. \tag{2.4.3}
$$

Using the integral

$$
\int_{-\infty}^{\infty}\mathrm{d}x\ \frac{1}{\sqrt{2\pi}\,\sigma}\ \mathrm{e}^{-x^2/2\sigma^2}=1\ , \tag{2.4.4}
$$

where σ in the two square brackets of the above equation are $\sqrt{m/\mathrm{i}\hbar T}$ and $\sqrt{1/\mathrm{i}\hbar m\omega^2 T}$, respectively, then for $n=0$ becomes

$$
\exp\left\{\mathrm{i}\hbar T\left[\frac{1}{2m}\left(\frac{\partial}{\partial S_0}\right)^2+\frac{m\omega^2}{2}\left(\frac{\partial}{\partial F_0}\right)^2\right]\right\}(2\pi\hbar)\ \delta(S_0)\ \delta(F_0)\ \Bigg|
$$

$$
=\frac{\hbar}{2\pi}(2\pi)\sqrt{\frac{m}{\mathrm{i}\hbar T}}\ \sqrt{\frac{1}{\mathrm{i}\hbar m\omega^2 T}}
$$

$$
=\frac{1}{\mathrm{i}\omega T}\ . \tag{2.4.5}
$$

On the other hand, for $n\neq 0$, consider the $n=1$ and -1 terms in Eq. (2.4.2):

$$
\exp\left(\frac{\mathrm{i}\hbar T}{2m}\ \frac{\partial}{\partial S_1}\ \frac{\partial}{\partial S_{-1}}+\frac{\mathrm{i}\hbar T m\omega^2}{2}\ \frac{\partial}{\partial F_1}\ \frac{\partial}{\partial F_{-1}}\right)\exp\left[\frac{1}{\hbar}\ \frac{S_1\ F_1}{2\pi(1)}\right]
$$

$$
\times\exp\left(\frac{\mathrm{i}\hbar T}{2m}\ \frac{\partial}{\partial S_{-1}}\ \frac{\partial}{\partial S_1}+\frac{\mathrm{i}\hbar T m\omega^2}{2}\ \frac{\partial}{\partial F_{-1}}\ \frac{\partial}{\partial F_1}\right)\exp\left[\frac{1}{\hbar}\ \frac{S_{-1}F_{-1}}{2\pi(-1)}\right]
$$

$$
=\ \exp\left(\frac{\mathrm{i}\hbar T}{m}\ \frac{\partial}{\partial S_1}\ \frac{\partial}{\partial S_{-1}}+\mathrm{i}\hbar T m\omega^2\ \frac{\partial}{\partial F_1}\ \frac{\partial}{\partial F_{-1}}\right)
$$

$$
\times\exp\left[\frac{1}{\hbar}\ \frac{S_1\ F_1}{2\pi(1)}\right]\exp\left[\frac{1}{\hbar}\ \frac{S_{-1}F_{-1}}{2\pi(-1)}\right]. \tag{2.4.6}
$$

Or

$$\exp\left(\frac{\mathrm{i}\hbar T}{2m}\,\frac{\partial}{\partial S_1}\,\frac{\partial}{\partial S_{-1}}+\frac{\mathrm{i}\hbar T m\omega^2}{2}\,\frac{\partial}{\partial F_1}\,\frac{\partial}{\partial F_{-1}}\right)\exp\left[\frac{1}{\hbar}\,\frac{S_1\,F_1}{2\pi(1)}\right]$$

$$\times\exp\left(\frac{\mathrm{i}\hbar T}{2m}\,\frac{\partial}{\partial S_{-1}}\,\frac{\partial}{\partial S_1}+\frac{\mathrm{i}\hbar T m\omega^2}{2}\,\frac{\partial}{\partial F_{-1}}\,\frac{\partial}{\partial F_1}\right)\exp\left[\frac{1}{\hbar}\,\frac{S_{-1}F_{-1}}{2\pi(-1)}\right]$$

$$=\exp\left(\frac{\mathrm{i}\hbar T}{m}\,\frac{\partial}{\partial S_1}\,\frac{\partial}{\partial S_{-1}}\right)\exp\left(\mathrm{i}\hbar T m\omega^2\,\frac{\partial}{\partial F_{-1}}\,\frac{\partial}{\partial F_1}\right)$$

$$\times\exp\left[\frac{1}{\hbar}\,\frac{S_1\,F_1}{2\pi(1)}\right]\exp\left[\frac{1}{\hbar}\,\frac{S_{-1}F_{-1}}{2\pi(-1)}\right]. \tag{2.4.7}$$

Finally we use the translation operation property

$$\left[\exp\left(a\,\frac{\mathrm{d}}{\mathrm{d}x}\right)\right]f(x)\;=\;f(x+a)\,, \tag{2.4.8}$$

to determine $K(T)$.

Using Eq. (2.4.8), the second and the third terms in the right-hand side of Eq. (2.4.7), become

$$\exp\left(\frac{\mathrm{i}\hbar T}{m}\,\frac{\partial}{\partial S_1}\,\frac{\partial}{\partial S_{-1}}\right)\exp\left[\frac{S_1}{\hbar 2\pi(1)}\left(F_1+\mathrm{i}\hbar T m\omega^2\frac{\partial}{\partial F_{-1}}\right)\right]\exp\left[\frac{1}{\hbar}\,\frac{S_{-1}F_{-1}}{2\pi(-1)}\right]$$

$$=\exp\left(\frac{\mathrm{i}\hbar T}{m}\,\frac{\partial}{\partial S_1}\,\frac{\partial}{\partial S_{-1}}\right)\exp\left[\frac{S_1\,F_1}{\hbar 2\pi(1)}\right]\exp\left[\frac{\mathrm{i}T m\omega^2}{2\pi(1)}\,S_1\,\frac{\partial}{\partial F_{-1}}\right]\exp\left[\frac{S_{-1}F_{-1}}{\hbar 2\pi(-1)}\right]$$

$$=\exp\left(\frac{\mathrm{i}\hbar T}{m}\,\frac{\partial}{\partial S_1}\,\frac{\partial}{\partial S_{-1}}\right)\exp\left[\frac{S_1\,F_1}{\hbar 2\pi(1)}\right]\exp\left[\frac{S_{-1}}{\hbar 2\pi(-1)}\left(F_{-1}+\frac{\mathrm{i}T m\omega^2}{2\pi(1)}\,S_1\right)\right]$$

$$=\exp\left(\frac{\mathrm{i}\hbar T}{m}\,\frac{\partial}{\partial S_1}\,\frac{\partial}{\partial S_{-1}}\right)\underbrace{\exp\left[\frac{S_1\,F_1}{\hbar 2\pi(1)}\right]\exp\left[\frac{S_{-1}F_{-1}}{\hbar 2\pi(-1)}\right]}_{=1}\exp\left[\frac{\mathrm{i}T m\omega^2}{\hbar 2\pi(-1)2\pi(1)}\,S_{-1}S_1\right]$$

$$=\exp\left(\frac{\mathrm{i}\hbar T}{m}\,\frac{\partial}{\partial S_1}\,\frac{\partial}{\partial S_{-1}}\right)\exp\left[\frac{\mathrm{i}T m\omega^2}{\hbar 2\pi(-1)2\pi(1)}\,S_{-1}S_1\right]. \tag{2.4.9}$$

Hence we may obviously write by a similar reasoning as above:

$$(\mathrm{i}\omega T)K(T) \;=\; \prod_{n=1}^{\infty} \exp\left(\frac{\mathrm{i}\hbar T}{m}\,\frac{\partial}{\partial S_n}\,\frac{\partial}{\partial S_{-n}}\right)\exp\left(-\frac{\mathrm{i}Tm\omega^2}{\hbar(2\pi n)^2}\,S_n\,S_{-n}\right)\Bigg| \,. \qquad (2.4.10)$$

To carry out the differentiations with respect to S_n, we may use the convenient representation

$$\mathrm{e}^{-\mathrm{i}\beta S_n\,S_{-n}} = \int_{-\infty}^{\infty}\mathrm{d}\lambda_2\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda_1}{2\pi}\,\mathrm{e}^{\mathrm{i}\lambda_1\lambda_2}\,\mathrm{e}^{-\mathrm{i}\lambda_2 S_{-n}}\,\mathrm{e}^{-\mathrm{i}\beta\lambda_1 S_n}\,, \qquad (2.4.11)$$

with

$$\beta_n = \frac{Tm\omega^2}{\hbar(2\pi n)^2}\,. \qquad (2.4.12)$$

That is, Eq. (2.4.10) becomes

$$\begin{aligned}
(\mathrm{i}\omega T)K(T) \;&=\; \prod_{n=1}^{\infty} \exp\left(\frac{\mathrm{i}\hbar T}{m}\,\frac{\partial}{\partial S_n}\,\frac{\partial}{\partial S_{-n}}\right)\exp\left(-\frac{\mathrm{i}Tm\omega^2}{\hbar(2\pi n)^2}\,S_n\,S_{-n}\right)\Bigg| \\
&=\; \prod_{n=1}^{\infty} \exp\left(\frac{\mathrm{i}\hbar T}{m}\,\frac{\partial}{\partial S_n}\,\frac{\partial}{\partial S_{-n}}\right)\int_{-\infty}^{\infty}\mathrm{d}\lambda_2\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda_1}{2\pi}\,\mathrm{e}^{\mathrm{i}\lambda_1\lambda_2}\,\mathrm{e}^{-\mathrm{i}\lambda_2 S_{-n}}\,\mathrm{e}^{-\mathrm{i}\beta_n\lambda_1 S_n} \\
&=\; \prod_{n=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\lambda_2\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda_1}{2\pi}\;\exp(\mathrm{i}\lambda_1\lambda_2) \\
&\qquad\times\exp\left(\frac{\mathrm{i}\hbar T}{m}\,\frac{\partial}{\partial S_n}\,\frac{\partial}{\partial S_{-n}}\right)\;\exp\left(-\mathrm{i}\lambda_2 S_{-n}\right)\;\exp\left(-\mathrm{i}\beta_n\lambda_1 S_n\right) \\
&=\; \prod_{n=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\lambda_2\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda_1}{2\pi}\;\exp(\mathrm{i}\lambda_1\lambda_2) \\
&\qquad\times\exp\left[-\mathrm{i}\lambda_2\left(S_{-n}+\frac{\mathrm{i}\hbar T}{m}\,\frac{\partial}{\partial S_n}\right)\right]\;\exp\left(-\mathrm{i}\beta_n\lambda_1 S_n\right)
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\lambda_2 \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda_1}{2\pi}\ \exp(\mathrm{i}\lambda_1\lambda_2)\ \underbrace{\exp\left(-\mathrm{i}\lambda_2 S_{-n}\right)}_{=1} \\
&\qquad \times \exp\left(\lambda_2 \frac{\hbar T}{m}\frac{\partial}{\partial S_n}\right)\ \exp\left(-\mathrm{i}\beta_n\lambda_1 S_n\right) \\
&= \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\lambda_2 \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda_1}{2\pi}\ \exp(\mathrm{i}\lambda_1\lambda_2)\ \exp\left[-\mathrm{i}\beta_n\lambda_1\left(S_n + \lambda_2\frac{\hbar T}{m}\right)\right] \\
&= \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\lambda_2 \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda_1}{2\pi}\ \exp(\mathrm{i}\lambda_1\lambda_2) \\
&\qquad \times \underbrace{\exp\left(-\mathrm{i}\beta_n\lambda_1 S_n\right)}_{=1}\ \exp\left(-\mathrm{i}\beta_n\lambda_1\lambda_2\frac{\hbar T}{m}\right) \\
&= \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\lambda_2 \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda_1}{2\pi}\ \exp(\mathrm{i}\lambda_1\lambda_2)\ \exp\left(-\mathrm{i}\beta_n\lambda_1\lambda_2\frac{\hbar T}{m}\right) \\
&= \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\lambda_2 \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda_1}{2\pi}\ \exp\left[\mathrm{i}\lambda_1\left(\lambda_2 - \beta_n\lambda_2\frac{\hbar T}{m}\right)\right] \\
&= \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\lambda_2\ \delta\left(\lambda_2\left[1 - \beta_n\frac{\hbar T}{m}\right]\right). \qquad (2.4.13)
\end{aligned}
$$

The latter integrates out simply to give

$$
(\mathrm{i}\omega T)K(T) = \prod_{n=1}^{\infty} \frac{1}{\left[1 - \beta_n\frac{\hbar T}{m}\right]}\,. \qquad (2.4.14)
$$

We now substitute $\beta_n = Tm\omega^2/\hbar(2\pi n)^2$ in the right-hand side of Eq. (2.4.14), to get

$$\frac{1}{\left[1-\beta_n\,\dfrac{\hbar T}{m}\right]} = \frac{1}{\left[1-\dfrac{Tm\omega^2}{\hbar(2\pi n)^2}\,\dfrac{\hbar T}{m}\right]} = \frac{1}{\left[1-\left(\dfrac{T\omega}{2\pi n}\right)^2\right]}\,. \tag{2.4.15}$$

Accordingly Eq. (2.4.10), becomes

$$(\mathrm{i}\omega T)K(T) = \prod_{n=1}^{\infty}\left[1-\left(\frac{T\omega}{2\pi n}\right)^2\right]^{-1}\,. \tag{2.4.16}$$

The latter is the infinite product representation of $(T\omega/2)/\sin(T\omega/2)$, i.e.,

$$\prod_{n=1}^{\infty}\left[1-\left(\frac{T\omega}{2\pi n}\right)^2\right]^{-1} = \frac{\mathrm{i}T\omega\,\mathrm{e}^{-\mathrm{i}T\omega/2}}{1-\mathrm{e}^{-\mathrm{i}T\omega}} = \mathrm{i}T\omega\sum_{n=0}^{\infty}\,\exp\left[-\mathrm{i}\frac{T}{\hbar}\,\hbar\omega\left(n+\frac{1}{2}\right)\right]. \tag{2.4.17}$$

The above equation then gives the following final expression for:

$$K(T) = \frac{1}{\mathrm{i}\omega T}(\mathrm{i}T\omega)\sum_{n=0}^{\infty}\,\exp\left[-\mathrm{i}\frac{T}{\hbar}\,\hbar\omega\left(n+\frac{1}{2}\right)\right] = \sum_{n=0}^{\infty}\,\exp\left[-\frac{\mathrm{i}T}{\hbar}\,\hbar\omega\left(n+\frac{1}{2}\right)\right]. \tag{2.4.18}$$

All told, Eq. (2.2.51) leads to

$$N(\xi) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{d}T}{T-\mathrm{i}\epsilon}\,\mathrm{e}^{\mathrm{i}\xi T/\hbar}\,K(T) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{d}T}{T-\mathrm{i}\epsilon}\,\mathrm{e}^{\mathrm{i}\xi T/\hbar}\sum_{n=0}^{\infty}\,\exp\left[-\frac{\mathrm{i}T}{\hbar}\,\hbar\omega\left(n+\frac{1}{2}\right)\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon} \; \exp\left\{ \mathrm{i} \left[\xi - \hbar\omega \left(n + \frac{1}{2} \right) \right] \frac{T}{\hbar} \right\}$$

$$N(\xi) = \sum_{n=0}^{\infty} \Theta \left(\xi - \hbar\omega (n + \frac{1}{2}) \right), \tag{2.4.19}$$

and Eq. (2.2.52) leads to

$$N[\xi] = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon} \, \mathrm{e}^{\mathrm{i}\xi T/\hbar} \, \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}T} \, K(T)$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon} \, \mathrm{e}^{\mathrm{i}\xi T/\hbar} \, \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}T} \sum_{n=0}^{\infty} \exp \left[-\frac{\mathrm{i}T}{\hbar} \, \hbar\omega \left(n + \frac{1}{2} \right) \right]$$

$$= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon} \, \mathrm{e}^{\mathrm{i}\xi T/\hbar} \, \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}T} \; \exp \left[-\frac{\mathrm{i}T}{\hbar} \, \hbar\omega \left(n + \frac{1}{2} \right) \right]$$

$$= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon} \, \mathrm{e}^{\mathrm{i}\xi T/\hbar} \, \mathrm{i}\hbar \left[-\frac{\mathrm{i}}{\hbar} \, \hbar\omega \left(n + \frac{1}{2} \right) \right] \; \exp \left[-\frac{\mathrm{i}T}{\hbar} \, \hbar\omega \left(n + \frac{1}{2} \right) \right]$$

$$= \sum_{n=0}^{\infty} \hbar\omega \left(n + \frac{1}{2} \right) \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon} \; \exp \left(\mathrm{i}\xi \frac{T}{\hbar} \right) \exp \left[-\frac{\mathrm{i}T}{\hbar} \, \hbar\omega \left(n + \frac{1}{2} \right) \right]$$

$$= \sum_{n=0}^{\infty} \hbar\omega \left(n + \frac{1}{2} \right) \int_{-\infty}^{\infty} \frac{\mathrm{d}T}{T - \mathrm{i}\epsilon} \; \exp\left\{ \mathrm{i} \left[\xi - \hbar\omega \left(n + \frac{1}{2} \right) \right] \frac{T}{\hbar} \right\},$$

$$N[\xi] = \sum_{n=0}^{\infty} \hbar\omega \left(n + \frac{1}{2} \right) \Theta \left(\xi - \hbar\omega (n + \frac{1}{2}) \right). \tag{2.4.20}$$

as expected, where we have used the integral respresentation of the step function after carrying out the differentiation with respect to T in Eq. (2.2.52) to obtain Eq. (2.4.20).

CHAPTER III

CONSTRAINTS, DEPENDENT FIELDS AND THE QUANTUM DYNAMICAL PRINCIPLE: ENLARGEMENT OF PHASE SPACE

3.1 Introduction

The development of constrained dynamics in quantum physics in this chapter was inspired by the situation occurring in quantum electrodynamics in the Coulomb gauge. In this gauge, the vector potential components A_i, $i = 1, 2, 3$, are related by

$$\partial_i A_i \;=\; 0, \tag{3.1.1}$$

from which we may, for example, solve for A_3 as follows

$$A_3 \;=\; -\partial_3^{-1}(\partial_a A_a), \tag{3.1.2}$$

with $a = 1, 2$, and treat A_1, A_2 as independent variables, while A_3 as a dependent one. The canonical conjugate momenta π^1, π^2 of A_1, A_2 are given by (see Eqs. (5.3.22) and (5.3.23))

$$\pi^a \;=\; \partial_3^{-1}(\partial^a F^{03} - \partial^3 F^{0a}) \quad , \quad a = 1, 2 \tag{3.1.3}$$

where

$$F^{\mu\nu} \;=\; \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{3.1.4}$$

$\mu, \nu = 0, 1, 2, 3$. By definition, the canonical conjugate momentum π^3 of A_3 is zero with the latter being a dependent variable. Accordingly, we can extend Eq. (3.1.3) form $a = 1, 2$ trivially to $i = 1, 2, 3$ by rewriting Eq. (3.1.3) as

$$\pi^i = \partial_3^{-1}(\partial^i F^{03} - \partial^3 F^{0i}) \quad , \quad i = 1, 2, 3, \tag{3.1.5}$$

since for $i = 3$, we simply obtain $0 = 0$ as is easily checked from Eq. (3.1.5), giving

$$\pi^3 = \partial_3^{-1}(\partial^3 F^{03} - \partial^3 F^{03}) = 0 \tag{3.1.6}$$

In the Hamiltonian formalism this suggests to develop a formalism such that a Hamiltonian $H(\mathbf{q}, \mathbf{p})$, as a function of independent variables $\mathbf{q} = (q_1, \ldots, q_n)$ and their canonical conjugate momenta $\mathbf{p} = (p_1, \ldots, p_n)$, is obtained from Hamiltonians $\{\tilde{H}(\mathbf{q}, \mathbf{Q}, \mathbf{p}, \mathbf{P})\}$ with constraints $\mathbf{Q} = \mathbf{G}(\mathbf{q}, \mathbf{p})$ and $\mathbf{P} = 0$.

The purpose of this chapter is therefore to show within the functional *differential* treatment of quantum systems (e.g., Schwinger, 1951, 1953, 1954,1972; Manoukian, 1985, 1986,1987, 2006; Manoukian and Siranan, 2005; Limboonsong and Manoukian, 2006), also known as the quantum dynamical principle (QDP), given independent pairs of canonical conjugate variables $\{q_i(t), p_i(t),\ i = 1, \ldots, n\} \equiv \{\mathbf{q}(t), \mathbf{p}(t)\}$ and a Hamiltonian $H(\mathbf{q}, \mathbf{p})$, and a given set of pairwise commuting operator functions $\{G_j(\mathbf{q}(t), \mathbf{p}(t)),\ j = 1, \ldots, k\}$ of these variables defined, *transformation* functions may be then explicitly given for constrained dynamical systems describing the dynamics of a system with a Hamiltonian defined, *a priori*, in a *larger* phase space of dimensionality $> 2n$, for which constrained are imposed. This is spelled out below. These transformation functions are expressed as functional differential operations, involving functional differentiations with respect to external sources, applied to a given functional of these sources written in closed form. The very elegant QDP has been indisputably recognized as a powerful tool over the years. There has been a renewed interest recently in Schwinger's action principle (see, e.g., Das and Scherer, 2005; Kawai, 2005;

Schweber, 2005; Iliev, 2003; Faddeev and Popov, 1967; Fradkin and Tyutin, 1970) emphasizing generally, however, operator aspects, as deriving, for example, commutation relations, rather than dealing with computational ones related directly to transformation functions as done here. We note that in the functional differential formalism external sources are, *a priori*, necessarily introduced to generate transformation functions and matrix elements of various operators. It will be understood throughout the bulk of this communication, that all these sources will eventually be set equal to zero after all the relevant functional differentiations with respect to them have been carried out. The connection of this work to the so-called Faddeev-Popov technique in path integrals will be pointed out.

Our procedure, as well as the main results of this chapter may be summarized as follows.

Suppose we are given a Hamiltonian $H(\mathbf{q},\mathbf{p})$ as a function of independent pairs of canonical conjugate variables $\{q_i, p_i, i = 1,\ldots,n\} \equiv \{\mathbf{q},\mathbf{p}\}$, that is, it is defined in a phase space of dimensionality equal to $2n$. We are also given a set of pairwise commuting operator functions $\{G_j(\mathbf{q}(t),\mathbf{p}(t)), j = 1,\ldots,k\}$ of these variables. These allow us to describe the dynamics of any Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})$ in, *a priori*, $(2n+2k)$ dimensional phase space in which constraints are imposed given by

$$Q_j(t) - G_j(\mathbf{q}(t),\mathbf{p}(t)) = 0, \quad j = 1,\ldots,k, \tag{3.1.7}$$

with $\mathbf{Q} = (Q_1,\ldots,Q_k)$, for which $\mathbf{P} = \mathbf{0}$, *such that*

$$\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{G}(\mathbf{q},\mathbf{p}),0) \;=\; H(\mathbf{q},\mathbf{p}). \tag{3.1.8}$$

The transformation functions $\langle \mathbf{q},\mathbf{Q},t\,|\,\mathbf{q}',\mathbf{Q}',t'\rangle$ of the constrained dynamics is given by

$$\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle = \exp\Big(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\tilde{H}'(\tau)\Big)\,\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle_0^{\wedge}\,\Big|, \tag{3.1.9}$$

where

$$\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle_0^{\wedge} = \delta^{(k)}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\cdot)} - \mathbf{G}'(\cdot)\right)\delta^{(k)}\left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)}\frac{\delta}{\delta\mathbf{s}(\cdot)}\right)\langle \mathbf{q}\,t\,|\,\mathbf{q}'t'\rangle_0\ A, \tag{3.1.10}$$

$$A = \delta^k\left(\mathbf{Q} - \mathbf{Q}' - \int_{t'}^{t}\mathrm{d}\tau\,\mathbf{s}(\tau)\right)\exp\left(\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{f}(\tau)\right)$$

$$\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{\tau}\mathrm{d}\tau'\,\mathbf{s}(\tau)\cdot\mathbf{f}(\tau)\right), \tag{3.1.11}$$

$$\langle \mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle_0 = \delta^n\left(\mathbf{q} - \mathbf{q}' - \int_{t'}^{t}\mathrm{d}\tau\,\mathbf{S}(\tau)\right)\exp\left(\frac{\mathrm{i}}{\hbar}\,\mathbf{q}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{F}(\tau)\right)$$

$$\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{\tau}\mathrm{d}\tau'\,\mathbf{S}(\tau)\cdot\mathbf{F}(\tau')\right), \tag{3.1.12}$$

and the vertical bar $|$ in Eq. (3.1.9) refers to the fact that all the external sources are to be set to zero after all the relevant functional differentiations have been carried out. $\delta^{(k)}(-\mathrm{i}\hbar\delta/\delta\mathbf{f}(\cdot) - \mathbf{G}'(\cdot))$ and $\delta^{(k)}(\mathrm{i}\hbar\delta/(2\pi\hbar)\delta\mathbf{s}(\cdot))$ in Eq. (3.1.10), as arising from the conditions in Eqs. (3.2.44) and (3.2.45), refer, each, to the product of *k*-dimensional deltas with τ running over all points in the interval $[t', t]$, i.e.,

$$\delta^{(k)}(D(\cdot)) \;=\; \prod_{t'\le\tau\le t}\delta^k(D(\tau)). \tag{3.1.13}$$

The numericals $\mathbf{Q}$, $\mathbf{Q}'$ are defined as follows:

$$\mathbf{Q} \;=\; \mathbf{Q}^{\mathrm{c}}(\tau)\Big|_{\tau\to t},$$

$$\mathbf{Q}' \;=\; \mathbf{Q}^{\mathrm{c}}(\tau)\Big|_{\tau\to t'},$$

where $\mathbf{Q}^{\mathrm{c}}(\tau)$ is the *classical* function

$$\mathbf{Q}^{\mathrm{c}}(\tau) \;=\; \frac{\langle \mathbf{q}t\,|\mathbf{G}(\mathbf{q}(\tau),\mathbf{p}(\tau))|\mathbf{q}'t'\rangle}{\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle}\Bigg|,$$

and in detail

$$\mathbf{Q}^{\mathrm{c}}(\tau) \;=\; \frac{1}{\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle}\,G'(\tau)\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau'\,H'(\tau')\right)\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle_0\Bigg|,$$

where

$$H'(\tau') \;=\; H\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau')},\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau')}\right), \tag{3.1.14}$$

obtained from $H(\mathbf{q},\mathbf{p})$ by replacing $\mathbf{q}$, $\mathbf{p}$, respectively, by $-\mathrm{i}\hbar\delta/\delta\mathbf{F}(\tau')$, $\mathrm{i}\hbar\delta/\delta\mathbf{S}(\tau')$, while

$$\tilde{H}'(\tau') \;=\; \tilde{H}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau')},\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau')},-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\tau')},\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{s}(\tau')}\right),$$

as similarly obtained from a Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})$. We note that because of the equality in Eq. (3.1.8), we may replace $\tilde{H}'(\tau)$ in Eq. (3.1.9) by $H'(\tau)$ as a consequence of the constraints imposed by the delta functionals:

$$\delta^{(k)}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\cdot)}-\mathbf{G}'(\cdot)\right)\delta^{(k)}\left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)}\frac{\delta}{\delta\mathbf{s}(\cdot)}\right), \tag{3.1.15}$$

in Eq. (3.1.10).

The procedure for describing the dynamics of a Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})$ with constraints may be summarized through the following:

$$\underbrace{H(\mathbf{q},\mathbf{p})}_{\substack{\text{Phase Space of}\\ \dim(2n)}} \quad \longrightarrow \quad \underbrace{\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})}_{\substack{\text{Phase Space of}\\ \dim(2n+2k)}} \quad \longrightarrow \quad \tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})\Big|_{\text{constraints}}, \qquad (3.1.16)$$

with the transformations functions of the constrained dynamics with Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})$ given in Eq. (3.1.9).

3.2 Functional Differentiations, Dependent Field and Transformation Functions

Consider a Hamiltonian $H(\mathbf{q},\mathbf{p})$ as a function of independent pairs of canonical conjugate variables $\{q_i, p_i,\ i = 1,\ldots,n\} \equiv \{\mathbf{q},\mathbf{p}\}$. We introduce external sources $\{F_i(\tau), S_i(\tau),\ i = 1,\ldots,n\} \equiv \{\mathbf{F}(\tau),\mathbf{S}(\tau)\}$ and define the extended Hamiltonian $\underline{H}(\tau)$, in the presence of these sources, by

$$\underline{H}(\tau) = H(\mathbf{q},\mathbf{p}) - \mathbf{q}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau), \qquad (3.2.1)$$

with $\mathbf{F}(\tau)$, $\mathbf{S}(\tau)$ vanishing outside an interval $[t', t]$ with $t' < t$. Of physical interest are the transformation functions $\langle \mathbf{q}\, t \,|\, \mathbf{q}' t' \rangle$, in particular, in the limit of vanishing external sources. To obtain these transformation functions, we first multiply $H(\mathbf{q},\mathbf{p})$ in Eq. (3.2.1) by a parameter λ which we eventually set equal to one and define

$$\underline{H}(\tau,\lambda) \;=\; \lambda H(\mathbf{q},\mathbf{p}) - \mathbf{q}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau). \qquad (3.2.2)$$

The explicit functional derivative expression for the transformation functions is well known (see, e.g., Manoukian, 2006, sect. 11.2) and is given by

$$\frac{\partial}{\partial\lambda}\langle\mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle = -\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,\langle\mathbf{q}\,t\,|H(\mathbf{q},\mathbf{p})|\mathbf{q}'\,t'\rangle\,, \tag{3.2.3}$$

$$\frac{\partial}{\partial\lambda}\langle\mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle = -\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,H'(\tau)\,\langle\mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle\,, \tag{3.2.4}$$

$$\delta\,\langle\mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle = -\frac{\mathrm{i}}{\hbar}\,\mathrm{d}\lambda\int_{t'}^{t}\mathrm{d}\tau\,H'(\tau)\,\langle\mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle\,, \tag{3.2.5}$$

which integrates out to

$$\langle\mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,H'(\tau)\right]\langle\mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle_0\,\Bigg|, \tag{3.2.6}$$

where $H'(\tau)$ is the functional differential operator

$$H'(\tau) = H\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)},\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right), \tag{3.2.7}$$

obtained from $H(\mathbf{q},\mathbf{p})$ in Eq. (3.2.1), by replacing $\mathbf{q}$, $\mathbf{p}$, respectively, by $-\mathrm{i}\hbar\delta/\delta\mathbf{F}(\tau)$, $\mathrm{i}\hbar\delta/\delta\mathbf{S}(\tau)$ as elaborated upon earlier. For the simple Hamiltonian

$$\hat{H} = -\mathbf{q}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau), \tag{3.2.8}$$

the Heisenberg equations are

$$\dot{\mathbf{q}}(\tau) = \mathbf{S}(\tau), \tag{3.2.9}$$

$$\dot{\mathbf{p}}(\tau) = \mathbf{F}(\tau). \tag{3.2.10}$$

These equations may be integrated to

$$\int_{\tau}^{t}\mathrm{d}\mathbf{q}(\tau') = \int_{\tau}^{t}\mathrm{d}\tau'\,\mathbf{S}(\tau'), \tag{3.2.11}$$

$$\mathbf{q}(t) - \mathbf{q}(\tau) = \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau' - \tau) \, \mathbf{S}(\tau'), \tag{3.2.12}$$

$$\mathbf{q}(\tau) = \mathbf{q}(t) - \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau' - \tau) \, \mathbf{S}(\tau'), \tag{3.2.13}$$

and

$$\int_{t'}^{\tau} \mathrm{d}\mathbf{p}(\tau) = \int_{t'}^{\tau} \mathrm{d}\tau \, \mathbf{F}(\tau), \tag{3.2.14}$$

$$\mathbf{p}(\tau) - \mathbf{p}(t') = \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau') \, \mathbf{F}(\tau'), \tag{3.2.15}$$

$$\mathbf{p}(\tau) = \mathbf{p}(t') + \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau') \, \mathbf{F}(\tau'), \tag{3.2.16}$$

and taking the matrix element between $\langle \mathbf{q}\, t|$ and $|\mathbf{p}t'\rangle$ for $\lambda = 0$, we obtain

$$\langle \mathbf{q}\, t|\mathbf{q}(\tau)|\mathbf{p}t'\rangle_0 = \left[\mathbf{q}(t) - \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau' - \tau) \, \mathbf{S}(\tau')\right] \langle \mathbf{q}\, t\,|\,\mathbf{p}'\, t'\rangle_0 \,, \tag{3.2.17}$$

$$\langle \mathbf{q}\, t|\mathbf{p}(\tau)|\mathbf{p}t'\rangle_0 = \left[\mathbf{p}(t') + \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau') \, \mathbf{F}(\tau')\right] \langle \mathbf{q}\, t\,|\,\mathbf{p}'\, t'\rangle_0 \,, \tag{3.2.18}$$

where $\mathbf{q}$ and $\mathbf{p}$ within the square brackets on the right-hand sides of the above two equations are c-numbers, and we have used the relations

$${}_0\langle \mathbf{q}\, t|\, \mathbf{q}(t) = \mathbf{q}\, {}_0\langle \mathbf{q}\, t| \,, \tag{3.2.19}$$

$$\mathbf{p}(t')\, |\mathbf{p}\, t'\rangle_0 = \mathbf{p}\, |\mathbf{p}\, t'\rangle_0 \,, \tag{3.2.20}$$

for $\lambda = 0$ at coincident times. Eqs. (3.2.17) and (3.2.18) may be rewritten as

$$-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{F}(\tau)} \langle \mathbf{q}\, t\,|\,\mathbf{p}\, t'\rangle_0 = \left[\mathbf{q} - \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau' - \tau) \, \mathbf{S}(\tau')\right] \langle \mathbf{q}\, t\,|\,\mathbf{p}\, t'\rangle_0 \,, \tag{3.2.21}$$

$$\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\,\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0=\left[\mathbf{p}+\int_{t'}^{t}\mathrm{d}\tau'\;\Theta(\tau-\tau')\;\mathbf{F}(\tau')\right]\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0\,. \tag{3.2.22}$$

These equations may be integrated to yield

$$-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}\,\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0=\left[\mathbf{q}-\int_{t'}^{t}\mathrm{d}\tau'\;\Theta(\tau'-\tau)\;\mathbf{S}(\tau')\right]\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0\,, \tag{3.2.23}$$

$$\int\frac{\delta\,\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0}{\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0}=\frac{\mathrm{i}}{\hbar}\int\delta\mathbf{F}(\tau)\left[\mathbf{q}-\int_{t'}^{t}\mathrm{d}\tau'\;\Theta(\tau'-\tau)\;\mathbf{S}(\tau')\right], \tag{3.2.24}$$

$$\ln\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0=\frac{\mathrm{i}}{\hbar}\,\mathbf{q}\cdot\int_{t'}^{t}\mathbf{F}(\tau)\,\mathrm{d}\tau$$
$$-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\;\mathbf{F}(\tau)\cdot\Theta(\tau'-\tau)\;\mathbf{S}(\tau'), \tag{3.2.25}$$

$$\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0=A\;\exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{q}\cdot\int_{t'}^{t}\mathbf{F}(\tau)\,\mathrm{d}\tau\right]$$
$$\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\;\mathbf{F}(\tau)\cdot\Theta(\tau'-\tau)\;\mathbf{S}(\tau')\right], \tag{3.2.26}$$

where A is a normalization factor, and

$$\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\,\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0=\left[\mathbf{p}+\int_{t'}^{t}\mathrm{d}\tau'\;\Theta(\tau-\tau')\;\mathbf{F}(\tau')\right]\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0\,, \tag{3.2.27}$$

$$\int\frac{\delta\,\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0}{\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0}=-\frac{\mathrm{i}}{\hbar}\int\delta\mathbf{S}(\tau)\left[\mathbf{p}+\int_{t'}^{t}\mathrm{d}\tau'\;\Theta(\tau-\tau')\;\mathbf{F}(\tau')\right], \tag{3.2.28}$$

$$\ln\langle\mathbf{q}\,t\,|\,\mathbf{p}\,t'\rangle_0=-\frac{\mathrm{i}}{\hbar}\,\mathbf{p}\cdot\int_{t'}^{t}\mathbf{S}(\tau)\,\mathrm{d}\tau$$
$$-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\;\mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\;\mathbf{F}(\tau'), \tag{3.2.29}$$

$$\langle \mathbf{q}\, t | \mathbf{p}\, t' \rangle_0 = B \, \exp\left[-\frac{\mathrm{i}}{\hbar}\, \mathbf{p} \cdot \int_{t'}^{t} \mathbf{S}(\tau)\, \mathrm{d}\tau \right]$$

$$\times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau' \, \mathbf{S}(\tau) \cdot \Theta(\tau - \tau')\, \mathbf{F}(\tau') \right], \quad (3.2.30)$$

to find A and B in Eqs. (3.2.26) and (3.2.30), respectively, we have

$$\text{Eq.}(3.2.26) \;=\; \text{Eq.}(3.2.30). \qquad (3.2.31)$$

Or

$$A \, \exp\left[\frac{\mathrm{i}}{\hbar}\, \mathbf{q} \cdot \int_{t'}^{t} \mathbf{F}(\tau)\, \mathrm{d}\tau \right] \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau' \, \mathbf{F}(\tau) \cdot \Theta(\tau' - \tau)\, \mathbf{S}(\tau') \right]$$

$$= \; B \, \exp\left[-\frac{\mathrm{i}}{\hbar}\, \mathbf{p} \cdot \int_{t'}^{t} \mathbf{S}(\tau)\, \mathrm{d}\tau \right] \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau' \, \mathbf{S}(\tau) \cdot \Theta(\tau - \tau')\, \mathbf{F}(\tau') \right], \quad (3.2.32)$$

we have,

$$A = \exp\left[-\frac{\mathrm{i}}{\hbar}\, \mathbf{p} \cdot \int_{t'}^{t} \mathbf{S}(\tau)\, \mathrm{d}\tau \right] \quad , \quad B = \exp\left[\frac{\mathrm{i}}{\hbar}\, \mathbf{q} \cdot \int_{t'}^{t} \mathbf{F}(\tau)\, \mathrm{d}\tau \right]. \qquad (3.2.33)$$

In the absence of external sources: $\mathbf{F} = 0$, $\mathbf{S} = 0$, $\hat{H} \to 0$, and

$$\underline{H}(\tau)\Big| \;=\; H(\mathbf{q}, \mathbf{p})\,, \qquad (3.2.34)$$

in an obvious notation. When $t = t'$

$$\begin{aligned} \langle \mathbf{q}\, t | \mathbf{p}\, t \rangle \;&=\; \big\langle \mathbf{q} \big| U(t) U^{\dagger}(t) \big| \mathbf{p} \big\rangle \\ &=\; \langle \mathbf{q} | \mathbf{p} \rangle \\ &=\; \exp\left(\frac{\mathrm{i}}{\hbar}\, \mathbf{q} \cdot \mathbf{p} \right). \end{aligned} \qquad (3.2.35)$$

This gives

$$\langle \mathbf{q}\, t\,|\, \mathbf{p}\, t' \rangle_0 = \exp\left[\frac{\mathrm{i}}{\hbar}\, \mathbf{q}\cdot \int_{t'}^{t} \mathrm{d}\tau\, \mathbf{F}(\tau)\right] \exp\left[-\frac{\mathrm{i}}{\hbar}\, \mathbf{p}\cdot \int_{t'}^{t} \mathrm{d}\tau\, \mathbf{S}(\tau)\right] \exp\left(\frac{\mathrm{i}}{\hbar}\, \mathbf{q}\cdot\mathbf{p}\right)$$
$$\times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\, \mathbf{F}(\tau')\right]. \qquad (3.2.36)$$

To obtain the expression for $\langle \mathbf{q}\, t\,|\, \mathbf{q}'\, t' \rangle_0$, we multiply Eq. (3.2.36) by $\langle \mathbf{p}\, t'\,|\, \mathbf{q}'\, t' \rangle = \exp(-\mathrm{i}\, \mathbf{q}'\cdot\mathbf{p}/\hbar)$ and integrate over $\mathbf{p}$, with measure $\mathrm{d}^n\mathbf{p}/(2\pi\hbar)^n$, to obtain

$$\begin{aligned}
\langle \mathbf{q}\, t\,|\, \mathbf{q}'\, t' \rangle_0 &= \int \frac{\mathrm{d}^n\mathbf{p}}{(2\pi\hbar)^n}\, \langle \mathbf{q}\, t\,|\, \mathbf{p}\, t' \rangle_0\, \langle \mathbf{p}\, t'\,|\, \mathbf{q}'\, t' \rangle\,, \\
&= \int \frac{\mathrm{d}^n\mathbf{p}}{(2\pi\hbar)^n}\, \exp\left[\frac{\mathrm{i}}{\hbar}\, \mathbf{q}\cdot \int_{t'}^{t} \mathrm{d}\tau\, \mathbf{F}(\tau)\right] \exp\left[-\frac{\mathrm{i}}{\hbar}\, \mathbf{p}\cdot \int_{t'}^{t} \mathrm{d}\tau\, \mathbf{S}(\tau)\right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\, \mathbf{F}(\tau')\right] \\
&\qquad \times \exp\left(\frac{\mathrm{i}}{\hbar}\, \mathbf{q}\cdot\mathbf{p}\right) \exp\left(-\frac{\mathrm{i}}{\hbar}\, \mathbf{q}'\cdot\mathbf{p}\right), \\
&= \int \frac{\mathrm{d}^n\mathbf{p}}{(2\pi\hbar)^n}\, \exp\left[\frac{\mathrm{i}}{\hbar}\, \mathbf{q}\cdot \int_{t'}^{t} \mathrm{d}\tau\, \mathbf{F}(\tau)\right] \\
&\qquad \times \exp\left[\mathrm{i}\left(\mathbf{q}-\mathbf{q}'-\int_{t'}^{t} \mathrm{d}\tau\, \mathbf{S}(\tau)\right)\cdot\frac{\mathbf{p}}{\hbar}\right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\, \mathbf{F}(\tau')\right], \\
&= \delta^n\left(\mathbf{q}-\mathbf{q}'-\int_{t'}^{t} \mathrm{d}\tau\, \mathbf{S}(\tau)\right) \exp\left[\frac{\mathrm{i}}{\hbar}\, \mathbf{q}\cdot \int_{t'}^{t} \mathrm{d}\tau\, \mathbf{F}(\tau)\right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau'\, \mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\, \mathbf{F}(\tau')\right],
\end{aligned}$$

$$\langle \mathbf{q}\, t \,|\, \mathbf{q}'\, t' \rangle_0 \;=\; \delta^n\left(\mathbf{q} - \mathbf{q}' - \int_{t'}^{t} \mathrm{d}\tau\, \mathbf{S}(\tau)\right) \exp\left(\frac{\mathrm{i}}{\hbar}\, \mathbf{q} \cdot \int_{t'}^{t} \mathrm{d}\tau\, \mathbf{F}(\tau)\right)$$

$$\times \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{\tau} \mathrm{d}\tau'\, \mathbf{S}(\tau) \cdot \mathbf{F}(\tau')\right), \tag{3.2.37}$$

defining a functional of $\mathbf{F}(\tau)$, $\mathbf{S}(\tau)$. Here we recall that the vertical bar $|$ in Eq. (3.2.6) means to set these external sources equal to zero after the functional differentiations with respect to them, as defined in the exponential expression in Eq. (3.2.6) are carried out.

Given a set of operator functions $\{G_j(\mathbf{q}(\tau), \mathbf{p}(\tau)),\ j = 1, \ldots, k\}$, as mentioned in the introductory section, with time development given by the Hamiltonian $H(\mathbf{q}, \mathbf{p})$, we may introduce the following c-functions

$$Q_j^c(\tau) = \frac{\langle \mathbf{q}\, t \,| G_j\left(\mathbf{q}(\tau), \mathbf{p}(\tau)\right) | \mathbf{q}'\, t' \rangle \,\big|}{\langle \mathbf{q}\, t \,|\, \mathbf{q}'\, t' \rangle \,\big|}, \tag{3.2.38}$$

for τ in the interval $[t', t]$. We may promote the $Q_j^c(\tau)$ to quantum variables $Q_j(\tau)$ by noting:

(A) The canonical conjugate momenta $P_j(\tau)$ of dependent fields $Q_j(\tau)$ must vanish, by definition.

(B) We may introduce external sources $\mathbf{f}(\tau)$, $\mathbf{s}(\tau)$ to generate functionals of the latter fields as done for the $q_j(\tau)$, $p_j(\tau)$ fields and, in the process, make use of Eq. (3.2.37).

(C) $H(\mathbf{q}, \mathbf{p})$ in Eq. (3.2.1) is a function of independent pairs of the canonical conjugate variables in $\{q_i, p_i,\ i = 1, \ldots, n\}$, and hence no explicit functional differentiation operations with respect to the sources $\mathbf{f}(\tau)$, $\mathbf{s}(\tau)$ appear in $H'(\tau)$. We define the Hamiltonian $\underline{\tilde{H}}(\tau, \lambda)$, in the presence of these sources, by

$$\underline{\tilde{H}}(\tau, \lambda) = \lambda H(\mathbf{q}, \mathbf{p}) - \mathbf{q} \cdot \mathbf{F}(\tau) + \mathbf{p} \cdot \mathbf{S}(\tau). \tag{3.2.39}$$

To the above end, we note that Eq. (3.2.38) may be rewritten as

$$Q^c_j(\tau)\,\langle \mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle\Big| = \langle \mathbf{q}\,t|G_j\left(\mathbf{q}(\tau),\mathbf{p}(\tau)\right)|\mathbf{q}'\,t'\rangle|\,, \tag{3.2.40}$$

$$Q^c_j(\tau)\,\langle \mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle\Big| - \langle \mathbf{q}\,t|G_j\left(\mathbf{q}(\tau),\mathbf{p}(\tau)\right)|\mathbf{q}'\,t'\rangle| = 0, \tag{3.2.41}$$

$$\left[Q^c_j(\tau) - G'_j(\tau)\right]\langle \mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle\Big| = 0, \tag{3.2.42}$$

with

$$G'_j(\tau) = G_j\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right), \tag{3.2.43}$$

by an immediate application of the QDP. By promoting the $Q^c_j(\tau)$ to quantum variables, with $\langle \mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle$ generalized to $\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle$, we must have from (A), (B), (C), above, and Eq. (3.2.42)

$$\begin{aligned} &\left[-\mathrm{i}\hbar\frac{\delta}{\delta f_j(\tau)} - G'_j(\tau)\right]\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle\Bigg| \\ = \;& \langle \mathbf{q}\mathbf{Q}t\,|Q_j(\tau)|\mathbf{q}'\mathbf{Q}'t'\rangle\Bigg| - \langle \mathbf{q}\mathbf{Q}t\,|G_j(\mathbf{q}(\tau),\mathbf{p}(\tau))|\mathbf{q}'\mathbf{Q}'t'\rangle\Bigg| \\ = \;& \langle \mathbf{q}\mathbf{Q}t\,|Q_j(\tau) - G_j(\mathbf{q}(\tau),\mathbf{p}(\tau))|\mathbf{q}'\mathbf{Q}'t'\rangle\Bigg| \\ = \;& 0, \end{aligned} \tag{3.2.44}$$

and

$$\begin{aligned} \mathrm{i}\hbar\frac{\delta}{\delta s_j(\tau)}\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle\Bigg| &= \langle \mathbf{q}\mathbf{Q}t\,|P_j(\tau)|\mathbf{q}'\mathbf{Q}'t'\rangle\Bigg| \\ &= 0, \end{aligned} \tag{3.2.45}$$

for all $t' \le \tau \le t$. Since a relation $xg(x) = 0$, implies that $g(x)$ involves the factor $\delta(x)$. We note from Eqs. (3.2.44), (3.2.45) and (3.2.37), and finally from Eq. (3.2.6), by following a procedure as in deriving Eqs. (3.2.3) and (3.2.4), that

$$\frac{\partial}{\partial\lambda}\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle = -\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,\langle \mathbf{q}\mathbf{Q}t\,|H(\mathbf{q}(\tau),\mathbf{p}(\tau))|\mathbf{q}'\mathbf{Q}'t'\rangle$$

$$= -\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,H'(\tau)\,\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle\,, \qquad (3.2.46)$$

$$\delta\,\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle = -\frac{\mathrm{i}}{\hbar}\,\mathrm{d}\lambda\int_{t'}^{t}\mathrm{d}\tau\,H'(\tau)\,\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle\,, \qquad (3.2.47)$$

where $H'(\tau)$ is defined in Eq. (3.2.7). Eq. (3.2.47) integrates out to

$$\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle\Big| = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,H'(\tau)\right]\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle_0^{\wedge}\Big|, \qquad (3.2.48)$$

where $\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle_0^{\wedge}$ is determined below in Eq. (3.2.86). From Eq. (3.2.37)

$$\begin{aligned}\langle \mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle_0 \;=\;& \delta^n\left(\mathbf{q}-\mathbf{q}'-\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{S}(\tau)\right)\exp\left(\frac{\mathrm{i}}{\hbar}\,\mathbf{q}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{F}(\tau)\right)\\ &\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right).\end{aligned}$$

The conditions in Eqs. (3.2.44) and (3.2.45), as mentioned earlier, that in a relation $x\,g(x)=0$, as applied to

$$\left[-\mathrm{i}\hbar\frac{\delta}{\delta f_i(\tau)} - G'_j(\tau)\right]\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle\Big| = 0, \qquad (3.2.49)$$

with the formal substitutions

$$\left[-\mathrm{i}\hbar\frac{\delta}{\delta f_i(\tau)} - G'_j(\tau)\right] \;\longrightarrow\; x, \qquad (3.2.50)$$

$$\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle\Big| \;\longrightarrow\; g(x), \qquad (3.2.51)$$

imply that $\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle$ must involve delta functionals as given below in Eq. (3.2.55).

On the other hand from Eq. (3.2.45),

$$\mathrm{i}\hbar\frac{\delta}{\delta s_j(\tau)}\left\langle \mathbf{q}\mathbf{Q}t \,|\, \mathbf{q}'\mathbf{Q}'t'\right\rangle \Bigg| = 0, \tag{3.2.52}$$

we may make the formal substitutions

$$\mathrm{i}\hbar\frac{\delta}{\delta s_j(\tau)} \quad \longrightarrow \quad x, \tag{3.2.53}$$

$$\left\langle \mathbf{q}\mathbf{Q}t \,|\, \mathbf{q}'\mathbf{Q}'t'\right\rangle \Bigg| \quad \longrightarrow \quad g(x), \tag{3.2.54}$$

implying finally that $\langle \mathbf{q}\mathbf{Q}t \,|\, \mathbf{q}'\mathbf{Q}'t'\rangle$ must involve the product of delta functionals

$$\delta^{(k)}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\cdot)} - \mathbf{G}'(\cdot)\right)\delta^{(k)}\left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)}\frac{\delta}{\delta\mathbf{s}(\cdot)}\right). \tag{3.2.55}$$

For the simple Hamiltonian

$$\tilde{H} = -\mathbf{Q}\cdot\mathbf{f}(\tau) + \mathbf{P}\cdot\mathbf{s}(\tau), \tag{3.2.56}$$

the Heisenberg equations are

$$\dot{\mathbf{Q}}(\tau) \quad = \quad \mathbf{s}(\tau), \tag{3.2.57}$$

$$\dot{\mathbf{P}}(\tau) \quad = \quad \mathbf{f}(\tau). \tag{3.2.58}$$

These equations may be integrated to

$$\int_{\tau}^{t} \mathrm{d}\mathbf{Q}(\tau) \quad = \quad \int_{\tau}^{t} \mathrm{d}\tau\, \mathbf{s}(\tau), \tag{3.2.59}$$

$$\mathbf{Q}(t) - \mathbf{Q}(\tau) \quad = \quad \int_{t'}^{t} \mathrm{d}\tau'\, \Theta(\tau' - \tau)\, \mathbf{s}(\tau'), \tag{3.2.60}$$

$$\mathbf{Q}(\tau) \;=\; \mathbf{Q}(t) - \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau' - \tau) \, \mathbf{s}(\tau'), \tag{3.2.61}$$

and

$$\int_{t'}^{\tau} \mathrm{d}\mathbf{P}(\tau) \;=\; \int_{t'}^{\tau} \mathrm{d}\tau \, \mathbf{f}(\tau), \tag{3.2.62}$$

$$\mathbf{P}(\tau) - \mathbf{P}(t') \;=\; \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau') \, \mathbf{f}(\tau'), \tag{3.2.63}$$

$$\mathbf{P}(\tau) \;=\; \mathbf{P}(t') + \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau') \, \mathbf{f}(\tau'), \tag{3.2.64}$$

and taking the matrix element between $\langle \mathbf{Q} \, t|$ and $|\mathbf{P}' \, t'\rangle$ for $\lambda = 0$, we obtain

$$\langle \mathbf{Q} \, t|\mathbf{Q}(\tau)|\mathbf{P}' \, t'\rangle_0 \;=\; \left[\mathbf{Q}(t) - \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau' - \tau) \, \mathbf{s}(\tau')\right] \langle \mathbf{Q} \, t|\mathbf{Q}' \, t'\rangle_0 \,, \tag{3.2.65}$$

$$\langle \mathbf{Q} \, t|\mathbf{P}(\tau)|\mathbf{P}' \, t'\rangle_0 \;=\; \left[\mathbf{P}(t') + \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau') \, \mathbf{f}(\tau')\right] \langle \mathbf{Q} \, t|\mathbf{Q}' \, t'\rangle_0 \,. \tag{3.2.66}$$

We have used the relations

$${}_0\langle \mathbf{Q} \, t| \, \mathbf{Q}(t) \;=\; \mathbf{Q} \, {}_0\langle \mathbf{Q} \, t| \,, \tag{3.2.67}$$

$$\mathbf{P}(t') \, |\mathbf{P} \, t'\rangle_0 \;=\; \mathbf{P} \, |\mathbf{P} \, t'\rangle_0 \,, \tag{3.2.68}$$

for $\lambda = 0$ at coincident times. Eqs. (3.2.65) and (3.2.66) may be rewritten as

$$-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{f}(\tau)} \, \langle \mathbf{Q} \, t|\mathbf{P} \, t'\rangle_0 \;=\; \left[\mathbf{Q} - \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau' - \tau) \, \mathbf{s}(\tau')\right] \langle \mathbf{Q} \, t|\mathbf{P} \, t'\rangle_0 \,, \tag{3.2.69}$$

$$\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{s}(\tau)} \, \langle \mathbf{Q} \, t|\mathbf{P} \, t'\rangle_0 \;=\; \left[\mathbf{P} + \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau') \, \mathbf{f}(\tau')\right] \langle \mathbf{Q} \, t|\mathbf{P} \, t'\rangle_0 \,. \tag{3.2.70}$$

These equations may be integrated as before to yield

$$-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\tau)}\left\langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t'\right\rangle_0=\left[\mathbf{Q}-\int_{t'}^{t}\mathrm{d}\tau'\,\Theta(\tau'-\tau)\,\mathbf{s}(\tau')\right]\left\langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t'\right\rangle_0, \tag{3.2.71}$$

$$\int\frac{\delta\left\langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t'\right\rangle_0}{\left\langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t'\right\rangle_0}=\frac{\mathrm{i}}{\hbar}\int\delta\mathbf{f}(\tau)\left[\mathbf{Q}-\int_{t'}^{t}\mathrm{d}\tau'\,\Theta(\tau'-\tau)\,\mathbf{s}(\tau')\right], \tag{3.2.72}$$

$$\ln\left\langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t'\right\rangle_0=\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\int_{t'}^{t}\mathbf{f}(\tau)\,\mathrm{d}\tau$$

$$-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{f}(\tau)\cdot\Theta(\tau'-\tau)\,\mathbf{s}(\tau'), \tag{3.2.73}$$

$$\left\langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t'\right\rangle_0=C\,\exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\int_{t'}^{t}\mathbf{f}(\tau)\,\mathrm{d}\tau\right]$$

$$\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{f}(\tau)\cdot\Theta(\tau'-\tau)\,\mathbf{s}(\tau')\right], \tag{3.2.74}$$

and

$$\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{s}(\tau)}\left\langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t'\right\rangle_0=\left[\mathbf{P}+\int_{t'}^{t}\mathrm{d}\tau'\,\Theta(\tau-\tau')\,\mathbf{f}(\tau')\right]\left\langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t'\right\rangle_0, \tag{3.2.75}$$

$$\int\frac{\delta\left\langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t'\right\rangle_0}{\left\langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t'\right\rangle_0}=-\frac{\mathrm{i}}{\hbar}\int\delta\mathbf{s}(\tau)\left[\mathbf{P}+\int_{t'}^{t}\mathrm{d}\tau'\,\Theta(\tau-\tau')\,\mathbf{f}(\tau')\right], \tag{3.2.76}$$

$$\ln\left\langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t'\right\rangle_0=-\frac{\mathrm{i}}{\hbar}\,\mathbf{P}\cdot\int_{t'}^{t}\mathbf{s}(\tau)\,\mathrm{d}\tau$$

$$-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{s}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{f}(\tau'), \tag{3.2.77}$$

$$\left\langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t'\right\rangle_0=D\,\exp\left[-\frac{\mathrm{i}}{\hbar}\,\mathbf{P}\cdot\int_{t'}^{t}\mathbf{s}(\tau)\,\mathrm{d}\tau\right]$$

$$\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{s}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{f}(\tau')\right], \tag{3.2.78}$$

to find C and D in Eqs. (3.2.74) and (3.2.78), respectively, we have

$$\text{Eq.}(3.2.74) \;=\; \text{Eq.}(3.2.78). \tag{3.2.79}$$

Or

$$\begin{aligned} & C\;\exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\int_{t'}^{t}\mathbf{f}(\tau)\,\mathrm{d}\tau\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{f}(\tau)\cdot\Theta(\tau'-\tau)\,\mathbf{s}(\tau')\right] \\ = \; & D\;\exp\left[-\frac{\mathrm{i}}{\hbar}\,\mathbf{P}\cdot\int_{t'}^{t}\mathbf{s}(\tau)\,\mathrm{d}\tau\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{s}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{f}(\tau')\right], \end{aligned} \tag{3.2.80}$$

we have,

$$C=\exp\left[-\frac{\mathrm{i}}{\hbar}\,\mathbf{P}\cdot\int_{t'}^{t}\mathbf{s}(\tau)\,\mathrm{d}\tau\right]\quad,\quad D=\exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\int_{t'}^{t}\mathbf{f}(\tau)\,\mathrm{d}\tau\right]. \tag{3.2.81}$$

In the absence of the external sources: $\mathbf{f}=0$, $\mathbf{s}=0$, $\tilde{H}\to 0$, and Eq. (3.2.39) reduces to

$$\underline{\tilde{H}}(\tau,\lambda)\to\underline{H}(\tau,\lambda), \tag{3.2.82}$$

with $\underline{H}(\tau,\lambda)$ given in Eq. (3.2.3). For $t=t'$

$$\begin{aligned} \langle\mathbf{Q}\,t\,|\,\mathbf{P}\,t\rangle &= \big\langle\mathbf{Q}\big|U(t)U^{\dagger}(t)\big|\mathbf{P}\big\rangle \\ &= \langle\mathbf{Q}\,|\,\mathbf{P}\rangle \\ &= \exp\left(\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\mathbf{P}\right), \end{aligned} \tag{3.2.83}$$

then we obtain

$$\langle \mathbf{Q}\, t\,|\,\mathbf{P}\, t'\rangle_0 = \exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{f}(\tau)\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\,\mathbf{P}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{s}(\tau)\right]\exp\left(\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\mathbf{P}\right)$$
$$\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{s}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{f}(\tau')\right]. \qquad (3.2.84)$$

To obtain the expression for $\langle \mathbf{Q}t\,|\,\mathbf{Q}'t'\rangle_0$, we multiply Eq. (3.2.84) by $\langle \mathbf{P}t'\,|\,\mathbf{Q}'t'\rangle = \exp(-\mathrm{i}\mathbf{Q}'\cdot\mathbf{P}/\hbar)$ and integrate over $\mathbf{P}$, with measure $\mathrm{d}^k\mathbf{P}/(2\pi\hbar)^k$. This gives

$$\begin{aligned}
\langle \mathbf{Q}\, t\,|\,\mathbf{Q}'\, t'\rangle_0 &= \int\frac{\mathrm{d}^k\mathbf{P}}{(2\pi\hbar)^k}\,\langle \mathbf{Q}\, t\,|\,\mathbf{P}\, t'\rangle_0\,\langle \mathbf{P}\, t'\,|\,\mathbf{Q}'\, t'\rangle\,,\\
&= \int\frac{\mathrm{d}^k\mathbf{P}}{(2\pi\hbar)^k}\,\exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{f}(\tau)\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\,\mathbf{P}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{s}(\tau)\right]\\
&\qquad\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{s}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{f}(\tau')\right]\\
&\qquad\times\exp\left(\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\mathbf{P}\right)\exp\left(-\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}'\cdot\mathbf{P}\right),\\
&= \int\frac{\mathrm{d}^k\mathbf{P}}{(2\pi\hbar)^k}\,\exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{f}(\tau)\right]\\
&\qquad\times\exp\left[\mathrm{i}\left(\mathbf{Q}-\mathbf{Q}'-\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{s}(\tau)\right)\cdot\frac{\mathbf{P}}{\hbar}\right]\\
&\qquad\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{s}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{f}(\tau')\right],\\
&= \delta^k\left(\mathbf{Q}-\mathbf{Q}'-\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{s}(\tau)\right)\exp\left[\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{f}(\tau)\right]\\
&\qquad\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{s}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{f}(\tau')\right],
\end{aligned}$$

$$\langle \mathbf{Q}\, t \,|\, \mathbf{Q}'\, t' \rangle_0 \;=\; \delta^k \left(\mathbf{Q} - \mathbf{Q}' - \int_{t'}^{t} \mathrm{d}\tau\, \mathbf{s}(\tau) \right) \exp\left(\frac{\mathrm{i}}{\hbar}\, \mathbf{Q} \cdot \int_{t'}^{t} \mathrm{d}\tau\, \mathbf{f}(\tau) \right)$$

$$\times \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau'\, \mathbf{s}(\tau) \cdot \Theta(\tau - \tau')\, \mathbf{f}(\tau') \right). \quad (3.2.85)$$

From $\langle \mathbf{q}\, t \,|\, \mathbf{q}' t' \rangle_0$ in equation Eq. (3.2.37), the conditions in Eqs. (3.2.44), (3.2.45), (3.2.49) - (3.2.55) and $\langle \mathbf{Q} t \,|\, \mathbf{Q}' t' \rangle_0$ in Eq. (3.2.85), we then have

$$\langle \mathbf{q}\mathbf{Q} t \,|\, \mathbf{q}'\mathbf{Q}' t' \rangle_0^{\wedge} = \delta^{(k)} \left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{f}(\cdot)} - \mathbf{G}'(\cdot) \right) \delta^{(k)} \left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)} \frac{\delta}{\delta \mathbf{s}(\cdot)} \right) \langle \mathbf{q}\, t \,|\, \mathbf{q}' t' \rangle_0 \; A, \quad (3.2.86)$$

where

$$A \;=\; \delta^k \left(\mathbf{Q} - \mathbf{Q}' - \int_{t'}^{t} \mathrm{d}\tau\, \mathbf{s}(\tau) \right) \exp\left(\frac{\mathrm{i}}{\hbar}\, \mathbf{Q} \cdot \int_{t'}^{t} \mathrm{d}\tau\, \mathbf{f}(\tau) \right)$$

$$\times \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{\tau} \mathrm{d}\tau'\, \mathbf{s}(\tau) \cdot \mathbf{f}(\tau) \right). \quad (3.2.87)$$

We recall that the vertical bar $|$ in Eq. (3.2.48) refers to the fact that all the external sources are to be set to zero after all the relevant functional differentiations have been carried out. $\delta^{(k)}(-\mathrm{i}\hbar\delta/\delta\mathbf{f}(\cdot) - \mathbf{G}'(\cdot))$ and $\delta^{(k)}(\mathrm{i}\hbar\delta/(2\pi\hbar)\delta\mathbf{s}(\cdot))$ in Eq. (3.2.86), as arising from the conditions in Eqs. (3.2.44) and (3.2.45), refer, each, to the product of k-dimensional deltas with τ running over all points in the interval $[t', t]$, i.e., $\delta^{(k)}(D(\cdot)) = \prod_{t' \leq \tau \leq t} \delta^k(D(\tau))$. We also note that functional differentiation operations with respect to external sources commute, unlike quantum operators, showing the power of the underlying formalism.

Accordingly, the transformation function $\langle \mathbf{q}\mathbf{Q} t \,|\, \mathbf{q}'\mathbf{Q}' t' \rangle$ for the constrained dynamics may be written as

$$\langle \mathbf{q}\mathbf{Q} t \,|\, \mathbf{q}'\mathbf{Q}' t' \rangle \;=\; \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau\, H'(\tau) \right) \langle \mathbf{q}\mathbf{Q} t \,|\, \mathbf{q}'\mathbf{Q}' t' \rangle_0^{\wedge} \bigg|\,, \quad (3.2.88)$$

with $\langle \mathbf{q}\mathbf{Q}t \,|\, \mathbf{q}'\mathbf{Q}'t' \rangle_0^\wedge$ defined in Eqs. (3.2.86), (3.2.87) and (3.2.37), and the bar sign | on its right-hand side in Eq. (3.2.88) means to set all the external sources equal to zero *after all the functional differential operations in Eq. (3.2.86) are carried out*. Due to the delta functionals

$$\delta^{(k)}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\cdot)} - \mathbf{G}'(\cdot)\right)\delta^{(k)}\left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)}\frac{\delta}{\delta\mathbf{s}(\cdot)}\right), \tag{3.2.89}$$

we may *replace* $H'(\tau)$ in Eq. (3.2.88), and defined in Eq. (3.2.7) from the Hamiltonian $H(\mathbf{q},\mathbf{p})$, by any Hamiltonian operator functional $\tilde{H}'(\tau)$ which coincides with $H'(\tau)$ for $-\mathrm{i}\hbar\delta/\delta\mathbf{f}(\tau)$ replaced by $\mathbf{G}'(\tau)$ and $\mathrm{i}\hbar\delta/\delta\mathbf{s}(\tau)$ by 0. That is, the transformation functions in Eq. (3.2.88) correspond to the dynamics of any Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})$ for which

$$\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{G}(\mathbf{q},\mathbf{p}),0) \;=\; H(\mathbf{q},\mathbf{p}), \tag{3.2.90}$$

corresponding to constraints $\mathbf{Q} - \mathbf{G}(\mathbf{q},\mathbf{p}) = 0$ for which $\mathbf{P} = 0$.

3.3 Contact with the Faddeev-Popov Technique

Eq. (3.2.48) for $\langle \mathbf{q}\mathbf{Q}t \,|\, \mathbf{q}'\mathbf{Q}'t' \rangle$ gives the expression for the transformation functions with the constraints given in Eqs. (3.2.44) and (3.2.45) for all τ in $[t', t]$. They involve functional differential operations, with respect to external sources, to be applied to the functional $[\langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_0 \;\; A]$ with the latter given in closed form in Eqs. (3.2.86), (3.2.37) and (3.2.87).

Finally we may make contact with the Faddeev-Popov technique, in the path integral formalism, by noting that the path integral representation for $[\langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_0 \, A]$ (see, Manoukian, 2006, sect 11.4) on the extreme right-hand side of Eq. (3.2.86) is given by

$$[\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle_0\, A]$$

$$= \int_{\mathbf{q}(t')=\mathbf{q}',\mathbf{Q}(t')=\mathbf{Q}'}^{\mathbf{q}(t)=\mathbf{q},\mathbf{Q}(t)=\mathbf{Q}} \mathscr{D}(\mathbf{q}(\cdot),\mathbf{p}(\cdot))\mathscr{D}(\mathbf{Q}(\cdot),\mathbf{P}(\cdot)) \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau\; v(\tau)\right), \tag{3.3.1}$$

where

$$\begin{aligned} v(\tau) \;=\; & [\mathbf{p}(\tau)\cdot\dot{\mathbf{q}}(\tau) + \mathbf{P}(\tau)\cdot\dot{\mathbf{Q}}(\tau) + \mathbf{q}(\tau)\cdot\mathbf{F}(\tau) \\ & -\mathbf{p}(\tau)\cdot\mathbf{S}(\tau) + \mathbf{Q}(\tau)\cdot\mathbf{f}(\tau) - \mathbf{P}(\tau)\cdot\mathbf{s}(\tau)], \end{aligned} \tag{3.3.2}$$

and we have to carry out the explicit functional differentiations with respect to the external sources in Eqs. (3.2.88) and (3.2.86), and finally set the external sources equal to zero. Here we note that the Hamiltonian of the system describing its time evolution *appears* in the first factor on the right-hand side of Eq. (3.2.88) as a functional differential operator with respect to external sources as defined in Eq. (3.2.7).

By carrying out the functional differential operations in Eqs. (3.2.88) and (3.2.86), and using the expression in Eq. (3.3.1), contact will be made with the Faddeev-Popov form in Eq. (3.2.48) for the dynamical systems described by Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})$ with constraints as given in Eq. (3.2.90) This is shown below.

From Eqs. (3.2.88), (3.2.86) and (3.3.1), we obtain

$$\begin{aligned} \langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle = &\; \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\; H'(\tau)\right)\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle_0^{\wedge}\Big| \\ &= \delta^{(k)}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\cdot)} - \mathbf{G}'(\cdot)\right)\delta^{(k)}\left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)}\frac{\delta}{\delta\mathbf{s}(\cdot)}\right) \\ &\times \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\; H'(\tau)\right)[\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle_0\; A]\Big| \end{aligned}$$

$$
\begin{aligned}
&= \delta^{(k)}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\cdot)} - \mathbf{G}'(\cdot)\right)\delta^{(k)}\left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)}\frac{\delta}{\delta\mathbf{s}(\cdot)}\right) \\
&\quad\times \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\; H'(\tau)\right) \\
&\quad\times \int_{\mathbf{q}(t')=\mathbf{q}',\mathbf{Q}(t')=\mathbf{Q}'}^{\mathbf{q}(t)=\mathbf{q},\mathbf{Q}(t)=\mathbf{Q}} \mathscr{D}(\mathbf{q}(\cdot),\mathbf{p}(\cdot))\mathscr{D}(\mathbf{Q}(\cdot),\mathbf{P}(\cdot))\exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\; v(\tau)\right)\bigg| \\
&= \delta^{(k)}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\cdot)} - \mathbf{G}'(\cdot)\right)\delta^{(k)}\left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)}\frac{\delta}{\delta\mathbf{s}(\cdot)}\right) \\
&\quad\times \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\; H'(\tau)\right) \\
&\quad\times \int_{\mathbf{q}(t')=\mathbf{q}',\mathbf{Q}(t')=\mathbf{Q}'}^{\mathbf{q}(t)=\mathbf{q},\mathbf{Q}(t)=\mathbf{Q}} \mathscr{D}(\mathbf{q}(\cdot),\mathbf{p}(\cdot))\mathscr{D}(\mathbf{Q}(\cdot),\mathbf{P}(\cdot)) \\
&\quad\times \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\;[\mathbf{p}(\tau)\cdot\dot{\mathbf{q}}(\tau) + \mathbf{P}(\tau)\cdot\dot{\mathbf{Q}}(\tau) + \mathbf{q}(\tau)\cdot\mathbf{F}(\tau)\right. \\
&\qquad\qquad \left. - \mathbf{p}(\tau)\cdot\mathbf{S}(\tau) + \mathbf{Q}(\tau)\cdot\mathbf{f}(\tau) - \mathbf{P}(\tau)\cdot\mathbf{s}(\tau)]\right)\bigg| \\
&= \delta^{(k)}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\cdot)} - \mathbf{G}'(\cdot)\right)\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\; H'(\tau)\right) \\
&\quad\times \int_{\mathbf{q}(t')=\mathbf{q}',\mathbf{Q}(t')=\mathbf{Q}'}^{\mathbf{q}(t)=\mathbf{q},\mathbf{Q}(t)=\mathbf{Q}} \mathscr{D}(\mathbf{q}(\cdot),\mathbf{p}(\cdot))\mathscr{D}(\mathbf{Q}(\cdot),\mathbf{P}(\cdot))\delta^{(k)}\left(\frac{\mathrm{i}\hbar}{2\pi\hbar}\frac{-\mathrm{i}}{\hbar}\;\mathbf{P}(\cdot)\right) \\
&\quad\times \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\;[\mathbf{p}(\tau)\cdot\dot{\mathbf{q}}(\tau) + \mathbf{P}(\tau)\cdot\dot{\mathbf{Q}}(\tau) + \mathbf{q}(\tau)\cdot\mathbf{F}(\tau)\right. \\
&\qquad\qquad \left. - \mathbf{p}(\tau)\cdot\mathbf{S}(\tau) + \mathbf{Q}(\tau)\cdot\mathbf{f}(\tau) - \mathbf{P}(\tau)\cdot\mathbf{s}(\tau)]\right)\bigg|
\end{aligned}
$$

$$= \exp\left(\frac{-\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau\, H'(\tau)\right)\int_{\mathbf{q}(t')=\mathbf{q}',\mathbf{Q}(t')=\mathbf{Q}'}^{\mathbf{q}(t)=\mathbf{q},\mathbf{Q}(t)=\mathbf{Q}} \mathscr{D}(\mathbf{q}(\cdot),\mathbf{p}(\cdot))\mathscr{D}(\mathbf{Q}(\cdot),\mathbf{P}(\cdot))$$

$$\times\, \delta^{(k)}\left(\frac{1}{2\pi\hbar}\,\mathbf{P}(\cdot)\right)\delta^{(k)}\left(-\mathrm{i}\hbar\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}(\cdot)-\mathbf{G}(\cdot)\right)$$

$$\times \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,[\mathbf{p}(\tau)\cdot\dot{\mathbf{q}}(\tau)+\mathbf{P}(\tau)\cdot\dot{\mathbf{Q}}(\tau)+\mathbf{q}(\tau)\cdot\mathbf{F}(\tau)\right.$$

$$\left.\left. -\,\mathbf{p}(\tau)\cdot\mathbf{S}(\tau)+\mathbf{Q}(\tau)\cdot\mathbf{f}(\tau)-\mathbf{P}(\tau)\cdot\mathbf{s}(\tau)]\right)\right|$$

$$\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle = \exp\left(\frac{-\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau\, H'(\tau)\right)\int_{\mathbf{q}(t')=\mathbf{q}',\mathbf{Q}(t')=\mathbf{Q}'}^{\mathbf{q}(t)=\mathbf{q},\mathbf{Q}(t)=\mathbf{Q}} \mathscr{D}(\mathbf{q}(\cdot),\mathbf{p}(\cdot))\mathscr{D}(\mathbf{Q}(\cdot),\mathbf{P}(\cdot))$$

$$\times\, \delta^{(k)}\left(\frac{1}{2\pi\hbar}\,\mathbf{P}(\cdot)\right)\delta^{(k)}\left(\mathbf{Q}(\cdot)-\mathbf{G}(\cdot)\right)$$

$$\times \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,[\mathbf{p}(\tau)\cdot\dot{\mathbf{q}}(\tau)+\mathbf{P}(\tau)\cdot\dot{\mathbf{Q}}(\tau)+\mathbf{q}(\tau)\cdot\mathbf{F}(\tau)\right.$$

$$\left.\left. -\,\mathbf{p}(\tau)\cdot\mathbf{S}(\tau)+\mathbf{Q}(\tau)\cdot\mathbf{f}(\tau)-\mathbf{P}(\tau)\cdot\mathbf{s}(\tau)]\right)\right|, \tag{3.3.3}$$

which upon integration over $\mathbf{Q}(\cdot), \mathbf{P}(\cdot)$ this is equal to

$$\exp\left(\frac{-\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau\, H'(\tau)\right)\int_{\mathbf{q}(t')=\mathbf{q}',\mathbf{Q}(t')=\mathbf{Q}'}^{\mathbf{q}(t)=\mathbf{q},\mathbf{Q}(t)=\mathbf{Q}} \mathscr{D}(\mathbf{q}(\cdot),\mathbf{p}(\cdot))$$

$$\times \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,[\mathbf{p}(\tau)\cdot\dot{\mathbf{q}}(\tau)+\mathbf{0}\cdot\dot{\mathbf{Q}}(\tau)+\mathbf{q}\cdot\mathbf{F}(\tau)\right.$$

$$\left.\left. -\,\mathbf{p}(\tau)\cdot\mathbf{S}(\tau)+\mathbf{G}(\tau)\cdot\mathbf{f}(\tau)-\mathbf{0}\cdot\mathbf{s}(\tau)]\right)\right|\,, \tag{3.3.4}$$

or

$$\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau\; H'(\tau)\right) \int_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t)=\mathbf{q}} \mathscr{D}(\mathbf{q}(\cdot),\mathbf{p}(\cdot))$$

$$\times \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau\; [\mathbf{p}(\tau)\cdot\dot{\mathbf{q}}(\tau) + \mathbf{q}(\tau)\cdot\mathbf{F}(\tau)\right.$$

$$\left. - \mathbf{p}(\tau)\cdot\mathbf{S}(\tau) + \mathbf{G}(\tau)\cdot\mathbf{f}(\tau)]\right)\Bigg| . \qquad (3.3.5)$$

After setting all the external sources equal to zero, we then have

$$\langle \mathbf{q}\mathbf{Q}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle \;=\; \int_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t)=\mathbf{q}} \mathscr{D}(\mathbf{q}(\cdot),\mathbf{p}(\cdot)) \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau\; [\mathbf{p}\cdot\dot{\mathbf{q}}(\tau) - H(\mathbf{q}(t),\mathbf{p}(t))]\right). \qquad (3.3.6)$$

3.4 Application

Consider the Hamiltonian in four-dimensional phase space in quantum mechanics

$$H(\mathbf{q},\mathbf{p}) \;=\; \frac{p_1^2 + p_2^2 + p_1 p_2}{3m} + f(q_1,q_2), \qquad (3.4.1)$$

where $f(q_1,q_2)$ is an arbitrary real function of (q_1,q_2) and m is a mass parameter.

Hamilton's equations for the $\dot{q}_i$ are

$$\dot{q}_i \;=\; \frac{\partial H}{\partial p_i} . \qquad (3.4.2)$$

That is,

$$\dot{q}_1 \;=\; \frac{\partial}{\partial p_1}\left[\frac{p_1^2 + p_2^2 + p_1 p_2}{3m} + f(q_1,q_2)\right]$$

$$=\; \frac{1}{3m}(2p_1 + p_2)\,, \qquad (3.4.3)$$

and

$$\dot{q}_2 = \frac{\partial}{\partial p_2}\left[\frac{p_1^2 + p_2^2 + p_1 p_2}{3m} + f(q_1, q_2)\right]$$

$$= \frac{1}{3m}(2p_2 + p_1) \; . \tag{3.4.4}$$

From Eqs. (3.4.3) and (3.4.4) we then obtain

$$3m\dot{q}_1 = 2p_1 + p_2 \; , \tag{3.4.5}$$

$$2p_1 = 3m\dot{q}_1 - p_2 \; , \tag{3.4.6}$$

$$p_1 = \frac{3m\dot{q}_1 - p_2}{2} \; , \tag{3.4.7}$$

and

$$3m\dot{q}_2 = 2p_2 + p_1 \; , \tag{3.4.8}$$

$$p_1 = 3m\dot{q}_2 - 2p_2 \; . \tag{3.4.9}$$

Upon comparing Eqs. (3.4.7) and (3.4.9), we have

$$\frac{3m\dot{q}_1 - p_2}{2} = 3m\dot{q}_2 - 2p_2 \; , \tag{3.4.10}$$

$$3m\dot{q}_1 - p_2 = 6m\dot{q}_2 - 4p_2 \; , \tag{3.4.11}$$

$$-p_2 + 4p_2 = 6m\dot{q}_2 - 3m\dot{q}_1 \; , \tag{3.4.12}$$

$$3p_2 = 3m(2\dot{q}_2 - \dot{q}_1) \; , \tag{3.4.13}$$

$$p_2 = m(2\dot{q}_2 - \dot{q}_1) \; . \tag{3.4.14}$$

Substituting Eq. (3.4.14) into Eq. (3.4.9), gives

$$\begin{aligned}
p_1 &= 3m\dot{q}_2 - 2m(2\dot{q}_2 - \dot{q}_1) \\
&= 3m\dot{q}_2 - 4m\dot{q}_2 + 2m\dot{q}_1 \\
&= 2m\dot{q}_1 - m\dot{q}_2 \ , \\
p_1 &= m(2\dot{q}_1 - \dot{q}_2) \ . \qquad (3.4.15)
\end{aligned}$$

The Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ of the system is defined by

$$\begin{aligned}
L(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2}\sum_{a=1,2}(p_a\dot{q}_a + \dot{q}_a p_a) - H(\mathbf{q}, \mathbf{p}) \\
&= \frac{1}{2}(p_1\dot{q}_1 + \dot{q}_1 p_1) + \frac{1}{2}(p_2\dot{q}_2 + \dot{q}_2 p_2) - H(\mathbf{q}, \mathbf{p}) \\
&= \frac{1}{2}\big[m(2\dot{q}_1 - \dot{q}_2)\dot{q}_1 + \dot{q}_1 m(2\dot{q}_1 - \dot{q}_2)\big] \\
&\quad + \frac{1}{2}\big[m(2\dot{q}_2 - \dot{q}_1)\dot{q}_2 + \dot{q}_2 m(2\dot{q}_2 - \dot{q}_1)\big] - H(\mathbf{q}, \mathbf{p}) \\
&= \frac{1}{2}(2m\dot{q}_1^2 - m\dot{q}_2\dot{q}_1 + 2m\dot{q}_1^2 - m\dot{q}_1\dot{q}_2) \\
&\quad + \frac{1}{2}(2m\dot{q}_2^2 - m\dot{q}_1\dot{q}_2 + 2m\dot{q}_2^2 - m\dot{q}_2\dot{q}_1) - H(\mathbf{q}, \mathbf{p}) \\
&= \frac{1}{2}(4m\dot{q}_1^2 + 4m\dot{q}_2^2 - 2m\dot{q}_1\dot{q}_2 - 2m\dot{q}_2\dot{q}_1) - H(\mathbf{q}, \mathbf{p}) \ , \\
L(\mathbf{q}, \dot{\mathbf{q}}) &= 2m\dot{q}_1^2 + 2m\dot{q}_2^2 - m\dot{q}_1\dot{q}_2 - m\dot{q}_2\dot{q}_1 - H(\mathbf{q}, \mathbf{p}) \ , \qquad (3.4.16)
\end{aligned}$$

which has to be expressed as a function of $\mathbf{q}$, $\dot{\mathbf{q}}$ only. To this end we note that we may write

$$\begin{aligned}
H(\mathbf{q},\mathbf{p}) &= \frac{p_1^2+p_2^2+p_1p_2}{3m} + f(q_1,q_2) \\
&= \frac{1}{3m}\, m^2\big[(2\dot{q}_1-\dot{q}_2)^2+(2\dot{q}_2-\dot{q}_1)^2+(2\dot{q}_1-\dot{q}_2)(2\dot{q}_2-\dot{q}_1)\big] + f(q_1,q_2) \\
&= \frac{m}{3}(4\dot{q}_1^2-4\dot{q}_1\dot{q}_2+\dot{q}_2^2)+\frac{m}{3}(4\dot{q}_2^2-4\dot{q}_2\dot{q}_1+\dot{q}_1^2) \\
&\quad +\frac{m}{3}(4\dot{q}_1\dot{q}_2-2\dot{q}_1^2-2\dot{q}_2^2+\dot{q}_2\dot{q}_1)+f(q_1,q_2) \\
&= \frac{m}{3}(3\dot{q}_1^2+3\dot{q}_2^2-3\dot{q}_2\dot{q}_1)+f(q_1,q_2)\ , \\
H(\mathbf{q},\mathbf{p}) &= m(\dot{q}_1^2+\dot{q}_2^2-\dot{q}_2\dot{q}_1)+f(q_1,q_2)\ . \qquad (3.4.17)
\end{aligned}$$

expressed formally as a function of $\mathbf{q}_a$, $\dot{\mathbf{q}}_a$. Upon substituting the expression for $H(\mathbf{q},\mathbf{p})$ in Eq. (3.4.17) into Eq. (3.4.16) gives

$$\begin{aligned}
L(\mathbf{q},\dot{\mathbf{q}}) &= m(2\dot{q}_1^2+2\dot{q}_2^2-\dot{q}_1\dot{q}_2-\dot{q}_2\dot{q}_1)-m(\dot{q}_1^2+\dot{q}_2^2-\dot{q}_2\dot{q}_1)-f(q_1,q_2) \\
&= m(\dot{q}_1^2+\dot{q}_2^2-\dot{q}_1\dot{q}_2)-f(q_1,q_2)\ . \qquad (3.4.18)
\end{aligned}$$

This may be rewritten in a more interesting way as

$$L(\mathbf{q},\dot{\mathbf{q}}) = \frac{m\dot{q}_1^2}{2}+\frac{m\dot{q}_2^2}{2}+\frac{m}{2}(\dot{q}_1-\dot{q}_2)^2-f(q_1,q_2)\ , \qquad (3.4.19)$$

for the independent variables q_1, q_2, where we note from Eqs. (3.4.3) and (3.4.4), that due to the independence of the variables p_1, p_2, that is $[p_1,p_2]=0$, $\dot{q}_1$ and $\dot{q}_2$ commute. The expression in Eq. (3.4.19) is highly suggestive as constrained dynamics of systems with Lagrangians

$$L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) \;=\; \frac{m\dot{q}_1^2}{2} + \frac{m\dot{q}_2^2}{2} + \frac{m\dot{q}_3^2}{2} - V(q_{1,2}\,, q_3)\;, \qquad (3.4.20)$$

with constraint

$$q_3 - (q_1 - q_2) \;=\; 0\;, \qquad (3.4.21)$$

for *all* times, and $V(q_1, q_2, q_3)$ are any real function such that

$$V(q_1, q_2, q_3)\Big|_{q_3 = q_1 - q_2} \;=\; f(q_1, q_2)\;. \qquad (3.4.22)$$

Due to the fact that q_1, q_2 are independent variables with canonical conjugate momenta p_1, p_2, we have the commutators:

$$[q_1, p_1] \;=\; \mathrm{i}\hbar\;, \qquad (3.4.23)$$

$$[q_2, p_2] \;=\; \mathrm{i}\hbar\;, \qquad (3.4.24)$$

and more generally

$$[q_a, p_b] \;=\; \mathrm{i}\hbar\,\delta_{ab} \quad , \quad a, b = 1, 2\;. \qquad (3.4.25)$$

From Eqs. (3.4.14) and (3.4.15) we note that we may rewrite p_1, p_2 as

$$p_1 \;=\; m\dot{q}_1 + m(\dot{q}_1 - \dot{q}_2)\;, \qquad (3.4.26)$$

$$p_2 \;=\; m\dot{q}_2 - m(\dot{q}_1 - \dot{q}_2)\;. \qquad (3.4.27)$$

On the other hand, $q_3 = q_1 - q_2$ being a dependent variable, its canonical conjugate momentum p_3 is zero by definition. Accordingly, as in quantum electrodynamics (see the introductory section to this chapter), we may trivially extend Eqs. (3.4.26) and

(3.4.27) from $a = 1, 2$, to $i = 1, 2, 3$ by setting

$$p_3 \;=\; m\dot{q}_3 - m(\dot{q}_1 - \dot{q}_2)\ , \tag{3.4.28}$$

since $q_3 = q_1 - q_2$ at *all* times implies that the right-hand side of Eq. (3.4.28) is zero, and Eq. (3.4.28) gives the trivial equation that $0 = 0$.

Thus we have succeeded, in extending Eqs. (3.4.26) and (3.4.27) from $a = 1, 2$ to $i = 1, 2, 3$. Since, $\dot{q}_3 = \dot{q}_1 - \dot{q}_2$ as a consequence of the fact that $q_3 = q_1 - q_2$ at all times, we may use the latter equality for $\dot{q}_3 = \dot{q}_1 - \dot{q}_2$, together with the equalities in Eqs. (3.4.3) and (3.4.4) and the commutation relations in Eq. (3.4.25) to derive explicitly that

$$\begin{aligned}
[q_1, \dot{q}_1] &= q_1\dot{q}_1 - \dot{q}_1 q_1\ , \\
&= q_1\frac{1}{3m}(2p_1 + p_2) - \frac{1}{3m}(2p_1 + p_2)q_1\ , \\
&= \frac{2}{3m}q_1p_1 + \frac{1}{3m}q_1p_2 - \frac{2}{3m}p_1q_1 - \frac{1}{3m}p_2q_1\ , \\
&= \frac{2}{3m}(q_1p_1 - p_1q_1) + \frac{1}{3m}(q_1p_2 - p_2q_1)\ , \\
&= \frac{2}{3m}[q_1, p_1] + \frac{1}{3m}[q_1, p_2]\ , \\
[q_1, \dot{q}_1] &= \frac{2}{3m}(\mathrm{i}\hbar)\ , \qquad (3.4.29) \\
[q_1, \dot{q}_2] &= q_1\dot{q}_2 - \dot{q}_2 q_1\ , \\
&= q_1\frac{1}{3m}(2p_2 + p_1) - \frac{1}{3m}(2p_2 + p_1)q_1\ , \\
&= \frac{2}{3m}q_1p_2 + \frac{1}{3m}q_1p_1 - \frac{2}{3m}p_2q_1 - \frac{1}{3m}p_1q_1\ ,
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3m}(q_1p_2 - p_2q_1) + \frac{1}{3m}(q_1p_1 - p_1q_1) \; , \\
&= \frac{2}{3m}[q_1, p_2] + \frac{1}{3m}[q_1, p_1] \; , \\
[q_1, \dot{q}_2] &= \frac{1}{3m}(\mathrm{i}\hbar) \; , \qquad (3.4.30) \\
[q_1, \dot{q}_3] &= q_1\dot{q}_3 - \dot{q}_3q_1 \; , \\
&= q_1(\dot{q}_1 - \dot{q}_2) - (\dot{q}_1 - \dot{q}_2)q_1 \; , \\
&= q_1\dot{q}_1 - q_1\dot{q}_2 - \dot{q}_1q_1 + \dot{q}_2q_1 \; , \\
&= (q_1\dot{q}_1 - \dot{q}_1q_1) - (q_1\dot{q}_2 - \dot{q}_2q_1) \; , \\
&= [q_1, \dot{q}_1] - [q_1, \dot{q}_2] \; , \\
&= \frac{2}{3m}(\mathrm{i}\hbar) - \frac{1}{3m}(\mathrm{i}\hbar) \; , \\
[q_1, \dot{q}_3] &= \frac{1}{3m}(\mathrm{i}\hbar) \; , \qquad (3.4.31) \\
[q_2, \dot{q}_1] &= q_2\dot{q}_1 - \dot{q}_1q_2 \; , \\
&= q_2\frac{1}{3m}(2p_1 + p_2) - \frac{1}{3m}(2p_1 + p_2)q_2 \; , \\
&= \frac{2}{3m}q_2p_1 + \frac{1}{3m}q_2p_2 - \frac{2}{3m}p_1q_2 - \frac{1}{3m}p_2q_2 \; , \\
&= \frac{2}{3m}(q_2p_1 - p_1q_2) + \frac{1}{3m}(q_2p_2 - p_2q_2) \; , \\
&= \frac{2}{3m}[q_2, p_1] + \frac{1}{3m}[q_2, p_2] \; ,
\end{aligned}
$$

$$
\begin{aligned}
[q_2, \dot{q}_1] &= \frac{1}{3m}(\mathrm{i}\hbar) \;, && (3.4.32) \\
[q_2, \dot{q}_2] &= q_2\dot{q}_2 - \dot{q}_2 q_2 \;, \\
&= q_2 \frac{1}{3m}(2p_2 + p_1) - \frac{1}{3m}(2p_2 + p_1) q_2 \;, \\
&= \frac{2}{3m} q_2 p_2 + \frac{1}{3m} q_2 p_1 - \frac{2}{3m} p_2 q_2 - \frac{1}{3m} p_1 q_2 \;, \\
&= \frac{2}{3m}(q_2 p_2 - p_2 q_2) + \frac{1}{3m}(q_2 p_1 - p_1 q_2) \;, \\
&= \frac{2}{3m}[q_2, p_2] + \frac{1}{3m}[q_2, p_1] \;, \\
[q_2, \dot{q}_2] &= \frac{2}{3m}(\mathrm{i}\hbar) \;, && (3.4.33) \\
[q_2, \dot{q}_3] &= q_2\dot{q}_3 - \dot{q}_3 q_2 \;, \\
&= q_2(\dot{q}_1 - \dot{q}_2) - (\dot{q}_1 - \dot{q}_2) q_2 \;, \\
&= q_2\dot{q}_1 - q_2\dot{q}_2 - \dot{q}_1 q_2 + \dot{q}_2 q_2 \;, \\
&= (q_2\dot{q}_1 - \dot{q}_1 q_2) - (q_2\dot{q}_2 - \dot{q}_2 q_2) \;, \\
&= [q_2, \dot{q}_1] - [q_2, \dot{q}_2] \;, \\
&= \frac{1}{3m}(\mathrm{i}\hbar) - \frac{2}{3m}(\mathrm{i}\hbar) \;, \\
[q_2, \dot{q}_3] &= -\frac{1}{3m}(\mathrm{i}\hbar) \;, && (3.4.34) \\
[q_3, \dot{q}_1] &= q_3\dot{q}_1 - \dot{q}_1 q_3 \;, \\
&= (q_1 - q_2)\dot{q}_1 - \dot{q}_1(q_1 - q_2) \;,
\end{aligned}
$$

$$
\begin{aligned}
&= q_1\dot{q}_1 - q_2\dot{q}_1 - \dot{q}_1 q_1 + \dot{q}_1 q_2 \ , \\
&= (q_1\dot{q}_1 - \dot{q}_1 q_1) - (q_2\dot{q}_1 - \dot{q}_1 q_2) \ , \\
&= [q_1, \dot{q}_1] - [q_2, \dot{q}_1] \ , \\
&= \frac{2}{3m}(\mathrm{i}\hbar) - \frac{1}{3m}(\mathrm{i}\hbar) \ , \\
[q_3, \dot{q}_1] &= \frac{1}{3m}(\mathrm{i}\hbar) \ , \qquad (3.4.35) \\
[q_3, \dot{q}_2] &= q_3\dot{q}_2 - \dot{q}_2 q_3 \ , \\
&= (q_1 - q_2)\dot{q}_2 - \dot{q}_2(q_1 - q_2) \ , \\
&= q_1\dot{q}_2 - q_2\dot{q}_2 - \dot{q}_2 q_1 + \dot{q}_2 q_2 \ , \\
&= (q_1\dot{q}_2 - \dot{q}_2 q_1) - (q_2\dot{q}_2 - \dot{q}_2 q_2) \ , \\
&= [q_1, \dot{q}_2] - [q_2, \dot{q}_2] \ , \\
&= \frac{1}{3m}(\mathrm{i}\hbar) - \frac{2}{3m}(\mathrm{i}\hbar) \ , \\
[q_3, \dot{q}_2] &= -\frac{1}{3m}(\mathrm{i}\hbar) \ , \qquad (3.4.36) \\
[q_3, \dot{q}_3] &= q_3\dot{q}_3 - \dot{q}_3 q_3 \ , \\
&= q_3(\dot{q}_1 - \dot{q}_2) - (\dot{q}_1 - \dot{q}_2) q_3 \ , \\
&= q_3\dot{q}_1 - q_3\dot{q}_2 - \dot{q}_1 q_3 + \dot{q}_2 q_3 \ , \\
&= (q_3\dot{q}_1 - \dot{q}_1 q_3) - (q_3\dot{q}_2 - \dot{q}_2 q_3) \ ,
\end{aligned}
$$

$$
\begin{aligned}
&= [q_3, \dot{q}_1] - [q_3, \dot{q}_2] \;, \\
&= \frac{1}{3m}(\mathrm{i}\hbar) - \left(-\frac{1}{3m}\right)(\mathrm{i}\hbar) \;, \\
[q_3, \dot{q}_3] &= \frac{2}{3m}(\mathrm{i}\hbar) \;. \qquad (3.4.37)
\end{aligned}
$$

These may be combined in a matrix notation as

$$
\Big([q_i, \dot{q}_j]\Big) = \frac{\mathrm{i}\hbar}{3m}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} . \qquad (3.4.38)
$$

In the *Hamiltonian* formalism, we then have

$$
H(q_1, q_2, p_1, p_2) = \frac{p_1^2 + p_2^2 + 2p_1p_2}{3m} + f(q_1, q_2) \;, \qquad (3.4.39)
$$

as the Hamiltonian of a system described, *in particular*, by a Hamiltonian

$$
H(q_1, q_2, q_3, p_1, p_2, p_3) = \frac{p_1^2}{3m} + \frac{p_2^2}{3m} + \frac{p_1p_2}{3m} + \frac{p_3^2}{2m} + V(q_1, q_2, q_3) \;, \qquad (3.4.40)
$$

with constraints

$$
q_3 - (q_1 - q_2) = 0 \;, \qquad (3.4.41)
$$

$$
p_3 = 0 \;, \qquad (3.4.42)
$$

such that

$$
V(q_1, q_2, q_3)\Big|_{q_3 = q_1 - q_2} = f(q_1, q_2) \;. \qquad (3.4.43)
$$

In our general notation we identify Q with q_3, P with p_3 and G with $q_1 - q_2$.

From the Hamiltonian in Eq. (3.4.1) we are following from Eqs. (3.2.38) - (3.2.87), firstly, from Eq. (3.2.38) and define $G(\tau) = q_1(\tau) - q_2(\tau)$, we obtain

$$Q^{\mathrm{c}}(\tau) = \frac{\langle \mathbf{q}t\,|(q_1(\tau) - q_2(\tau))|\mathbf{q}'t'\rangle \Big|}{\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle \Big|}\ , \tag{3.4.44}$$

$$Q^{\mathrm{c}}(\tau)\,\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle \Big| = \langle \mathbf{q}t\,|(q_1(\tau) - q_2(\tau))|\mathbf{q}'t'\rangle \Big|\ , \tag{3.4.45}$$

$$Q^{\mathrm{c}}(\tau)\,\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle \Big| - \langle \mathbf{q}t\,|(q_1(\tau) - q_2(\tau))|\mathbf{q}'t'\rangle \Big| = 0\ , \tag{3.4.46}$$

$$[Q^{\mathrm{c}}(\tau) - (q_1'(\tau) - q_2'(\tau))]\,\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle \Big| = 0\ . \tag{3.4.47}$$

with $q_1'(\tau) = -\mathrm{i}\hbar\delta/\delta S_1(\tau)$, $q_2'(\tau) = -\mathrm{i}\hbar\delta/\delta S_2(\tau)$. By promoting the $Q^{\mathrm{c}}(\tau)$ to a quantum variable $Q(\tau)$, with $\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle$ generalized to $\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle$, as applied to the problem at hand, we must have from the conditions (A), (B) and (C) in Sect. 3.2 and Eq. (3.4.47)

$$\begin{aligned}
&\left[-\mathrm{i}\hbar\frac{\delta}{\delta f(\tau)} - (q_1'(\tau) - q_2'(\tau))\right]\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle \Big| \\
&= \langle \mathbf{q}Qt\,|Q(\tau)|\mathbf{q}'Q't'\rangle \Big| - \langle \mathbf{q}Qt\,|(q_1(\tau) - q_2(\tau))|\mathbf{q}'Q't'\rangle \Big| \\
&= \langle \mathbf{q}Qt\,|Q(\tau) - (q_1(\tau) - q_2(\tau))|\mathbf{q}'Q't'\rangle \Big| \\
&= 0\ ,
\end{aligned} \tag{3.4.48}$$

and

$$\begin{aligned}
\mathrm{i}\hbar\frac{\delta}{\delta s(\tau)}\,\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle \Big| &= \langle \mathbf{q}Qt\,|P(\tau)|\mathbf{q}'Q't'\rangle \Big| \\
&= 0\ ,
\end{aligned} \tag{3.4.49}$$

for all $t' \leq \tau \leq t$. Since a relation $xg(x) = 0$, implies that $g(x)$ involves the factor $\delta(x)$, we note from Eqs. (3.4.48), (3.4.49) and (3.2.37), and finally from Eq. (3.2.6), by following a procedure as in deriving Eqs. (3.2.3) and (3.2.4), that

$$\frac{\partial}{\partial\lambda}\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle = -\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,\langle \mathbf{q}Qt\,|H(\mathbf{q},\mathbf{p})|\mathbf{q}'Q't'\rangle$$

$$= -\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,H'(\tau)\,\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle\ , \tag{3.4.50}$$

$$\delta\,\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle = -\frac{\mathrm{i}}{\hbar}\,\mathrm{d}\lambda\int_{t'}^{t}\mathrm{d}\tau\,H'(\tau)\,\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle\ , \tag{3.4.51}$$

which integrates out to

$$\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle\Big| = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,H'(\tau)\right]\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle_0^{\wedge}\Big|\,. \tag{3.4.52}$$

From Eq. (3.2.37)

$$\langle \mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle_0 = \delta^n\left(\mathbf{q}-\mathbf{q}'-\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{S}(\tau)\right)\exp\left(\frac{\mathrm{i}}{\hbar}\,\mathbf{q}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{F}(\tau)\right)$$
$$\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\,\mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\,\mathbf{F}(\tau')\right).$$

We note from the condition in Eqs. (3.4.48) and (3.4.49), that as in a relation $xg(x) = 0$, as applied to

$$\left[-\mathrm{i}\hbar\frac{\delta}{\delta f(\tau)}-(q_1(\tau)-q_2(\tau))\right]\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle\Big| = 0\,, \tag{3.4.53}$$

with the formal substitutions

$$\left[-\mathrm{i}\hbar\frac{\delta}{\delta f(\tau)}-(q_1(\tau)-q_2(\tau))\right] \longrightarrow x\ , \tag{3.4.54}$$

$$\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle \bigg| \quad \longrightarrow \quad g(x)\ . \tag{3.4.55}$$

On the other hand from Eq. (3.4.49),

$$\mathrm{i}\hbar\frac{\delta}{\delta s(\tau)}\,\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle \bigg| \;=\; 0\ , \tag{3.4.56}$$

we may make the formal substitutions

$$\mathrm{i}\hbar\frac{\delta}{\delta s(\tau)} \quad \longrightarrow \quad x\ , \tag{3.4.57}$$

$$\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle \bigg| \quad \longrightarrow \quad g(x)\ , \tag{3.4.58}$$

implying that $\langle \mathbf{q}Qt\,|\,\mathbf{q}'Q't'\rangle$ must involve delta functionals $\delta^{(1)}(-\mathrm{i}\hbar\delta/\delta f(\cdot) - G'(\cdot))\delta^{(1)}(\mathrm{i}\hbar\delta/(2\pi\hbar)\delta s(\cdot))$.

For the simple Hamiltonian

$$\tilde{H} = -Qf(\tau) + Ps(\tau)\ , \tag{3.4.59}$$

the Heisenberg equations are

$$\dot{Q}(\tau) \;=\; s(\tau)\ , \tag{3.4.60}$$

$$\dot{P}(\tau) \;=\; f(\tau)\ . \tag{3.4.61}$$

These equations may be integrated to

$$\int_\tau^t \mathrm{d}Q(\tau) \;=\; \int_\tau^t \mathrm{d}\tau\; s(\tau)\ , \tag{3.4.62}$$

$$Q(t) - Q(\tau) \;=\; \int_{t'}^t \mathrm{d}\tau'\;\Theta(\tau'-\tau)\;s(\tau')\ , \tag{3.4.63}$$

$$Q(\tau) \;=\; Q(t) - \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau' - \tau) \, s(\tau') \,, \tag{3.4.64}$$

and

$$\int_{t'}^{\tau} \mathrm{d}P(\tau) \;=\; \int_{t'}^{\tau} \mathrm{d}\tau \, f(\tau) \,, \tag{3.4.65}$$

$$P(\tau) - P(t') \;=\; \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau') \, f(\tau') \,, \tag{3.4.66}$$

$$P(\tau) \;=\; P(t') + \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau') \, f(\tau') \,, \tag{3.4.67}$$

and taking the matrix element between $\langle Q\, t|$ and $|P\, t'\rangle$ for $\lambda = 0$, we obtain

$$\langle Q\, t|Q(\tau)|P\, t'\rangle_0 \;=\; \left[Q(t) - \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau' - \tau) \, s(\tau') \right] \langle Q\, t|\, P\, t'\rangle_0 \;, \tag{3.4.68}$$

$$\langle Q\, t|P(\tau)|P\, t'\rangle_0 \;=\; \left[P(t') + \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau') \, f(\tau') \right] \langle Q\, t|\, P\, t'\rangle_0 \;. \tag{3.4.69}$$

We have used the relations

$${}_0\langle Q\, t|\, Q(t) \;=\; Q \, {}_0\langle Q\, t| \;, \tag{3.4.70}$$

$$P(t')\, |P\, t'\rangle_0 \;=\; P\, |P\, t'\rangle_0 \;, \tag{3.4.71}$$

for $\lambda = 0$ at coincident times. Eqs. (3.4.68) and (3.4.69) may be rewritten as

$$-\mathrm{i}\hbar \frac{\delta}{\delta f(\tau)} \langle Q\, t|\, P\, t'\rangle_0 \;=\; \left[Q - \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau' - \tau) \, s(\tau') \right] \langle Q\, t|\, P\, t'\rangle_0 \;, \tag{3.4.72}$$

$$\mathrm{i}\hbar \frac{\delta}{\delta s(\tau)} \langle Q\, t|\, P\, t'\rangle_0 \;=\; \left[P + \int_{t'}^{t} \mathrm{d}\tau' \, \Theta(\tau - \tau') \, f(\tau') \right] \langle Q\, t|\, P\, t'\rangle_0 \;. \tag{3.4.73}$$

These equations may be integrated to yield

$$-\mathrm{i}\hbar\frac{\delta}{\delta f(\tau)}\,\langle Q\,t\,|\,P\,t'\rangle_0 = \left[Q - \int_{t'}^{t} \mathrm{d}\tau'\;\Theta(\tau'-\tau)\;S(\tau')\right]\langle Q\,t\,|\,P\,t'\rangle_0\;, \tag{3.4.74}$$

$$\int \frac{\delta\,\langle Q\,t\,|\,P\,t'\rangle_0}{\langle Q\,t\,|\,P\,t'\rangle_0} = \frac{\mathrm{i}}{\hbar}\int \delta f(\tau)\left[Q - \int_{t'}^{t} \mathrm{d}\tau'\;\Theta(\tau'-\tau)\;s(\tau')\right], \tag{3.4.75}$$

$$\begin{aligned}\ln\langle Q\,t\,|\,P\,t'\rangle_0 &= \frac{\mathrm{i}}{\hbar}\,Q\int_{t'}^{t} f(\tau)\,\mathrm{d}\tau \\ &\quad - \frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\;f(\tau)\Theta(\tau'-\tau)\;s(\tau')\;,\end{aligned} \tag{3.4.76}$$

$$\begin{aligned}\langle Q\,t\,|\,P\,t'\rangle_0 &= E\;\exp\left[\frac{\mathrm{i}}{\hbar}\,Q\int_{t'}^{t} f(\tau)\,\mathrm{d}\tau\right] \\ &\quad\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\;f(\tau)\Theta(\tau'-\tau)\;s(\tau')\right],\end{aligned} \tag{3.4.77}$$

and

$$\mathrm{i}\hbar\frac{\delta}{\delta s(\tau)}\,\langle Q\,t\,|\,P\,t'\rangle_0 = \left[P + \int_{t'}^{t} \mathrm{d}\tau'\;\Theta(\tau-\tau')\;f(\tau')\right]\langle Q\,t\,|\,P\,t'\rangle_0\;, \tag{3.4.78}$$

$$\int \frac{\delta\,\langle Q\,t\,|\,P\,t'\rangle_0}{\langle Q\,t\,|\,P\,t'\rangle_0} = -\,\frac{\mathrm{i}}{\hbar}\int \delta s(\tau)\left[P + \int_{t'}^{t} \mathrm{d}\tau'\;\Theta(\tau-\tau')\;f(\tau')\right], \tag{3.4.79}$$

$$\begin{aligned}\ln\langle Q\,t\,|\,P\,t'\rangle_0 &= -\,\frac{\mathrm{i}}{\hbar}\,P\int_{t'}^{t} s(\tau)\,\mathrm{d}\tau \\ &\quad - \frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\;s(\tau)\Theta(\tau-\tau')\;f(\tau')\;,\end{aligned} \tag{3.4.80}$$

$$\begin{aligned}\langle Q\,t\,|\,P\,t'\rangle_0 &= F\;\exp\left[-\frac{\mathrm{i}}{\hbar}\,P\int_{t'}^{t} s(\tau)\,\mathrm{d}\tau\right] \\ &\quad\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\;s(\tau)\Theta(\tau-\tau')\;f(\tau')\right].\end{aligned} \tag{3.4.81}$$

To find E and F in Eqs. (3.4.77) and (3.4.81), respectively, we note that

$$\text{Eq.}(3.4.77) \ = \ \text{Eq.}(3.4.81) \tag{3.4.82}$$

and hence

$$E \ \exp\left[\frac{\mathrm{i}}{\hbar}\, Q \int_{t'}^{t} f(\tau)\, \mathrm{d}\tau\right] \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau'\, f(\tau)\Theta(\tau'-\tau)\, s(\tau')\right]$$

$$= \ F \ \exp\left[-\frac{\mathrm{i}}{\hbar}\, P \int_{t'}^{t} s(\tau)\, \mathrm{d}\tau\right] \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau'\, s(\tau)\Theta(\tau-\tau')\, f(\tau')\right] . \tag{3.4.83}$$

This gives,

$$E = \exp\left[-\frac{\mathrm{i}}{\hbar}\, P \int_{t'}^{t} s(\tau)\, \mathrm{d}\tau\right] \quad , \quad F = \exp\left[\frac{\mathrm{i}}{\hbar}\, Q \int_{t'}^{t} f(\tau)\, \mathrm{d}\tau\right] . \tag{3.4.84}$$

In the absence of the external sources: $f = 0$, $s = 0$, $\underline{\tilde{H}} \to 0$, Eq. (3.2.39) reduces to

$$\underline{\tilde{H}}(\tau, \lambda) \to \lambda H(\mathbf{q}, \mathbf{p}) - \mathbf{q}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau) \ . \tag{3.4.85}$$

For $t = t'$

$$\begin{aligned} \langle Q\, t \,|\, P\, t\rangle \ &= \ \big\langle Q \,\big|U(t)U^{\dagger}(t)\big|\, P\big\rangle \\ &= \ \langle Q \,|\, P\rangle \\ &= \ \exp\left(\frac{\mathrm{i}}{\hbar}\, QP\right) . \end{aligned} \tag{3.4.86}$$

Then we obtain

$$\langle Q\, t\,|\,P\, t'\rangle_0 = \exp\left[\frac{\mathrm{i}}{\hbar}\, Q \int_{t'}^{t} \mathrm{d}\tau\; f(\tau)\right] \exp\left[-\frac{\mathrm{i}}{\hbar}\, P \int_{t'}^{t} \mathrm{d}\tau\; s(\tau)\right] \exp\left(\frac{\mathrm{i}}{\hbar}\, QP\right)$$

$$\times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau'\; s(\tau)\Theta(\tau-\tau')\; f(\tau')\right]. \qquad (3.4.87)$$

To obtain the expression for $\langle Qt\,|\,Q't'\rangle_0$, we multiply Eq. (3.4.87) by $\langle Pt'\,|\,Q't'\rangle = \exp(-\mathrm{i}Q'P/\hbar)$ and integrate over P, with measure $\mathrm{d}P/(2\pi\hbar)$, to obtain

$$\begin{aligned}
\langle Q\, t\,|\,Q'\, t'\rangle_0 \;&=\; \int \frac{\mathrm{d}P}{(2\pi\hbar)}\, \langle Q\, t\,|\,P\, t'\rangle_0\, \langle P\, t'\,|\,Q'\, t'\rangle \\
&=\; \int \frac{\mathrm{d}P}{(2\pi\hbar)}\; \exp\left[\frac{\mathrm{i}}{\hbar}\, Q \int_{t'}^{t} \mathrm{d}\tau\; f(\tau)\right] \exp\left[-\frac{\mathrm{i}}{\hbar}\, P \int_{t'}^{t} \mathrm{d}\tau\; s(\tau)\right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau'\; s(\tau)\; \Theta(\tau-\tau')\; f(\tau')\right] \\
&\qquad \times \exp\left(\frac{\mathrm{i}}{\hbar}\, QP\right) \exp\left(-\frac{\mathrm{i}}{\hbar}\, Q'P\right), \\
&=\; \int \frac{\mathrm{d}P}{(2\pi\hbar)}\; \exp\left[\frac{\mathrm{i}}{\hbar}\, Q \int_{t'}^{t} \mathrm{d}\tau\; f(\tau)\right] \\
&\qquad \times \exp\left[\mathrm{i}\left(Q - Q' - \int_{t'}^{t} \mathrm{d}\tau\; s(\tau)\right)\frac{P}{\hbar}\right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau'\; s(\tau)\; \Theta(\tau-\tau')\; f(\tau')\right], \\
&=\; \delta\left(Q - Q' - \int_{t'}^{t} \mathrm{d}\tau\; s(\tau)\right) \exp\left[\frac{\mathrm{i}}{\hbar}\, Q \int_{t'}^{t} \mathrm{d}\tau\; f(\tau)\right] \\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau'\; s(\tau)\; \Theta(\tau-\tau')\; f(\tau')\right],
\end{aligned}$$

$$\langle Q\, t\,|\, Q'\, t'\rangle_0 \;=\; \delta\left(Q - Q' - \int_{t'}^{t} \mathrm{d}\tau\; s(\tau)\right) \exp\left(\frac{\mathrm{i}}{\hbar}\, Q \int_{t'}^{t} \mathrm{d}\tau\; f(\tau)\right)$$

$$\times \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau'\; s(\tau)\; \Theta(\tau - \tau')\; f(\tau')\right). \quad (3.4.88)$$

From $\langle \mathbf{q}\, t\,|\, \mathbf{q}' t'\rangle_0$ in equation Eq. (3.2.37), the conditions in Eqs. (3.4.48), (3.4.49), (3.4.53) - (3.4.58) and $\langle Qt\,|\, Q't'\rangle_0$ in Eq. (3.4.88), we then have

$$\langle \mathbf{q} Q t\,|\, \mathbf{q}' Q' t'\rangle_0^{\wedge}$$

$$= \delta^{(1)}\left[-\mathrm{i}\hbar \frac{\delta}{\delta f(\cdot)} - \left((-\mathrm{i}\hbar)\frac{\delta}{\delta F_1(\cdot)} - (-\mathrm{i}\hbar)\frac{\delta}{\delta F_2(\cdot)}\right)\right] \delta^{(1)}\left(\frac{\mathrm{i}\hbar}{2\pi\hbar}\frac{\delta}{\delta s(\cdot)}\right) \langle \mathbf{q}\, t\,|\, \mathbf{q}' t'\rangle_0\, A\,, \quad (3.4.89)$$

where

$$A \;=\; \delta\left(Q - Q' - \int_{t'}^{t} \mathrm{d}\tau\; s(\tau)\right) \exp\left(\frac{\mathrm{i}}{\hbar}\, Q \int_{t'}^{t} \mathrm{d}\tau\; f(\tau)\right)$$

$$\times \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{\tau} \mathrm{d}\tau'\; s(\tau) f(\tau)\right). \quad (3.4.90)$$

CHAPTER IV

CONSTRAINTS, DEPENDENT FIELDS AND THE QUANTUM DYNAMICAL PRINCIPLE: REDUCTION OF PHASE SPACE

4.1 Introduction

The approach in developing the dynamics of constrained systems in quantum physics, in the functional *differential* formalism via the application of the quantum dynamical principle, in this chapter is based on the following. Given a Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p})$ as a function of independent variables $\mathbf{q} = (q_1,\ldots,q_n)$ and their canonical conjugate momenta $\mathbf{p} = (p_1,\ldots,p_n)$, we consider a new system by defining constraint operator functions

$$\mathbf{G}(\mathbf{q}(\tau),\mathbf{p}(\tau)) \;=\; \big\{G_1(\mathbf{q}(\tau),\mathbf{p}(\tau),\ldots,G_k(\mathbf{q}(\tau),\mathbf{p}(\tau))\big\}\;, \tag{4.1.1}$$

as of pairwise commuting operator functions $G_j(\mathbf{q}(\tau),\mathbf{p}(\tau))$, which together we introduce canonical conjugate momenta for them

$$\hat{\mathbf{G}}(\mathbf{q}(\tau),\mathbf{p}(\tau)) \;=\; \big\{\hat{G}_1(\mathbf{q}(\tau),\mathbf{p}(\tau)),\ldots,\hat{G}_k(\mathbf{q}(\tau),\mathbf{p}(\tau))\big\}\;, \tag{4.1.2}$$

such that

$$\mathbf{G}(\mathbf{q}(\tau),\mathbf{p}(\tau)) \;=\; \mathbf{0}\,, \tag{4.1.3}$$

$$\hat{\mathbf{G}}(\mathbf{q}(\tau),\mathbf{p}(\tau)) \;=\; \mathbf{0}\,, \tag{4.1.4}$$

for *all* τ in the interval $[t', t]$.

The new Hamiltonian of the constrained dynamics is then defined by

$$H(\mathbf{q}^*, \mathbf{p}^*) \;=\; \tilde{H}(\mathbf{q}, \mathbf{p})\Big|_{\mathbf{G}=\mathbf{0}, \hat{\mathbf{G}}=\mathbf{0}} \,, \tag{4.1.5}$$

and Eqs. (4.1.3) and (4.1.4) define the constraints, and with $(\mathbf{q}, \mathbf{p}) \to (\mathbf{q}^*, \mathbf{p}^*, \mathbf{G}, \hat{\mathbf{G}})$ defining a canonical transformation, i.e., the Jacobian of the transformation is unity:

$$J \;=\; \left| \frac{\partial(\mathbf{q}, \mathbf{p})}{\partial(\mathbf{q}^*, \mathbf{p}^*, \mathbf{G}, \hat{\mathbf{G}})} \right| \;=\; 1 \,, \tag{4.1.6}$$

as obtained within a classical context.

The procedure for describing the dynamics of the new constrained dynamics with Hamiltonian $H(\mathbf{q}^*, \mathbf{p}^*)$ may be then summarized through the following:

$$\underbrace{\tilde{H}(\mathbf{q}, \mathbf{p})}_{\substack{\text{Phase Space of} \\ \text{dim } (2n)}} \qquad \longrightarrow \qquad \underbrace{\tilde{H}(\mathbf{q}, \mathbf{p})\Big|_{\mathbf{G}=\mathbf{0}, \hat{\mathbf{G}}=\mathbf{0}} \equiv H(\mathbf{q}^*, \mathbf{p}^*)}_{\substack{\text{Phase Space of} \\ \text{dim } (2(n-k))}} \,. \tag{4.1.7}$$

Given the Hamiltonian $\tilde{H}(\mathbf{q}, \mathbf{p})$ with the constraints Eqs. (4.1.3) and (4.1.4) now imposed, the transformation function $\langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_{\mathrm{C}}$, with the $\mathbf{q}$ (and similarly the $\mathbf{q}'$) not necessarily independent variables is then given by

$$\begin{aligned} \langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_{\mathrm{C}} \;=\; & \delta^{(k)}\left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{f}(\cdot)} - \mathbf{G}'(\cdot) \right) \delta^{(k)}\left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{f}(\cdot)} \right) \\ & \times \delta^{(k)}\left(\frac{1}{(2\pi\hbar)} \left(\mathrm{i}\hbar \, \frac{\delta}{\delta \mathbf{s}(\cdot)} - \hat{\mathbf{G}}'(\cdot) \right) \right) \delta^{(k)}\left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)} \frac{\delta}{\delta \mathbf{s}(\cdot)} \right) \\ & \times \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \; \tilde{H}'(\tau) \right) \langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle \, \langle \mathbf{Q}t \,|\, \mathbf{Q}'t' \rangle \,, \end{aligned} \tag{4.1.8}$$

where

$$\tilde{H}'(\tau) \;= \tilde{H}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}}\right), \tag{4.1.9}$$

$$\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle \;= \delta^k\left(\mathbf{q}-\mathbf{q}'-\int_{t'}^{t}\mathrm{d}\tau\;\mathbf{S}(\tau)\right)\exp\left(\frac{\mathrm{i}}{\hbar}\mathbf{q}\cdot\int_{t'}^{t}\mathrm{d}\tau\;\mathbf{F}(\tau)\right)$$

$$\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\;\mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\mathbf{F}(\tau')\right), \tag{4.1.10}$$

$$\langle \mathbf{Q}t\,|\,\mathbf{Q}'t'\rangle \;= \delta^k\left(\mathbf{Q}-\mathbf{Q}'-\int_{t'}^{t}\mathrm{d}\tau\;\mathbf{s}(\tau)\right)\exp\left(\frac{\mathrm{i}}{\hbar}\mathbf{Q}\cdot\int_{t'}^{t}\mathrm{d}\tau\;\mathbf{f}(\tau)\right)$$

$$\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\;\mathbf{s}(\tau)\cdot\Theta(\tau-\tau')\mathbf{f}(\tau')\right), \tag{4.1.11}$$

and the numericals $\mathbf{Q}$, $\mathbf{Q}'$ are defined as follows:

$$\mathbf{Q} \;=\; \mathbf{Q}^{\mathrm{c}}(\tau)\Big|_{\tau\to t} \;=\; \mathbf{0}\,, \tag{4.1.12}$$

$$\mathbf{Q}' \;=\; \mathbf{Q}^{\mathrm{c}}(\tau)\Big|_{\tau\to t'} \;=\; \mathbf{0}\,, \tag{4.1.13}$$

where $\mathbf{Q}^{\mathrm{c}}(\tau)$ is the classical function having the expression

$$\mathbf{Q}^{\mathrm{c}}(\tau) \;=\; \frac{\langle \mathbf{q}t\,|\mathbf{G}(\mathbf{q}(\tau),\mathbf{p}(\tau))|\mathbf{q}'t'\rangle}{\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle} \;=\; 0\,. \tag{4.1.14}$$

4.2 Transformation Functions

Given a Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p})$ with corresponding transformation functions $\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle$, we may implement the constraints Eqs. (4.1.3) and (4.1.4) as follows:

$$\mathbf{Q}^{\mathrm{c}}(\tau) \;=\; \frac{\langle \mathbf{q}t\,|\mathbf{G}(\mathbf{q}(\tau),\mathbf{p}(\tau))|\mathbf{q}'t'\rangle}{\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle} \;=\; 0\,, \tag{4.2.1}$$

$$\mathbf{P}^{\mathrm{c}}(\tau) \;=\; \frac{\left\langle \mathbf{q}t \,\middle|\, \hat{\mathbf{G}}(\mathbf{q}(\tau), \mathbf{p}(\tau)) \,\middle|\, \mathbf{q}'t' \right\rangle}{\langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle} \;=\; 0 \, , \tag{4.2.2}$$

for all τ in the interval of interest $[t', t]$.

Eqs. (4.2.1) and (4.2.2) suggest first to promote $\mathbf{Q}^{\mathrm{c}}$ and $\mathbf{P}^{\mathrm{c}}$ to quantum variables by $\mathbf{Q}$, $\mathbf{P}$ by coupling them to external sources $\mathbf{f}$ and $\mathbf{s}$ and specialize $\langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle$, accordingly, to a new transformation function $\langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_{\mathrm{C}}$, whose expression will be obtained below, such that

$$\langle \mathbf{q}t \,|\mathbf{Q}(\tau)|\mathbf{q}'t' \rangle_{\mathrm{C}} \;=\; (-\mathrm{i}\hbar)\frac{\delta}{\delta \mathbf{f}(\tau)} \, \langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_{\mathrm{C}}^{\wedge} \Big| \;=\; 0 \, , \tag{4.2.3}$$

$$\langle \mathbf{q}t \,|\mathbf{P}(\tau)|\mathbf{q}'t' \rangle_{\mathrm{C}} \;=\; (\mathrm{i}\hbar)\frac{\delta}{\delta \mathbf{s}(\tau)} \, \langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_{\mathrm{C}}^{\wedge} \Big| \;=\; 0 \, , \tag{4.2.4}$$

where the bar signs $\big|$, as usual, mean to set all the external sources equal to zero.

The conditions Eqs. (4.2.1) - (4.2.4), then mean that $\langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_{\mathrm{C}}^{\wedge}$ necessarily involve the product of the delta functionals as a factor:

$$\delta^{(k)}\left(\mathbf{G}'(\cdot) - (-\mathrm{i}\hbar)\frac{\delta}{\delta \mathbf{f}(\cdot)}\right)\delta^{(k)}\left((-\mathrm{i}\hbar)\frac{\delta}{\delta \mathbf{f}(\cdot)}\right)$$
$$\times\, \delta^{(k)}\left(\frac{1}{2\pi\hbar}\left(\hat{\mathbf{G}}'(\cdot) - (\mathrm{i}\hbar)\frac{\delta}{\delta \mathbf{s}(\cdot)}\right)\right)\delta^{(k)}\left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)}\frac{\delta}{\delta \mathbf{s}(\cdot)}\right), \tag{4.2.5}$$

with the transformations functions $\langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_{\mathrm{C}}$ for the constrained dynamics given by Eq. (4.1.8).

4.3 Contact with the Faddeev-Popov Technique

From the path integral representation of

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau \; \tilde{H}'(\tau)\right) \langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle \, \langle \mathbf{Q}t \,|\, \mathbf{Q}'t' \rangle \;\; , \tag{4.3.1}$$

given by

$$\int_{\mathbf{q}(t')=\mathbf{q}',\mathbf{Q}(t')=\mathbf{Q}'}^{\mathbf{q}(t)=\mathbf{q},\mathbf{Q}(t)=\mathbf{Q}} \mathscr{D}\left(\mathbf{q}(\cdot),\mathbf{p}(\cdot)\right)\mathscr{D}\left(\mathbf{Q}(\cdot),\mathbf{P}(\cdot)\right) \exp -\frac{\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau\, [\tilde{H}(\tau) - v(\tau)]\,, \quad (4.3.2)$$

where

$$\begin{aligned} v(\tau) &= [\mathbf{p}(\tau)\cdot\dot{\mathbf{q}}(\tau) + \mathbf{P}(\tau)\cdot\dot{\mathbf{Q}}(\tau) + \mathbf{q}(\tau)\cdot\mathbf{F}(\tau) \\ &\quad -\mathbf{p}(\tau)\cdot\mathbf{S}(\tau) + \mathbf{Q}(\tau)\cdot\mathbf{f}(\tau) - \mathbf{P}(\tau)\cdot\mathbf{s}(\tau)]\,, \qquad (4.3.3) \\ \tilde{H}(\tau) &= \tilde{H}(\mathbf{q}(\tau),\mathbf{p}(\tau))\,, \qquad (4.3.4) \end{aligned}$$

we have,

$$\begin{aligned} \langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle_{\mathrm{C}} = & \int_{\mathbf{q}(t')=\mathbf{q}',\mathbf{Q}(t')=\mathbf{Q}'}^{\mathbf{q}(t)=\mathbf{q},\mathbf{Q}(t)=\mathbf{Q}} \mathscr{D}\left(\mathbf{q}(\cdot),\mathbf{p}(\cdot)\right)\mathscr{D}\left(\mathbf{Q}(\cdot),\mathbf{P}(\cdot)\right) \\ & \times \delta^{(k)}(\mathbf{Q}(\cdot)-\mathbf{G}(\cdot))\, \delta^{(k)}\!\left(\frac{1}{(2\pi\hbar)}\Big(\mathbf{P}(\cdot)-\hat{\mathbf{G}}(\cdot)\Big)\right) \delta^{(k)}(\mathbf{Q}(\cdot)) \\ & \times \delta^{(k)}\!\left(\frac{1}{(2\pi\hbar)}\mathbf{P}(\cdot)\right) \exp -\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\, [\tilde{H}(\tau) - \mathbf{p}(\tau)\cdot\dot{\mathbf{q}}(\tau) - \mathbf{P}(\tau)\cdot\dot{\mathbf{Q}}(\tau) \\ & - \mathbf{q}(\tau)\cdot\mathbf{F}(\tau) + \mathbf{p}(\tau)\cdot\mathbf{S}(\tau) - \mathbf{Q}(\tau)\cdot\mathbf{f}(\tau) + \mathbf{P}(\tau)\cdot\mathbf{s}(\tau)]\Big|\,. \end{aligned} \qquad (4.3.5)$$

Integrating Eq. (4.3.5) over $\mathbf{Q}(\cdot)$ and $\mathbf{P}(\cdot)$, gives

$$\begin{aligned} \langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle_{\mathrm{C}} &= \int_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t)=\mathbf{q}} \mathscr{D}\left(\mathbf{q}(\cdot),\mathbf{p}(\cdot)\right)\, \delta^{(k)}(\mathbf{G}(\cdot))\, \delta^{(k)}\!\left(\frac{1}{(2\pi\hbar)}\,\hat{\mathbf{G}}(\cdot)\right) \\ &\quad \times \exp -\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\, [\tilde{H}(\tau) - \mathbf{p}(\tau)\cdot\dot{\mathbf{q}}(\tau)]\,. \qquad (4.3.6) \end{aligned}$$

On the other hand,

$$\tilde{H}(\tau)\Big|_{\mathbf{G}(\tau)=\mathbf{0},\hat{\mathbf{G}}(\tau)=\mathbf{0}} \;=\; H(\mathbf{q}^*(\tau), \mathbf{p}^*(\tau)) \;, \tag{4.3.7}$$

(see Eq. (4.1.7), and with $(\mathbf{q}, \mathbf{p}) \to (\mathbf{q}^*, \mathbf{p}^*; \mathbf{G}, \hat{\mathbf{G}})$ defining a canonical transformation, i.e., the Jacobian of the transformation is unity:

$$\left| \frac{\partial(\mathbf{q}, \mathbf{p})}{\partial(\mathbf{q}^*, \mathbf{p}^*, \mathbf{G}, \hat{\mathbf{G}})} \right| \;=\; 1 \;, \tag{4.3.8}$$

leading to the standard path integral expression for $\langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_{\mathrm{C}}$.

4.4 Application

As an explicit illustration following the procedure developed through Eqs. (4.1.8) - (4.2.2), consider a Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}^2) \;, \tag{4.4.1}$$

in 3D which is obviously rotationally invariant with the dynamics occurring in the three dimensional space, where V is arbitrary. Now suppose one is interested in developing the dynamics to be constrained to a fixed two-dimensional plane making a given angle α with the (q_1, q_3)-plane.

To do this, we introduce canonical conjugate variables which we eventually set equal to zero:

$$G_1 \;=\; q_1 \sin\alpha - q_2 \cos\alpha \;, \tag{4.4.2}$$

$$\hat{G}_1 \;=\; p_1 \sin\alpha - p_2 \cos\alpha \;. \tag{4.4.3}$$

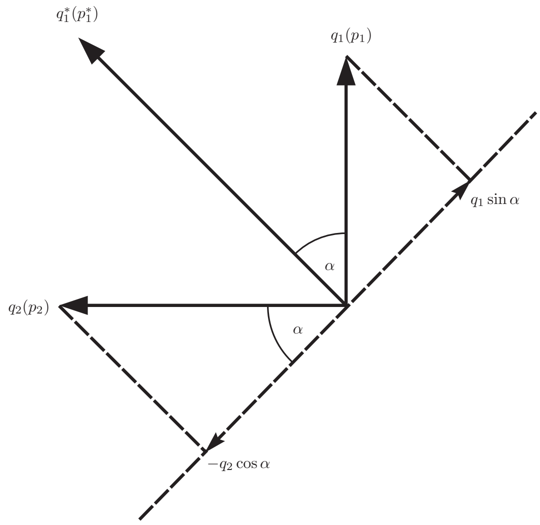

Figure 4.1 In 3D, the introduction of the constraint generates a two-dimensional plane (q_1, q_3)-plane with the parameter α, defining the angle between the $q_1^*, q_3(p_1^*, p_3)$ and $q_1, q_3(p_1, p_3)$ planes.

The commutation relation between G_1 and $\hat{G}_1$ is equal to i$\hbar$, i.e., in particular, we show that $[G_1, \hat{G}_1] \neq 0$, as follow from the commutation relations, $[q_a, p_a] = \mathrm{i}\hbar$ and $[q_a, p_b] = 0$ for $a \neq b$. To this end,

$$
\begin{aligned}
[G_1, \hat{G}_1] &= [(q_1 \sin\alpha - q_2 \cos\alpha), (p_1 \sin\alpha - p_2 \cos\alpha)] \\
&= (q_1 \sin\alpha - q_2 \cos\alpha)(p_1 \sin\alpha - p_2 \cos\alpha) \\
&\quad -(p_1 \sin\alpha - p_2 \cos\alpha)(q_1 \sin\alpha - q_2 \cos\alpha) \\
&= q_1 p_1 \sin^2\alpha - q_1 p_2 \sin\alpha\cos\alpha - q_2 p_1 \cos\alpha\sin\alpha + q_2 p_2 \cos^2\alpha \\
&\quad -(p_1 q_1 \sin^2\alpha - p_1 q_2 \sin\alpha\cos\alpha - p_2 q_1 \cos\alpha\sin\alpha + p_2 q_2 \cos^2\alpha) \\
&= (q_1 p_1 - p_1 q_1)\sin^2\alpha + (q_2 p_2 - p_2 q_2)\cos^2\alpha \\
&\quad -(q_2 p_1 - p_1 q_2)\sin\alpha\cos\alpha - (q_1 p_2 - p_2 q_1)\sin\alpha\cos\alpha
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{i}\hbar \sin^2\alpha + \mathrm{i}\hbar\cos^2\alpha \\
&= \mathrm{i}\hbar(\sin^2\alpha + \cos^2\alpha) \\
[G_1, \hat{G}_1] &= \mathrm{i}\hbar\,. \qquad (4.4.4)
\end{aligned}
$$

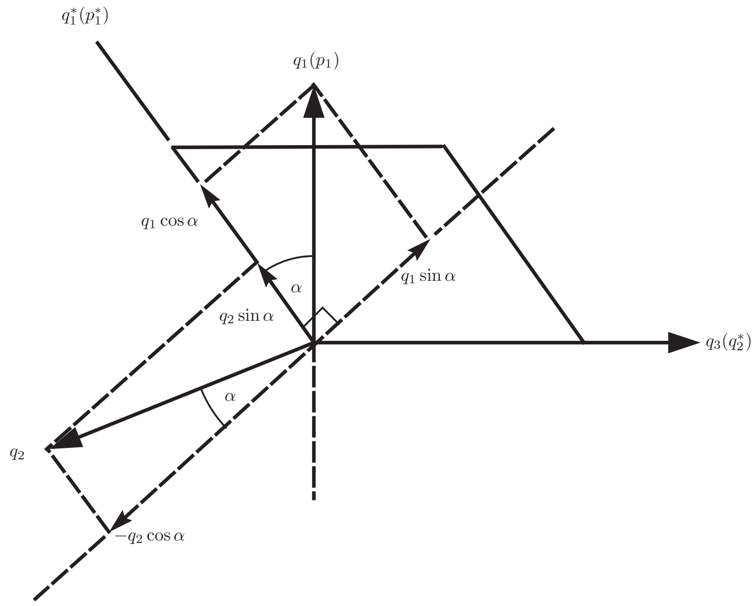

Figure 4.2 The projection of the (q_1, q_3)-plane, with defining angle α, into the (q_1^*, q_2^*)-plane.

The Hamiltonian of the dynamical system restricted to the two-dimensional plane described above is then given by

$$H(\mathbf{q}^*, \mathbf{p}^*) = \frac{\mathbf{p}^{*2}}{2m} + V(\mathbf{q}^{*2})\,, \qquad (4.4.5)$$

where $\mathbf{p}^* = (p_1^*, p_2^*)$, $\mathbf{q}^* = (q_1^*, q_2^*)$ with

$$p_1^* = p_1\cos\alpha + p_2\sin\alpha \quad , \quad p_2^* = p_3\,, \qquad (4.4.6)$$

$$q_1^* = q_1\cos\alpha + q_2\sin\alpha \quad , \quad q_2^* = q_3\,, \qquad (4.4.7)$$

and hence with the (q_1^*, q_2^*)-plane *making* an angle α with the (q_1, q_3)-plane corresponding to a rotation about the q_3-axis by the angle α.

Therefore it remains to show that the transformation

$$(q_1, p_1, q_2, p_2, q_3, p_3) \longrightarrow (q_1^*, p_1^*, q_2^*, p_2^*, G_1, \hat{G}_1) \tag{4.4.8}$$

is canonical. That is, the Jacobian

$$J = \left| \frac{\partial(q_1, p_1, q_2, p_2, q_3, p_3)}{\partial(q_1^*, p_1^*, q_2^*, p_2^*, G_1, \hat{G}_1)} \right| = 1 \, , \tag{4.4.9}$$

where $q_1^*, q_2^*, p_1^*, p_2^*, G_1, \hat{G}_1$ are defined, respectively, in Eqs. (4.4.7), (4.4.6), (4.4.2) and (4.4.3).

To the above end, we may solve for $(q_1, p_1, q_2, p_2, q_3, p_3)$ in terms of the variables $(q_1^*, p_1^*, q_2^*, p_2^*, G_1, \hat{G}_1)$ from just mentioned equations giving:

$$q_1 = q_1^* \cos\alpha + G_1 \sin\alpha \, , \tag{4.4.10}$$

$$p_1 = p_1^* \cos\alpha + \hat{G}_1 \sin\alpha \, , \tag{4.4.11}$$

$$q_2 = q_1^* \sin\alpha - G_1 \cos\alpha \, , \tag{4.4.12}$$

$$p_2 = p_1^* \sin\alpha - \hat{G}_1 \cos\alpha \, , \tag{4.4.13}$$

$$q_3 = q_2^* \, , \tag{4.4.14}$$

$$p_3 = p_2^* \, . \tag{4.4.15}$$

From the above equations, we then obtain,

$$\frac{\partial q_1}{\partial q_1^*} = \cos\alpha \, , \tag{4.4.16}$$

$$\frac{\partial q_1}{\partial p_1^*} = 0 \,, \tag{4.4.17}$$

$$\frac{\partial q_1}{\partial q_2^*} = 0 \,, \tag{4.4.18}$$

$$\frac{\partial q_1}{\partial p_2^*} = 0 \,, \tag{4.4.19}$$

$$\frac{\partial q_1}{\partial G_1} = \sin\alpha \,, \tag{4.4.20}$$

$$\frac{\partial q_1}{\partial \hat{G}_1} = 0 \,, \tag{4.4.21}$$

$$\frac{\partial q_2}{\partial q_1^*} = \sin\alpha \,, \tag{4.4.22}$$

$$\frac{\partial q_2}{\partial p_1^*} = 0 \,, \tag{4.4.23}$$

$$\frac{\partial q_2}{\partial q_2^*} = 0 \,, \tag{4.4.24}$$

$$\frac{\partial q_2}{\partial p_2^*} = 0 \,, \tag{4.4.25}$$

$$\frac{\partial q_2}{\partial G_1} = -\cos\alpha \,, \tag{4.4.26}$$

$$\frac{\partial q_2}{\partial \hat{G}_1} = 0 \,, \tag{4.4.27}$$

$$\frac{\partial q_3}{\partial q_1^*} = 0 \,, \tag{4.4.28}$$

$$\frac{\partial q_3}{\partial p_1^*} = 0 \,, \tag{4.4.29}$$

$$\frac{\partial q_3}{\partial q_2^*} = 1 \,, \tag{4.4.30}$$

$$\frac{\partial q_3}{\partial p_2^*} = 0 \,, \tag{4.4.31}$$

$$\frac{\partial q_3}{\partial G_1} = 0 \,, \tag{4.4.32}$$

$$\frac{\partial q_3}{\partial \hat{G}_1} = 0 \,, \tag{4.4.33}$$

$$\frac{\partial p_1}{\partial q_1^*} = 0 \,, \tag{4.4.34}$$

$$\frac{\partial p_1}{\partial p_1^*} = \cos\alpha \,, \tag{4.4.35}$$

$$\frac{\partial p_1}{\partial q_2^*} = 0 \,, \tag{4.4.36}$$

$$\frac{\partial p_1}{\partial p_2^*} = 0 \,, \tag{4.4.37}$$

$$\frac{\partial p_1}{\partial G_1} = 0 \,, \tag{4.4.38}$$

$$\frac{\partial p_1}{\partial \hat{G}_1} = \sin\alpha \,, \tag{4.4.39}$$

$$\frac{\partial p_2}{\partial q_1^*} = 0 \,, \tag{4.4.40}$$

$$\frac{\partial p_2}{\partial p_1^*} = \sin\alpha \,, \tag{4.4.41}$$

$$\frac{\partial p_2}{\partial q_2^*} = 0 \,, \tag{4.4.42}$$

$$\frac{\partial p_2}{\partial p_2^*} = 0 \,, \tag{4.4.43}$$

$$\frac{\partial p_2}{\partial G_1} = 0 \,, \tag{4.4.44}$$

$$\frac{\partial p_2}{\partial \hat{G}_1} = -\cos\alpha \, , \tag{4.4.45}$$

$$\frac{\partial p_3}{\partial q_1^*} = 0 \, , \tag{4.4.46}$$

$$\frac{\partial p_3}{\partial p_1^*} = 0 \, , \tag{4.4.47}$$

$$\frac{\partial p_3}{\partial q_2^*} = 0 \, , \tag{4.4.48}$$

$$\frac{\partial p_3}{\partial p_2^*} = 1 \, , \tag{4.4.49}$$

$$\frac{\partial p_3}{\partial G_1} = 0 \, , \tag{4.4.50}$$

$$\frac{\partial p_3}{\partial \hat{G}_1} = 0 \, . \tag{4.4.51}$$

The Jacobian of the transformation as defined in Eq. (4.4.9) is then given by

$$J = \left\| \begin{matrix} \frac{\partial q_1}{\partial q_1^*} & \frac{\partial q_1}{\partial p_1^*} & \cdots & \frac{\partial q_1}{\partial \hat{G}_1} \\ \frac{\partial p_1}{\partial p_1^*} & \frac{\partial p_1}{\partial p_1^*} & \cdots & \frac{\partial p_1}{\partial \hat{G}_1} \\ \vdots & & & \\ \frac{\partial p_3}{\partial q_1^*} & \frac{\partial p_3}{\partial p_1^*} & \cdots & \frac{\partial p_3}{\partial \hat{G}_1} \end{matrix} \right\| , \tag{4.4.52}$$

where $\| \cdot \|$ denotes the absolute value of the corresponding determinant $| \cdot |$.

Substituting Eqs. (4.4.16) - (4.4.51) into Eq. (4.4.52), we obtain

$$J = \left\| \begin{matrix} \cos\alpha & 0 & 0 & 0 & \sin\alpha & 0 \\ 0 & \cos\alpha & 0 & 0 & 0 & \sin\alpha \\ \sin\alpha & 0 & 0 & 0 & -\cos\alpha & 0 \\ 0 & \sin\alpha & 0 & 0 & 0 & -\cos\alpha \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix} \right\|, \tag{4.4.53}$$

Find the determinant of matrix J:

$$\begin{aligned} J &= \Big|(0)(-1)^{6+1}M_{61} + (0)(-1)^{6+2}M_{62} + (0)(-1)^{6+3}M_{63} \\ &\quad +(1)(-1)^{6+4}M_{64} + (0)(-1)^{6+5}M_{65} + (0)(-1)^{6+6}M_{66}\Big| \\ J &= \left|M_{64}\right|, \end{aligned} \tag{4.4.54}$$

and

$$M_{64} = \left| \begin{matrix} \cos\alpha & 0 & 0 & \sin\alpha & 0 \\ 0 & \cos\alpha & 0 & 0 & \sin\alpha \\ \sin\alpha & 0 & 0 & -\cos\alpha & 0 \\ 0 & \sin\alpha & 0 & 0 & -\cos\alpha \\ 0 & 0 & 1 & 0 & 0 \end{matrix} \right|, \tag{4.4.55}$$

$$\begin{aligned} J &= \Big|(0)(-1)^{5+1}N_{51} + (0)(-1)^{5+2}N_{52} + (1)(-1)^{5+3}N_{53} \\ &\quad +(0)(-1)^{5+4}N_{54} + (0)(-1)^{5+5}N_{55}\Big| \\ J &= \left|N_{53}\right|, \end{aligned} \tag{4.4.56}$$

and

$$N_{53} = \begin{vmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ \sin\alpha & 0 & -\cos\alpha & 0 \\ 0 & \sin\alpha & 0 & -\cos\alpha \end{vmatrix}, \tag{4.4.57}$$

$$J = \left|(0)(-1)^{4+1}O_{41} + \sin\alpha(-1)^{4+2}O_{42} + (0)(-1)^{4+3}O_{43} - \cos\alpha(-1)^{4+4}O_{44}\right|$$

$$J = \left|O_{42}\sin\alpha - O_{44}\cos\alpha\right| , \tag{4.4.58}$$

and

$$\begin{aligned} O_{42} &= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & \sin\alpha \\ \sin\alpha & -\cos\alpha & 0 \end{vmatrix}, \\ &= (0)(-1)^{2+1}P_{21} + (0)(-1)^{2+2}P_{22} + \sin\alpha(-1)^{2+3}P_{23} \end{aligned}$$

$$O_{42} = -P_{23}\sin\alpha , \tag{4.4.59}$$

$$\begin{aligned} O_{44} &= \begin{vmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & \cos\alpha & 0 \\ \sin\alpha & 0 & -\cos\alpha \end{vmatrix}, \\ &= (0)(-1)^{2+1}Q_{21} + \cos\alpha(-1)^{2+2}Q_{22} + (0)(-1)^{2+3}Q_{23} \end{aligned}$$

$$O_{44} = Q_{22}\cos\alpha , \tag{4.4.60}$$

Substituting Eqs. (4.4.59) and (4.4.60) into Eq. (4.4.58), we obtain

$$J = \Big| - P_{23}\sin^2\alpha - Q_{22}\cos^2\alpha \Big| \tag{4.4.61}$$

and

$$\begin{aligned}
P_{23} &= \begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix}, \\
&= -\cos^2\alpha - \sin^2\alpha\ , \\
&= -(\cos^2\alpha + \sin^2\alpha)\ , \\
P_{23} &= -1\ , \qquad (4.4.62) \\
Q_{22} &= \begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix}, \\
&= -\cos^2\alpha - \sin^2\alpha\ , \\
&= -(\cos^2\alpha + \sin^2\alpha)\ , \\
Q_{22} &= -1\ . \qquad (4.4.63)
\end{aligned}$$

Substituting Eqs. (4.4.62) and (4.4.63) into Eq. (4.4.61), we obtain

$$\begin{aligned}
J &= \Big| - (-1)\sin^2\alpha - (-1)\cos^2\alpha \Big|\ , \\
&= \sin^2\alpha + \cos^2\alpha\ , \\
J &= 1 \qquad (4.4.64)
\end{aligned}$$

thus confirming that the transformation in Eq. (4.4.8) is indeed canonical.

The power and simplicity of the functional *differential* formalism via the quantum dynamical principle evident. The constraints are implemented by functional differentiations, with respect to additional external sources, of transformation functions generalized to dynamical systems, in the presence of dependent degrees of freedom, which are readily spelled out from the corresponding systems with no constraints. We emphasize that in the functional *differential* treatment, via the action principle, external sources must, *a priori*, be introduced.

CHAPTER V

FUNCTIONAL CALCULUS AND DEPENDENT FIELDS

5.1 Maxwell's Lagrangian

The Lagrangian density for the photon field A^μ, in the presence of an external source J^μ, is given by

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu \ , \tag{5.1.1}$$

and the action for photon field is defined by the 4-dimensional integral:

$$\begin{aligned} \mathcal{W} &= \int (\mathrm{d}x)\, \mathscr{L} \\ &= \int (\mathrm{d}x) \left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu \right] , \end{aligned} \tag{5.1.2}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $\mu, \nu = 0, 1, 2, 3.$

5.2 The Coulomb Gauge

In the coulomb gauge, we have the constraint

$$\nabla \cdot \mathbf{A} \ \equiv \ \partial_k A^k \ = \ 0, \qquad k = 1, 2, 3 \ . \tag{5.2.1}$$

Equation (5.2.1) allows us to solve, for example, A^3 in terms of A^1, A^2:

$$\partial_k A^k \ = \ 0 \ ,$$

$$\partial_1 A^1 + \partial_2 A^2 + \partial_3 A^3 \;=\; 0 \,, \tag{5.2.2}$$

or

$$A^3 \;=\; -\frac{1}{\partial^3}\,\partial_a A^a, \qquad a = 1, 2\,. \tag{5.2.3}$$

For the variation of this field, we have

$$\delta A^3 \;=\; -\frac{1}{\partial^3}\,\partial_a(\delta A^a)\,, \tag{5.2.4}$$

while simply,

$$\delta A^a \;=\; \delta A^a\,. \tag{5.2.5}$$

We may combine Eqs. (5.2.4) and (5.2.5) into the form:

$$\delta A^i \;=\; \left(\delta^{ia} - \delta^{i3}\frac{\partial_a}{\partial_3}\right)\delta A^a, \qquad i = 1, 2, 3\,. \tag{5.2.6}$$

5.3 Modification of Maxwell's Equations for a Priori Non-Conserved Current

From Eq. (5.1.2):

$$\begin{aligned}
\delta\mathcal{W} &= \int (\mathrm{d}x)\left[-\frac{1}{4}\delta(F^{\mu\nu}F_{\mu\nu}) + J^\mu \delta A_\mu\right], \\
&= \int (\mathrm{d}x)\left[-\frac{1}{4}F^{\mu\nu}\delta F_{\mu\nu} - \frac{1}{4}F_{\mu\nu}\delta F^{\mu\nu} + J^\mu \delta A_\mu\right], \\
&= \int (\mathrm{d}x)\left[-\frac{1}{4}F^{\mu\nu}\delta(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{4}F_{\mu\nu}\delta(\partial^\mu A^\nu - \partial^\nu A^\mu) + J^\mu \delta A_\mu\right],
\end{aligned}$$

$$
\begin{aligned}
&= \int (\mathrm{d}x) \Big[-\frac{1}{4} F^{\mu\nu} \partial_\mu \delta A_\nu + \frac{1}{4} F^{\mu\nu} \partial_\nu \delta A_\mu - \frac{1}{4} F_{\mu\nu} \partial^\mu \delta A^\nu \\
&\qquad + \frac{1}{4} F_{\mu\nu} \partial^\nu \delta A^\mu + J^\mu \delta A_\mu \Big] , \\
&= \int (\mathrm{d}x) \Big[-\frac{1}{4} F^{\mu\nu} \partial_\mu \delta A_\nu + \frac{1}{4} F^{\nu\mu} \partial_\mu \delta A_\nu - \frac{1}{4} F_{\nu\mu} \partial^\nu \delta A^\mu \\
&\qquad + \frac{1}{4} F_{\mu\nu} \partial^\nu \delta A^\mu + J^\mu \delta A_\mu \Big] , \\
&= \int (\mathrm{d}x) \Big[-\frac{1}{4} F^{\mu\nu} \partial_\mu \delta A_\nu - \frac{1}{4} F^{\mu\nu} \partial_\mu \delta A_\nu + \frac{1}{4} F_{\mu\nu} \partial^\nu \delta A^\mu \\
&\qquad + \frac{1}{4} F_{\mu\nu} \partial^\nu \delta A^\mu + J^\mu \delta A_\mu \Big] , \\
&= \int (\mathrm{d}x) \left[-\frac{1}{2} F^{\mu\nu} \partial_\mu \delta A_\nu + \frac{1}{2} F_{\mu\nu} \partial^\nu \delta A^\mu + J^\mu \delta A_\mu \right] , \\
&= \int (\mathrm{d}x) \left[-\frac{1}{2} F^{\mu\nu} \partial_\mu \delta A_\nu + \frac{1}{2} F^{\mu\nu} \partial_\nu \delta A_\mu + J^\mu \delta A_\mu \right] , \\
&= \int (\mathrm{d}x) \left[-\frac{1}{2} F^{\mu\nu} \partial_\mu \delta A_\nu - \frac{1}{2} F^{\mu\nu} \partial_\mu \delta A_\nu + J^\mu \delta A_\mu \right] , \\
\delta \mathcal{W} &= \int (\mathrm{d}x) \left[-F^{\mu\nu} \partial_\mu \delta A_\nu + J^\mu \delta A_\mu \right] . \qquad (5.3.1)
\end{aligned}
$$

For the first term of the above equation, we use the fact that $A\delta B = \delta(AB) - (\delta A)B$, to obtain

$$
\begin{aligned}
\delta \mathcal{W} &= \int (\mathrm{d}x) \left[-\partial_\mu (F^{\mu\nu} \delta A_\nu) + (\partial_\mu F^{\mu\nu}) \delta A_\nu + J^\mu \delta A_\mu \right] , \qquad (5.3.2) \\
&= \oint \mathrm{d}\Sigma_\mu F^{\mu\nu} \delta A_\nu + \int (\mathrm{d}x) \left[(\partial_\mu F^{\mu\nu} + J^\nu) \delta A_\nu \right] , \qquad (5.3.3)
\end{aligned}
$$

and

$$\int (\mathrm{d}x)\partial_\mu(F^{\mu\nu}\delta A_\nu) \ , \tag{5.3.4}$$

is a surface term. Accordingly, we have

$$\delta\mathcal{W} \ = \ \int (\mathrm{d}x) \left[(\partial_\mu F^{\mu 0} + J^0)\, \delta A_0 + (\partial_\mu F^{\mu i} + J^i)\, \delta A_i\right] , \tag{5.3.5}$$

or

$$\delta\mathcal{W} \ = \ \int (\mathrm{d}x) \left[-(\partial_\mu F^{\mu 0} + J^0)\, \delta A^0 + (\partial_\mu F^{\mu i} + J^i)\, \delta A^i\right] . \tag{5.3.6}$$

Using Eq. (5.2.6), we get

$$\begin{aligned}
\delta\mathcal{W} \ &= \ \int (\mathrm{d}x) \left[-(\partial_\mu F^{\mu 0} + J^0)\, \delta A^0 + (\partial_\mu F^{\mu i} + J^i) \left(\delta^{ia} - \delta^{i3}\frac{\partial_a}{\partial_3}\right) \delta A^a\right] \\
&= \ \int (\mathrm{d}x)\left[- (\partial_\mu F^{\mu 0} + J^0)\, \delta A^0 + (\partial_\mu F^{\mu a} + J^a)\, \delta A^a - (\partial_\mu F^{\mu 3} + J^3)\frac{\partial_a}{\partial_3}\, \delta A^a\right] \\
&= \ -\int (\mathrm{d}x)(\partial_\mu F^{\mu 0} + J^0)\, \delta A^0 + \int (\mathrm{d}x)(\partial_\mu F^{\mu a} + J^a)\, \delta A^a \\
&\quad \ -\int (\mathrm{d}x)(\partial_\mu F^{\mu 3} + J^3)\frac{\partial_a}{\partial_3}\, \delta A^a \ .
\end{aligned} \tag{5.3.7}$$

Consider the third term given by:

$$\begin{aligned}
-\int (\mathrm{d}x)(\partial_\mu F^{\mu 3} + J^3)\frac{\partial_a}{\partial_3}\, \delta A^a \ &= \ -\int (\mathrm{d}x)\partial_a \left[(\partial_\mu F^{\mu 3} + J^3)\frac{1}{\partial_3}\, \delta A^a\right] \\
&\quad + \int (\mathrm{d}x) \left[\partial_a(\partial_\mu F^{\mu 3} + J^3)\right] \frac{1}{\partial_3}\, \delta A^a \\
&= \ -\oint \mathrm{d}\Sigma_a \left[(\partial_\mu F^{\mu 3} + J^3)\frac{1}{\partial_3}\, \delta A^a\right] \\
&\quad + \int (\mathrm{d}x) \left[\partial_a(\partial_\mu F^{\mu 3} + J^3)\right] \frac{1}{\partial_3}\, \delta A^a
\end{aligned}$$

$$
\begin{aligned}
&= \int (\mathrm{d}x) \left[\partial_a(\partial_\mu F^{\mu 3} + J^3)\right] \frac{1}{\partial_3}\, \delta A^a \\
&= \int (\mathrm{d}x) \left[\partial_3 \frac{\partial_a}{\partial_3}(\partial_\mu F^{\mu 3} + J^3)\right] \frac{1}{\partial_3}\, \delta A^a \\
&= \int (\mathrm{d}x) \partial_3 \left\{ \left[\frac{\partial_a}{\partial_3}(\partial_\mu F^{\mu 3} + J^3)\right] \frac{1}{\partial_3}\, \delta A^a \right\} \\
&\quad - \int (\mathrm{d}x) \left[\frac{\partial_a}{\partial_3}(\partial_\mu F^{\mu 3} + J^3)\right] \partial_3 \frac{1}{\partial_3}\, \delta A^a \\
&= \oint \mathrm{d}\Sigma_3 \left\{ \left[\frac{\partial_a}{\partial_3}(\partial_\mu F^{\mu 3} + J^3)\right] \frac{1}{\partial_3}\, \delta A^a \right\} \\
&\quad - \int (\mathrm{d}x) \left[\frac{\partial_a}{\partial_3}(\partial_\mu F^{\mu 3} + J^3)\right] \partial_3 \frac{1}{\partial_3}\, \delta A^a \,,
\end{aligned}
$$

$$
- \int (\mathrm{d}x)(\partial_\mu F^{\mu 3} + J^3) \frac{\partial_a}{\partial_3}\, \delta A^a = - \int (\mathrm{d}x) \left[\frac{\partial_a}{\partial_3}(\partial_\mu F^{\mu 3} + J^3)\right] \delta A^a \,. \tag{5.3.8}
$$

Substitute Eq. (5.3.8) into Eq. (5.3.7), to get

$$
\begin{aligned}
\delta \mathcal{W} &= - \int (\mathrm{d}x)(\partial_\mu F^{\mu 0} + J^0)\, \delta A^0 + \int (\mathrm{d}x)(\partial_\mu F^{\mu a} + J^a)\, \delta A^a \\
&\quad - \int (\mathrm{d}x) \left[\frac{\partial_a}{\partial_3}\, (\partial_\mu F^{\mu 3} + J^3)\right] \delta A^a \\
&= - \int (\mathrm{d}x)(\partial_\mu F^{\mu 0} + J^0)\, \delta A^0 \\
&\quad + \int (\mathrm{d}x) \left[(\partial_\mu F^{\mu a} + J^a) - \frac{\partial_a}{\partial_3}\, (\partial_\mu F^{\mu 3} + J^3)\right] \delta A^a \,,
\end{aligned} \tag{5.3.9}
$$

or

$$
\frac{\delta \mathcal{W}}{\delta A^0} = -(\partial_\mu F^{\mu 0} + J^0) \,, \tag{5.3.10}
$$

$$\frac{\delta \mathcal{W}}{\delta A^a} = (\partial_\mu F^{\mu a} + J^a) - \frac{\partial_a}{\partial_3}(\partial_\mu F^{\mu 3} + J^3) \,. \tag{5.3.11}$$

The field equations are then given by

$$-\partial_\mu F^{\mu 0} = J^0 \,, \tag{5.3.12}$$

$$-\partial_\mu F^{\mu a} = J^a - \frac{\partial_a}{\partial_3}(\partial_\mu F^{\mu 3} + J^3) \,. \tag{5.3.13}$$

From Eq. (5.2.6):

$$\delta A^i = \left(\delta^{ia} - \delta^{i3}\frac{\partial_a}{\partial_3}\right)\delta A^a \,,$$

$$\delta(\partial_\mu A^i) = \left(\delta^{ia} - \delta^{i3}\frac{\partial_a}{\partial_3}\right)\delta(\partial_\mu A^a) \,. \tag{5.3.14}$$

On the other hand, from Eq. (5.3.6) we have:

$$\begin{aligned}
\delta \mathcal{W} &= \int (\mathrm{d}x) \left[-(\partial_\mu F^{\mu 0} + J^0)\,\delta A^0 + (\partial_\mu F^{\mu i} + J^i)\,\delta A^i\right] \\
&= \int (\mathrm{d}x) \left[(-\partial_k F^{k0})\,\delta A^0 - J^0 \delta A^0 + J^i \delta A^i + (\partial_\mu F^{\mu i})\,\delta A^i\right] \\
&= \int (\mathrm{d}x) \left[F^{k0}\delta(\partial_k A^0) - \partial_k(F^{k0}\delta A^0) - J^0 \delta A^0 + J^i \delta A^i + (\partial_\mu F^{\mu i})\,\delta A^i\right] \\
&= \int (\mathrm{d}x) \left[F^{k0}\delta(\partial_k A^0) - J^0 \delta A^0 + J^i \delta A^i + (\partial_\mu F^{\mu i})\,\delta A^i\right] \\
\delta \mathcal{W} &= \int (\mathrm{d}x) \left[J^0 \delta A^0 + J^i \delta A^i - F^{k0}\delta(\partial_k A_0) + (\partial_\mu F^{\mu i})\,\delta A^i\right] .
\end{aligned} \tag{5.3.15}$$

Consider the second term in Eq. (5.3.15) and use Eq. (5.2.6) to obtain

$$\begin{aligned}
\int(\mathrm{d}x)J^i\delta A^i &= \int(\mathrm{d}x)J^i\left(\delta^{ia}-\delta^{i3}\frac{\partial_a}{\partial_3}\right)\,\delta A^a \\
&= \int(\mathrm{d}x)J^a\delta A^a - \int(\mathrm{d}x)J^3\frac{\partial_a}{\partial_3}\,\delta A^a \\
&= \int(\mathrm{d}x)J^a\delta A^a - \int(\mathrm{d}x)\,\partial_a\left(J^3\frac{1}{\partial_3}\,\delta A^a\right) + \int(\mathrm{d}x)(\partial_a J^3)\frac{1}{\partial_3}\,\delta A^a \\
&= \int(\mathrm{d}x)J^a\delta A^a + \int(\mathrm{d}x)(\partial_a J^3)\frac{1}{\partial_3}\,\delta A^a \\
&= \int(\mathrm{d}x)J^a\delta A^a + \int(\mathrm{d}x)\left(J^3\frac{1}{\partial_3}\,\delta A^a\right) - \int(\mathrm{d}x)\left(\frac{\partial_a}{\partial_3}\,J^3\right)\delta A^a\,, \\
\int(\mathrm{d}x)J^i\delta A^i &= \int(\mathrm{d}x)J^a\delta A^a - \int(\mathrm{d}x)\left(\frac{\partial_a}{\partial_3}\,J^3\right)\delta A^a\,. \qquad (5.3.16)
\end{aligned}$$

That is,

$$\int(\mathrm{d}x)J^i\delta A^i = \int(\mathrm{d}x)\left(J^a-\frac{\partial_a}{\partial_3}\,J^3\right)\delta A^a\,. \qquad (5.3.17)$$

Consider the last term in Eq. (5.3.9) and use Eq. (5.3.14) to derive

$$\begin{aligned}
\int(\mathrm{d}x)(\partial_\mu F^{\mu i})\,\delta A^i &= \int(\mathrm{d}x)\,\partial_\mu(F^{\mu i}\delta A^i) - \int(\mathrm{d}x)\,F^{\mu i}\delta(\partial_\mu A^i) \\
&= -\int(\mathrm{d}x)\,F^{\mu i}\delta(\partial_\mu A^i) \\
&= -\int(\mathrm{d}x)\,F^{\mu i}\left(\delta^{ia}-\delta^{i3}\frac{\partial_a}{\partial_3}\right)\delta(\partial_\mu A^a) \\
&= -\int(\mathrm{d}x)\,F^{\mu a}\delta(\partial_\mu A^a) + \int(\mathrm{d}x)\,F^{\mu 3}\frac{\partial_a}{\partial_3}\,\delta(\partial_\mu A^a) \\
&= -\int(\mathrm{d}x)\,F^{\mu a}\delta(\partial_\mu A^a) + \int(\mathrm{d}x)\,\partial_a\left[F^{\mu 3}\frac{1}{\partial_3}\,\delta(\partial_\mu A^a)\right] \\
&\quad -\int(\mathrm{d}x)(\partial_a F^{\mu 3})\frac{1}{\partial_3}\,\delta(\partial_\mu A^a)
\end{aligned}$$

$$
\begin{aligned}
&= -\int (\mathrm{d}x)\, F^{\mu a}\delta(\partial_\mu A^a) - \int (\mathrm{d}x)(\partial_a F^{\mu 3})\frac{1}{\partial_3}\,\delta(\partial_\mu A^a) \\
&= -\int (\mathrm{d}x)\, F^{\mu a}\delta(\partial_\mu A^a) - \int (\mathrm{d}x)\,\partial_a \left[F^{\mu 3}\frac{1}{\partial_3}\,\delta(\partial_\mu A^a)\right] \\
&\quad + \int (\mathrm{d}x)(\partial_a F^{\mu 3})\frac{1}{\partial_3}\,\delta(\partial_\mu A^a)\,, \\
\int (\mathrm{d}x)(\partial_\mu F^{\mu i})\,\delta A^i &= -\int (\mathrm{d}x)\, F^{\mu a}\delta(\partial_\mu A^a) + \int (\mathrm{d}x)\left(\frac{\partial_a}{\partial_3}F^{\mu 3}\right)\delta(\partial_\mu A^a)\,,
\end{aligned}
\tag{5.3.18}
$$

or

$$
\int (\mathrm{d}x)(\partial_\mu F^{\mu i})\,\delta A^i \;=\; \int (\mathrm{d}x)\left(-F^{\mu a} + \frac{\partial_a}{\partial_3}F^{\mu 3}\right)\delta(\partial_\mu A^a)\,. \tag{5.3.19}
$$

Upon substituting Eqs. (5.3.17) and (5.3.19) into Eq. (5.3.15) this leads to:

$$
\begin{aligned}
\delta\mathcal{W} &= \int (\mathrm{d}x)\Bigg[J^0\delta A_0 + \left(J^a - \frac{\partial_a}{\partial_3}\,J^3\right)\delta A^a - F^{k0}\delta(\partial_k A_0) \\
&\quad + \left(-F^{\mu a} + \frac{\partial_a}{\partial_3}\,F^{\mu 3}\right)\delta(\partial_\mu A^a)\Bigg]\,.
\end{aligned}
\tag{5.3.20}
$$

The canonical conjugate momenta to A^μ are defined by

$$
\pi_\mu \;\equiv\; \pi[A^\mu] \;=\; \frac{\delta\mathcal{W}}{\delta\dot{A}^\mu} \;=\; \frac{\delta\mathcal{W}}{\delta(\partial_0 A^\mu)}\,. \tag{5.3.21}
$$

For the photon field, A^0 is a dependent field since its canonical momentum vanishes ($\pi^0 = 0$),

$$
\pi^1 \;\equiv\; \pi[A^1] \;=\; -F^{01} + \frac{\partial_1}{\partial_3}\,F^{03}\,, \tag{5.3.22}
$$

$$
\pi^2 \;\equiv\; \pi[A^2] \;=\; -F^{02} + \frac{\partial_2}{\partial_3}\,F^{03}\,, \tag{5.3.23}
$$

or

$$\pi_a \;\equiv\; \pi[A^a] \;=\; -F^{0a} + \frac{\partial_a}{\partial_3}\,F^{03} \qquad , \qquad a = 1, 2\;. \tag{5.3.24}$$

But $\pi^3 = 0$ since A^3 is not a dynamical variable. Also

$$\partial^3 F^{0a} + \partial^0 F^{a3} + \partial^a F^{30} \;=\; 0 \tag{5.3.25}$$

$$-\partial^3 F^{0a} - \partial^a F^{30} \;=\; \partial^0 F^{a3} \tag{5.3.26}$$

$$-\partial^3 F^{0a} + \partial^a F^{03} \;=\; -\partial_0 F^{a3} \tag{5.3.27}$$

$$-F^{0a} + \frac{\partial_a}{\partial_3}\,F^{03} \;=\; -\frac{\partial_0}{\partial_3}\,F^{a3}\;. \tag{5.3.28}$$

Therefore

$$\pi^a \;=\; -F^{0a} + \frac{\partial_a}{\partial_3}\,F^{03} \;=\; -\frac{\partial_0}{\partial_3}\,F^{a3}\;, \tag{5.3.29}$$

which may be generalized from $a = 1, 2$ to $k = 1, 2, 3$ through

$$\pi^k \;=\; -F^{0k} + \frac{\partial_k}{\partial_3}\,F^{03} \;=\; -\frac{\partial_0}{\partial_3}\,F^{k3}\;, \tag{5.3.30}$$

giving, in particular, for $k = 3$, $\pi^3 = 0$, as expected.

Also $\pi^0 = 0$ since A^0 is a dependent field. On the other hand

$$\pi_\mu \;=\; -F^0{}_\mu + g_\mu{}^k \frac{\partial_k}{\partial_3}\,F^{03} \;=\; -g_\mu{}^k \frac{\partial_0}{\partial_3}\,F^{k3}\;, \tag{5.3.31}$$

for $\mu = 0, 1, 2, 3$, giving

$$\begin{aligned} \partial^\mu \pi_\mu \;\equiv\; \partial_\mu \pi^\mu \;&=\; -\partial_\mu F^{0\mu} + g^{\mu k} \frac{\partial_\mu \partial_k}{\partial_3}\,F^{03}\;, \\ &=\; \partial_\mu F^{\mu 0} + \frac{\partial^k \partial_k}{\partial_3}\,F^{03}\;, \end{aligned} \tag{5.3.32}$$

while from Eq. (5.3.12) we have

$$\partial_\mu \pi^\mu \;=\; -J^0 + \frac{\nabla^2}{\partial_3}\, F^{03}\,, \tag{5.3.33}$$

or

$$F^{03} \;=\; \frac{\partial_3 \partial_\mu}{\nabla^2}\, \pi^\mu + \frac{\partial_3}{\nabla^2}\, J^0\,. \tag{5.3.34}$$

Substitute Eq. (5.3.34) into Eq. (5.3.31) to get

$$\begin{aligned}
F^{0\mu} &= -\pi^\mu + g^{\mu k}\frac{\partial_k}{\partial_3}\, F^{03} \\
&= -\pi^\mu + g^{\mu k}\frac{\partial_k}{\partial_3}\left(\frac{\partial_3 \partial_\nu}{\nabla^2}\, \pi^\nu + \frac{\partial_3}{\nabla^2}\, J^0\right) \\
&= -\pi^\mu + g^{\mu k}\frac{\partial_k \partial_\nu}{\nabla^2}\, \pi^\nu + g^{\mu k}\frac{\partial_k}{\nabla^2}\, J^0\,, \\
F^{0\mu} &= -\left(g^{\mu\nu} - g^{\mu k}\frac{\partial_k \partial^\nu}{\nabla^2}\right)\pi_\nu + g^{\mu k}\frac{\partial_k}{\nabla^2}\, J^0\,,
\end{aligned} \tag{5.3.35}$$

where $\mu, \nu = 0, 1, 2, 3$, $k = 1, 2, 3$.

Since $\pi^0 = 0$ and $\pi^3 = 0$, we have $g^{\mu\nu}\pi_\nu = g^{\mu a}\pi^a$ and $\partial_\nu \pi^\nu = \partial_a \pi^a$. Equation (5.3.35) gives

$$\begin{aligned}
F^{0\mu} &= -\left(g^{\mu a} - \frac{\partial^\mu \partial_a}{\nabla^2}\right)\pi^a + \frac{\partial^\mu}{\nabla^2}\, J^0\,, \\
F^{0k} &= -\left(\delta^{ka} - \frac{\partial_k \partial_a}{\nabla^2}\right)\pi^a + \frac{\partial_k}{\nabla^2}\, J^0\,,
\end{aligned} \tag{5.3.36}$$

or

$$\left(g^{k\mu} - \frac{\partial_k \partial^\mu}{\nabla^2}\right)\pi_\mu \;=\; \left(\delta^{ka} - \frac{\partial_k \partial_a}{\nabla^2}\right)\pi_a \;=\; -F^{0k} + \frac{\partial_k}{\nabla^2}\, J^0\,. \tag{5.3.37}$$

We may generalize Eq. (5.3.35) into

$$F^{\mu\nu} = \frac{1}{2}\left[g^{\mu 0}\left(g^{\nu\alpha} - g^{\nu k}\frac{\partial_k\partial^\alpha}{\nabla^2}\right) - g^{\nu 0}\left(g^{\mu\alpha} - g^{\mu k}\frac{\partial_k\partial^\alpha}{\nabla^2}\right)\right]\pi_\alpha$$

$$+ \frac{1}{2}(\delta^\mu{}_k F^{k\nu} - \delta^\nu{}_k F^{k\mu}) - \frac{1}{2}(g^{\mu 0}g^{\nu k} - g^{\nu 0}g^{\mu k})\frac{\partial_k}{\nabla^2}\, J^0 \, . \tag{5.3.38}$$

Or using a notation $b^\mu \equiv \partial^{\mu k}\partial_k/\nabla^2$ for $b^0 \equiv 0$, this gives

$$F^{\mu\nu} = \frac{1}{2}\left[(g^{\mu 0}g^{\nu\alpha} - g^{\nu 0}g^{\mu\alpha}) - (g^{\mu 0}b^\nu - g^{\nu 0}b^\mu)\partial^\alpha\right]\pi_\alpha$$

$$+ \frac{1}{2}(\delta^\mu{}_k F^{k\nu} - \delta^\nu{}_k F^{k\mu}) - \frac{1}{2}(g^{\mu 0}b^\nu - g^{\nu 0}b^\mu)J^0 \, . \tag{5.3.39}$$

We may write

$$\begin{aligned} \delta^\mu{}_k F^{k\nu} &= \delta^\mu{}_\alpha F^{\alpha\nu} - \delta^\mu{}_0 F^{0\nu} \\ &= F^{\mu\nu} - \delta^\mu{}_0 F^{0\nu} \, , \end{aligned} \tag{5.3.40}$$

which gives

$$\begin{aligned} \delta^\nu{}_k F^{k\mu} &= \delta^\nu{}_\beta F^{\beta\mu} - \delta^\nu{}_0 F^{0\mu} \\ &= F^{\nu\mu} - \delta^\nu{}_0 F^{0\mu} \, . \end{aligned} \tag{5.3.41}$$

Now combine Eqs. (5.3.40) and (5.3.41), to obtain

$$\delta^\mu{}_k F^{k\nu} - \delta^\nu{}_k F^{k\mu} = F^{\mu\nu} - F^{\nu\mu} - \delta^\mu{}_0 F^{0\nu} + \delta^\nu{}_0 F^{0\mu} \tag{5.3.42}$$

$$\delta^\mu{}_k F^{k\nu} - \delta^\nu{}_k F^{k\mu} = 2F^{\mu\nu} + (g^{\mu 0}F^{0\nu} - g^{\nu 0}F^{0\mu}) \tag{5.3.43}$$

$$2F^{\mu\nu} = (\delta^{\mu}{}_{k}F^{k\nu} - \delta^{\nu}{}_{k}F^{k\mu}) - (g^{\mu 0}F^{0\nu} - g^{\nu 0}F^{0\mu}) \tag{5.3.44}$$

$$F^{\mu\nu} = \frac{1}{2}(\delta^{\mu}{}_{k}F^{k\nu} - \delta^{\nu}{}_{k}F^{k\mu}) - \frac{1}{2}(g^{\mu 0}F^{0\nu} - g^{\nu 0}F^{0\mu}) \, . \tag{5.3.45}$$

On the other hand, using Eq. (5.3.35), the latter leads to

$$\begin{aligned} g^{\mu 0}F^{0\nu} - g^{\nu 0}F^{0\mu} &= g^{\mu 0}\left[-\left(g^{\nu\alpha} - g^{\nu k}\frac{\partial_k \partial^\alpha}{\nabla^2}\right)\pi_\alpha + g^{\nu k}\frac{\partial_k}{\nabla^2}J^0\right] \\ & -g^{\nu 0}\left[-\left(g^{\mu\alpha} - g^{\mu k}\frac{\partial_k \partial^\alpha}{\nabla^2}\right)\pi_\alpha + g^{\mu k}\frac{\partial_k}{\nabla^2}J^0\right] \\ &= -\left[g^{\mu 0}\left(g^{\nu\alpha} - g^{\nu k}\frac{\partial_k \partial^\alpha}{\nabla^2}\right) - g^{\nu 0}\left(g^{\mu\alpha} - g^{\mu k}\frac{\partial_k \partial^\alpha}{\nabla^2}\right)\right]\pi_\alpha \\ & +(g^{\mu 0}g^{\nu k} - g^{\nu 0}g^{\mu k})\frac{\partial_k}{\nabla^2}J^0 \, . \end{aligned} \tag{5.3.46}$$

Substitute Eq. (5.3.45) into Eq. (5.3.46), to obtain the expression.

$$\begin{aligned} F^{\mu\nu} &= \frac{1}{2}\left[g^{\mu 0}\left(g^{\nu\alpha} - g^{\nu k}\frac{\partial_k \partial^\alpha}{\nabla^2}\right) - g^{\nu 0}\left(g^{\mu\alpha} - g^{\mu k}\frac{\partial_k \partial^\alpha}{\nabla^2}\right)\right]\pi_\alpha \\ & + \frac{1}{2}(\delta^{\mu}{}_{k}F^{k\nu} - \delta^{\nu}{}_{k}F^{k\mu}) - \frac{1}{2}(g^{\mu 0}g^{\nu k} - g^{\nu 0}g^{\mu k})\frac{\partial_k}{\nabla^2}J^0 \, . \end{aligned}$$

In Coulomb gauge $A^3 = -\frac{1}{\partial_3}(\partial_b A^b)$, $b = 1, 2$, and

$$\begin{aligned} \pi^a &= -\frac{\partial_0}{\partial_3}\, F^{a3} \\ &= -\frac{\partial_0}{\partial_3}\,(\partial^a A^3 - \partial^3 A^a) \\ &= \partial_0 A^a - \frac{\partial_0 \partial^a}{\partial_3}\, A^3 \\ &= \dot{A}^a - \frac{\partial_0 \partial^a}{\partial_3}\left(-\frac{1}{\partial_3}\,\partial_b A^b\right), \end{aligned}$$

$$\pi^a \;=\; \dot{A}^a + \frac{\partial^a \partial_b}{(\partial_3)^2}\, \dot{A}^b \;. \tag{5.3.47}$$

Or

$$\pi^a \;=\; \left[\delta^{ab} + \frac{\partial^a \partial_b}{(\partial_3)^2}\right] \dot{A}^b \qquad , \qquad a, b = 1, 2 \;. \tag{5.3.48}$$

For $c = 1, 2$

$$\begin{aligned}
\left(\delta^{ca} - \frac{\partial_c \partial_a}{\nabla^2}\right) \pi^a \;&=\; \left(\delta^{ca} - \frac{\partial_c \partial_a}{\nabla^2}\right) \left[\delta^{ab} + \frac{\partial^a \partial_b}{(\partial_3)^2}\right] \dot{A}^b \\
&=\; \left[\delta^{cb} + \frac{\partial^c \partial_b}{(\partial_3)^2} - \frac{\partial_c \partial^b}{\nabla^2} - \frac{\partial_c \partial_a}{\nabla^2} \frac{\partial^a \partial_b}{(\partial_3)^2}\right] \dot{A}^b \\
&=\; \left[\delta^{cb} + \frac{\partial_c \partial_b}{(\partial_3)^2} - \frac{\partial_c \partial_b}{\nabla^2} - \frac{\partial_a \partial_a}{(\partial_3)^2} \frac{\partial_c \partial_b}{\nabla^2}\right] \dot{A}^b \;.
\end{aligned} \tag{5.3.49}$$

By definition, $\nabla^2 = \partial_a \partial_a + (\partial_3)^2$, and we may write

$$\begin{aligned}
\left(\delta^{ca} - \frac{\partial_c \partial_a}{\nabla^2}\right) \pi^a \;&=\; \left[\delta^{cb} + \frac{\partial_c \partial_b}{(\partial_3)^2} - \frac{\partial_c \partial_b}{\nabla^2} - \left(\frac{\nabla^2 - (\partial_3)^2}{(\partial_3)^2}\right) \frac{\partial_c \partial_b}{\nabla^2}\right] \dot{A}^b \\
&=\; \left[\delta^{cb} + \frac{\partial_c \partial_b}{(\partial_3)^2} - \frac{\partial_c \partial_b}{\nabla^2} - \left(\frac{\nabla^2}{(\partial_3)^2} - 1\right) \frac{\partial_c \partial_b}{\nabla^2}\right] \dot{A}^b \\
&=\; \left[\delta^{cb} + \frac{\partial_c \partial_b}{(\partial_3)^2} - \left(1 + \frac{\nabla^2}{(\partial_3)^2} - 1\right) \frac{\partial_c \partial_b}{\nabla^2}\right] \dot{A}^b \\
&=\; \left[\delta^{cb} + \frac{\partial_c \partial_b}{(\partial_3)^2} - \frac{\nabla^2}{(\partial_3)^2} \frac{\partial_c \partial_b}{\nabla^2}\right] \dot{A}^b \\
&=\; \left[\delta^{cb} + \frac{\partial_c \partial_b}{(\partial_3)^2} - \frac{\partial_c \partial_b}{(\partial_3)^2}\right] \dot{A}^b \\
&=\; \delta^{cb} \dot{A}^b \;,
\end{aligned} \tag{5.3.50}$$

giving the useful expression

$$\left(\delta^{ca} - \frac{\partial_c \partial_a}{\nabla^2}\right) \pi^a \;\; = \;\; \dot{A}^c \, . \tag{5.3.51}$$

We use Eq. (5.3.36) to obtain

$$F^{0k} \;\; = \;\; -\left(\delta^{ka} - \frac{\partial_k \partial_a}{\nabla^2}\right) \pi^a + \frac{\partial_k}{\nabla^2} J^0 \, ,$$

$$F^{0b} \;\; = \;\; -\left(\delta^{ba} - \frac{\partial_b \partial_a}{\nabla^2}\right) \pi^a + \frac{\partial_b}{\nabla^2} J^0 \, ,$$

$$F^{0b} \;\; = \;\; -\dot{A}^b + \frac{\partial_b}{\nabla^2} J^0 \, , \tag{5.3.52}$$

$$\dot{A}^b \;\; = \;\; -F^{0b} + \frac{\partial_b}{\nabla^2} J^0 \qquad , \qquad a, b = 1, 2 \, . \tag{5.3.53}$$

The functional derivative of the vacuum-to-vacuum transition amplitude with respect to an external source J^μ is

$$-\mathrm{i} \frac{\delta}{\delta J_\mu(x)} \langle 0_+ | 0_- \rangle \; = \; \langle 0_+ | A^\mu(x) | 0_- \rangle \;\; , \tag{5.3.54}$$

and for any operator-valued (q-number) function $\mathscr{O}(x)$

$$-\mathrm{i} \frac{\delta}{\delta J_\nu(x')} \langle 0_+ | \mathscr{O}(x) | 0_- \rangle \; = \; \left\langle 0_+ \left| (\mathscr{O}(x) A^\nu(x'))_+ \right| 0_- \right\rangle + \left\langle 0_+ \left| -\mathrm{i} \frac{\delta}{\delta J_\nu(x')} \, \mathscr{O}(x) \right| 0_- \right\rangle \, . \tag{5.3.55}$$

In terms of the expectation value of the photon field $A^\mu(x)$:

$$\langle A^\mu(x) \rangle \; \equiv \; \frac{\langle 0_+ | A^\mu(x) | 0_- \rangle}{\langle 0_+ | 0_- \rangle} \, , \tag{5.3.56}$$

we have

$$\langle A^\mu(x) \rangle \; = \; \frac{1}{\langle 0_+ | 0_- \rangle} \left(-\mathrm{i} \frac{\delta}{\delta J_\mu(x)} \right) \langle 0_+ | 0_- \rangle \;\; . \tag{5.3.57}$$

From the field equations, Eqs. (5.3.12) and (5.3.13), we have

$$-\partial_\mu F^{\mu 0} \;=\; J^0 \;,$$

and

$$-\partial_\mu F^{\mu a} \;=\; J^a - \frac{\partial_a}{\partial_3}(\partial_\mu F^{\mu 3} + J^3) \;,$$

for $a = 1, 2$. But Eq. (5.3.13) is also true for a replaced by 3:

$$-\partial_\mu F^{\mu i} \;=\; J^i - \frac{\partial_i}{\partial_3}(\partial_\mu F^{\mu 3} + J^3) \qquad , \qquad i = 1, 2, 3 \;. \tag{5.3.58}$$

We may combine Eqs. (5.3.12) and (5.3.58) as follows

$$-\partial_\mu(F^\mu 0 + F^\mu i) \;=\; J^0 + J^i - \frac{\partial_i}{\partial_3}(\partial_\mu F^{\mu 3} + J^3) \tag{5.3.59}$$

$$-\partial_\mu F^{\mu\nu} \;=\; J^0 + J^i - \frac{\partial_i}{\partial_3}(\partial_\mu F^{\mu 3} + J^3) \tag{5.3.60}$$

$$-\partial_\nu\partial_\mu F^{\mu\nu} \;=\; \partial_0 J^0 + \partial_0 J^i - \frac{\partial_i\partial_i}{\partial_3}(\partial_\mu F^{\mu 3} + J^3) \tag{5.3.61}$$

$$-\partial_\nu\partial_\mu F^{\mu\nu} \;=\; \partial_\mu J^\mu - \frac{\nabla^2}{\partial_3}(\partial_\mu F^{\mu 3} + J^3) \;. \tag{5.3.62}$$

Using $\partial_\nu\partial_\mu F^{\mu\nu} = 0$, we also have

$$0 \;=\; \partial_\mu J^\mu - \frac{\nabla^2}{\partial_3}(\partial_\mu F^{\mu 3} + J^3) \;, \tag{5.3.63}$$

$$\partial_\mu J^\mu \;=\; \frac{\nabla^2}{\partial_3}(\partial_\mu F^{\mu 3} + J^3) \;\neq\; 0 \;, \tag{5.3.64}$$

giving

$$\frac{1}{\partial_3}(\partial_\mu F^{\mu 3} + J^3) \;=\; \frac{1}{\nabla^2}\,\partial_\mu J^\mu \;. \tag{5.3.65}$$

Substituting Eq. (5.3.65) into Eq. (5.3.58) leads to

$$-\partial_\mu F^{\mu i} \;=\; J^i - \frac{\partial_i}{\nabla^2}\,\partial_\mu J^\mu \;, \tag{5.3.66}$$

which may be rewritten as

$$-\partial_\mu F^{\mu i} \;=\; \left(g^{i\alpha} - \frac{\partial^i \partial^\alpha}{\nabla^2}\right) J_\alpha \qquad , \qquad i = 1,2,3 \;. \tag{5.3.67}$$

We may combining Eqs. (5.3.12) and (5.3.66) into one equation of the form

$$-\partial_\mu F^{\mu 0} - \partial_\mu F^{\mu i} \;=\; J^0 + \left(g^{i\alpha} - \frac{\partial^i \partial^\alpha}{\nabla^2}\right) J_\alpha \;, \tag{5.3.68}$$

$$-\partial_\mu F^{\mu\nu} \;=\; \left(g^{\nu\alpha} - g^{\nu i}\,\frac{\partial_i \partial^\alpha}{\nabla^2}\right) J_\alpha \;, \tag{5.3.69}$$

leading to the *modified Maxwell's equations for a priori non-conserved current* given by:

$$-\partial_\mu F^{\mu\nu} \;=\; \left(g^{\nu\alpha} - b^\nu \partial^\alpha\right) J_\alpha \;, \tag{5.3.70}$$

where

$$b^\nu \;=\; g^{\nu i}\,\frac{\partial_i}{\nabla^2} \;. \tag{5.3.71}$$

For a conserved current $\partial_\alpha J^\alpha = 0$, we obtain the well known Maxwell's equations:

$$-\partial_\mu F^{\mu\nu}\Big|_{\partial_\alpha J^\alpha = 0} \;=\; J^\nu \;. \tag{5.3.72}$$

In the Coulomb gauge ($\partial_k A^k = 0$),

$$\begin{aligned} -\partial_\mu F^{\mu\nu} &= -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= -\partial_\mu \partial^\mu A^\nu + \partial^\nu \partial_\mu A^\mu \\ &= -\partial_\mu \partial^\mu A^\nu + \partial^\nu(\partial_0 A^0 + \partial_k A^k) \\ -\partial_\mu F^{\mu\nu} &= -\Box A^\nu + \partial^\nu \partial_0 A^0 \,, \end{aligned} \tag{5.3.73}$$

where $\Box = (\partial_0)^2 + \nabla^2$. Equations (5.3.73) and (5.3.69) then give

$$-\Box A^\nu + \partial^\nu \partial_0 A^0 = \left(g^{\nu\alpha} - g^{\nu i}\,\frac{\partial_i \partial^\alpha}{\nabla^2}\right) J_\alpha \,. \tag{5.3.74}$$

For $\nu = 0$:

$$-\Box A^0 + \partial^0 \partial_0 A^0 = \left(g^{0\alpha} - g^{0i}\,\frac{\partial_i \partial^\alpha}{\nabla^2}\right) J_\alpha \,, \tag{5.3.75}$$

$$-\nabla^2 A^0 = J^0 \qquad \text{and} \qquad A^0 = \frac{1}{-\nabla^2}\, J^0 \,. \tag{5.3.76}$$

From Eq. (5.3.76), we obtain the time derivative of A^0 as

$$\partial_0 A^0 = -\frac{\partial_0}{\nabla^2}\, J^0 \,. \tag{5.3.77}$$

Substitute this into Eq. (5.3.74) to get

$$\begin{aligned} -\Box A^\nu &= \left(g^{\nu\alpha} - g^{\nu i}\frac{\partial_i \partial^\alpha}{\nabla^2}\right) J_\alpha - \partial^\nu \partial_0 A^0 \\ &= \left(g^{\nu\alpha} - g^{\nu i}\frac{\partial_i \partial^\alpha}{\nabla^2}\right) J_\alpha + \frac{\partial^\nu \partial_0}{\nabla^2} J^0 \end{aligned}$$

$$
\begin{aligned}
&= J^\nu - g^{\nu i}\frac{\partial_i\partial^\alpha}{\nabla^2}J_\alpha + \frac{\partial^\nu\partial_0}{\nabla^2}J^0 \\
&= J^\nu + \frac{\partial^\nu\partial^\alpha}{\nabla^2}\,J_\alpha - g^{\nu i}\frac{\partial_i\partial^\alpha}{\nabla^2}\,J_\alpha - \frac{\partial^\nu\partial^\alpha}{\nabla^2}\,J_\alpha + \frac{\partial^\nu\partial^0}{\nabla^2}\,J_0 \\
&= J^\nu + \frac{\partial^\nu\partial^\alpha}{\nabla^2}\,J_\alpha - g^{\nu i}\frac{\partial_i\partial^\alpha}{\nabla^2}J_\alpha - \frac{\partial^\nu\partial^i}{\nabla^2}\,J_i \\
&= J^\nu + \frac{\partial^\nu\partial^\alpha}{\nabla^2}\,J_\alpha - g^{\nu i}\frac{\partial_i\partial^\alpha}{\nabla^2}J_\alpha - g^{\alpha i}\frac{\partial^\nu\partial^i}{\nabla^2}\,J_\alpha \\
&= \left[g^{\nu\alpha} + \frac{\partial^\nu\partial^\alpha}{\nabla^2} - g^{\nu i}\frac{\partial_i\partial^\alpha}{\nabla^2} - g^{\alpha i}\frac{\partial_i\partial^\nu}{\nabla^2}\right]J_\alpha\,, \\
-\Box A^\nu &= \left(g^{\nu\lambda} - g^{\lambda i}\frac{\partial_i\partial^\nu}{\nabla^2}\right)g_{\lambda\sigma}\left(g^{\sigma\alpha} - g^{\sigma k}\frac{\partial_k\partial^\alpha}{\nabla^2}\right)J_\alpha\,. \qquad (5.3.78)
\end{aligned}
$$

Or for *a priori* non-conserved current we have:

$$
\begin{aligned}
-\Box A^\nu &= \left[g^{\nu\alpha} + \frac{\partial^\nu\partial^\alpha}{\nabla^2} - b^\nu\partial^\alpha - b^\alpha\partial^\nu\right]J_\alpha \\
&= (g^{\nu\lambda} - b^\lambda\partial^\nu)g_{\lambda\sigma}(g^{\sigma\alpha} - b^\sigma\partial^\alpha)\;J_\alpha\,. \qquad (5.3.79)
\end{aligned}
$$

On the other hand for $\partial_\alpha J^\alpha = 0$, we obtain

$$
-\Box A^\nu\Big|_{\partial_\alpha J^\alpha=0} = \left(g^{\nu\alpha} - g^{\alpha i}\frac{\partial_i\partial^\nu}{\nabla^2}\right)J_\alpha\,. \qquad (5.3.80)
$$

In particular, for $\nu = k = 1,2,3$ this specializes to

$$
\begin{aligned}
-\Box A^k &= \left[g^{k\alpha} + \frac{\partial^k\partial^\alpha}{\nabla^2} - g^{ki}\frac{\partial_i\partial^\alpha}{\nabla^2} - g^{\alpha i}\frac{\partial^k\partial_i}{\nabla^2}\right]J_\alpha \\
&= J^k + \frac{\partial^k\partial^\alpha}{\nabla^2}J_\alpha - \delta^k{}_i\frac{\partial^i\partial^\alpha}{\nabla^2}J_\alpha - \frac{\partial^k\partial_i}{\nabla^2}J^i \\
&= J^k - \frac{\partial^k\partial_i}{\nabla^2}J^i\,, \qquad (5.3.81)
\end{aligned}
$$

or

$$-\Box A^k = \left(\delta^{ki} - \frac{\partial_k \partial_i}{\nabla^2}\right) J^i \ . \tag{5.3.82}$$

Equation (5.3.78) gives

$$\begin{aligned} -\Box A^\nu &= \left[g^{\nu\alpha} + \frac{\partial^\nu \partial^\alpha}{\nabla^2} - g^{\nu i}\frac{\partial_i \partial^\alpha}{\nabla^2} - g^{\alpha i}\frac{\partial_i \partial^\nu}{\nabla^2}\right] J_\alpha \\ -\Box \partial^\mu A^\nu &= \left[g^{\nu\alpha}\partial^\mu + \frac{\partial^\mu \partial^\nu \partial^\alpha}{\nabla^2} - g^{\nu i}\frac{\partial_i \partial^\mu \partial^\alpha}{\nabla^2} - g^{\alpha i}\frac{\partial_i \partial^\mu \partial^\nu}{\nabla^2}\right] J_\alpha \ , \end{aligned} \tag{5.3.83}$$

and

$$\begin{aligned} -\Box A^\mu &= \left[g^{\mu\alpha} + \frac{\partial^\mu \partial^\alpha}{\nabla^2} - g^{\mu i}\frac{\partial_i \partial^\alpha}{\nabla^2} - g^{\alpha i}\frac{\partial_i \partial^\mu}{\nabla^2}\right] J_\alpha \\ -\Box \partial^\nu A^\mu &= \left[g^{\mu\alpha}\partial^\nu + \frac{\partial^\nu \partial^\mu \partial^\alpha}{\nabla^2} - g^{\mu i}\frac{\partial_i \partial^\nu \partial^\alpha}{\nabla^2} - g^{\alpha i}\frac{\partial_i \partial^\nu \partial^\mu}{\nabla^2}\right] J_\alpha \ . \end{aligned} \tag{5.3.84}$$

Equations (5.3.83) and (5.3.84), lead to

$$-\Box(\partial^\mu A^\nu - \partial^\nu A^\mu) = \left[(g^{\nu\alpha}\partial^\mu - g^{\mu\alpha}\partial^\nu) - g^{\nu i}\frac{\partial_i \partial^\mu \partial^\alpha}{\nabla^2} + g^{\mu i}\frac{\partial_i \partial^\nu \partial^\alpha}{\nabla^2}\right] J_\alpha \ , \tag{5.3.85}$$

$$\begin{aligned} -\Box F^{\mu\nu} &= \left[(g^{\nu\alpha}\partial^\mu - g^{\mu\alpha}\partial^\nu) - (g^{\nu i}\partial^\mu - g^{\mu i}\partial^\nu)\frac{\partial_i \partial^\alpha}{\nabla^2}\right] J_\alpha \\ &= (g^{\nu\lambda}\partial^\mu - g^{\mu\lambda}\partial^\nu) g_{\lambda\sigma}\left(g^{\sigma\alpha} - g^{\sigma i}\frac{\partial_i \partial^\alpha}{\nabla^2}\right) J_\alpha \ , \end{aligned} \tag{5.3.86}$$

or

$$\begin{aligned} -\Box F^{\mu\nu} &= \left[(g^{\nu\alpha}\partial^\mu - g^{\mu\alpha}\partial^\nu) - (b^\nu \partial^\mu - b^\mu \partial^\nu)\partial^\alpha\right] J_\alpha \\ &= (g^{\nu\lambda}\partial^\mu - g^{\mu\lambda}\partial^\nu) g_{\lambda\sigma}(g^{\sigma\alpha} - b^\sigma \partial^\alpha) J_\alpha \ . \end{aligned} \tag{5.3.87}$$

For $\partial_\alpha J^\alpha = 0$, we obtain

$$\Box F^{\mu\nu} \;=\; \partial^\mu J^\nu - \partial^\nu J^\mu \,, \tag{5.3.88}$$

and Eq. (5.3.79) gives

$$-\Box A^\mu(x) \;=\; \left[g^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{\nabla^2} - b^\mu\partial^\nu - b^\nu\partial^\mu\right] J_\nu(x) \,. \tag{5.3.89}$$

We may solve for $A^\mu(x)$ from Eq. (5.3.89) in the form:

$$A^\mu(x) \;=\; \int (\mathrm{d}x') D_+^{\mu\nu}(x,x') J_\nu(x') \,, \tag{5.3.90}$$

$$-\Box A^\mu(x) \;=\; \int (\mathrm{d}x') \left[-\Box D_+^{\mu\nu}(x,x')\right] \, J_\nu(x') \,. \tag{5.3.91}$$

Using the integral

$$\int (\mathrm{d}x') \delta^4(x-x') f(x') \;=\; f(x) \,, \tag{5.3.92}$$

we may rewrite Eq. (5.3.89) as

$$-\Box A^\mu(x) \;=\; \int (\mathrm{d}x') \delta^4(x-x') \left[g^{\mu\nu} + \frac{\partial'^\mu\partial'^\nu}{\nabla^2} - b'^\mu\partial'^\nu - b'^\nu\partial'^\mu\right] J_\nu(x') \,. \tag{5.3.93}$$

Upon comparing Eq. (5.3.91) and Eq. (5.3.93), lead to

$$D_+^{\mu\nu}(x,x') \;=\; \left[g^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{\nabla^2} - b^\mu\partial^\nu - b^\nu\partial^\mu\right] \frac{1}{-\Box - \mathrm{i}\epsilon}\, \delta^4(x-x') \,, \tag{5.3.94}$$

or

$$D_+^{\mu\nu}(x,x') \;=\; (g^{\mu\alpha - b^\alpha \partial^\mu}) g_{\alpha\beta} (g^{\beta\nu} - b^\beta \partial^\nu) \frac{1}{-\Box - \mathrm{i}\epsilon}\, \delta^4(x-x') \,. \tag{5.3.95}$$

We note that

$$\begin{aligned}
\partial_\mu D_+^{\mu\nu}(x,x') &= \left[\partial^\nu + \frac{\Box\partial^\nu}{\nabla^2} - \partial_\mu g^{\mu k}\frac{\partial_k\partial^\nu}{\nabla^2} - g^{\nu k}\frac{\partial_k}{\nabla^2}\Box\right]\frac{1}{-\Box - \mathrm{i}\epsilon}\,\delta^4(x-x') \\
&= \left[\partial^\nu + \frac{\Box\partial^\nu}{\nabla^2} - \frac{\partial^k\partial_k}{\nabla^2}\,\partial^\nu - g^{\nu k}\frac{\partial_k}{\nabla^2}\Box\right]\frac{1}{-\Box - \mathrm{i}\epsilon}\,\delta^4(x-x') \\
&= \frac{\Box}{\nabla^2}(\partial^\nu - g^{\nu k}\partial_k)\frac{1}{-\Box - \mathrm{i}\epsilon}\,\delta^4(x-x')\,, \qquad (5.3.96)
\end{aligned}$$

or

$$\begin{aligned}
\partial_\mu D_+^{\mu\nu}(x,x') &= -(g^{\nu\alpha}\partial_\alpha - g^{\nu k}\partial_k)\frac{1}{\nabla^2}\,\delta^4(x-x') \\
&= -g^{\nu 0}\frac{\partial_0}{\nabla^2}\,\delta^4(x-x')\,. \qquad (5.3.97)
\end{aligned}$$

Upon multiplying Eq. (5.3.97) by ∂_ν, we have

$$\begin{aligned}
\partial_\nu\partial_\mu D_+^{\mu\nu}(x,x') &= -g^{\nu 0}\partial_\nu\frac{\partial_0}{\nabla^2}\,\delta^4(x-x') \\
&= -\frac{\partial^0\partial_0}{\nabla^2}\,\delta^4(x-x') \\
&= \frac{(\partial_0)^2}{\nabla^2}\,\delta^4(x-x')\,. \qquad (5.3.98)
\end{aligned}$$

Let

$$\begin{aligned}
\mathscr{A} &\equiv \left[\int(\mathrm{d}x)(\mathrm{d}x')\;J_\mu(x)\;D_+^{\mu\nu}(x,x')\;J_\nu(x')\right]_{\partial_\alpha J^\alpha = 0} \\
&= \left[\int(\mathrm{d}x)(\mathrm{d}x')\;J_\mu(x)\left(g^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{\nabla^2} - b^\mu\partial^\nu - b^\nu\partial^\mu\right)\right. \\
&\qquad \left.\times\frac{1}{-\Box - \mathrm{i}\epsilon}\,\delta^4(x-x')\;J_\nu(x')\right]_{\partial_\alpha J^\alpha = 0}\,, \qquad (5.3.99)
\end{aligned}$$

using Eq. (5.3.92), we may rewrite $\mathscr{A}$ as

$$\begin{aligned}
\mathscr{A} &= \left[\int (\mathrm{d}x)\; J_\mu(x) \left(g^{\mu\nu} + \frac{\partial^\mu \partial^\nu}{\nabla^2} - b^\mu \partial^\nu - b^\nu \partial^\mu\right) \frac{1}{-\Box - \mathrm{i}\epsilon}\; J_\nu(x)\right]_{\partial_\alpha J^\alpha = 0} \\
&= \left[\int (\mathrm{d}x)\; J_\mu(x) \left(g^{\mu\nu} - b^\nu \partial^\mu\right) \frac{1}{-\Box - \mathrm{i}\epsilon}\; J_\nu(x)\right]_{\partial_\alpha J^\alpha = 0} \\
&= \left[\int (\mathrm{d}x)\; J_\mu(x) \left(g^{\mu\nu} - g^{\nu i}\frac{\partial_i \partial^\mu}{\nabla^2}\right) \frac{1}{-\Box - \mathrm{i}\epsilon}\; J_\nu(x)\right]_{\partial_\alpha J^\alpha = 0} \\
&= \int (\mathrm{d}x)\; J_\mu(x) \frac{g^{\mu\nu}}{-\Box - \mathrm{i}\epsilon}\; J_\nu(x)\;, \qquad (5.3.100)
\end{aligned}$$

or

$$\mathscr{A} = \int (\mathrm{d}x)\; J_\mu(x)\; D_+^{\mu\nu}(x,x')\; J_\nu(x)\;. \qquad (5.3.101)$$

Therefore $\mathscr{A}$ is a gauge invariant quantity

$$\left[\int (\mathrm{d}x)(\mathrm{d}x')\; J_\mu(x)\; D_+^{\mu\nu}(x,x')\; J_\nu(x')\right]_{\partial_\alpha J^\alpha = 0} = \int (\mathrm{d}x)\; J_\mu(x) \frac{g^{\mu\nu}}{-\Box - \mathrm{i}\epsilon}\; J_\nu(x)\;. \qquad (5.3.102)$$

5.4 The QDP, Dependent Fields and Canonical Commutation Relations

We consider the matrix element of $A^\mu(x)$ defined by

$$\langle A^\mu(x)\rangle = \frac{\langle 0_+|A^\mu(x)|0_-\rangle}{\langle 0_+|0_-\rangle}\;, \qquad (5.4.1)$$

with $A^\mu(x)$ obtained in Eq. (5.3.90). $\big\langle A^\mu(x)\big\rangle$ is then given by

$$\big\langle A^\mu(x)\big\rangle = \int (\mathrm{d}x')\; D_+^{\mu\nu}(x,x')\; J_\nu(x')\;, \qquad (5.4.2)$$

where $D_+^{\mu\nu}(x,x')$ is the exact photon propagator in the presence of an external source J^μ in the Coulomb gauge. Obviously, for the vacuum expectation value of the photon field $A^\mu(x)$ in the absence of an external source J^μ this gives

$$\big\langle A^\mu(x)\big\rangle_{J=0} \;=\; 0\,. \tag{5.4.3}$$

The functional derivative of the vacuum-to-vacuum transition amplitude $\langle 0_+\,|\,0_-\rangle$ with respect to $J_\mu(x)$ may be obtained from Eqs. (5.3.54) and (5.4.2) to give

$$\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\,\langle 0_+\,|\,0_-\rangle \;=\; \int(\mathrm{d}x')\;D_+^{\mu\nu}(x,x')\;J_\nu(x')\,\langle 0_+\,|\,0_-\rangle \tag{5.4.4}$$

$$\int\frac{\delta\,\langle 0_+\,|\,0_-\rangle}{\langle 0_+\,|\,0_-\rangle} \;=\; \mathrm{i}\int\delta J_\mu(x)\int(\mathrm{d}x')\;D_+^{\mu\nu}(x,x')\;J_\nu(x') \tag{5.4.5}$$

$$\ln\langle 0_+\,|\,0_-\rangle \;=\; \frac{\mathrm{i}}{2}\int(\mathrm{d}x')(\mathrm{d}x)\;D_+^{\mu\nu}(x,x')\;J_\nu(x') \tag{5.4.6}$$

$$\langle 0_+\,|\,0_-\rangle \;=\; \exp\left[\frac{\mathrm{i}}{2}\int(\mathrm{d}x')(\mathrm{d}x)\;D_+^{\mu\nu}(x,x')\;J_\nu(x')\right]. \tag{5.4.7}$$

In taking the functional derivative of, the photon field $A^i(x)$ with respect to J^μ, we have to keep $A^i(x)$ fixed, but not for the dependent field $A^0(x)$ which depends on J^0 through Eq. (5.3.76).

In particular, we obtain

$$\frac{\delta A^0(x)}{\mathrm{i}\delta J^0(x')} \;=\; \frac{\mathrm{i}}{\nabla^2}\,\delta^4(x-x')\,, \tag{5.4.8}$$

which generalizes to

$$\frac{\delta A^\mu(x)}{\mathrm{i}\delta J_\nu(x')} \;=\; -\mathrm{i}g^{\mu0}g^{\nu0}\frac{1}{\nabla^2}\,\delta^4(x-x')\,, \tag{5.4.9}$$

and their right-hand sides are both c-numbers.

From Eq. (5.3.55), we note that

$$\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\langle 0_+|A^\mu(x)|0_-\rangle \;=\; \langle 0_+|(A^\mu(x)A^\nu(x'))_+|0_-\rangle + \left\langle 0_+\left|\frac{\delta A^\mu(x)}{\mathrm{i}\delta J_\nu(x')}\right|0_-\right\rangle . \tag{5.4.10}$$

The expectation value of A^μ depend on J^μ:

$$\begin{aligned}\big\langle A^\mu(x)\big\rangle \;&=\; \int (\mathrm{d}x')\, D_+^{\mu\nu}(x,x')\, J_\nu(x') ,\\ \frac{\delta}{\mathrm{i}\delta J_\nu(x')}\big\langle A^\mu(x)\big\rangle\Big|_{J=0} \;&=\; -\mathrm{i}\, D_+^{\mu\nu}(x,x') .\end{aligned} \tag{5.4.11}$$

Equation (5.4.10), gives

$$\begin{aligned}\frac{1}{\langle 0_+|0_-\rangle}\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\langle 0_+|A^\mu(x)|0_-\rangle &= \frac{\langle 0_+|(A^\mu(x)A^\nu(x'))_+|0_-\rangle}{\langle 0_+|0_-\rangle}\\ &\quad + \frac{\left\langle 0_+\left|\frac{\delta A^\mu(x)}{\mathrm{i}\delta J_\nu(x')}\right|0_-\right\rangle}{\langle 0_+|0_-\rangle}\\ &= \big\langle (A^\mu(x)A^\nu(x'))_+\big\rangle + \left\langle\frac{\delta A^\mu(x)}{\mathrm{i}\delta J_\nu(x')}\right\rangle ,\end{aligned} \tag{5.4.12}$$

and we note that

$$\begin{aligned}\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\big\langle A^\mu(x)\big\rangle \;&=\; \frac{\delta}{\mathrm{i}\delta J_\nu(x')}\frac{\langle 0_+|A^\mu(x)|0_-\rangle}{\langle 0_+|0_-\rangle}\\ &=\; \frac{1}{\langle 0_+|0_-\rangle}\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\langle 0_+|A^\mu(x)|0_-\rangle\\ &\quad -\frac{\langle 0_+|A^\mu(x)|0_-\rangle}{\langle 0_+|0_-\rangle}\frac{1}{\langle 0_+|0_-\rangle}\underbrace{\frac{\delta}{\mathrm{i}\delta J_\nu(x')}\langle 0_+|0_-\rangle}_{=\langle 0_+|A^\nu(x')|0_-\rangle}\end{aligned}$$

$$= \frac{1}{\langle 0_+ | 0_- \rangle} \frac{\delta}{\mathrm{i}\delta J_\nu(x')} \langle 0_+ | A^\mu(x) | 0_- \rangle$$

$$- \frac{\langle 0_+ | A^\mu(x) | 0_- \rangle}{\langle 0_+ | 0_- \rangle} \frac{\langle 0_+ | A^\nu(x') | 0_- \rangle}{\langle 0_+ | 0_- \rangle} ,$$

$$\frac{\delta}{\mathrm{i}\delta J_\nu(x')} \left\langle A^\mu(x) \right\rangle = \frac{1}{\langle 0_+ | 0_- \rangle} \frac{\delta}{\mathrm{i}\delta J_\nu(x')} \langle 0_+ | A^\mu(x) | 0_- \rangle$$

$$- \left\langle A^\mu(x) \right\rangle \left\langle A^\nu(x') \right\rangle . \quad (5.4.13)$$

Substituting Eq. (5.4.12) into Eq. (5.4.13), leads to

$$\frac{\delta}{\mathrm{i}\delta J_\nu(x')} \left\langle A^\mu(x) \right\rangle = \left\langle (A^\mu(x) A^\nu(x'))_+ \right\rangle + \left\langle \frac{\delta A^\mu(x)}{\mathrm{i}\delta J_\nu(x')} \right\rangle - \left\langle A^\mu(x) \right\rangle \left\langle A^\nu(x') \right\rangle . \quad (5.4.14)$$

For $J = 0$, $\left\langle A^\mu(x) \right\rangle_{J=0} = 0$, this specializes to

$$\frac{\delta}{\mathrm{i}\delta J_\nu(x')} \left\langle A^\mu(x) \right\rangle = \left\langle (A^\mu(x) A^\nu(x'))_+ \right\rangle_{J=0} + \left\langle \frac{\delta A^\mu(x)}{\mathrm{i}\delta J_\nu(x')} \right\rangle_{J=0} . \quad (5.4.15)$$

Equation (5.4.9), consistently gives

$$D_+^{\mu\nu}(x, x') = \mathrm{i} \left\langle (A^\mu(x) A^\nu(x'))_+ \right\rangle_{J=0} + g^{\mu 0} g^{\nu 0} \frac{1}{\nabla^2} \delta^4(x - x') . \quad (5.4.16)$$

From Eq. (5.3.86), we may write

$$-\Box F^{\mu\nu} = \left[(g^{\nu\alpha} \partial^\mu - g^{\mu\alpha} \partial^\nu) - (g^{\nu i} \partial^\mu - g^{\mu i} \partial^\nu) \tfrac{\partial_i \partial^\alpha}{\nabla^2} \right] J_\alpha , \quad (5.4.17)$$

or for its matrix element $\langle 0_+ | \cdot | 0_- \rangle$, we have

$$-\Box \langle 0_+ | F^{\mu\nu} | 0_- \rangle = \left[(g^{\nu\alpha} \partial^\mu - g^{\mu\alpha} \partial^\nu) - (g^{\nu i} \partial^\mu - g^{\mu i} \partial^\nu) \frac{\partial_i \partial^\alpha}{\nabla^2} \right] J_\alpha \langle 0_+ | 0_- \rangle , \quad (5.4.18)$$

$$\langle 0_+ | F^{\mu\nu} | 0_- \rangle = \frac{1}{-\Box - \mathrm{i}\epsilon} \left[(g^{\nu\alpha}\partial^\mu - g^{\mu\alpha}\partial^\nu) - (g^{\nu i}\partial^\mu - g^{\mu i}\partial^\nu) \frac{\partial_i \partial^\alpha}{\nabla^2} \right] J_\alpha \langle 0_+ | 0_- \rangle$$

$$= \mathcal{D}^{\mu\nu\alpha} J_\alpha(x) \langle 0_+ | 0_- \rangle \ , \tag{5.4.19}$$

where

$$\mathcal{D}^{\mu\nu\alpha} \equiv \frac{1}{-\Box - \mathrm{i}\epsilon} \left[(g^{\nu\alpha}\partial^\mu - g^{\mu\alpha}\partial^\nu) - (g^{\nu i}\partial^\mu - g^{\mu i}\partial^\nu) \frac{\partial_i \partial^\alpha}{\nabla^2} \right]. \tag{5.4.20}$$

Accordingly,

$$\frac{\langle 0_+ | F^{\mu\nu} | 0_- \rangle}{\langle 0_+ | 0_- \rangle} = \mathcal{D}^{\mu\nu\alpha} J_\alpha(x) = \left\langle F^{\mu\nu}(x) \right\rangle , \tag{5.4.21}$$

and

$$\frac{\delta}{\mathrm{i}\delta J_\nu(x')} \left\langle F^{\mu\nu}(x) \right\rangle = \frac{\mathcal{D}^{\mu\nu\alpha}}{\mathrm{i}} \delta^4(x - x')$$

$$= -\mathrm{i}\mathcal{D}^{\mu\nu\alpha} \delta^4(x - x') \, . \tag{5.4.22}$$

Equation (5.3.55), also gives

$$\frac{\delta}{\mathrm{i}\delta J_\nu(x')} \langle 0_+ | F^{\mu\nu}(x) | 0_- \rangle = \langle 0_+ | (F^{\mu\nu}(x) A^\nu(x'))_+ | 0_- \rangle$$

$$+ \left\langle 0_+ \left| \frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_\nu(x')} \right| 0_- \right\rangle , \tag{5.4.23}$$

$$\frac{1}{\langle 0_+ | 0_- \rangle} \frac{\delta}{\mathrm{i}\delta J_\nu(x')} \langle 0_+ | F^{\mu\nu}(x) | 0_- \rangle = \left\langle (F^{\mu\nu}(x) A^\nu(x'))_+ \right\rangle + \left\langle \frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_\nu(x')} \right\rangle . \tag{5.4.24}$$

In terms of the matrix element $\langle 0_+ | \cdot | 0_- \rangle \,/\, \langle 0_+ | 0_- \rangle$ of $F^{\mu\nu}$, we have

$$\begin{aligned}
\frac{\delta}{\mathrm{i}\delta J_\nu(x')} \left\langle F^{\mu\nu}(x) \right\rangle &= \frac{\delta}{\mathrm{i}\delta J_\nu(x')} \frac{\left\langle 0_+ \middle| F^{\mu\nu(x)} \middle| 0_- \right\rangle}{\langle 0_+ | 0_- \rangle} \\
&= \frac{1}{\langle 0_+ | 0_- \rangle} \frac{\delta}{\mathrm{i}\delta J_\nu(x')} \langle 0_+ | F^{\mu\nu}(x) | 0_- \rangle \\
&\quad - \frac{\left\langle 0_+ \middle| F^{\mu\nu(x)} \middle| 0_- \right\rangle}{\langle 0_+ | 0_- \rangle} \frac{1}{\langle 0_+ | 0_- \rangle} \underbrace{\frac{\delta}{\mathrm{i}\delta J_\nu(x')} \langle 0_+ | 0_- \rangle}_{= \langle 0_+ | A^\nu(x') | 0_- \rangle} ,
\end{aligned}$$

$$\begin{aligned}
\frac{\delta}{\mathrm{i}\delta J_\nu(x')} \left\langle F^{\mu\nu}(x) \right\rangle &= \frac{1}{\langle 0_+ | 0_- \rangle} \frac{\delta}{\mathrm{i}\delta J_\nu(x')} \langle 0_+ | F^{\mu\nu}(x) | 0_- \rangle \\
&\quad - \left\langle F^{\mu\nu}(x) \right\rangle \left\langle A^\nu(x') \right\rangle , \qquad (5.4.25)
\end{aligned}$$

or

$$\frac{1}{\langle 0_+ | 0_- \rangle} \frac{\delta}{\mathrm{i}\delta J_\nu(x')} \langle 0_+ | F^{\mu\nu}(x) | 0_- \rangle = \frac{\delta}{\mathrm{i}\delta J_\nu(x')} \left\langle F^{\mu\nu}(x) \right\rangle + \left\langle F^{\mu\nu}(x) \right\rangle \left\langle A^\nu(x') \right\rangle . \tag{5.4.26}$$

Therefore by equating Eq. (5.4.24) and Eq. (5.4.26), it follows that

$$\left\langle (F^{\mu\nu}(x) A^\nu(x'))_+ \right\rangle + \left\langle \frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_\nu(x')} \right\rangle = \frac{\delta}{\mathrm{i}\delta J_\nu(x')} \left\langle F^{\mu\nu}(x) \right\rangle + \left\langle F^{\mu\nu}(x) \right\rangle \left\langle A^\nu(x') \right\rangle , \tag{5.4.27}$$

$$\begin{aligned}
\left\langle (F^{\mu\nu}(x) A^\nu(x'))_+ \right\rangle &= \frac{\delta}{\mathrm{i}\delta J_\nu(x')} \left\langle F^{\mu\nu}(x) \right\rangle - \left\langle \frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_\nu(x')} \right\rangle \\
&\quad + \left\langle F^{\mu\nu}(x) \right\rangle \left\langle A^\nu(x') \right\rangle . \qquad (5.4.28)
\end{aligned}$$

In particular, for $J = 0$

$$\left\langle (F^{\mu\nu}(x)A^{\nu}(x'))_{+} \right\rangle_{J=0} = \left[\frac{\delta}{\mathrm{i}\delta J_{\nu}(x')} \left\langle F^{\mu\nu}(x) \right\rangle \right]_{J=0} - \left\langle \frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_{\nu}(x')} \right\rangle_{J=0} . \quad (5.4.29)$$

In the Coulomb gauge $A^3 = -\dfrac{1}{\partial_3}(\partial_b A^b)$ with a sum over $b = 1, 2$, and from Eq. (5.3.31) we may write

$$\begin{aligned} \pi^{\mu} &= -g^{\mu k} \frac{\partial_0}{\partial_3} F^{k3} \\ &= -g^{\mu k} \frac{\partial_0}{\partial_3} (\partial^k A^3 - \partial^3 A^k) \\ &= g^{\mu k} \partial_0 A^k - g^{\mu k} \frac{\partial_0 \partial^k}{\partial_3} A^3 \\ &= g^{\mu k} \dot{A}^k + g^{\mu k} \frac{\partial_0 \partial^k}{\partial_3} \left(\frac{\partial_b}{\partial_3} A^b \right) \\ &= g^{\mu k} \dot{A}^k + g^{\mu k} \frac{\partial_k \partial_b}{(\partial_3)^2} \dot{A}^b , \\ \pi^{\mu} &= g^{\mu k} \left[\dot{A}^k + \frac{\partial_k \partial_b}{(\partial_3)^2} \dot{A}^b \right] . \end{aligned} \quad (5.4.30)$$

Using the gauge constraint $A^3 = -\dfrac{1}{\partial_3}(\partial_b A^b)$, gives

$$\begin{aligned} \pi^{\mu} &= g^{\mu a} \left[\dot{A}^a + \frac{\partial_a \partial_b}{(\partial_3)^2} \dot{A}^b \right] \\ &= g^{\mu a} \left[\delta^{ab} + \frac{\partial_a \partial_b}{(\partial_3)^2} \right] \dot{A}^b , \end{aligned} \quad (5.4.31)$$

with $\mu = 0, 1, 2, 3$, $k = 1, 2, 3$ and $a, b = 1, 2$.

From Eq. (5.4.30), we may solve for $\partial_b \dot{A}^b$ as follows:

$$\begin{aligned}
\partial_\mu \pi^\mu &= g^{\mu k} \partial_\mu \left[\dot{A}^k + \frac{\partial_k \partial_b}{(\partial_3)^2} \, \dot{A}^b \right] \\
&= \partial_k \dot{A}^k + \frac{\partial^k \partial_k \partial_b}{(\partial_3)^2} \, \dot{A}^b \\
&= \frac{\nabla^2}{(\partial_3)^2} \, \partial_b \dot{A}^b \, ,
\end{aligned} \tag{5.4.32}$$

or

$$\partial_b \dot{A}^b \;=\; \frac{(\partial_3)^2}{\nabla^2} \, \partial_\mu \pi^\mu \;=\; \frac{(\partial_3)^2}{\nabla^2} \, \partial_a \pi^a \, . \tag{5.4.33}$$

Substituting this in Eq. (5.4.30) leads to

$$\begin{aligned}
\pi^\mu &= g^{\mu k} \dot{A}^k + g^{\mu k} \frac{\partial_k}{(\partial_3)^2} \, \partial_b \dot{A}^b \\
&= g^{\mu k} \dot{A}^k + g^{\mu k} \frac{\partial_k}{(\partial_3)^2} \, \frac{(\partial_3)^2}{\nabla^2} \, \partial_\nu \pi^\nu \\
&= g^{\mu k} \dot{A}^k + g^{\mu k} \frac{\partial_k \partial_\nu}{\nabla^2} \, \pi^\nu \, ,
\end{aligned} \tag{5.4.34}$$

or

$$\begin{aligned}
g^{\mu k} \dot{A}^k &= \pi^\mu - g^{\mu k} \frac{\partial_k \partial_\nu}{\nabla^2} \, \pi^\nu \\
&= \left(g^{\mu\nu} - g^{\mu k} \frac{\partial_k \partial^\nu}{\nabla^2} \right) \pi_\nu \, ,
\end{aligned} \tag{5.4.35}$$

and

$$g_{i\mu} \, g^{\mu k} \dot{A}^k \;=\; g_{i\mu} \left(g^{\mu\nu} - g^{\mu k} \frac{\partial_k \partial^\nu}{\nabla^2} \right) \pi_\nu \, . \tag{5.4.36}$$

By using Eq. (5.3.37), we may write

$$\begin{aligned}
\dot{A}^i &= \left(g^{i\nu} - \frac{\partial_i \partial^\nu}{\nabla^2}\right) \pi_\nu \\
&= g_{i\mu} \left(-F^{0\mu} + \frac{\partial^\mu}{\nabla^2}\, J^0\right) \\
&= -F^{0i} + \frac{\partial_i}{\nabla^2}\, J^0 \, , \qquad (5.4.37)
\end{aligned}$$

or

$$\begin{aligned}
\dot{A}^i &= \left(\delta^{ik} - \frac{\partial_i \partial_k}{\nabla^2}\right) \pi^k \\
&= -F^{0a} + \frac{\partial_a}{\nabla^2}\, J^0 \, , \qquad (5.4.38)
\end{aligned}$$

and

$$\begin{aligned}
\dot{A}^a &= \left(\delta^{ab} - \frac{\partial_a \partial_b}{\nabla^2}\right) \pi^b \\
&= -F^{0a} + \frac{\partial_a}{\nabla^2}\, J^0 \, . \qquad (5.4.39)
\end{aligned}$$

We may combine Eq. (5.4.37) with $\partial_0 A^0 = -\dfrac{1}{\boldsymbol{\partial}^2}\partial_0 J^0$:

$$\partial_0 A^0 \;=\; \dot{A}^0 \;=\; -\frac{\partial_0}{\nabla^2} J^0 \;=\; \frac{\partial^0 J^0}{\nabla^2}\, , \qquad (5.4.40)$$

to obtain

$$\begin{aligned}
g^{\mu i} \dot{A}_i &= g^{\mu i} \left(g_{i\nu} - \frac{\partial_i \partial_\nu}{\nabla^2}\right) \pi^\nu + g^{\mu 0} \frac{\partial_0}{\nabla^2}\, J^0 \\
&= g^{\mu i} \left(F_{0i} + \frac{\partial_i}{\nabla^2}\, J^0\right), \qquad (5.4.41)
\end{aligned}$$

$$\begin{aligned}\dot{A}^{\mu} &= g^{\mu i}\left(g_{i\nu} - \frac{\partial_i \partial_\nu}{\nabla^2}\right)\pi^{\nu} + g^{\mu 0}\frac{\partial_0}{\nabla^2}\, J^0 \\ &= -F^{0\mu} + \frac{\partial^{\mu}}{\nabla^2}\, J^0 \; . \end{aligned} \tag{5.4.42}$$

The canonical commutation relations, for $a, b = 1, 2$ are given by

$$\left[A^a(x), \pi^b(x')\right]\Big|_{x^0 = x'^0} = \mathrm{i}\delta^{ab}\delta^3(\mathbf{x} - \mathbf{x}') \; , \tag{5.4.43}$$

and upon using the equality

$$\delta(x^0 - x'^0)\delta^3(\mathbf{x} - \mathbf{x}') = \delta^4(x - x') \; , \tag{5.4.44}$$

we have from Eq. (5.4.43)

$$\delta(x^0 - x'^0)\left[A^a(x), \pi^b(x')\right] = \mathrm{i}\delta^{ab}\delta^4(x - x') \; , \tag{5.4.45}$$

for the quantization rule of the physical degrees of freedom A^1 and A^2.

Since $\pi^0 = 0$ and $\pi^3 = 0$, we may generalize Eq. (5.4.45) to

$$\delta(x^0 - x'^0)\left[A^a(x), g^{\nu b}\pi^b(x')\right] = \mathrm{i}\delta^{ab} g^{\nu b}\delta^4(x - x') \; , \tag{5.4.46}$$

or

$$\delta(x^0 - x'^0)\left[A^a(x), \pi^{\nu}(x')\right] = \mathrm{i}g^{\nu a}\delta^4(x - x') \; , \tag{5.4.47}$$

and using $A^3 = -\dfrac{1}{\partial_3}(\partial_a A^a)$, we get

$$\left[A^a(x), \pi^{\nu}(x')\right]\Big|_{x^0 = x'^0} = \mathrm{i}g^{\nu a}\delta^3(\mathbf{x} - \mathbf{x}') \; , \tag{5.4.48}$$

$$\left[-\frac{\partial_a}{\partial_3}A^a(x), \pi^{\nu}(x')\right]\Big|_{x^0 = x'^0} = -\mathrm{i}g^{\nu a}\frac{\partial_a}{\partial_3}\,\delta^3(\mathbf{x} - \mathbf{x}') \; , \tag{5.4.49}$$

$$\left[A^3(x), \pi^\nu(x')\right]\Big|_{x^0=x'^0} \;=\; -\mathrm{i}g^{\nu a}\frac{\partial_a}{\partial_3}\,\delta^3(\mathbf{x}-\mathbf{x}')\;, \tag{5.4.50}$$

or

$$\delta(x^0-x'^0)\left[A^3(x), \pi^\nu(x')\right] \;=\; -\mathrm{i}g^{\nu a}\frac{\partial_a}{\partial_3}\,\delta^4(x-x')\;. \tag{5.4.51}$$

We may combine Eqs. (5.4.47) and (5.4.51), in the form:

$$\delta(x^0-x'^0)\left[A^a(x)+A^3(x), \pi^\nu(x')\right] = \mathrm{i}g^{\nu a}\delta^4(x-x') - \mathrm{i}g^{\nu a}\frac{\partial_a}{\partial_3}\,\delta^4(x-x')\;, \tag{5.4.52}$$

$$\delta(x^0-x'^0)\left[A^i(x), \pi^\nu(x')\right] = \mathrm{i}g^{\nu a}\left(\delta^{ia}-\delta^{i3}\frac{\partial_a}{\partial_3}\right)\delta^4(x-x')\;, \tag{5.4.53}$$

with $i=1,2,3$ and $a=1,2$.

Since $A^0 = -\dfrac{1}{\boldsymbol{\partial}^2}J^0$ is just a c-number,

$$\delta(x^0-x'^0)\left[g^{\mu i}A^i(x), \pi^\nu(x')\right] \;=\; \mathrm{i}g^{\mu i}g^{\nu a}\left(\delta^{ia}-\delta^{i3}\frac{\partial_a}{\partial_3}\right)\delta^4(x-x')\;, \tag{5.4.54}$$

we note that

$$g^{\nu k}\left(\delta^{ik}-\delta^{i3}\frac{\partial_k}{\partial_3}\right) \;=\; g^{\nu a}\left(\delta^{ia}-\delta^{i3}\frac{\partial_a}{\partial_3}\right), \tag{5.4.55}$$

and Eq. (5.4.54) becomes,

$$\delta(x^0-x'^0)\left[A^\mu(x), \pi^\nu(x')\right] \;=\; \mathrm{i}g^{\mu i}g^{\nu k}\left(\delta^{ik}-\delta^{i3}\frac{\partial_k}{\partial_3}\right)\delta^4(x-x')\;. \tag{5.4.56}$$

Using Eq. (5.4.42) for $j=1,2,3$, gives

$$\dot{A}^\nu(x') = g^{\nu i}\left(g_{j\alpha}-\frac{\partial'_j\partial'_\alpha}{\nabla'^2}\right)\pi^\alpha + g^{\nu 0}\frac{\partial_0}{\nabla^2}J^0\;,$$

$$\left[A^\mu(x), \dot{A}^\nu(x')\right]\Big|_{x^0=x'^0} = g^{\nu i}\left(g_{j\alpha} - \frac{\partial'_j\partial'_\alpha}{\nabla'^2}\right)\left[A^\mu(x), \pi^\alpha(x')\right]\Big|_{x^0=x'^0}$$

$$= g^{\nu i}\left(g_{j\alpha} - \frac{\partial_j\partial_\alpha}{\nabla^2}\right) \mathrm{i} g^{\mu i} g^{\alpha k}\left(\delta^{ik} - \delta^{i3}\frac{\partial_k}{\partial_3}\right)\delta^3(\mathbf{x}-\mathbf{x}')$$

$$= \mathrm{i} g^{\mu i} g^{\nu i}\left(\delta^{jk} - \frac{\partial_j\partial_k}{\nabla^2}\right)\left(\delta^{ik} - \delta^{i3}\frac{\partial_k}{\partial_3}\right)\delta^3(\mathbf{x}-\mathbf{x}')\ , \quad (5.4.57)$$

where

$$\partial'_\alpha \ \equiv \ \frac{\partial}{\partial x'^\alpha}\ , \quad (5.4.58)$$

and

$$\nabla'^2 \ \equiv \ \delta^{ij}\,\frac{\partial}{\partial x'^i}\,\frac{\partial}{\partial x'^j}\ . \quad (5.4.59)$$

Consider the expression

$$\left(\delta^{jk} - \frac{\partial_j\partial_k}{\nabla^2}\right)\left(\delta^{ik} - \delta^{i3}\frac{\partial_k}{\partial_3}\right) = \delta^{jk}\delta^{ik} - \delta^{jk}\delta^{i3}\frac{\partial_k}{\partial_3} - \delta^{ik}\frac{\partial_j\partial_k}{\nabla^2} + \delta^{i3}\frac{\partial_j}{\partial_3}\frac{\partial_k\partial_k}{\nabla^2}$$

$$= \delta^{ik}\delta^{jk} - \delta^{i3}\frac{\partial_j}{\partial_3} - \delta^{ik}\frac{\partial_j\partial_k}{\nabla^2} + \delta^{i3}\frac{\partial_j}{\partial_3}$$

$$= \delta^{ik}\delta^{jk} - \delta^{ik}\frac{\partial_j\partial_k}{\nabla^2}\ , \quad (5.4.60)$$

which may be rewritten as

$$\left(\delta^{jk} - \frac{\partial_j\partial_k}{\nabla^2}\right)\left(\delta^{ik} - \delta^{i3}\frac{\partial_k}{\partial_3}\right) \ = \ \delta^{ik}\left(\delta^{jk} - \frac{\partial_j\partial_k}{\nabla^2}\right)\ . \quad (5.4.61)$$

In terms of the transverse delta function

$$\delta_\perp^{ij}(\mathbf{x}-\mathbf{x}') \equiv \left(\delta^{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)\delta^3(\mathbf{x}-\mathbf{x}')$$

$$= \left(\delta^{ij} - b_i\partial_j\right)\delta^3(\mathbf{x}-\mathbf{x}')\,, \qquad (5.4.62)$$

and

$$\delta_\perp^{ij}(x-x') \equiv \delta(x^0 - x'^0)\,\delta_\perp^{ij}(\mathbf{x}-\mathbf{x}')\,, \qquad (5.4.63)$$

we obtain

$$\big[A^\mu(x), \dot{A}^\nu(x')\big]\Big|_{x^0=x'^0} = \mathrm{i}g^{\mu i}g^{\nu j}\delta^{ik}\left(\delta^{jk} - \frac{\partial_j\partial_k}{\nabla^2}\right)\delta^3(\mathbf{x}-\mathbf{x}')$$

$$= \mathrm{i}g^{\mu i}g^{\nu j}\delta^{ik}\,\delta_\perp^{jk}(\mathbf{x}-\mathbf{x}')$$

$$= \mathrm{i}g^{\mu k}g^{\nu j}\,\delta_\perp^{jk}(\mathbf{x}-\mathbf{x}')\,. \qquad (5.4.64)$$

Or equivalently,

$$\delta(x^0-x'^0)\big[A^\mu(x), \dot{A}^\nu(x')\big] = \mathrm{i}g^{\mu k}g^{\nu j}\,\delta_\perp^{jk}(x-x')$$

$$= \mathrm{i}g^{\mu i}g^{\nu k}\,\delta_\perp^{ik}(x-x')\,. \qquad (5.4.65)$$

Equation (5.4.42) provides the equality

$$\dot{A}^\nu = -F^{0\nu} + \frac{\partial^\nu}{\nabla^2}J^0\,,$$

this leading to

$$\delta(x^0-x'^0)\big[A^\mu(x), F^{0\nu}(x')\big] = -\mathrm{i}g^{\mu i}g^{\nu k}\delta_\perp^{ik}(x-x')\,. \qquad (5.4.66)$$

We may generalize Eq. (5.4.66) for all the components of $F^{\alpha\beta} = -F^{\beta\alpha}$ as follows:

$$\delta(x^0 - x'^0)\big[A^\mu(x), F^{\alpha\beta}(x')\big] = \mathrm{i}g^{\mu i}(g^{\beta k}g^{\alpha 0} - g^{\alpha k}g^{\beta 0})\,\delta_\perp^{ik}(x - x')\,, \tag{5.4.67}$$

and

$$\delta(x^0 - x'^0)\big[F^{\mu\nu}(x), F^{\alpha\beta}(x')\big] = \mathrm{i}\big[(g^{\beta k}g^{\alpha 0} - g^{\alpha k}g^{\beta 0})(g^{\nu i}\partial^\mu - g^{\mu i}\partial^\nu) - (g^{\nu k}g^{\mu 0} - g^{\mu k}g^{\nu 0})(g^{\beta i}\partial^\alpha - g^{\alpha i}\partial^\beta)\big]\delta_\perp^{ik}(x - x')\,, \tag{5.4.68}$$

We note that

$$\delta(x^0 - x'^0)\big[F^{0i}(x), F^{0j}(x')\big] = 0\,, \tag{5.4.69}$$

$$\delta(x^0 - x'^0)\big[F^{ij}(x), F^{mn}(x')\big] = 0\,, \tag{5.4.70}$$

$$\begin{aligned}\delta(x^0 - x'^0)\big[F^{ij}(x), F^{mn}(x')\big] &= +\,\mathrm{i}(\delta^{in}\partial^m - \delta^{im}\partial^n)\delta_\perp^{ik}(x - x')\\ &= +\,\mathrm{i}(\delta^{kn}\partial^m - \delta^{km}\partial^n)\delta^4(x - x')\,,\end{aligned} \tag{5.4.71}$$

$$\begin{aligned}\delta(x^0 - x'^0)\big[F^{mn}(x), F^{0i}(x')\big] &= -\,\mathrm{i}(\delta^{in}\partial^m - \delta^{im}\partial^n)\delta_\perp^{ik}(x - x')\\ &= -\,\mathrm{i}(\delta^{kn}\partial^m - \delta^{km}\partial^n)\delta^4(x - x')\,.\end{aligned} \tag{5.4.72}$$

For the bosonic q-number fields $\mathscr{A}(x)$ and $\mathscr{B}(x')$, the time-order product is defined by

$$(\mathscr{A}(x)\mathscr{B}(x'))_+ = \Theta(x^0 - x'^0)\mathscr{A}(x)\mathscr{B}(x') + \Theta(x'^0 - x^0)\mathscr{B}(x')\mathscr{A}(x)\,, \tag{5.4.73}$$

and hence

$$\big\langle(\mathscr{A}(x)\mathscr{B}(x'))_+\big\rangle \;=\; \Theta(x^0-x'^0)\big\langle\mathscr{A}(x)\mathscr{B}(x')\big\rangle+\Theta(x'^0-x^0)\big\langle\mathscr{B}(x')\mathscr{A}(x)\big\rangle\;. \tag{5.4.74}$$

Using the fundamental identity,

$$\frac{\partial}{\partial t}\,\Theta(t-t') \;=\; \delta(t-t')\;, \tag{5.4.75}$$

with

$$\partial_0' \;\equiv\; \frac{\partial}{\partial x'^0}\;, \tag{5.4.76}$$

gives

$$\begin{aligned}
\partial_0'\big\langle(\mathscr{A}(x)\mathscr{B}(x'))_+\big\rangle &= \Theta(x^0-x'^0)\big\langle\mathscr{A}(x)\partial_0'\mathscr{B}(x')\big\rangle \\
&\quad+\big\langle\mathscr{A}(x)\mathscr{B}(x')\big\rangle\partial_0'\Theta(x^0-x'^0) \\
&\quad+\Theta(x'^0-x^0)\big\langle\partial_0'\mathscr{B}(x')\mathscr{A}(x)\big\rangle \\
&\quad+\big\langle\mathscr{B}(x')\mathscr{A}(x)\big\rangle\partial_0'\Theta(x'^0-x^0)\;, \\
\partial_0'\big\langle(\mathscr{A}(x)\mathscr{B}(x'))_+\big\rangle &= \Theta(x^0-x'^0)\big\langle\mathscr{A}(x)\partial_0'\mathscr{B}(x')\big\rangle \\
&\quad-\delta(x^0-x'^0)\big\langle\mathscr{A}(x)\mathscr{B}(x')\big\rangle \\
&\quad+\Theta(x'^0-x^0)\big\langle\partial_0'\mathscr{B}(x')\mathscr{A}(x)\big\rangle \\
&\quad+\delta(x'^0-x^0)\big\langle\mathscr{B}(x')\mathscr{A}(x)\big\rangle\;,
\end{aligned} \tag{5.4.77}$$

or to the equations

$$\partial'_0 \big\langle (\mathscr{A}(x)\mathscr{B}(x'))_+ \big\rangle \;=\; \big\langle (\mathscr{A}(x)\partial'_0\mathscr{B}(x'))_+ \big\rangle - \;\delta(x^0 - x'^0)\big\langle [\mathscr{A}(x), \mathscr{B}(x')] \big\rangle \;, \quad (5.4.78)$$

and

$$\partial'_k \big\langle (\mathscr{A}(x)\mathscr{B}(x'))_+ \big\rangle \;=\; \big\langle (\mathscr{A}(x)\partial'_k\mathscr{B}(x'))_+ \big\rangle \;. \quad (5.4.79)$$

We may combine Eqs. (5.4.78) and (5.4.79) through the following:

$$\begin{aligned} -\, g_{\mu 0}\, \partial'_0 \big\langle (\mathscr{A}(x)\mathscr{B}(x'))_+ \big\rangle = &-\; g_{\mu 0} \big\langle (\mathscr{A}(x)\partial'_k\mathscr{B}(x'))_+ \big\rangle \\ &+\; g_{\mu 0}\, \delta(x^0 - x'^0)\big\langle [\mathscr{A}(x), \mathscr{B}(x')] \big\rangle \;, \end{aligned} \quad (5.4.80)$$

$$\begin{aligned} \partial'_\mu \big\langle (\mathscr{A}(x)\mathscr{B}(x'))_+ \big\rangle = &\big\langle (\mathscr{A}(x)\partial'_\mu\mathscr{B}(x'))_+ \big\rangle \\ &+\; g_{\mu 0}\, \delta(x^0 - x'^0)\big\langle [\mathscr{A}(x), \mathscr{B}(x')] \big\rangle \;, \end{aligned} \quad (5.4.81)$$

or

$$\begin{aligned} \big\langle (\mathscr{A}(x)\partial'_\mu\mathscr{B}(x'))_+ \big\rangle \;=\; &\partial'_\mu \big\langle (\mathscr{A}(x)\mathscr{B}(x'))_+ \big\rangle \\ &-\, g_{\mu 0}\, \delta(x^0 - x'^0)\big\langle [\mathscr{A}(x), \mathscr{B}(x')] \big\rangle \;. \end{aligned} \quad (5.4.82)$$

Equation (5.4.28) gives:

$$\begin{aligned} \big\langle (F^{\mu\nu}(x)A^\alpha(x'))_+ \big\rangle \;=\; &\frac{\delta}{\mathrm{i}\delta J_\alpha(x')} \big\langle F^{\mu\nu}(x) \big\rangle - \left\langle \frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_\nu(x')} \right\rangle \\ &+ \big\langle F^{\mu\nu}(x) \big\rangle \big\langle A^\alpha(x') \big\rangle \;, \end{aligned} \quad (5.4.83)$$

and we have

$$\big\langle (F^{\mu\nu}(x)\partial'^{\beta}A^{\alpha}(x'))_{+}\big\rangle = \partial'^{\beta}\big\langle (F^{\mu\nu}(x)A^{\alpha}(x'))_{+}\big\rangle$$

$$+\ g^{\beta 0}\ \delta(x^0 - x'^0)\big\langle [F^{\mu\nu}(x), A^{\alpha}(x')]_{+}\big\rangle\ . \tag{5.4.84}$$

Substitute Eq. (5.4.83) into Eq. (5.4.84), to obtain

$$\begin{aligned}
\big\langle (F^{\mu\nu}(x)\partial'^{\beta}A^{\alpha}(x'))_{+}\big\rangle &= \partial'^{\beta}\frac{\delta}{\mathrm{i}\delta J_{\alpha}(x')}\big\langle F^{\mu\nu}(x)\big\rangle - \partial'^{\beta}\Big\langle \frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_{\alpha}(x')}\Big\rangle \\
&\quad + \partial'^{\beta}\Big(\big\langle F^{\mu\nu}(x)\big\rangle\big\langle A^{\alpha}(x')\big\rangle\Big) \\
&\quad + g^{\beta 0}\ \delta(x^0 - x'^0)\big\langle [F^{\mu\nu}(x), A^{\alpha}(x')]\big\rangle\ , \\
\big\langle (F^{\mu\nu}(x)\partial'^{\beta}A^{\alpha}(x'))_{+}\big\rangle &= \partial'^{\beta}\frac{\delta}{\mathrm{i}\delta J_{\alpha}(x')}\big\langle F^{\mu\nu}(x)\big\rangle - \Big\langle \partial'^{\beta}\frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_{\alpha}(x')}\Big\rangle \\
&\quad + \big\langle F^{\mu\nu}(x)\big\rangle\partial'^{\beta}\big\langle A^{\alpha}(x')\big\rangle \\
&\quad + g^{\beta 0}\ \delta(x^0 - x'^0)\big\langle [F^{\mu\nu}(x), A^{\alpha}(x')]\big\rangle\ .
\end{aligned} \tag{5.4.85}$$

By defining the functional differential operator

$$A'^{\mu} \equiv \frac{\delta}{\mathrm{i}\delta J_{\mu}(x)}\ , \tag{5.4.86}$$

and the operator

$$\begin{aligned}
F'^{\mu\nu}(x) &\equiv \partial^{\mu}A'^{\nu}(x) - \partial^{\nu}A'^{\mu}(x) \\
&= \partial^{\mu}\frac{\delta}{\mathrm{i}\delta J_{\nu}(x)} - \partial^{\nu}\frac{\delta}{\mathrm{i}\delta J_{\mu}(x)}\ ,
\end{aligned} \tag{5.4.87}$$

we may write the quantum dynamical principle in Eq. (5.3.54) as

$$A'^{\mu}(x)\,\langle 0_+\,|\,0_-\rangle \;\;=\;\; \langle 0_+\,|A^{\mu}(x)|0_-\rangle \;\;, \tag{5.4.88}$$

and write the equation (5.4.14) as

$$\begin{aligned} A'^{\nu}(x')\big\langle A^{\mu}(x)\big\rangle \;\;&=\;\; \big\langle (A^{\mu}(x)A^{\nu}(x'))_+\big\rangle - \big\langle A^{\mu}(x)\big\rangle\big\langle A^{\nu}(x')\big\rangle \\ &\quad + \big\langle A'^{\alpha}(x')A^{\mu}(x)\big\rangle\;, \end{aligned} \tag{5.4.89}$$

where

$$\big\langle A^{\mu}(x)\big\rangle \;\;=\;\; \frac{1}{\langle 0_+\,|\,0_-\rangle}\;A'^{\mu}(x)\,\langle 0_+\,|\,0_-\rangle\;\;. \tag{5.4.90}$$

We may rewrite Eq. (5.4.83) as

$$\begin{aligned} \big\langle (F^{\mu\nu}(x)A^{\alpha}(x'))_+\big\rangle \;\;&=\;\; A'^{\alpha}(x')\big\langle F^{\mu\nu}(x)\big\rangle - \big\langle A'^{\alpha}(x')F^{\mu\nu}(x)\big\rangle \\ &\quad + \big\langle F^{\mu\nu}(x)\big\rangle\big\langle A^{\alpha}(x')\big\rangle\;, \end{aligned} \tag{5.4.91}$$

as a consequence of the fact that $F^{\alpha\beta}(x') = \partial'^{\alpha}A^{\beta}(x') - \partial'^{\beta}A^{\alpha}(x')$. From Eq. (5.4.85), we then obtain

$$\begin{aligned} \big\langle (F^{\mu\nu}(x)F^{\alpha\beta}(x'))_+\big\rangle \;\;&=\;\; \big\langle (F^{\mu\nu}(x)\partial'^{\alpha}A^{\beta}(x'))_+\big\rangle - \big\langle (F^{\mu\nu}(x)\partial'^{\beta}A^{\alpha}(x'))_+\big\rangle \\ &=\;\; \big[\partial'^{\alpha}A'^{\beta}(x') - \partial'^{\beta}A'^{\alpha}(x')\big]\big\langle F^{\mu\nu}(x)\big\rangle \\ &\quad - \;\big\langle\big[\partial'^{\alpha}A'^{\beta}(x') - \partial'^{\beta}A'^{\alpha}(x')\big]F^{\mu\nu}(x)\big\rangle \\ &\quad + \big\langle F^{\mu\nu}(x)\big\rangle\,\big[\partial'^{\alpha}\big\langle A^{\beta}(x')\big\rangle - \partial'^{\beta}\big\langle A^{\alpha}(x')\big\rangle\big] \end{aligned}$$

$$
\begin{aligned}
&+ g^{\alpha 0}\, \delta(x^0 - x'^0)\big\langle [F^{\mu\nu}(x), A^{\beta}(x')]\big\rangle \\
&- g^{\beta 0}\, \delta(x^0 - x'^0)\big\langle [F^{\mu\nu}(x), A^{\alpha}(x')]\big\rangle \,,
\end{aligned}
$$

$$
\begin{aligned}
\big\langle (F^{\mu\nu}(x) F^{\alpha\beta}(x'))_+ \big\rangle \;=\;& F'^{\alpha\beta}(x')\big\langle F^{\mu\nu}(x)\big\rangle - \big\langle F'^{\alpha\beta}(x') F^{\mu\nu}(x)\big\rangle \\
&+ \big\langle F^{\mu\nu}(x)\big\rangle\big\langle F^{\alpha\beta}(x')\big\rangle \\
&+ g^{\alpha 0}\, \delta(x^0 - x'^0)\big\langle [F^{\mu\nu}(x), A^{\beta}(x')]\big\rangle \\
&- g^{\beta 0}\, \delta(x^0 - x'^0)\big\langle [F^{\mu\nu}(x), A^{\alpha}(x')]\big\rangle \,,
\end{aligned}
\tag{5.4.92}
$$

where

$$
\big\langle F^{\mu\nu}(x)\big\rangle \;=\; \frac{1}{\langle 0_+ \,|\, 0_- \rangle}\, F'^{\mu\nu}(x)\, \langle 0_+ \,|\, 0_- \rangle \;. \tag{5.4.93}
$$

We note that

$$
\begin{aligned}
\frac{\delta}{\mathrm{i}\delta J_\beta(x')}\, \big\langle F^{\mu\nu}(x)\big\rangle \;=\;& \frac{1}{\langle 0_+ \,|\, 0_- \rangle}\, \frac{\delta}{\mathrm{i}\delta J_\beta(x')}\, F'^{\mu\nu}(x)\, \langle 0_+ \,|\, 0_- \rangle \\
&- \frac{1}{\langle 0_+ \,|\, 0_- \rangle}\, F'^{\mu\nu}(x)\, \langle 0_+ \,|\, 0_- \rangle \left[\frac{1}{\langle 0_+ \,|\, 0_- \rangle}\, \frac{\delta}{\mathrm{i}\delta J_\beta(x')}\, \langle 0_+ \,|\, 0_- \rangle \right] \\
=\;& \frac{1}{\langle 0_+ \,|\, 0_- \rangle}\, \frac{\delta}{\mathrm{i}\delta J_\beta(x')}\, F'^{\mu\nu}(x)\, \langle 0_+ \,|\, 0_- \rangle \\
&- \big\langle A^{\beta}(x')\big\rangle\big\langle F^{\mu\nu}(x)\big\rangle \,,
\end{aligned}
\tag{5.4.94}
$$

or

$$
\begin{aligned}
F'^{\alpha\beta}(x')\big\langle F^{\mu\nu}(x)\big\rangle &= \frac{1}{\langle 0_+\,|\,0_-\rangle}\, F'^{\alpha\beta}(x')F'^{\mu\nu}(x)\,\langle 0_+\,|\,0_-\rangle \\
&\quad - \big\langle F^{\mu\nu}(x)\big\rangle \left[\frac{1}{\langle 0_+\,|\,0_-\rangle}\, F'^{\alpha\beta}(x')\,\langle 0_+\,|\,0_-\rangle\right] \\
&= \frac{1}{\langle 0_+\,|\,0_-\rangle}\, F'^{\alpha\beta}(x')F'^{\mu\nu}(x)\,\langle 0_+\,|\,0_-\rangle \\
&\quad - \big\langle F^{\mu\nu}(x)\big\rangle\big\langle F^{\alpha\beta}(x')\big\rangle\,. \qquad (5.4.95)
\end{aligned}
$$

We may then rewrite Eq. (5.4.92) as

$$
\begin{aligned}
\big\langle (F^{\mu\nu}(x)F^{\alpha\beta}(x'))_+\big\rangle &= \frac{1}{\langle 0_+\,|\,0_-\rangle}\, F'^{\alpha\beta}(x')F'^{\mu\nu}(x)\,\langle 0_+\,|\,0_-\rangle - \big\langle F^{\mu\nu}(x)\big\rangle\big\langle F^{\alpha\beta}(x')\big\rangle \\
&\quad - \big\langle F'^{\alpha\beta}(x')F^{\mu\nu}(x)\big\rangle + \big\langle F^{\mu\nu}(x)\big\rangle\big\langle F^{\alpha\beta}(x')\big\rangle \\
&\quad + g^{\alpha 0}\,\delta(x^0 - x'^0)\big\langle [F^{\mu\nu}(x), A^{\beta}(x')]\big\rangle \\
&\quad - g^{\beta 0}\,\delta(x^0 - x'^0)\big\langle [F^{\mu\nu}(x), A^{\alpha}(x')]\big\rangle\,, \qquad (5.4.96)
\end{aligned}
$$

where $[F^{\mu\nu}(x), A^{\alpha}(x')]$ is a c-numbers, and we have $\big\langle [F^{\mu\nu}(x), A^{\alpha}(x')]\big\rangle = [F^{\mu\nu}(x), A^{\alpha}(x')]$. Accordingly, we get

$$
\begin{aligned}
\big\langle (F^{\mu\nu}(x)F^{\alpha\beta}(x'))_+\big\rangle &= \frac{1}{\langle 0_+\,|\,0_-\rangle}\, F'^{\alpha\beta}(x')F'^{\mu\nu}(x)\,\langle 0_+\,|\,0_-\rangle - \big\langle F'^{\alpha\beta}(x')F^{\mu\nu}(x)\big\rangle \\
&\quad + g^{\alpha 0}\,\delta(x^0 - x'^0)\big\langle [F^{\mu\nu}(x), A^{\beta}(x')]\big\rangle \\
&\quad - g^{\beta 0}\,\delta(x^0 - x'^0)\big\langle [F^{\mu\nu}(x), A^{\alpha}(x')]\big\rangle\,. \qquad (5.4.97)
\end{aligned}
$$

Equation (5.3.39) leads to:

$$F^{\mu\nu} = \frac{1}{2}\left[(g^{\mu 0}g^{\nu\alpha} - g^{\nu 0}g^{\mu\alpha}) - (g^{\mu 0}b^{\nu} - g^{\nu 0}b^{\mu})\partial^{\alpha}\right]\pi_{\alpha}$$

$$+ \frac{1}{2}(\delta^{\mu}{}_{k}F^{k\nu} - \delta^{\nu}{}_{k}F^{k\mu}) - \frac{1}{2}(g^{\mu 0}b^{\nu} - g^{\nu 0}b^{\mu})J^{0} ,$$

giving

$$\frac{\delta F^{\mu\nu}(x)}{\mathrm{i}\delta J_{\alpha}(x')} = -\frac{1}{2\mathrm{i}}\, g^{\alpha 0}(g^{\mu 0}b^{\nu} - g^{\nu 0}b^{\mu})\, \delta^{4}(x-x')$$

$$= \frac{\mathrm{i}}{2}\, g^{\alpha 0}(g^{\mu 0}b^{\nu} - g^{\nu 0}b^{\mu})\, \delta^{4}(x-x') , \qquad (5.4.98)$$

or to

$$A'^{\alpha}(x')F^{\mu\nu}(x) = \frac{\mathrm{i}}{2}\, g^{\alpha 0}(g^{\mu 0}b^{\nu} - g^{\nu 0}b^{\mu})\, \delta^{4}(x-x') , \qquad (5.4.99)$$

and

$$\partial'^{\beta}A'^{\alpha}(x')F^{\mu\nu}(x) = \frac{\mathrm{i}}{2}\, g^{\alpha 0}\partial'^{\beta}(g^{\mu 0}b^{\nu} - g^{\nu 0}b^{\mu})\, \delta^{4}(x-x') . \qquad (5.4.100)$$

Using the identity

$$F'^{\alpha\beta}(x') = \partial'^{\alpha}A'^{\beta}(x') - \partial'^{\beta}A'^{\alpha}(x') , \qquad (5.4.101)$$

we get

$$F'^{\alpha\beta}(x')F^{\mu\nu}(x) = \partial'^{\alpha}A'^{\beta}(x')F^{\mu\nu}(x) - \partial'^{\beta}A'^{\alpha}(x')F^{\mu\nu}(x)$$

$$= \frac{\mathrm{i}}{2}(g^{\beta 0}\partial'^{\alpha} - g^{\alpha 0}\partial'^{\beta})(g^{\mu 0}b^{\nu} - g^{\nu 0}b^{\mu})\, \delta^{4}(x-x')$$

$$= \frac{\mathrm{i}}{2}(g^{\beta 0}\partial^{\alpha} - g^{\alpha 0}\partial^{\beta})(g^{\mu 0}b^{\nu} - g^{\nu 0}b^{\mu})\, \delta^4(x - x')\,, \qquad (5.4.102)$$

which is just a c-number, and $\left\langle F'^{\alpha\beta}(x')F^{\mu\nu}(x)\right\rangle = F'^{\alpha\beta}(x')F^{\mu\nu}(x)$.

CHAPTER VI
ACTION PRINCIPLE AND MODIFICATION OF THE FADDEEV-POPOV FACTOR IN GAUGE THEORIES

6.1 Introduction

Over the years (Manoukian, 1986, 1987; Manoukian and Siranan, 2005), we have seen that the quantum action (dynamical) principle (Schwinger, 1951, 1953, 1954, 1972, 1973; Lam, 1965; Manoukian, 1985) may be used to quantize gauge theories in constructing the vacuum-to-vacuum transition amplitude and the Faddeev-Popov (FP) factor (Faddeev and Popov, 1967), encountered in non-abelian gauge theories (e.g., Abers and Lee, 1973; Rivers, 1987; 't Hooft, 2000; Veltman, 2000; Gross, (2005); Politzer, 2005; Wilczek, 2005), may be obtained *directly* from the action principle without much effort. No appeal was made to path integrals, and there was not even the need to go into the well-known complicated structure of the Hamiltonian (Fradkin and Tyutin, 1970) in non-abelian gauge theories. For extensive references on the gauge problem in gauge theories see Manoukian and Siranan. The latter reference traces its historical development from early papers to most recent ones.

In the present investigation, we consider the generic non-abelian gauge theory Lagrangian density

$$\mathscr{L}_T = \mathscr{L} + \mathscr{L}_S \, , \qquad (6.1.1)$$

and modifications thereof, where

$$\mathscr{L} = -\frac{1}{4}\, G^a_{\mu\nu} G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}[(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi] - m_0\bar{\psi}\psi$$
$$+ g_0\bar{\psi}\gamma_\mu A^\mu\psi\ , \tag{6.1.2}$$

$$\mathscr{L}_S = \bar{\eta}\psi + \bar{\psi}\eta + J^\mu_a A^a_\mu\ , \tag{6.1.3}$$

$$A_\mu = A^a_\mu t_a \quad , \quad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g_0[A_\mu, A_\nu]\ , \tag{6.1.4}$$

$$G_{\mu\nu} = G^a_{\mu\nu} t_a\ , \tag{6.1.5}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 f^{abc} A^b_\mu A^c_\nu\ . \tag{6.1.6}$$

The t^a are generators of the underlying algebra, and the f^{abc}, totally antisymmetric, are the structure constants satisfying the Jacobi identity, $[t^a, t^b] = \mathrm{i}f^{abc}t^c$. Note that A_μ is a matrix. $\mathscr{L}_S$ is the source term with the J^a_μ classical functions, while η, $\bar{\eta}$ are so-called anti-commuting Grassmann variables.

The Lagrangian density $\mathscr{L}$ in Eq. (6.1.2) is invariant under simultaneous local gauge transformations:

$$\psi \longrightarrow U\psi, \qquad \bar{\psi} \longrightarrow \bar{\psi}U^{-1}\ , \tag{6.1.7}$$

$$A_\mu \longrightarrow UA_\mu U^{-1} + \frac{\mathrm{i}}{g_0}U\partial_\mu U^{-1}\ , \tag{6.1.8}$$

$$G_{\mu\nu} \longrightarrow UG_{\mu\nu}U^{-1}\ , \tag{6.1.9}$$

where

$$U = U(\theta) = \exp[\,\mathrm{i}g_0\theta^a t^a] \quad , \quad \theta = \theta^a t^a \quad , \quad \theta = \theta(x)\ . \tag{6.1.10}$$

Upon setting

$$\nabla_\mu \;=\; \partial_\mu - \mathrm{i}g_0 A_\mu \;, \tag{6.1.11}$$

with $f^{abc} = \mathrm{i}(t^a)^{bc} \to f^{cab} = \mathrm{i}(t^c)^{ab}$, we have

$$\begin{aligned}
\nabla_\mu &= \partial_\mu - \mathrm{i}g_0 A_\mu \\
&= \partial_\mu - \mathrm{i}g_0 A^c_\mu t^c \;, && (6.1.12)\\
\nabla^{ab}_\mu &= \delta^{ab}\partial_\mu - \mathrm{i}g_0 A^c_\mu (t^c)^{ab} \\
&= \delta^{ab}\partial_\mu - g_0 f^{cab} A^c_\mu \;, \\
\nabla^{ab}_\mu &= \delta^{ab}\partial_\mu + g_0 f^{acb} A^c_\mu \;, && (6.1.13)
\end{aligned}$$

We next consider the basic commutator defined by

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu] &= \nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu \\
&= (\partial_\mu - \mathrm{i}g_0 A_\mu)(\partial_\nu - \mathrm{i}g_0 A_\nu) - (\partial_\nu - \mathrm{i}g_0 A_\nu)(\partial_\mu - \mathrm{i}g_0 A_\mu) \\
&= \partial_\mu\partial_\nu - \mathrm{i}g_0 \partial_\mu A_\nu - \mathrm{i}g_0 A_\mu \partial_\nu - g_0^2 A_\mu A_\nu \\
&\quad - \underbrace{\partial_\nu\partial_\mu}_{=\partial_\mu\partial_\nu} + \mathrm{i}g_0 \partial_\nu A_\mu + \underbrace{\mathrm{i}g_0 A_\nu \partial_\mu}_{=\mathrm{i}g_0 A_\mu \partial_\nu} + g_0^2 A_\nu A_\mu \\
&= -\mathrm{i}g_0 \partial_\mu A_\nu + \mathrm{i}g_0 \partial_\nu A_\mu - g_0^2 A_\mu A_\mu + g_0^2 A_\nu A_\mu \\
&= -\mathrm{i}g_0 \partial_\mu A^a_\nu t_a + \mathrm{i}g_0 \partial_\nu A^a_\mu t_a - g_0^2 (A_\mu A_\nu - A_\nu A_\mu)
\end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{i}g_0\partial_\mu A^a_\nu t_a + \mathrm{i}g_0\partial_\nu A^a_\mu t_a - g_0^2[A_\mu, A_\nu] \\
&= -\mathrm{i}g_0\partial_\mu A^a_\nu t_a + \mathrm{i}g_0\partial_\nu A^a_\mu t_a - g_0^2[t^b, t^c]A^b_\mu A^c_\nu \\
&= -\mathrm{i}g_0\partial_\mu A^a_\nu t_a + \mathrm{i}g_0\partial_\nu A^a_\mu t_a - g_0^2(\mathrm{i}f^{bca}t^a)A^b_\mu A^c_\nu \\
&= -\mathrm{i}g_0\partial_\mu A^a_\nu t_a + \mathrm{i}g_0\partial_\nu A^a_\mu t_a - g_0^2(\mathrm{i}f^{abc}t^a)A^b_\mu A^c_\nu \\
&= -\mathrm{i}g_0\partial_\mu A^a_\nu t_a + \mathrm{i}g_0\partial_\nu A^a_\mu t_a - \mathrm{i}g_0^2 f^{abc}t_a A^b_\mu A^c_\nu \,, \\
[\nabla_\mu, \nabla_\nu] &= -\mathrm{i}g_0(\partial_\mu A^a_\nu t_a - \partial_\nu A^a_\mu t_a + g_0 f^{abc}t_a A^b_\mu A^c_\nu) \,.
\end{aligned}
$$

That is

$$
[\nabla_\mu, \nabla_\nu] = -\mathrm{i}g_0 G_{\mu\nu} \,. \tag{6.1.14}
$$

We also prove the following identity:

$$
\begin{aligned}
(\nabla_\mu\nabla_\nu)^{ac} &= \frac{1}{2}[(\nabla_\mu\nabla_\nu) - (\nabla_\nu\nabla_\mu)]^{ac} + \frac{1}{2}[(\nabla_\mu\nabla_\nu) + (\nabla_\nu\nabla_\mu)]^{ac} \\
&= \frac{1}{2}([\nabla_\mu, \nabla_\nu])^{ac} + \frac{1}{2}(\{\nabla_\mu, \nabla_\nu\})^{ac} \,, \qquad (6.1.15) \\
(\nabla_\mu\nabla_\nu)^{ac}G^{\mu\nu}_c &= \frac{1}{2}([\nabla_\mu, \nabla_\nu])^{ac}\, G^{\mu\nu}_c + \frac{1}{2}(\underbrace{\{\nabla_\mu, \nabla_\nu\}}_{\text{symmetric}})^{ac}\, \underbrace{G^{\mu\nu}_c}_{\text{antisymmetric}} \\
&= \frac{1}{2}([\nabla_\mu, \nabla_\nu])^{ac}\, G^{\mu\nu}_c + 0 \\
&= \frac{1}{2}\left[-\mathrm{i}g_0\, G^b_{\mu\nu}(t_b)^{ac}\, G^{\mu\nu}_c\right] \\
&= \frac{1}{2}\left[-\mathrm{i}g_0\, G^b_{\mu\nu}(-\mathrm{i}f^{bac})\, G^{\mu\nu}_c\right]
\end{aligned}
$$

$$= -\frac{g_0}{2} \underbrace{G^b_{\mu\nu} f^{bac} G^{\mu\nu}_c}_{=0} ,$$

$$(\nabla_\mu \nabla_\nu)^{ac} G^{\mu\nu}_c = 0 . \tag{6.1.16}$$

That is,

$$\nabla^{ab}_\mu \nabla^{bc}_\nu G^{\mu\nu}_c = 0 . \tag{6.1.17}$$

[The latter generalizes the elementary identity $\partial_\mu \partial_\nu F^{\mu\nu} = 0$, in abelian gauge theory, to non-abelian ones, where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.]

We consider gauge invariant (Sect. 6.3) as well as gauge non-invariant (Sect. 6.4) modifications of the Lagrangian density and show by a systematic use of the quantum action principle that the familiar FP factor needs to be modified in more general cases and explicit expressions for these modifications are derived. In particular, we show that a gauge invariant theory does *not* necessarily imply the familiar FP factor for proper quantization, as may be perhaps expected (cf. Rivers, 1987, p. 204), and modifications thereof may be necessary. Before doing so, however, we use the action principle to derive, in Sect. 6.2, the FP factor and investigate its origin for the classic Lagrangian density $\mathscr{L}$, without recourse to path integrals, as an anticipation of what to expect in more general cases. Throughout, we work in the celebrated Coulomb gauge $\partial_k A^k_a = 0$, $k = 1, 2, 3$ as always.

6.2 Action Principle and the Origin of the FP Factor

To obtain the expression for the vacuum-to-vacuum transition amplitude $\langle 0_+ \,|\, 0_- \rangle$, in the presence of external sources J^a_μ, η^a, $\bar{\eta}^a$, as the generator of all the Green functions of the theory, *no* restrictions may be set, in particular, on the external current J^a_μ, coupled to the gauge fields A^μ_a, such as $\partial^\mu J^a_\mu = 0$, so that *variations of the components of J^a_μ may be carried out independently*, until the entire analysis is completed, and all functional differentiations are carried out to generate Green functions. This point cannot be overemphasized. As we will see, the *generality* condition that must be adopted on the external current J^a_μ together with the presence of *dependent* gauge field components in (A^μ_a), as a result of the structure of the Lagrangian density $\mathscr{L}$ in Eq. (6.1.2) and the gauge constraint, are responsible for the *origin* and the presence of the FP factor in the theory for a proper quantization in the realm of the quantum action principle.

We define the Green operator $D^{ab}(x,x')$ satisfying the differential equation

$$[\delta^{ac}\boldsymbol{\partial}^2 + g_0 f^{abc} A^b_k \partial_k] D^{cd}(x,x') \;=\; \delta^4(x,x')\delta^{ad}\,. \tag{6.2.1}$$

Since the differential operator on the left-hand side of $D^{cd}(x,x')$ is independent of the time derivative, $D^{cd}(x,x')$ involves a $\delta(x^0 - x'^0)$ factor. Using the gauge constraint, one may, for example, eliminate A^3_a in favor of A^1_a, A^2_a. That is, we may treat the A^3_a as dependent fields.

The action in question is defined by

$$\begin{aligned}\mathcal{W}_T \;&=\; \int (\mathrm{d}x)\,\mathscr{L}\\ &=\; \int (\mathrm{d}x)\bigg\{ -\frac{1}{4}\,G^a_{\mu\nu}G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\big] - m_0\bar{\psi}\psi\\ &\qquad + g_0\bar{\psi}\gamma_\mu A^\mu\psi + \bar{\eta}\psi + \bar{\psi}\eta + J^\mu_a A^a_\mu\bigg\}\,,\end{aligned} \tag{6.2.2}$$

and for its variation we have

$$\begin{aligned}\delta\mathcal{W}_T \;&=\; \int (\mathrm{d}x)\bigg\{ -\frac{1}{4}\,\delta(G^a_{\mu\nu}G^{\mu\nu}_a) + \frac{1}{2\mathrm{i}}\,\delta\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\big] - m_0\delta(\bar{\psi}\psi)\\ &\qquad + g_0\delta(\bar{\psi}\gamma_\mu A^\mu\psi) + \bar{\eta}\delta\psi + (\delta\bar{\psi})\eta + J^\mu_a\delta A^a_\mu\bigg\}\,.\end{aligned} \tag{6.2.3}$$

The first term in the integrand in Eq. (6.2.3) is worked out as

$$\begin{aligned}
\delta(G^a_{\mu\nu}G^{\mu\nu}_a) &= G^a_{\mu\nu}\delta G^{\mu\nu}_a + G^{\mu\nu}_a\delta G^a_{\mu\nu} \\
&= G^a_{\mu\nu}\delta(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g_0 f^{abc}A^\mu_b A^\nu_c) \\
&\quad + G^{\mu\nu}_a\delta(\partial_\mu a^a_\nu - \partial_\nu A^a_\mu + g_0 f^{abc}A^b_\mu A^c_\nu) \\
&= G^a_{\mu\nu}\partial^\mu\delta A^\nu_a - G^a_{\mu\nu}\partial^\nu\delta A^\mu_a + g_0 f^{abc}G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) \\
&\quad + G^{\mu\nu}_a\partial_\mu\delta A^a_\nu - G^{\mu\nu}_a\partial_\nu\delta A^a_\mu + g_0 f^{abc}G^{\mu\nu}_a\delta(A^b_\mu A^c_\nu) \\
&= G^{\mu\nu}_a\partial_\mu\delta A^a_\nu + G^{\mu\nu}_a\partial_\mu\delta A^a_\nu + g_0 f^{abc}G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) \\
&\quad + G^{\mu\nu}_a\partial_\mu\delta A^a_\nu + G^{\mu\nu}_a\partial_\mu\delta A^a_\nu + g_0 f^{abc}G^{\mu\nu}_a\delta(A^b_\mu A^c_\nu)\,, \\
\delta(G^a_{\mu\nu}G^{\mu\nu}_a) &= 4G^{\mu\nu}_a\partial_\mu\delta A^a_\nu + g_0 f^{abc}G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) + g_0 f^{abc}G^{\mu\nu}_a\delta(A^b_\mu A^c_\nu)\,.
\end{aligned} \tag{6.2.4}$$

Using $A\delta B = \delta(AB) - (\delta A)B$, we obtain

$$G^{\mu\nu}_a\partial_\mu\delta A^a_\nu = \partial_\mu(G^{\mu\nu}_a\delta A^a_\nu) - (\partial_\mu G^{\mu\nu}_a)\delta A^a_\nu\,, \tag{6.2.5}$$

and

$$\int(\mathrm{d}x)\partial_\mu(G^{\mu\nu}\delta A^a_\nu) = \oint \mathrm{d}\Sigma_\mu\, G^{\mu\nu}\delta A^a_\nu = 0\,, \tag{6.2.6}$$

and we have

$$G^{\mu\nu}_a\partial_\mu\delta A^a_\nu = -(\partial_\mu G^{\mu\nu}_a)\delta A^a_\nu\,. \tag{6.2.7}$$

Upon substituting Eq. (6.2.7) into the first term on the right-hand side of

Eq. (6.2.4), gives

$$\begin{aligned}
\delta(G^a_{\mu\nu}G^{\mu\nu}_a) &= -4(\partial_\mu G^{\mu\nu}_a)\delta A^a_\nu + g_0 f^{abc} G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) + g_0 f^{abc} G^{\mu\nu}_a \delta(A^b_\mu A^c_\nu) \\
&= -4(\partial_\mu G^{\mu\nu}_a)\delta A^a_\nu + g_0 f^{abc} G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) + g_0 f^{abc} G^a_{\mu\nu} \delta(A^\mu_b A^\nu_c)\ , \\
\delta(G^a_{\mu\nu}G^{\mu\nu}_a) &= -4(\partial_\mu G^{\mu\nu}_a)\delta A^a_\nu + 2g_0 f^{abc} G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c)\ . \qquad (6.2.8)
\end{aligned}$$

On the other hand for the second term on the right-hand side of the above equation we may write:

$$\begin{aligned}
2g_0 f^{abc} G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) &= 2g_0 f^{abc} G^a_{\mu\nu}\left[A^\mu_b \delta A^\nu_c + (\delta A^\mu_b) A^\nu_c\right] \\
&= 2g_0 f^{abc} G^a_{\mu\nu} A^\mu_b \delta A^\nu_c + 2g_0 f^{abc} G^a_{\mu\nu} (\delta A^\mu_b) A^\nu_c \\
&= 2g_0 f^{abc} A^\mu_b G^a_{\mu\nu} \delta A^\nu_c + 2g_0 f^{abc} (\delta A^\mu_b) G^a_{\mu\nu} A^\nu_c \\
&= 2g_0 f^{bca} A^\mu_c G^b_{\mu\nu} \delta A^\nu_a + 2g_0 f^{bac} (\delta A^\mu_a) G^b_{\mu\nu} A^\nu_c \\
&= -2g_0 f^{acb} A^\mu_c G^b_{\mu\nu} \delta A^\nu_a + 2g_0 f^{acb} (\delta A^\mu_a) G^b_{\mu\nu} A^\nu_c \\
&= -2g_0 f^{acb} A^\mu_c G^b_{\mu\nu} \delta A^\nu_a - 2g_0 f^{acb} (\delta A^\nu_a) G^b_{\mu\nu} A^\mu_c\ , \\
2g_0 f^{abc} G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) &= -4g_0 f^{acb} A^\mu_c G^b_{\mu\nu} \delta A^\nu_a\ . \qquad (6.2.9)
\end{aligned}$$

Substituting Eq. (6.2.9) into Eq. (6.2.8), then gives

$$\begin{aligned}
\delta(G^a_{\mu\nu}G^{\mu\nu}_a) &= -4(\partial_\mu G^{\mu\nu}_a)\delta A^a_\nu - 4g_0 f^{acb} A^\mu_c G^b_{\mu\nu}\delta A^\nu_a\ , \qquad (6.2.10) \\
-\frac{1}{4}\,\delta(G^a_{\mu\nu}G^{\mu\nu}_a) &= (\partial_\mu G^{\mu\nu}_a)\delta A^a_\nu + g_0 f^{acb} A^\mu_c G^b_{\mu\nu}\delta A^\nu_a\ . \qquad (6.2.11)
\end{aligned}$$

The second term may be rewritten as:

$$
\begin{aligned}
+\frac{1}{2\mathrm{i}}\,\delta\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi-\bar{\psi}\gamma^\mu\partial_\mu\psi\big] &= \frac{1}{2\mathrm{i}}\left\{\delta\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi\big]-\delta(\bar{\psi}\gamma^\mu\partial_\mu\psi)\right\}\\
&= \frac{1}{2\mathrm{i}}\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi+\delta(\partial_\mu\bar{\psi})\gamma^\mu\psi\\
&\qquad -\bar{\psi}\gamma^\mu\delta(\partial_\mu\psi)-\delta\bar{\psi}\gamma^\mu(\partial_\mu\psi)\;, \qquad (6.2.12)
\end{aligned}
$$

while for the third term we have,

$$
\begin{aligned}
-m_0\delta(\bar{\psi}\psi) &= -m_0\big[\bar{\psi}\delta\psi+(\delta\bar{\psi})\psi\big]\\
&= -m_0\bar{\psi}\delta\psi-m_0(\delta\bar{\psi})\psi\;. \qquad (6.2.13)
\end{aligned}
$$

On the other hand, the forth term may be expressed as

$$
\begin{aligned}
g_0\delta(\bar{\psi}\gamma_\mu A^\mu\psi) &= g_0\big[\bar{\psi}\gamma_\mu A^\mu\delta\psi+\delta(\bar{\psi}\gamma_\mu A^\mu)\psi\big]\\
&= g_0\Big\{\bar{\psi}\gamma_\mu A^\mu\delta\psi+\big[\bar{\psi}\gamma_\mu(\delta A^\mu)+(\delta\bar{\psi})\gamma_\mu A^\mu\big]\psi\Big\}\\
&= g_0\big[\bar{\psi}\gamma_\mu A^\mu\delta\psi+\bar{\psi}\gamma_\mu(\delta A^\mu)\psi+(\delta\bar{\psi})\gamma_\mu A^\mu\psi\big]\;,\\
g_0\delta(\bar{\psi}\gamma_\mu A^\mu\psi) &= g_0\bar{\psi}\gamma_\mu A^\mu\delta\psi+g_0\bar{\psi}\gamma_\mu(\delta A^\mu)\psi+g_0(\delta\bar{\psi})\gamma_\mu A^\mu\psi\;. \qquad (6.2.14)
\end{aligned}
$$

Upon substituting Eqs. (6.2.11) - (6.2.14) into Eq. (6.2.3), we obtain

$$\begin{aligned}\delta\mathcal{W}_T &= \int(\mathrm{d}x)\bigg\{(\partial_\mu G_a^{\mu\nu})\delta A_\nu^a + g_0 f^{acb}A_c^\mu G_{\mu\nu}^b\delta A_a^\nu - m_0\bar{\psi}\delta\psi - m_0(\delta\bar{\psi})\psi \\ &\quad +\frac{1}{2\mathrm{i}}\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi + \delta(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\delta(\partial_\mu\psi) - \delta\bar{\psi}\gamma^\mu(\partial_\mu\psi)\big] \\ &\quad +g_0\bar{\psi}\gamma_\mu A^\mu\delta\psi + g_0\bar{\psi}\gamma_\mu(\delta A^\mu)\psi + g_0(\delta\bar{\psi})\gamma_\mu A^\mu\psi \\ &\quad +\bar{\eta}\delta\psi + (\delta\bar{\psi})\eta + J_a^\mu\delta A_\mu^a\bigg\}\,, \qquad (6.2.15)\end{aligned}$$

and

$$\begin{aligned}\frac{\delta\mathcal{W}_T}{\delta\psi} &= -m_0\bar{\psi} + \frac{1}{2\mathrm{i}}(\partial_\mu\bar{\psi})\gamma^\mu + \frac{1}{2\mathrm{i}}(\partial_\mu\bar{\psi})\gamma^\mu + g_0\bar{\psi}\gamma_\mu A^\mu + \bar{\eta} \\ &= 0\,, \qquad (6.2.16)\end{aligned}$$

leading to

$$-m_0\bar{\psi} + (\partial_\mu\bar{\psi})\frac{\gamma^\mu}{\mathrm{i}} + g_0\bar{\psi}\gamma_\mu A^\mu = -\bar{\eta}\,, \qquad (6.2.17)$$

$$\bar{\psi}\left(\gamma^\mu\frac{\partial_\mu}{\mathrm{i}} + g_0\gamma_\mu A^\mu - m_0\right) = -\bar{\eta}\,, \qquad (6.2.18)$$

$$\bar{\psi}\left(\gamma^\mu\frac{\nabla_\mu^*}{\mathrm{i}} - m_0\right) = -\bar{\eta}\,, \qquad (6.2.19)$$

and

$$\begin{aligned}\frac{\delta\mathcal{W}_T}{\delta\bar{\psi}} &= -m_0\psi - \frac{1}{2\mathrm{i}}\,\gamma^\mu(\partial_\mu\psi) - \frac{1}{2\mathrm{i}}\,\gamma^\mu(\partial_\mu\psi) + g_0\gamma_\mu A^\mu\psi + \eta \\ &= 0\,. \qquad (6.2.20)\end{aligned}$$

Or

$$
\begin{aligned}
m_0\psi + \frac{\gamma^\mu}{\mathrm{i}}(\partial_\mu\bar{\psi}) - g_0\gamma_\mu A^\mu\psi &= \eta\,, &(6.2.21)\\
\left(\frac{\gamma^\mu}{\mathrm{i}}\,\partial_\mu - g_0\gamma_\mu A^\mu + m_0\right)\psi &= \eta\,, &(6.2.22)\\
\left(\gamma^\mu\frac{\nabla_\mu}{\mathrm{i}} + m_0\right)\psi &= \eta\,. &(6.2.23)
\end{aligned}
$$

with $\mu,\nu = 0,1,2,3$ and where ∇_μ is defined in Eq. (6.1.11).

Consider the terms not containing $\delta\psi$ and $\delta\bar{\psi}$ in Eq. (6.2.15). The first in this is given by

$$
\begin{aligned}
(\partial_\mu G_a^{\mu\nu})\delta A_\nu^a &= (\partial_\mu G_a^{\mu 0})\delta A_0^a + (\partial_\mu G_a^{\mu k})\delta A_k^a\\
&= (\partial_\mu G_a^{\mu 0})\delta A_0^a + (\partial_\mu G_a^{\mu k})\left(\delta^{ki} - \delta^{k3}\frac{\partial^i}{\partial_3}\right)\delta A_i^a\\
&= (\partial_\mu G_a^{\mu 0})\delta A_0^a + (\partial_\mu G_a^{\mu i})\delta A_i^a - \frac{\partial^i}{\partial_3}(\partial_\mu G_a^{\mu 3})\delta A_i^a\,,\\
(\partial_\mu G_a^{\mu\nu})\delta A_\nu^a &= \delta^{ab}\partial_k G_b^{k0}\delta A_0^a + \delta^{ab}\partial_\mu G_b^{\mu i}\delta A_i^a - \frac{\partial^i}{\partial_3}\,\delta^{ab}\partial_\mu G_b^{\mu 3}\delta A_i^a\,, \quad (6.2.24)
\end{aligned}
$$

while for the second term we have

$$
\begin{aligned}
g_0 f^{acb}A_c^\mu G_{\mu\nu}^b\delta A_a^\nu &= g_0 f^{acb}A_\mu^c G_b^{\mu\nu}\delta A_\nu^a\\
&= g_0 f^{acb}(A_\mu^c G_b^{\mu 0}\delta A_0^a + A_\mu^c G_b^{\mu k}\delta A_k^a)\\
&= g_0 f^{acb}\left[A_k^c G_b^{k0}\delta A_0^a + A_\mu^c G_b^{\mu k}\left(\delta^{ki} - \delta^{k3}\frac{\partial^i}{\partial_3}\right)\delta A_i^a\right]\\
&= g_0 f^{acb}\left(A_k^c G_b^{k0}\delta A_0^a + A_\mu^c G_b^{\mu i}\delta A_i^a - \frac{\partial^i}{\partial_3}A_\mu^c G_b^{\mu 3}\delta A_i^a\right),
\end{aligned}
$$

$$
\begin{aligned}
g_0 f^{acb} A^{\mu}_c G^b_{\mu\nu} \delta A^{\nu}_a &= g_0 f^{acb} A^c_k G^{k0}_b \delta A^a_0 + g_0 f^{acb} A^c_\mu G^{\mu i}_b \delta A^a_i \\
&\quad - \frac{\partial^i}{\partial_3}\, g_0 f^{acb} A^c_\mu G^{\mu 3}_b \delta A^a_i \ . \qquad (6.2.25)
\end{aligned}
$$

The third may be rewritten as

$$
\begin{aligned}
g_0 \bar{\psi} \gamma_\mu t^a (\delta A^\mu_a) \psi &= g_0 \bar{\psi} t^a \psi \gamma_\mu (\delta A^\mu_a) \\
&= g_0 \bar{\psi} t^a \psi \big[\gamma_0 (\delta A^0_a) + \gamma_k (\delta A^k_a) \big] \\
&= g_0 \bar{\psi} t^a \psi \left[\gamma_0 (\delta A^0_a) + \gamma_k \left(\delta^{ki} - \delta^{k3} \frac{\partial_i}{\partial_3} \right) \delta A^i_a \right] \\
&= g_0 \bar{\psi} t^a \psi \left[\gamma_0 (\delta A^0_a) + \left(\delta^{ki} \gamma_k - \delta^{k3} \gamma_k \frac{\partial_i}{\partial_3} \right) \delta A^i_a \right] \\
&= g_0 \bar{\psi} t^a \psi \left[\gamma_0 (\delta A^0_a) + \gamma^i \delta A^i_a - \gamma^3 \frac{\partial_i}{\partial_3}\, \delta A^i_a \right], \\
g_0 \bar{\psi} \gamma_\mu t^a (\delta A^\mu_a) \psi &= g_0 \bar{\psi} \gamma_0 t^a (\delta A^0_a) \psi + g_0 \bar{\psi} \gamma^i t^a (\delta A^i_a) \psi \\
&\quad - g_0 \bar{\psi} \gamma^3 t^a \frac{\partial_i}{\partial_3} (\delta A^i_a) \psi \ . \qquad (6.2.26)
\end{aligned}
$$

Finally for the last term we have,

$$
\begin{aligned}
J^\mu_a \delta A^a_\mu &= J^0_a \delta A^a_0 + J^k_a \delta A^a_k \\
&= J^0_a \delta A^a_0 + J^k_a \left(\delta^{ki} - \delta^{k3} \frac{\partial^i}{\partial_3} \right) \delta A^a_i \\
&= J^0_a \delta A^a_0 + \delta^{ki} J^k_a \delta A^a_i - \delta^{k3} J^k_a \frac{\partial^i}{\partial_3}\, \delta A^a_i \ , \\
J^\mu_a \delta A^a_\mu &= J^0_a \delta A^a_0 + J^i_a \delta A^a_i - J^3_a \frac{\partial^i}{\partial_3}\, \delta A^a_i \ . \qquad (6.2.27)
\end{aligned}
$$

Substituting Eqs. (6.2.24) - (6.2.27) into Eq. (6.2.15) leads to

$$\begin{aligned}
\delta\mathcal{W}_T &= \int(\mathrm{d}x)\Big\{\delta^{ab}\partial_k G_b^{k0}\delta A_0^a + \delta^{ab}\partial_\mu G_b^{\mu i}\delta A_i^a - \frac{\partial^i}{\partial_3}\,\delta^{ab}\partial_\mu G_b^{\mu 3}\delta A_i^a \\
&\quad + g_0 f^{acb}A_k^c G_b^{k0}\delta A_0^a + g_0 f^{acb}A_\mu^c G_b^{\mu i}\delta A_i^a - \frac{\partial^i}{\partial_3}\,g_0 f^{acb}A_\mu^c G_b^{\mu 3}\delta A_i^a \\
&\quad + \frac{1}{2\mathrm{i}}\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi + \delta(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\delta(\partial_\mu\psi) - \delta\bar{\psi}\gamma^\mu(\partial_\mu\psi)\big] \\
&\quad + g_0\bar{\psi}\gamma_0 t^a(\delta A_a^0)\psi + g_0\bar{\psi}\gamma^i t^a(\delta A_a^i)\psi - g_0\bar{\psi}\gamma^3 t^a\frac{\partial_i}{\partial_3}(\delta A_a^i)\psi \\
&\quad - m_0\bar{\psi}\delta\psi - m_0(\delta\bar{\psi})\psi + g_0\bar{\psi}\gamma_\mu A^\mu\delta\psi + g_0(\delta\bar{\psi})\gamma_\mu A^\mu\psi \\
&\quad + \bar{\eta}\delta\psi + (\delta\bar{\psi})\eta + J_a^0\delta A_0^a + J_a^i\delta A_i^a - J_a^3\frac{\partial^i}{\partial_3}\,\delta A_i^a\Big\}\,, \qquad (6.2.28)
\end{aligned}$$

$$\begin{aligned}
\frac{\delta\mathcal{W}_T}{\delta A_0^a(x)} &= \delta^{ab}\partial_k G_b^{k0} + g_0 f^{acb}A_k^c G_b^{k0} + g_0\bar{\psi}\gamma^0 t^a\psi + J_a^0\,, \\
&= 0\,. \qquad (6.2.29)
\end{aligned}$$

These equations lead to the following

$$\delta^{ab}\partial_k G_b^{k0} + g_0 f^{acb}A_k^c G_b^{k0} = -g_0\bar{\psi}\gamma^0 t^a\psi - J_a^0\,, \qquad (6.2.30)$$

$$\big(\delta^{ab}\partial_k + g_0 f^{acb}A_k^c\big)G_b^{k0} = -g_0\bar{\psi}\gamma^0 t^a\psi - J_a^0\,, \qquad (6.2.31)$$

$$\nabla_k^{ab}G_b^{k0} = -g_0\bar{\psi}\gamma^0 t^a\psi - J_a^0\,. \qquad (6.2.32)$$

On the other hand,

$$
\begin{aligned}
\frac{\delta \mathcal{W}_T}{\delta A_i^a(x)} &= \delta^{ab}\partial_\mu G_b^{\mu i} - \frac{\partial_i}{\partial_3}\,\delta^{ab}\partial_\mu G_b^{\mu 3} + g_0 f^{acb} A_\mu^c G_b^{\mu i} - \frac{\partial_i}{\partial_3}\,g_0 f^{acb} A_\mu^c G_b^{\mu 3} \\
&\quad + g_0\bar{\psi}\gamma^i t^a\psi - \frac{\partial_i}{\partial_3}\,g_0\bar{\psi}\gamma^3 t^a\psi + J_a^i - \frac{\partial_i}{\partial_3} J_a^3 \\
&= 0\,, \qquad (6.2.33)
\end{aligned}
$$

gives

$$
\begin{aligned}
0 &= \left(\delta^{ab}\partial_\mu G_b^{\mu i} + g_0 f^{acb} A_\mu^c G_b^{\mu i}\right) - \left(\frac{\partial_i}{\partial_3}\,\delta^{ab}\partial_\mu G_b^{\mu 3} + \frac{\partial_i}{\partial_3}\,g_0 f^{acb} A_\mu^c G_b^{\mu 3}\right) \\
&\quad - \left(\frac{\partial_i}{\partial_3}\,g_0\bar{\psi}\gamma^3 t^a\psi + \frac{\partial_i}{\partial_3} J_a^3\right) + g_0\bar{\psi}\gamma^i t^a\psi + J_a^i\,, \qquad (6.2.34)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(\delta^{ab}\partial_\mu + g_0 f^{acb} A_\mu^c\right) G_b^{\mu i} - \frac{\partial_i}{\partial_3}\Big[\left(\delta^{ab}\partial_\mu + g_0 f^{acb} A_\mu^c\right) G_b^{\mu 3}\Big] \\
&\quad - \frac{\partial_i}{\partial_3}\left(\,g_0\bar{\psi}\gamma^3 t^a\psi + J_a^3\right) + g_0\bar{\psi}\gamma^i t^a\psi + J_a^i\,. \qquad (6.2.35)
\end{aligned}
$$

That is we have,

$$
\nabla_\mu^{ab} G_b^{\mu i} - \frac{\partial_i}{\partial_3}\left(\nabla_\mu^{ab} G_b^{\mu 3} + g_0\bar{\psi}\gamma^3 t^a\psi + J_a^3\right) = g_0\bar{\psi}\gamma^i t^a\psi + J_a^i\,, \qquad (6.2.36)
$$

where

$$
\nabla_\mu^{ab} = \delta^{ab}\partial_\mu + g_0 f^{acb} A_\mu^c\,. \qquad (6.2.37)
$$

The canonical conjugate momenta to A^μ are defined by

$$
\pi^\mu = \pi[A^\mu] = \frac{\delta \mathcal{W}}{\delta \dot{A}_\mu} = \frac{\delta \mathcal{W}}{\delta(\partial_0 A_\mu)}\,. \qquad (6.2.38)
$$

To obtain the expressions for the latter, consider the first term on the right-hand side of Eq. (6.2.15). This is given by

$$\begin{aligned}
\int(\mathrm{d}x)\,(\partial_\mu G_a^{\mu\nu})\,\delta A_\nu^a &= \int(\mathrm{d}x)\,\left[\left(\partial_\mu G_a^{\mu 0}\right)\delta A_0^a + \left(\partial_\mu G_a^{\mu k}\right)\delta A_k^a\right]\,, \quad k=1,2,3 \\
&= \int(\mathrm{d}x)\left[\left(\partial_\mu G_a^{\mu 0}\right)\delta A_0^a + \left(\partial_\mu G_a^{\mu k}\right)\left(\delta^{ki}-\delta^{k3}\frac{\partial^i}{\partial_3}\right)\delta A_i^a\right] \\
&= \int(\mathrm{d}x)\left[\left(\partial_\mu G_a^{\mu 0}\right)\delta A_0^a + \left(\partial_\mu G_a^{\mu i}-\frac{\partial^i}{\partial_3}\,\partial_\mu G_a^{\mu 3}\right)\delta A_i^a\right], \\
\int(\mathrm{d}x)\,(\partial_\mu G_a^{\mu\nu})\,\delta A_\nu^a &= \int(\mathrm{d}x)\left[-G_a^{\mu 0}\delta\left(\partial_\mu A_0^a\right) - \left(G_a^{\mu i}-\frac{\partial^i}{\partial_3}\,G_a^{\mu 3}\right)\delta(\partial_\mu A_i^a)\right].
\end{aligned} \tag{6.2.39}$$

The canonical conjugate variables to A_a^1, A_a^2, are then given by

$$\begin{aligned}
\pi_a^i &= -G_a^{0i} + \frac{\partial^i}{\partial_3}\,G_a^{03} \\
&= G_a^{i0} - \frac{\partial^i}{\partial_3}\,G_a^{30}\,,
\end{aligned} \tag{6.2.40}$$

with

$$\pi_a^0 = 0 \qquad , \qquad \pi_a^3 = 0\,. \tag{6.2.41}$$

We may rewrite Eq. (6.2.40) as

$$\pi_a^\mu = G_a^{\mu 0} - \partial_3^{-1} g^{\mu k}\partial_k G_a^{30}\,. \tag{6.2.42}$$

Upon multiplying Eq. (6.2.42) by ∇_μ^{ba}, we get

$$\nabla_\mu^{ba}\pi_a^\mu = \nabla_\mu^{ba} G_a^{\mu 0} - \nabla_\mu^{ba}\partial_3^{-1} g^{\mu k}\partial_k G_a^{30}\,. \tag{6.2.43}$$

We make use of the field equation

$$\nabla^{ab}_{\mu} G^{\mu\nu}_{b} = -\left(\delta^{\nu}{}_{\sigma}\delta^{ac} - g^{\nu k}\partial_k D^{ab}\nabla^{bc}_{\sigma}\right)\left[J^{\sigma}_{c} + g_0\bar{\psi}\gamma^{\sigma}t_c\psi\right], \qquad (6.2.44)$$

$$\begin{aligned}\nabla^{ba}_{\mu} G^{\mu 0}_{a} &= -\left(\delta^{0}{}_{\sigma}\delta^{bc} - g^{0k}\partial_k D^{ba}\nabla^{ac}_{\sigma}\right)\left[J^{\sigma}_{c} + g_0\bar{\psi}\gamma^{\sigma}t_c\psi\right] \\ &= -J^0_b - g_0\bar{\psi}\gamma^0 t_b\psi\ , \end{aligned} \qquad (6.2.45)$$

and substitute the above equation into Eq. (6.2.43), to obtain

$$\nabla^{ba}_{\mu}\pi^{\mu}_{a} = -J^0_b - g_0\bar{\psi}\gamma^0 t_b\psi - \nabla^{ba}_{k}\partial_3^{-1}\partial_k G^{30}_{a}\ , \qquad (6.2.46)$$

$$\nabla^{ba}_{k}\partial_3^{-1}\partial_k G^{30}_{a} = -\left[J^0_b + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla^{ba}_{\mu}\pi^{\mu}_{a}\right]. \qquad (6.2.47)$$

Let $\nabla^{ba}_{k}\partial_k \equiv \mathscr{O}^{ba}(x)$, $\partial_3^{-1}G^{30}_{a} \equiv A_a(x)$ to obtain from Eq. (6.2.47):

$$\mathscr{O}^{ba}(x)A_a(x) = \tilde{J}^b(x)\ , \qquad (6.2.48)$$

where

$$\tilde{J}^b = -[J^0_b + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla^{ba}_{\mu}\pi^{\mu}_{a}]\ . \qquad (6.2.49)$$

We now define the Green operator $D^{ab}(x,x')$ satisfying the differential equation

$$\nabla^{ac}_{k}\partial_k D^{cd}(x,x') = \delta^4(x,x')\delta^{ad}\ , \qquad (6.2.50)$$

to obtain from Eqs. (6.2.48), (6.2.47)

$$\partial_3^{-1}G^{30}_{a} = -D_{ab}\left[J^0_b + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla^{ba}_{\mu}\pi^{\mu}_{a}\right]. \qquad (6.2.51)$$

Finally we make use of Eq. (6.2.47) to solve for π^μ_a as follows:

$$\pi^\mu_a \;=\; G^{\mu 0}_a + g^{\mu k}\partial_k D_{ab}\left[J^0_b + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla^{bc}_\nu \pi^\nu_c\right] . \tag{6.2.52}$$

In particular, one may then readily express $G^{\mu 0}_a$ as in the form

$$G^{\mu 0}_a \;=\; \pi^\mu_a - g^{\mu k}\partial_k D_{ab}\left[J^0_b + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla^{bc}_\nu \pi^\nu_c\right] . \tag{6.2.53}$$

We note that the right-hand side of Eq. (6.2.53) is expressed in terms of the independent fields A^1_a, A^2_a, and their canonical conjugate momenta and involves no time derivatives. Here we recall that A^3_a is expressed in terms of A^1_a, A^2_a with no time derivative. Accordingly, with the (independent) fields and their canonical conjugate momenta kept *fixed*, we obtain the following functional derivative of $G^{\mu 0}_a(x)$ with respect to $J^\nu_b(x')$ to be

$$\begin{aligned}
\frac{\delta}{\delta J^\nu_b(x')}\, G^{\mu 0}_a(x) \;&=\; \frac{\delta}{\delta J^\nu_b(x')}\left[\pi^\mu_a - g^{\mu k}\partial_k D_{ab}\left(J^0_b + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla^{bc}_\nu \pi^\nu_c\right)\right] \\
&=\; \frac{\delta}{\delta J^\nu_b(x')}\left[\pi^\mu_a - g^{\mu k}\partial_k D_{ab}\left(\delta^0{}_\nu J^\nu_b + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla^{bc}_\nu \pi^\nu_c\right)\right] ,
\end{aligned}$$

or

$$\frac{\delta}{\delta J^\nu_b(x')}\, G^{\mu 0}_a(x) \;=\; -g^{\mu k}\delta^0{}_\nu \partial_k D_{ab}(x, x') , \tag{6.2.54}$$

where $\mu, \nu = 0, 1, 2, 3$ and $k = 1, 2, 3$. On the other hand, $G^{kl}_a = \partial^k A^l_a - \partial^l A^k_a + g_0 f^{abc} A^b_k A^c_l$, $k, l = 1, 2, 3$, may be expressed in terms of the independent fields A^1_a, A^2_a and involves no time derivatives. Accordingly with A^1_a, A^2_a and their canonical conjugate variables kept fixed, we also have

$$\frac{\delta}{\delta J^\nu_b(x')}\, G^{kl}_a(x) \;=\; 0 . \tag{6.2.55}$$

Similarly, with ψ and $\bar{\psi}$ kept fixed, we have the obvious functional derivative expression

$$\frac{\delta}{\delta J_b^\nu(x')}\left[\bar{\psi}(x)\gamma^\mu t^a\psi(x)\right] \;=\; 0\,. \tag{6.2.56}$$

The action principle gives

$$\frac{\partial}{\partial g_0}\langle 0_+\,|\,0_-\rangle \;=\; \mathrm{i}\left\langle 0_+\left|\int(\mathrm{d}x)\;\hat{\mathscr{L}}_I\right|0_-\right\rangle\,, \tag{6.2.57}$$

where

$$\hat{\mathscr{L}}_I \;=\; \frac{\partial}{\partial g_0}\,\mathscr{L}\,. \tag{6.2.58}$$

The first term on the right-hand side of Eq. (6.1.2) may be expressed as

$$\begin{aligned}
-\frac{1}{4}\,G^a_{\mu\nu}G^{\mu\nu}_a \;&=\; -\frac{1}{4}\left(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g_0 f^{abc}A^b_\mu A^c_\nu\right)\left(\partial^\mu A^\nu_a-\partial^\nu A^\mu_a+g_0 f^{abc}A^\mu_b A^\nu_c\right),\\
&=\; -\frac{1}{4}\Big[\partial_\mu A^a_\nu\partial^\mu A^\nu_a-\partial_\mu A^a_\nu\partial^\nu A^\mu_a+\left(\partial_\mu A^a_\nu\right)g_0 f^{abc}A^\mu_b A^\nu_c\\
&\qquad -\partial_\nu A^a_\mu\partial^\mu A^\nu_a+\partial_\nu A^a_\mu\partial^\nu A^\mu_a-\left(\partial_\nu A^a_\mu\right)g_0 f^{abc}A^\mu_b A^\nu_c\\
&\qquad +g_0 f^{abc}A^b_\mu A^c_\nu\left(\partial^\mu A^\nu_a\right)-g_0 f^{abc}A^b_\mu A^c_\nu\left(\partial^\nu A^\mu_a\right)\\
&\qquad +\left(g_0 f^{abc}\right)^2 A^b_\mu A^c_\nu A^\mu_b A^\nu_c\Big]\,,
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{4}\,\partial_\mu A^a_\nu \partial^\mu A^\nu_a + \frac{1}{4}\,\partial_\mu A^a_\nu \partial^\nu A^\mu_a - \frac{1}{4}\left(\partial_\mu A^a_\nu\right) g_0 f^{abc} A^\mu_b A^\nu_c \\
&\quad +\frac{1}{4}\,\partial_\nu A^a_\mu \partial^\mu A^\nu_a - \frac{1}{4}\,\partial_\nu A^a_\mu \partial^\nu A^\mu_a + \frac{1}{4}\left(\partial_\nu A^a_\mu\right) g_0 f^{abc} A^\mu_b A^\nu_c \\
&\quad -\frac{1}{4}\, g_0 f^{abc} A^b_\mu A^c_\nu \left(\partial^\mu A^\nu_a\right) + \frac{1}{4}\, g_0 f^{abc} A^b_\mu A^c_\nu \left(\partial^\nu A^\mu_a\right) \\
&\quad -\frac{1}{4}\left(g_0 f^{abc}\right)^2 A^b_\mu A^c_\nu A^\mu_b A^\nu_c \,, \\
&= -\frac{1}{4}\,\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{4}\,\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{4}\left(\partial^\mu A^\nu_a\right) g_0 f^{abc} A^b_\mu A^c_\nu \\
&\quad -\frac{1}{4}\,\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{4}\,\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{4}\left(\partial^\mu A^\nu_a\right) g_0 f^{abc} A^b_\mu A^c_\nu \\
&\quad +\frac{1}{4}\, g_0 f^{abc} A^b_\mu A^c_\nu \left(\partial^\nu A^\mu_a\right) + \frac{1}{4}\, g_0 f^{abc} A^b_\mu A^c_\nu \left(\partial^\nu A^\mu_a\right) \\
&\quad -\frac{1}{4}\left(g_0 f^{abc}\right)^2 A^b_\mu A^c_\nu A^\mu_b A^\nu_c \,,
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{4}\, G^a_{\mu\nu} G^{\mu\nu}_a &= -\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{2}\left(\partial^\mu A^\nu_a\right) g_0 f^{abc} A^b_\mu A^c_\nu \\
&\quad +\frac{1}{2}\, g_0 f^{abc} A^b_\mu A^c_\nu \left(\partial^\nu A^\mu_a\right) - \frac{1}{4}\left(g_0 f^{abc}\right)^2 A^b_\mu A^c_\nu A^\mu_b A^\nu_c \,. \qquad (6.2.59)
\end{aligned}
$$

Substituting this into Eq. (6.1.2), gives for the Lagrangian density

$$
\begin{aligned}
\mathscr{L} &= -\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{2}\left(\partial^\mu A^\nu_a\right) g_0 f^{abc} A^b_\mu A^c_\nu \\
&\quad +\frac{1}{2}\, g_0 f^{abc} A^b_\mu A^c_\nu \left(\partial^\nu A^\mu_a\right) - \frac{1}{4}\left(g_0 f^{abc}\right)^2 A^b_\mu A^c_\nu A^\mu_b A^\nu_c \\
&\quad +\frac{1}{2\mathrm{i}}\left[(\partial_\mu \bar{\psi})\gamma^\mu \psi - \bar{\psi}\gamma^\mu \partial_\mu \psi\right] - m_0 \bar{\psi}\psi + g_0 \bar{\psi}\gamma_\mu A^\mu \psi \,. \qquad (6.2.60)
\end{aligned}
$$

The derivative of the Lagrangian with respect to the coupling parent g_0 is given by

$$\begin{aligned}\frac{\partial}{\partial g_0}\mathscr{L} &= \frac{\partial}{\partial g_0}\Big\{-\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{2}\left(\partial^\mu A^\nu_a\right) g_0 f^{abc} A^b_\mu A^c_\nu \\ &\quad +\frac{1}{2}\, g_0 f^{abc} A^b_\mu A^c_\nu \left(\partial^\nu A^\mu_a\right) - \frac{1}{4}\left(g_0 f^{abc}\right)^2 A^b_\mu A^c_\nu A^\mu_b A^\nu_c \\ &\quad +\frac{1}{2\mathrm{i}}\left[(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\right] - m_0\bar{\psi}\psi + g_0\bar{\psi}\gamma_\mu A^\mu\psi\Big\} \\ &= -\frac{1}{2}\left(\partial^\mu A^\nu_a\right) f^{abc} A^b_\mu A^c_\nu + \frac{1}{2} f^{abc} A^b_\mu A^c_\nu \left(\partial^\nu A^\mu_a\right) \\ &\quad -(2)\frac{1}{4}\, g_0 f^{abc} f^{abc} A^b_\mu A^c_\nu A^\mu_b A^\nu_c + \bar{\psi}\gamma_\mu A^\mu\psi\,, \\ \frac{\partial}{\partial g_0}\mathscr{L} &= -\frac{1}{2} f^{abc} A^b_\mu A^c_\nu \left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g_0 f^{abc} A^\mu_b A^\nu_c\right) + \bar{\psi}\gamma_\mu A^\mu\psi\,,\end{aligned}$$

or

$$\frac{\partial}{\partial g_0}\mathscr{L} = -\frac{1}{2} f^{abc} A^b_\mu A^c_\nu G^{\mu\nu}_a + \bar{\psi}\gamma_\mu A^\mu\psi\,. \tag{6.2.61}$$

We may also write

$$\begin{aligned} f^{abc} A^b_\mu A^c_\nu G^{\mu\nu}_a &= f^{abc} A^b_0 A^c_\nu G^{0\nu}_a + f^{abc} A^b_k A^c_\nu G^{k\nu}_a \\ &= f^{abc} A^b_0 A^c_0 G^{00}_a + f^{abc} A^b_0 A^c_l G^{0l}_a \\ &\quad + f^{abc} A^b_k A^c_0 G^{k0}_a + f^{abc} A^b_k A^c_l G^{kl}_a \\ &= f^{abc} A^b_l A^c_0 G^{l0}_a + f^{abc} A^b_k A^c_0 G^{k0}_a + f^{abc} A^b_k A^c_l G^{kl}_a\,, \\ f^{abc} A^b_\mu A^c_\nu G^{\mu\nu}_a &= 2 f^{abc} A^b_k A^c_0 G^{k0}_a + f^{abc} A^b_k A^c_l G^{kl}_a\,. \end{aligned} \tag{6.2.62}$$

In the sequel, we set $(-\mathrm{i})\delta/\delta J^\mu_a = A'^a_\mu$, $(-\mathrm{i})\delta/\delta\bar{\eta} = \psi'$, $(-\mathrm{i})\delta/\delta\eta = \bar{\psi}'$. [Here we note that $G'^a_{\mu\nu}$ on the right-hand side of Eq. (5.30) of Manoukian (1986) should be replaced by $F'^a_{\mu\nu} = \partial_\mu A'^a_\nu - \partial_\nu A'^a_\mu$.]

Now we use the rule of functional differentiations (cf. Manoukian, 2006, Chap. 11; Manoukian et al., 2007) that for an operator $\mathscr{O}(x)$

$$(-\mathrm{i})\frac{\delta}{\delta J^\mu_a(x')}\langle 0_+|\mathscr{O}(x)|0_-\rangle \;=\; \left\langle 0_+\middle|(A^a_\mu(x')\mathscr{O}(x))_+\middle|0_-\right\rangle - \mathrm{i}\left\langle 0_+\middle|\frac{\delta}{\delta J^\mu_a(x')}\,\mathscr{O}(x)\middle|0_-\right\rangle, \tag{6.2.63}$$

where $(\ldots)_+$ denotes the time-ordered product, and the functional derivative of $\mathscr{O}(x)$ in the second term on the right-hand of Eq. (6.2.63) is taken as in Eqs. (6.2.54) - (6.2.56) with the (independent) fields and their canonical conjugate momenta kept fixed. Here we recall that A^3_a may be expressed in terms of A^1_a, A^2_a and involves no time derivatives.

From Eqs. (6.2.57) and (6.2.61), the action principle gives:

$$\frac{\partial}{\partial g_0}\langle 0_+|0_-\rangle \;=\; \mathrm{i}\left\langle 0_+\middle|\int(\mathrm{d}x)\left[-\frac{1}{2}f^{abc}A^b_\mu A^c_\nu G^{\mu\nu}_a + \bar{\psi}\gamma^\mu A_\mu\psi\right]\middle|0_-\right\rangle\;. \tag{6.2.64}$$

Thus Eq. (6.2.62), leads to

$$\begin{aligned}\frac{\partial}{\partial g_0}\langle 0_+|0_-\rangle \;&=\; \mathrm{i}\Big\langle 0_+\Big|\int(\mathrm{d}x)\Big[-\frac{1}{2}\left(2f^{abc}A^b_kA^c_0G^{k0}_a + f^{abc}A^b_kA^c_lG^{kl}_a\right)\\&\qquad\qquad +\bar{\psi}\gamma^\mu A_\mu\psi\Big]\Big|0_-\Big\rangle\\&=\; \mathrm{i}\Big\langle 0_+\Big|\int(\mathrm{d}x)\Big(-f^{abc}A^b_kA^c_0G^{k0}_a - \frac{1}{2}f^{abc}A^b_kA^c_lG^{kl}_a\\&\qquad\qquad +\bar{\psi}\gamma^\mu A_\mu\psi\Big)\Big|0_-\Big\rangle\,,\end{aligned}$$

$$\frac{\partial}{\partial g_0}\langle 0_+|0_-\rangle = -\mathrm{i}\int(\mathrm{d}x)\left\langle 0_+\middle|\left(f^{abc}A^b_kA^c_0G^{k0}_a\right)\middle|0_-\right\rangle$$
$$-\frac{\mathrm{i}}{2}\int(\mathrm{d}x)\left\langle 0_+\middle|f^{abc}A^b_kA^c_lG^{kl}_a\middle|0_-\right\rangle$$
$$+\mathrm{i}\int(\mathrm{d}x)\left\langle 0_+\middle|\bar{\psi}\gamma^\mu A_\mu\psi\middle|0_-\right\rangle\ . \tag{6.2.65}$$

Now we use the rule of functional differentiations of an operator $\mathscr{O}(x)$, and for the first term of the right-hand side of Eq. (6.2.65), we obtain

$$-\mathrm{i}\int(\mathrm{d}x)\left\langle 0_+\middle|\left(f^{abc}A^b_kA^c_0G^{k0}_a\right)\middle|0_-\right\rangle = -\mathrm{i}\int(\mathrm{d}x)f^{abc}A'^b_k\left\langle 0_+\middle|A^c_0G^{k0}_a\middle|0_-\right\rangle$$
$$= -\mathrm{i}\int(\mathrm{d}x)f^{abc}A'^b_k\Bigg[(-\mathrm{i})\frac{\delta}{\delta J^0_c(x')}\left\langle 0_+\middle|G^{k0}_a\middle|0_-\right\rangle$$
$$+\mathrm{i}\left\langle 0_+\middle|\frac{\delta}{\delta J^0_c(x')}G^{k0}_a\middle|0_-\right\rangle\Bigg]\ . \tag{6.2.66}$$

Using Eq. (6.2.54) in the last term of above equation and denoting $(-\mathrm{i})\delta/\delta J^\mu_a = A'^a_\mu$, as always, we obtain

$$-\mathrm{i}\int(\mathrm{d}x)\left\langle 0_+\middle|\left(f^{abc}A^b_kA^c_0G^{k0}_a\right)\middle|0_-\right\rangle$$
$$= -\mathrm{i}\int(\mathrm{d}x)f^{abc}A'^b_k\Big[A'^c_0G'^{k0}_a\langle 0_+|0_-\rangle + \mathrm{i}\,\langle 0_+|-\partial_kD_{ac}(x,x')|0_-\rangle\Big]$$
$$= -\mathrm{i}\int(\mathrm{d}x)f^{abc}A'^b_k\Big[A'^c_0G'^{k0}_a\langle 0_+|0_-\rangle - \mathrm{i}\partial_kD'_{ac}(x,x)\langle 0_+|0_-\rangle\Big]$$
$$= \int(\mathrm{d}x)\Big[-\mathrm{i}f^{abc}A'^b_kA'^c_0G'^{k0}_a\langle 0_+|0_-\rangle - f^{abc}A'^b_k\partial_kD'_{ac}(x,x)\langle 0_+|0_-\rangle\Big]$$
$$= \int(\mathrm{d}x)\Big[-\mathrm{i}f^{abc}A'^b_kA'^c_0G'^{k0}_a\langle 0_+|0_-\rangle - f^{bca}A'^b_k\partial_kD'_{ac}(x,x)\langle 0_+|0_-\rangle\Big]\ . \tag{6.2.67}$$

The second term on the right-hand side of Eq. (6.2.65) may be expressed as

$$-\frac{\mathrm{i}}{2}\int(\mathrm{d}x)\left\langle 0_+\left|f^{abc}A_k^b A_l^c G_a^{kl}\right|0_-\right\rangle$$

$$=-\frac{\mathrm{i}}{2}\int(\mathrm{d}x)f^{abc}A_k'^b\left\langle 0_+\left|A_l^c G_a^{kl}\right|0_-\right\rangle$$

$$=-\frac{\mathrm{i}}{2}\int(\mathrm{d}x)f^{abc}A_k'^b\left[(-\mathrm{i})\frac{\delta}{\delta J_c^l(x')}\left\langle 0_+\left|G_a^{kl}\right|0_-\right\rangle+\mathrm{i}\left\langle 0_+\left|\frac{\delta}{\delta J_c^l(x')}\,G_a^{kl}\right|0_-\right\rangle\right]. \tag{6.2.68}$$

Using Eq. (6.2.55) this simplifies to

$$-\frac{\mathrm{i}}{2}\int(\mathrm{d}x)\left\langle 0_+\left|f^{abc}A_k^b A_l^c G_a^{kl}\right|0_-\right\rangle = -\frac{\mathrm{i}}{2}\int(\mathrm{d}x)f^{abc}A_k'^b A_l'^c G_a'^{kl}\left\langle 0_+\left|0_-\right\rangle\right.$$

$$=\int(\mathrm{d}x)\left(-\frac{\mathrm{i}}{2}f^{abc}A_k'^b A_l'^c G_a'^{kl}\right)\left\langle 0_+\left|0_-\right\rangle\right.. \tag{6.2.69}$$

For the last term on the right-hand side of Eq. (6.2.65) we get

$$+\,\mathrm{i}\int(\mathrm{d}x)\left\langle 0_+\left|\bar{\psi}\gamma^\mu A_\mu\psi\right|0_-\right\rangle$$

$$=+\,\mathrm{i}\int(\mathrm{d}x)\left\langle 0_+\left|\bar{\psi}\gamma^\mu A_\mu^a t_a\psi\right|0_-\right\rangle$$

$$=+\,\mathrm{i}\int(\mathrm{d}x)\left[(-\mathrm{i})\frac{\delta}{\delta J_a^\mu(x')}\left\langle 0_+\left|\bar{\psi}\gamma^\mu t_a\psi\right|0_-\right\rangle+\mathrm{i}\left\langle 0_+\left|\frac{\delta}{\delta J_a^\mu(x')}\,(\bar{\psi}\gamma^\mu t_a\psi)\right|0_-\right\rangle\right], \tag{6.2.70}$$

and using Eq. (6.2.56) this again simplifies to the following

$$+\mathrm{i}\int(\mathrm{d}x)\left\langle 0_+\left|\bar{\psi}\gamma^\mu A_\mu\psi\right|0_-\right\rangle = \mathrm{i}\int(\mathrm{d}x)\bar{\psi}'\gamma^\mu A'^a_\mu t_a\psi'\left\langle 0_+\,|\,0_-\right\rangle$$

$$= \int(\mathrm{d}x)\;\mathrm{i}\bar{\psi}'\gamma^\mu A'^a_\mu t_a\psi'\left\langle 0_+\,|\,0_-\right\rangle\;. \quad (6.2.71)$$

Equations (6.2.67), (6.2.69) and (6.2.71), may be then used to rewrite Eq. (6.2.65) in the form

$$\frac{\partial}{\partial g_0}\left\langle 0_+\,|\,0_-\right\rangle = \int(\mathrm{d}x)\Big[-\mathrm{i}f^{abc}A'^b_k A'^c_0 G'^{k0}_a - \frac{\mathrm{i}}{2}f^{abc}A'^b_k A'^c_l G'^{kl}_a + \mathrm{i}\bar{\psi}'\gamma^\mu A'^a_\mu t_a\psi'$$

$$-f^{bca}A'^b_k\partial_k D'_{ac}(x,x)\Big]\left\langle 0_+\,|\,0_-\right\rangle$$

$$= \int(\mathrm{d}x)\big[\mathrm{i}\hat{\mathscr{L}}_I{}'(x) - f^{bca}A'^b_k\partial_k D'_{ac}(x,x)\big]\left\langle 0_+\,|\,0_-\right\rangle\;. \quad (6.2.72)$$

Using a matrix notation

$$D^{ab}(x,x') = \left[\left\langle x\left|\left(\frac{1}{\boldsymbol{\partial}^2 - \mathrm{i}g_0 A_k\partial_k}\right)\right|x'\right\rangle\right]^{ab}\;, \quad (6.2.73)$$

the notation

$$\mathrm{Tr}[f] = \int(\mathrm{d}x)\;f^{aa}(x,x)\;, \quad (6.2.74)$$

and the fact that $f^{bca}A^b_k = \mathrm{i}(A_k)^{ca}$, we may rewrite the second factor within the square brackets in Eq. (6.2.72) as

$$\frac{\partial}{\partial g_0}\left\langle 0_+\,|\,0_-\right\rangle = \int(\mathrm{d}x)\big[\mathrm{i}\hat{\mathscr{L}}_I{}'(x) - f^{bca}A'^b_k\partial_k D'_{ac}(x,x)\big]\left\langle 0_+\,|\,0_-\right\rangle$$

$$= \left\{\int(\mathrm{d}x)\;\mathrm{i}\hat{\mathscr{L}}_I{}'(x) - \int(\mathrm{d}x)\frac{\mathrm{i}A'_k\partial_k}{[\boldsymbol{\partial}^2 - \mathrm{i}g_0 A_l\partial_l]}\right\}\left\langle 0_+\,|\,0_-\right\rangle$$

$$= \left\{\mathrm{i}\int(\mathrm{d}x)\;\hat{\mathscr{L}}_I{}'(x) + \mathrm{Tr}\,\frac{-\mathrm{i}A'_k\partial_k}{[\boldsymbol{\partial}^2 - \mathrm{i}g_0 A'_l\partial^l]}\right\}\left\langle 0_+\,|\,0_-\right\rangle\;, \quad (6.2.75)$$

$$\int \delta \langle 0_+ | 0_- \rangle = \int \delta g_0 \left[\mathrm{i} \int (\mathrm{d}x)\, \hat{\mathscr{L}}_I{}'(x) + \mathrm{Tr}\, \frac{-\mathrm{i} A_k' \partial_k}{[\boldsymbol{\partial}^2 - \mathrm{i} g_0 A_l' \partial^l]} \right] \langle 0_+ | 0_- \rangle \,. \quad (6.2.76)$$

which integrates out to

$$\langle 0_+ | 0_- \rangle_{g_0} = \exp \int_0^{g_0} \mathrm{d}g_0 \left[\mathrm{i} \int (\mathrm{d}x)\, \hat{\mathscr{L}}_I{}'(x) + \mathrm{Tr}\, \frac{-\mathrm{i} A_k' \partial_k}{[\boldsymbol{\partial}^2 - \mathrm{i} g_0 A_l' \partial^l]} \right] \langle 0_+ | 0_- \rangle_0 \,, \quad (6.2.77)$$

where

$$\hat{\mathscr{L}}_I{}'(x) \;=\; -\frac{1}{2} f^{abc} A_\mu'^b A_\nu'^c G_a'^{\mu\nu} + \bar{\psi}' \gamma_\mu A'^\mu \psi' \,. \quad (6.2.78)$$

That is,

$$\langle 0_+ | 0_- \rangle_{g_0} = \left[\exp \mathrm{i} \int (\mathrm{d}x)\, \mathscr{L}_I{}'(x) \right] \exp \mathrm{Tr} \ln \left[1 - \mathrm{i} g_0 \frac{1}{\boldsymbol{\partial}^2} A_k' \partial^k \right] \langle 0_+ | 0_- \rangle_0 \,, \quad (6.2.79)$$

where

$$\mathscr{L}_I{}'(x) = -\frac{1}{4} G_{\mu\nu}'^a G_a'^{\mu\nu} + \frac{1}{2\mathrm{i}} [(\partial_\mu \bar{\psi}') \gamma^\mu \psi' - \bar{\psi}' \gamma^\mu \partial_\mu \psi'] - m_0 \bar{\psi}' \psi' + g_0 \bar{\psi}' \gamma_\mu A'^\mu \psi' \,. \quad (6.2.80)$$

Using the identity $\det M = \exp \mathrm{Tr} \ln M$, we may rewrite Eq. (6.2.79) as

$$\langle 0_+ | 0_- \rangle \;=\; \left[\exp \mathrm{i} \int (\mathrm{d}x)\, \mathscr{L}_I{}'(x) \right] \det \left[1 - \mathrm{i} g_0 \frac{1}{\boldsymbol{\partial}^2} A_k' \partial^k \right] \langle 0_+ | 0_- \rangle_0 \,, \quad (6.2.81)$$

where for simplicity of the notation we have written the exact expression for $\langle 0_+ | 0_- \rangle_{g_0}$ simply as $\langle 0_+ | 0_- \rangle$.

Thus we learn that the naïve Feynman rules are modified by the presence of the multiplicative factor

$$\det \left[1 - \mathrm{i} g_0 \frac{1}{\boldsymbol{\partial}^2} A_k' \partial^k \right] , \quad (6.2.82)$$

as a functional differential operation in the expression for $\langle 0_+ | 0_- \rangle_{g_0}$.

6.3 Gauge Invariance and Modification of the FP Factor

Now consider the modification of the Lagrangian density $\mathscr{L}$ in Eq. (6.1.2):

$$\mathscr{L} \to \mathscr{L} + \lambda\bar{\psi}\psi G^a_{\mu\nu}G^{\mu\nu}_a \equiv \mathscr{L}_1 \ , \tag{6.3.1}$$

which is obviously gauge invariant under the simultaneous local gauge transformations in Eqs. (6.1.7) - (6.1.9).

The Lagrangian density $\mathscr{L}_{1T} = \mathscr{L}_1 + \mathscr{L}_S$, where $\mathscr{L}_S$ is defined in Eq. (6.1.3) , is given by

$$\begin{aligned}
\mathscr{L}_{1T} &= \mathscr{L}_1 + \mathscr{L}_S \\
&= -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\big] - m_0\bar{\psi}\psi + g_0\bar{\psi}\gamma_\mu A^\mu\psi \\
&\quad +\lambda\bar{\psi}\psi G^a_{\mu\nu}G^{\mu\nu}_a + \bar{\eta}\psi + \bar{\psi}\eta + J^\mu_a A^a_\mu \ .
\end{aligned} \tag{6.3.2}$$

The action in question is defined by

$$\begin{aligned}
\mathcal{W} &= \int(\mathrm{d}x)\ \mathscr{L}_{1T} \\
&= \int(\mathrm{d}x)\bigg\{ -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\big] - m_0\bar{\psi}\psi \\
&\qquad +g_0\bar{\psi}\gamma_\mu A^\mu\psi + \lambda\bar{\psi}\psi G^a_{\mu\nu}G^{\mu\nu}_a + \bar{\eta}\psi + \bar{\psi}\eta + J^\mu_a A^a_\mu\bigg\} \\
&= \int(\mathrm{d}x)\bigg\{ -\frac{1}{4}(1-4\lambda\bar{\psi}\psi)\ G^a_{\mu\nu}G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\big] \\
&\qquad -m_0\bar{\psi}\psi + g_0\bar{\psi}\gamma_\mu A^\mu\psi + \bar{\eta}\psi + \bar{\psi}\eta + J^\mu_a A^a_\mu\bigg\} \ ,
\end{aligned} \tag{6.3.3}$$

$$\delta\mathcal{W} = \int(\mathrm{d}x)\Big\{ -\frac{1}{4}(1-4\lambda\bar{\psi}\psi)\,\delta G^a_{\mu\nu}G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}\delta\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\big]$$
$$-m_0\delta(\bar{\psi}\psi) + g_0\delta(\bar{\psi}\gamma_\mu A^\mu\psi) + \bar{\eta}\delta\psi + (\delta\bar{\psi})\eta + J^\mu_a\delta A^a_\mu\Big\}\,. \tag{6.3.4}$$

Corresponding to the first term in Eq. (6.3.4), we have the variation:

$$\begin{aligned}
\delta\left[(1-4\lambda\bar{\psi}\psi)G^a_{\mu\nu}G^{\mu\nu}_a\right] &= (1-4\lambda\bar{\psi}\psi)\delta\left(G^a_{\mu\nu}G^{\mu\nu}_a\right) + \delta(1-4\lambda\bar{\psi}\psi)G^a_{\mu\nu}G^{\mu\nu}_a \\
&= (1-4\lambda\bar{\psi}\psi)\left[G^a_{\mu\nu}\delta G^{\mu\nu}_a + \delta G^a_{\mu\nu}(G^{\mu\nu}_a)\right] \\
&\quad -4\lambda\left[\bar{\psi}\delta\psi + (\delta\bar{\psi})\psi\right]G^a_{\mu\nu}G^{\mu\nu}_a \\
&= (1-4\lambda\bar{\psi}\psi)\Big[G^a_{\mu\nu}\delta\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g_0 f^{abc}A^\mu_b A^\nu_c\right) \\
&\quad +\delta\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 f^{abc}A^b_\mu A^c_\nu\right)G^{\mu\nu}_a\Big] \\
&\quad -4\lambda\bar{\psi}(\delta\psi)G^a_{\mu\nu}G^{\mu\nu}_a - 4\lambda(\delta\bar{\psi})\psi G^a_{\mu\nu}G^{\mu\nu}_a \\
&= (1-4\lambda\bar{\psi}\psi)\Big[G^a_{\mu\nu}\partial^\mu\delta A^\nu_a - G^a_{\mu\nu}\partial^\nu\delta A^\mu_a \\
&\quad +g_0 f^{abc}G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) + \partial_\mu(\delta A^a_\nu)G^{\mu\nu}_a \\
&\quad -\partial_\nu(\delta A^a_\mu)G^{\mu\nu}_a + g_0 f^{abc}\delta(A^b_\mu A^c_\nu)G^{\mu\nu}_a\Big] \\
&\quad -4\lambda\bar{\psi}(\delta\psi)G^a_{\mu\nu}G^{\mu\nu}_a - 4\lambda(\delta\bar{\psi})\psi G^a_{\mu\nu}G^{\mu\nu}_a
\end{aligned}$$

$$
\begin{aligned}
= \;& G^a_{\mu\nu}\partial^\mu \delta A^\nu_a - G^a_{\mu\nu}\partial^\nu \delta A^\mu_a + g_0 f^{abc} G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) \\
& +\partial_\mu(\delta A^a_\nu)G^{\mu\nu}_a - \partial_\nu(\delta A^a_\mu)G^{\mu\nu}_a + g_0 f^{abc}\delta(A^b_\mu A^c_\nu)G^{\mu\nu}_a \\
& -4\lambda\bar{\psi}\psi G^a_{\mu\nu}\partial^\mu \delta A^\nu_a + 4\lambda\bar{\psi}\psi G^a_{\mu\nu}\partial^\nu \delta A^\mu_a \\
& -4\lambda\bar{\psi}\psi g_0 f^{abc} G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) - 4\lambda\bar{\psi}\psi\partial_\mu(\delta A^a_\nu)G^{\mu\nu}_a \\
& +4\lambda\bar{\psi}\psi\partial_\nu(\delta A^a_\mu)G^{\mu\nu}_a - 4\lambda\bar{\psi}\psi g_0 f^{abc}\delta(A^b_\mu A^c_\nu)G^{\mu\nu}_a \\
& -4\lambda\bar{\psi}(\delta\psi)G^a_{\mu\nu}G^{\mu\nu}_a - 4\lambda(\delta\bar{\psi})\psi G^a_{\mu\nu}G^{\mu\nu}_a \\
= \;& G^{\mu\nu}_a\partial_\mu(\delta A^a_\nu) + G^{\mu\nu}_a\partial_\mu \delta A^a_\nu + g_0 f^{abc} G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) \\
& +G^{\mu\nu}_a\partial_\mu(\delta A^a_\nu) + G^{\mu\nu}_a\partial_\mu \delta A^a_\nu + g_0 f^{abc} G^{\mu\nu}_a\delta(A^b_\mu A^c_\nu) \\
& -4\lambda\bar{\psi}\psi G^{\mu\nu}_a\partial_\mu(\delta A^a_\nu) - 4\lambda\bar{\psi}\psi G^{\mu\nu}_a\partial_\mu(\delta A^a_\nu) \\
& -4\lambda\bar{\psi}\psi G^{\mu\nu}_a\partial_\mu(\delta A^a_\nu) - 4\lambda\bar{\psi}\psi G^{\mu\nu}_a\partial_\mu(\delta A^a_\nu) \\
& -4\lambda\bar{\psi}\psi g_0 f^{abc} G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) - 4\lambda\bar{\psi}\psi g_0 f^{abc} G^{\mu\nu}_a\delta(A^b_\mu A^c_\nu) \\
& -4\lambda\bar{\psi}(\delta\psi)G^a_{\mu\nu}G^{\mu\nu}_a - 4\lambda(\delta\bar{\psi})\psi G^a_{\mu\nu}G^{\mu\nu}_a \\
= \;& 4G^{\mu\nu}_a\partial_\mu(\delta A^a_\nu) - 16\lambda\bar{\psi}\psi G^{\mu\nu}_a\partial_\mu(\delta A^a_\nu) \\
& +(1-4\lambda\bar{\psi}\psi)g_0 f^{abc} G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) \\
& +(1-4\lambda\bar{\psi}\psi)g_0 f^{abc} G^{\mu\nu}_a\delta(A^b_\mu A^c_\nu) \\
& -4\lambda\bar{\psi}(\delta\psi)G^a_{\mu\nu}G^{\mu\nu}_a - 4\lambda(\delta\bar{\psi})\psi G^a_{\mu\nu}G^{\mu\nu}_a \,,
\end{aligned}
$$

or

$$\begin{aligned}
\delta\left[(1-4\lambda\bar{\psi}\psi)G^a_{\mu\nu}G^{\mu\nu}_a\right] &= 4(1-4\lambda\bar{\psi}\psi)G^{\mu\nu}_a\partial_\mu(\delta A^a_\nu) \\
&\quad +2(1-4\lambda\bar{\psi}\psi)g_0 f^{abc}G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) \\
&\quad -4\lambda\bar{\psi}(\delta\psi)G^a_{\mu\nu}G^{\mu\nu}_a - 4\lambda(\delta\bar{\psi})\psi G^a_{\mu\nu}G^{\mu\nu}_a \ . \qquad (6.3.5)
\end{aligned}$$

From Eqs. (6.2.5) - (6.2.7), we also obtain

$$\begin{aligned}
f^{abc}G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) &= f^{abc}G^a_{\mu\nu}\left[A^\mu_b\delta A^\nu_c + (\delta A^\mu_b)A^\nu_c\right] \\
&= f^{abc}G^a_{\mu\nu}A^\mu_b\delta A^\nu_c + f^{abc}G^a_{\mu\nu}(\delta A^\mu_b)A^\nu_c \\
&= f^{abc}A^\mu_b G^a_{\mu\nu}\delta A^\nu_c + f^{abc}(\delta A^\mu_b)G^a_{\mu\nu}A^\nu_c \\
&= f^{bca}A^\mu_c G^b_{\mu\nu}\delta A^\nu_a + f^{bac}(\delta A^\mu_a)G^b_{\mu\nu}A^\nu_c \\
&= -f^{acb}A^\mu_c G^b_{\mu\nu}\delta A^\nu_a + f^{acb}(\delta A^\mu_a)G^b_{\mu\nu}A^\nu_c \\
&= -f^{acb}A^\mu_c G^b_{\mu\nu}\delta A^\nu_a - f^{acb}(\delta A^\nu_a)G^b_{\mu\nu}A^\mu_c \ , \\
f^{abc}G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) &= -2f^{abc}A^\mu_c G^b_{\mu\nu}\delta A^\nu_a \ . \qquad (6.3.6)
\end{aligned}$$

Substituting Eqs. (6.2.7) and (6.3.6) into Eq. (6.3.5), gives

$$\begin{aligned}
\delta\left[(1-4\lambda\bar{\psi}\psi)G^a_{\mu\nu}G^{\mu\nu}_a\right] = &-4(1-4\lambda\bar{\psi}\psi)\left(\partial_\mu G^{\mu\nu}_a\right)\delta A^a_\nu \\
&-4(1-4\lambda\bar{\psi}\psi)g_0 f^{acb}A^\mu_c G^b_{\mu\nu}\delta A^\nu_a \\
&-4\lambda\bar{\psi}(\delta\psi)G^a_{\mu\nu}G^{\mu\nu}_a - 4\lambda(\delta\bar{\psi})\psi G^a_{\mu\nu}G^{\mu\nu}_a \ , \qquad (6.3.7)
\end{aligned}$$

$$-\frac{1}{4}\,\delta\left[(1-4\lambda\bar{\psi}\psi)G^a_{\mu\nu}G^{\mu\nu}_a\right] = (1-4\lambda\bar{\psi}\psi)\,(\partial_\mu G^{\mu\nu}_a)\,\delta A^a_\nu$$

$$+\,(1-4\lambda\bar{\psi}\psi)g_0 f^{acb}A^\mu_c G^b_{\mu\nu}\delta A^\nu_a$$

$$+\,\lambda\bar{\psi}(\delta\psi)G^a_{\mu\nu}G^{\mu\nu}_a + \lambda(\delta\bar{\psi})\psi G^a_{\mu\nu}G^{\mu\nu}_a\,. \qquad (6.3.8)$$

Using Eqs. (6.3.8) and (6.2.12) - (6.2.14), we can rewrite Eq. (6.3.4) as

$$\begin{aligned}
\delta\mathcal{W}_{1T} \;=\; & \int(\mathrm{d}x)\Big\{(1-4\lambda\bar{\psi}\psi)\,(\partial_\mu G^{\mu\nu}_a)\,\delta A^a_\nu + (1-4\lambda\bar{\psi}\psi)g_0 f^{acb}A^\mu_c G^b_{\mu\nu}\delta A^\nu_a \\
& +\lambda\bar{\psi}(\delta\psi)G^a_{\mu\nu}G^{\mu\nu}_a + \lambda(\delta\bar{\psi})\psi G^a_{\mu\nu}G^{\mu\nu}_a \\
& +\frac{1}{2\mathrm{i}}\left[(\partial_\mu\bar{\psi})\gamma^\mu\delta\psi + \delta(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\delta(\partial_\mu\psi) - (\delta\bar{\psi})\gamma^\mu\partial_\mu\psi\right] \\
& -m_0\bar{\psi}\delta\psi - m_0(\delta\bar{\psi})\psi + g_0\bar{\psi}\gamma_\mu A^\mu\delta\psi + g_0\bar{\psi}\gamma_\mu(\delta A^\mu)\psi \\
& +g_0(\delta\bar{\psi})\gamma_\mu A^\mu\psi + \bar{\eta}\delta\psi + (\delta\bar{\psi})\eta + J^\mu_a\delta A^a_\mu\Big\}\,, \qquad (6.3.9)
\end{aligned}$$

$$\begin{aligned}
\frac{\delta\mathcal{W}_{1T}}{\delta\psi} \;=\; & \lambda\bar{\psi}G^a_{\mu\nu}G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}(\partial_\mu\bar{\psi})\gamma^\mu + \frac{1}{2\mathrm{i}}\,\gamma^\mu\partial_\mu\bar{\psi} - m_0\bar{\psi} + g_0\bar{\psi}\gamma_\mu A^\mu + \bar{\eta} \\
=\; & 0\,, \qquad (6.3.10)
\end{aligned}$$

or

$$\lambda\bar{\psi}G^a_{\mu\nu}G^{\mu\nu}_a + \frac{1}{\mathrm{i}}\,\gamma^\mu\partial_\mu\bar{\psi} - m_0\bar{\psi} + g_0\bar{\psi}\gamma_\mu A^\mu \;=\; -\bar{\eta}\,, \qquad (6.3.11)$$

$$\bar{\psi}\left(\lambda G^a_{\mu\nu}G^{\mu\nu}_a + \gamma^\mu\frac{\partial_\mu}{\mathrm{i}} - m_0 + g_0\gamma_\mu A^\mu\right) \;=\; -\bar{\eta}\,, \qquad (6.3.12)$$

$$\bar{\psi}\left[\gamma^\mu\frac{\nabla^*_\mu}{\mathrm{i}} + \lambda G^a_{\mu\nu}G^{\mu\nu}_a - m_0\right] \;=\; -\bar{\eta}\,, \qquad (6.3.13)$$

and

$$\frac{\delta \mathcal{W}_{1T}}{\delta \bar{\psi}} = \lambda \psi G^a_{\mu\nu} G^{\mu\nu}_a - \frac{1}{2\mathrm{i}} \gamma^\mu \partial_\mu \psi - \frac{1}{2\mathrm{i}} \gamma^\mu \partial_\mu \psi - m_0 \psi + g_0 \gamma_\mu A^\mu \psi + \eta$$

$$= 0 \,. \tag{6.3.14}$$

Thus we obtain

$$-\lambda \psi G^a_{\mu\nu} G^{\mu\nu}_a + \frac{1}{\mathrm{i}} \gamma^\mu \partial_\mu \psi + m_0 \psi - g_0 \gamma_\mu A^\mu \psi = \eta \,, \tag{6.3.15}$$

$$\left(-\lambda G^a_{\mu\nu} G^{\mu\nu}_a + \gamma^\mu \frac{\partial_\mu}{\mathrm{i}} + m_0 - g_0 \gamma_\mu A^\mu \right) \psi = \eta \,, \tag{6.3.16}$$

$$\left[\gamma^\mu \frac{\nabla_\mu}{\mathrm{i}} - \lambda G^a_{\mu\nu} G^{\mu\nu}_a + m_0 \right] \psi = \eta \,, \tag{6.3.17}$$

with $\mu, \nu = 0, 1, 2, 3$ and where ∇_μ is defined in Eq. (6.1.11).

Consider first the terms not containing $\delta\psi$ and $\delta\bar{\psi}$ in Eq. (6.3.9). The first term in this combination is given by

$$\left(\partial_\mu G^{\mu\nu}_a\right) \delta A^a_\nu = \left(\partial_\mu G^{\mu 0}_a\right) \delta A^a_0 + \left(\partial_\mu G^{\mu k}_a\right) \delta A^a_k$$

$$= \left(\partial_\mu G^{\mu 0}_a\right) \delta A^a_0 + \left(\partial_\mu G^{\mu k}_a\right) \left(\delta^{ki} - \delta^{k3} \frac{\partial^i}{\partial_3} \right) \delta A^a_i$$

$$= \left(\partial_\mu G^{\mu 0}_a\right) \delta A^a_0 + \left(\partial_\mu G^{\mu i}_a\right) \delta A^i_a - \frac{\partial^i}{\partial_3} \left(\partial_\mu G^{\mu 3}_a\right) \delta A^a_i \,,$$

$$\left(\partial_\mu G^{\mu\nu}_a\right) \delta A^a_\nu = \delta^{ab} \partial_k G^{k0}_b \delta A^a_0 + \delta^{ab} \partial_\mu G^{\mu i}_b \delta A^a_i - \frac{\partial^i}{\partial_3} \delta^{ab} \partial_\mu G^{\mu 3}_b \delta A^a_i \,, \tag{6.3.18}$$

and

$$(1-4\lambda\bar{\psi}\psi)\left(\partial_\mu G_a^{\mu\nu}\right)\delta A_\nu^a = (1-4\lambda\bar{\psi}\psi)\Big[\delta^{ab}\partial_k G_b^{k0}\delta A_0^a + \delta^{ab}\partial_\mu G_b^{\mu i}\delta A_i^a - \frac{\partial^i}{\partial_3}\,\delta^{ab}\partial_\mu G_b^{\mu 3}\delta A_i^a\Big]. \tag{6.3.19}$$

For second term in this combination we have

$$\begin{aligned}
&(1-4\lambda\bar{\psi}\psi)g_0 f^{acb}A_c^\mu G_{\mu\nu}^b\delta A_a^\nu \\
=\;& (1-4\lambda\bar{\psi}\psi)g_0 f^{acb}A_\mu^c G_b^{\mu\nu}\delta A_\nu^a \\
=\;& (1-4\lambda\bar{\psi}\psi)g_0 f^{acb}\left(A_\mu^c G_b^{\mu 0}\delta A_0^a + A_\mu^c G_b^{\mu k}\delta A_k^a\right) \\
=\;& (1-4\lambda\bar{\psi}\psi)g_0 f^{acb}\left[A_k^c G_b^{k0}\delta A_0^a + A_\mu^c G_b^{\mu k}\left(\delta^{ki} - \delta^{k3}\frac{\partial^i}{\partial_3}\right)\delta A_i^a\right] \\
=\;& (1-4\lambda\bar{\psi}\psi)g_0 f^{acb}\left(A_k^c G_b^{k0}\delta A_0^a + A_\mu^c G_b^{\mu i}\delta A_i^a - \frac{\partial^i}{\partial_3}A_\mu^c G_b^{\mu 3}\delta A_i^a\right) \\
=\;& (1-4\lambda\bar{\psi}\psi)\left[g_0 f^{acb}A_k^c G_b^{k0}\delta A_0^a + g_0 f^{acb}A_\mu^c G_b^{\mu i}\delta A_i^a - \frac{\partial^i}{\partial_3}g_0 f^{acb}A_\mu^c G_b^{\mu 3}\delta A_i^a\right].
\end{aligned} \tag{6.3.20}$$

On the other hand for the third in this combination we obtain

$$\begin{aligned}
g_0\bar{\psi}\gamma_\mu(\delta A^\mu)\psi &= g_0\bar{\psi}\gamma_\mu t^a(\delta A_a^\mu)\psi \\
&= g_0\bar{\psi}t^a\psi\gamma_\mu\left(\delta A_a^\mu\right) \\
&= g_0\bar{\psi}t^a\psi\left[\gamma_0\left(\delta A_a^0\right) + \gamma_k\left(\delta A_a^k\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= g_0\bar{\psi}t^a\psi\left[\gamma_0\left(\delta A_a^0\right)+\gamma_k\left(\delta^{ki}-\delta^{k3}\frac{\partial_i}{\partial_3}\right)\delta A_a^i\right] \\
&= g_0\bar{\psi}t^a\psi\left[\gamma_0\left(\delta A_a^0\right)+\left(\delta^{ki}\gamma_k-\delta^{k3}\gamma_k\frac{\partial_i}{\partial_3}\right)\delta A_a^i\right] \\
&= g_0\bar{\psi}t^a\psi\left[\gamma_0\left(\delta A_a^0\right)+\gamma^i\delta A_a^i-\gamma^3\frac{\partial_i}{\partial_3}\,\delta A_a^i\right],
\end{aligned}
$$

$$
g_0\bar{\psi}\gamma_\mu(\delta A^\mu)\psi = g_0\bar{\psi}\gamma_0 t^a\left(\delta A_a^0\right)\psi+g_0\bar{\psi}\gamma^i t^a\left(\delta A_a^i\right)\psi-g_0\bar{\psi}t^a\psi\gamma^3\frac{\partial_i}{\partial_3}\,\delta A_a^i\,. \tag{6.3.21}
$$

Finally for the last in this combination we get

$$
\begin{aligned}
J_a^\mu\delta A_\mu^a &= J_a^0\delta A_0^a+J_a^k\delta A_k^a \\
&= J_a^0\delta A_0^a+J_a^k\left(\delta^{ki}-\delta^{k3}\frac{\partial^i}{\partial_3}\right)\delta A_i^a \\
&= J_a^0\delta A_0^a+\delta^{ki}J_a^k\delta A_i^a-\delta^{k3}J_a^k\frac{\partial^i}{\partial_3}\,\delta A_i^a\,,
\end{aligned}
$$

$$
J_a^\mu\delta A_\mu^a = J_a^0\delta A_0^a+J_a^i\delta A_i^a-J_a^3\frac{\partial^i}{\partial_3}\,\delta A_i^a\,. \tag{6.3.22}
$$

We now substitute Eqs. (6.3.19) - (6.3.22) into Eq. (6.3.9), to obtain

$$
\begin{aligned}
\delta\mathcal{W}_{1T} = \int(\mathrm{d}x)\Bigg\{&(1-4\lambda\bar{\psi}\psi)\left[\delta^{ab}\partial_k G_b^{k0}\delta A_0^a+\delta^{ab}\partial_\mu G_b^{\mu i}\delta A_i^a-\frac{\partial^i}{\partial_3}\,\delta^{ab}\partial_\mu G_b^{\mu 3}\delta A_i^a\right] \\
&+(1-4\lambda\bar{\psi}\psi)\Bigg(g_0 f^{acb}A_k^c G_b^{k0}\delta A_0^a+g_0 f^{acb}A_\mu^c G_b^{\mu i}\delta A_i^a \\
&-\frac{\partial^i}{\partial_3}\,g_0 f^{acb}A_\mu^c G_b^{\mu 3}\delta A_i^a\Bigg)+\lambda\bar{\psi}(\delta\psi)G_{\mu\nu}^a G_a^{\mu\nu}+\lambda(\delta\bar{\psi})\psi G_{\mu\nu}^a G_a^{\mu\nu} \\
&+\frac{1}{2\mathrm{i}}\left[(\partial_\mu\bar{\psi})\gamma^\mu\delta\psi+\delta(\partial_\mu\bar{\psi})\gamma^\mu\psi-\bar{\psi}\gamma^\mu\delta(\partial_\mu\psi)-(\delta\bar{\psi})\gamma^\mu\partial_\mu\psi\right]
\end{aligned}
$$

$$-m_0\bar{\psi}\delta\psi - m_0(\delta\bar{\psi})\psi + g_0\bar{\psi}\gamma_\mu A^\mu\delta\psi + g_0\bar{\psi}\gamma_0 t^a\left(\delta A_a^0\right)\psi$$

$$+g_0\bar{\psi}\gamma^i t^a\left(\delta A_a^i\right)\psi - g_0\bar{\psi}t^a\psi\gamma^3\frac{\partial_i}{\partial_3}\,\delta A_a^i + g_0(\delta\bar{\psi})\gamma_\mu A^\mu\psi$$

$$\left.+\bar{\eta}\delta\psi + (\delta\bar{\psi})\eta + J_a^0\delta A_0^a + J_a^i\delta A_i^a - J_a^3\frac{\partial^i}{\partial_3}\,\delta A_i^a\right\}, \qquad (6.3.23)$$

$$\begin{aligned}\frac{\delta\mathcal{W}_{1T}}{\delta A_0^a(x)} &= (1-4\lambda\bar{\psi}\psi)\delta^{ab}\partial_k G_b^{k0} + (1-4\lambda\bar{\psi}\psi)g_0 f^{acb}A_k^c G_b^{k0} + g_0\bar{\psi}\gamma_0 t^a\psi + J_a^0 \\ &= (1-4\lambda\bar{\psi}\psi)\big(\delta^{ab}\partial_k G_b^{k0} + g_0 f^{acb}A_k^c G_b^{k0}\big) + g_0\bar{\psi}\gamma_0 t^a\psi + J_a^0 \\ &= 0\,. \end{aligned} \qquad (6.3.24)$$

Or

$$(1-4\lambda\bar{\psi}\psi)\big(\delta^{ab}\partial_k G_b^{k0} + g_0 f^{acb}A_k^c G_b^{k0}\big) = -g_0\bar{\psi}\gamma_0 t^a\psi - J_a^0\,, \qquad (6.3.25)$$

$$(1-4\lambda\bar{\psi}\psi)\big(\delta^{ab}\partial_k + g_0 f^{acb}A_k^c\big)G_b^{k0} = -g_0\bar{\psi}\gamma_0 t^a\psi - J_a^0\,, \qquad (6.3.26)$$

$$(1-4\lambda\bar{\psi}\psi)\nabla_k^{ab}G_b^{k0} = -g_0\bar{\psi}\gamma_0 t^a\psi - J_a^0\,. \qquad (6.3.27)$$

For the variation with respect to $A_i^a(x)$ we derive

$$\begin{aligned}\frac{\delta\mathcal{W}_{1T}}{\delta A_i^a(x)} &= (1-4\lambda\bar{\psi}\psi)\left(\delta^{ab}\partial_\mu G_b^{\mu i} - \frac{\partial^i}{\partial_3}\,\delta^{ab}\partial_\mu G_b^{\mu 3}\right) \\ &\quad +(1-4\lambda\bar{\psi}\psi)\left(g_0 f^{acb}A_\mu^c G_b^{\mu i} - \frac{\partial^i}{\partial_3}\,g_0 f^{acb}A_\mu^c G_b^{\mu 3}\right) \\ &\quad +g_0\bar{\psi}\gamma^i t^a\psi - g_0\bar{\psi}t^a\psi\gamma^3\frac{\partial_i}{\partial_3} + J_a^i - J_a^3\frac{\partial^i}{\partial_3} \\ &= 0\,, \end{aligned} \qquad (6.3.28)$$

or

$$\begin{aligned}
0 &= (1-4\lambda\bar{\psi}\psi)\left(\delta^{ab}\partial_\mu G_b^{\mu i} + g_0 f^{acb} A_\mu^c G_b^{\mu i}\right) \\
&\quad -(1-4\lambda\bar{\psi}\psi)\left(\frac{\partial^i}{\partial_3}\,\delta^{ab}\partial_\mu G_b^{\mu 3} + \frac{\partial^i}{\partial_3}\,g_0 f^{acb} A_\mu^c G_b^{\mu 3}\right) \\
&\quad -\left(g_0\bar{\psi}t^a\psi\gamma^3\frac{\partial_i}{\partial_3} + J_a^3\frac{\partial^i}{\partial_3}\right) + g_0\bar{\psi}\gamma^i t^a\psi + J_a^i \\
&= (1-4\lambda\bar{\psi}\psi)\left(\delta^{ab}\partial_\mu + g_0 f^{acb} A_\mu^c\right) G_b^{\mu i} \\
&\quad -(1-4\lambda\bar{\psi}\psi)\,\frac{\partial^i}{\partial_3}\left(\delta^{ab}\partial_\mu + g_0 f^{acb} A_\mu^c\right) G_b^{\mu 3} \\
&\quad -\frac{\partial_i}{\partial_3}\left(g_0\bar{\psi}t^a\psi\gamma^3 + J_a^3\right) + g_0\bar{\psi}\gamma^i t^a\psi + J_a^i \\
&= (1-4\lambda\bar{\psi}\psi)\nabla_\mu^{ab} G_b^{\mu i} - (1-4\lambda\bar{\psi}\psi)\,\frac{\partial^i}{\partial_3}\nabla_\mu^{ab} G_b^{\mu 3} \\
&\quad -\frac{\partial_i}{\partial_3}\left(g_0\bar{\psi}t^a\psi\gamma^3 + J_a^3\right) + g_0\bar{\psi}\gamma^i t^a\psi + J_a^i\,, \\
0 &= (1-4\lambda\bar{\psi}\psi)\left(\nabla_\mu^{ab} G_b^{\mu i} - \frac{\partial^i}{\partial_3}\nabla_\mu^{ab} G_b^{\mu 3}\right) \\
&\quad -\frac{\partial_i}{\partial_3}\left(g_0\bar{\psi}t^a\psi\gamma^3 + J_a^3\right) + g_0\bar{\psi}\gamma^i t^a\psi + J_a^i\,. \qquad (6.3.29)
\end{aligned}$$

Thus we have

$$(1-4\lambda\bar{\psi}\psi)\left(\nabla_\mu^{ab} G_b^{\mu i} - \frac{\partial^i}{\partial_3}\nabla_\mu^{ab} G_b^{\mu 3}\right) - \frac{\partial_i}{\partial_3}\left(g_0\bar{\psi}t^a\psi\gamma^3 + J_a^3\right) = -g_0\bar{\psi}\gamma^i t^a\psi - J_a^i\,. \qquad (6.3.30)$$

From Eq. (6.3.9), we will obtain the canonical conjugate momenta to A_a^1, A_a^2, using the expression for the canonical conjugate momenta to A^μ are defined by

$$\pi^\mu \equiv \pi[A^\mu] = \frac{\delta\mathcal{W}}{\delta\dot{A}_\mu} = \frac{\delta\mathcal{W}}{\delta(\partial_0 A_\mu)}\,.$$

Consider the first term on the right-hand side of Eq. (6.3.9):

$$\begin{aligned}
&\int(\mathrm{d}x)(1-4\lambda\bar{\psi}\psi)\,(\partial_\mu G_a^{\mu\nu})\,\delta A_\nu^a \\
=\;& \int(\mathrm{d}x)(1-4\lambda\bar{\psi}\psi)\,\left[\left(\partial_\mu G_a^{\mu 0}\right)\delta A_0^a + \left(\partial_\mu G_a^{\mu k}\right)\delta A_k^a\right] \qquad ; \qquad k=1,2,3 \\
=\;& \int(\mathrm{d}x)(1-4\lambda\bar{\psi}\psi)\left[\left(\partial_\mu G_a^{\mu 0}\right)\delta A_0^a + \left(\partial_\mu G_a^{\mu k}\right)\left(\delta^{ki}-\delta^{k3}\frac{\partial^i}{\partial_3}\right)\delta A_i^a\right]\;;\; i=1,2 \\
=\;& \int(\mathrm{d}x)(1-4\lambda\bar{\psi}\psi)\left[\left(\partial_\mu G_a^{\mu 0}\right)\delta A_0^a + \left(\partial_\mu G_a^{\mu i}-\frac{\partial^i}{\partial_3}\,\partial_\mu G_a^{\mu 3}\right)\delta A_i^a\right] \\
=\;& \int(\mathrm{d}x)(1-4\lambda\bar{\psi}\psi)\left[-G_a^{\mu 0}\delta(\partial_\mu A_0^a) - \left(G_a^{\mu i}-\frac{\partial^i}{\partial_3}\,G_a^{\mu 3}\right)\delta(\partial_\mu A_i^a)\right]. \qquad (6.3.31)
\end{aligned}$$

The canonical conjugate momenta to A_a^1, A_a^2 are then worked out to be

$$\begin{aligned}
\pi_a^i &= \frac{\delta\mathcal{W}_{1T}}{\delta(\partial_0 A_i)} \\
&= -(1-4\lambda\bar{\psi}\psi)\left(G_a^{0i}-\frac{\partial^i}{\partial_3}\,G_a^{03}\right), \\
\pi_a^i &= (1-4\lambda\bar{\psi}\psi)\,G_a^{i0}-\frac{\partial^i}{\partial_3}(1-4\lambda\bar{\psi}\psi)\,G_a^{30}\;;\; i=1,2\,, \qquad (6.3.32)
\end{aligned}$$

while for the dependent field A_a^0, A_a^3 we have

$$\pi_a^0 = 0 \qquad , \qquad \pi_a^3 = 0\,. \qquad (6.3.33)$$

We note that Eq. (6.3.32) is also valid if we replace i by 3. That is, for $k=1,2,3$, we may rewrite

$$\pi_a^k \;=\; (1-4\lambda\bar{\psi}\psi)\, G_a^{k0} - \partial_3^{-1}\partial^k(1-4\lambda\bar{\psi}\psi)\, G_a^{30}\,. \tag{6.3.34}$$

Upon multiplying Eq. (6.3.34) by ∇_k^{ba} this gives

$$\nabla_k^{ba}\pi_a^k \;=\; (1-4\lambda\bar{\psi}\psi)\nabla_k^{ba}G_a^{k0} - (1-4\lambda\bar{\psi}\psi)\nabla_k^{ba}\partial_3^{-1}\partial^k G_a^{30}\,. \tag{6.3.35}$$

We make use of the field equation

$$\nabla_\mu^{ab}\left[(1-4\lambda\bar{\psi}\psi)\, G_b^{\mu\nu}\right] = -\left(\delta^\nu{}_\sigma\delta^{ac} - g^{\nu k}\partial_k D^{ab}\nabla_\sigma^{bc}\right)\left[J_c^\sigma + g_0\bar{\psi}\gamma^\sigma t_c\psi\right], \tag{6.3.36}$$

$$\nabla_k^{ba}\left[(1-4\lambda\bar{\psi}\psi)\, G_a^{k0}\right] = -\left(\delta^0{}_\sigma\delta^{bc} - g^{0k}\partial_k D^{ba}\nabla_\sigma^{ac}\right)\left[J_c^\sigma + g_0\bar{\psi}\gamma^\sigma t_c\psi\right]$$

$$\nabla_k^{ba}\left[(1-4\lambda\bar{\psi}\psi)\, G_a^{k0}\right] = -\, J_b^0 - g_0\bar{\psi}\gamma^0 t^b\psi\,, \tag{6.3.37}$$

and substitute the above into the right-hand side of Eq. (6.3.35), to obtain

$$\nabla_k^{ba}\pi_a^k \;=\; -J_b^0 - g_0\bar{\psi}\gamma^0 t^b\psi - (1-4\lambda\bar{\psi}\psi)\nabla_k^{ba}\partial_3^{-1}\partial^k G_a^{30}\,, \tag{6.3.38}$$

$$(1-4\lambda\bar{\psi}\psi)\nabla_k^{ba}\partial_3^{-1}\partial^k G_a^{30} \;=\; -\left[J_b^0 + g_0\bar{\psi}\gamma^0 t^b\psi + \nabla_k^{ba}\pi_a^k\right]. \tag{6.3.39}$$

Let $\nabla_k^{ba}\partial^k \equiv \mathscr{O}^{ba}(x)$, $[1-4\lambda\bar{\psi}\psi]\partial_3^{-1}G_a^{30} \equiv \tilde{A}_a(x)$ and set

$$\mathscr{O}^{ba}(x)\tilde{A}_a(x) \;=\; \tilde{J}^b(x)\,,$$

where

$$\tilde{J}^b \;=\; -[J_b^0 + g_0\bar{\psi}\gamma^0 t^b\psi + \nabla_k^{ba}\pi_a^k]\,. \tag{6.3.40}$$

Using the Green operator equation:

$$\nabla_k^{ac}\partial_k D^{cd}(x,x') \;=\; \delta^4(x,x')\,\delta^{ad}\,, \tag{6.3.41}$$

we then obtain

$$(1-4\lambda\bar{\psi}\psi)\partial_3^{-1}G_a^{30} \;=\; -D^{ad}[J_b^0 + g_0\bar{\psi}\gamma^0 t^b\psi + \nabla_k^{ba}\pi_a^k]\,. \tag{6.3.42}$$

Upon substituting the above equation into Eq. (6.3.34), gives

$$\pi_a^k \;=\; (1-4\lambda\bar{\psi}\psi)\,G_a^{k0} + \partial^k D^{ad}[J_b^0 + g_0\bar{\psi}\gamma^0 t^b\psi + \nabla_k^{ba}\pi_a^k]\,, \tag{6.3.43}$$

or

$$\begin{aligned}[1-4\lambda\bar{\psi}(x)\psi(x)]\,G_a^{k0}(x) \;&=\; \pi_a^k(x) - \partial_k\int(\mathrm{d}x')\,D_{ab}(x,x')\big[J_b^0(x') \\ &\quad + g_0\bar{\psi}(x')\gamma^0 t_b\psi(x') + \nabla_j'^{bc}\pi_c^j(x')\big]\,,\end{aligned} \tag{6.3.44}$$

$$\begin{aligned}[1-4\lambda\bar{\psi}(x)\psi(x)]\,G_a^{k0}(x) \;&=\; \pi_a^k(x) - \partial_k\int(\mathrm{d}x')\,D_{ab}(x,x')\big[\delta^0{}_\nu J_b^\nu(x') \\ &\quad + g_0\bar{\psi}(x')\gamma^0 t_b\psi(x') + \nabla_j'^{bc}\pi_c^j(x')\big]\,,\end{aligned} \tag{6.3.45}$$

$k=1,2,3$ and with π_a^3 set equal to zero.

With the (independent) fields and their canonical conjugate momenta kept fixed, we then have the following functional derivative expression

$$[1-4\lambda\bar{\psi}(x)\psi(x)]\frac{\delta}{\delta J_b^\nu(x')}\,G_a^{k0}(x) \;=\; -\partial_k D_{ab}(x,x')\,\delta^0{}_\nu\,. \tag{6.3.46}$$

The canonical commutation relations of $A^a(x)$, $\pi^b(x')$ are given by

$$\big[A^a(x),\pi^b(x')\big] \;=\; \mathrm{i}\delta^{ab}\delta^3(\mathbf{x}-\mathbf{x}')\,. \tag{6.3.47}$$

Using

$$\delta(x^0 - x'^0)\,\delta^3(\mathbf{x} - \mathbf{x}') \;=\; \delta^4(x - x')\,, \tag{6.3.48}$$

the equal time commutation relations of the independent fields $A_a^1(x)$, $A_a^2(x)$ may be then rewritten in the form

$$\delta(x^0 - x'^0)\left[A_a^i(x), \pi_b^j(x')\right] \;=\; \mathrm{i}\delta_{ab}\delta^{ij}\delta^4(x - x')\,, \tag{6.3.49}$$

$$\delta(x^0 - x'^0)\left[A_a^i(x), g^{lj}\pi_b^j(x')\right] \;=\; \mathrm{i}\delta_{ab}\delta^{ij}g^{lj}\delta^4(x - x')\,, \tag{6.3.50}$$

$$\delta(x^0 - x'^0)\left[A_a^i(x), \pi_b^l(x')\right] \;=\; \mathrm{i}\delta_{ab}g^{li}\delta^4(x - x')\,. \tag{6.3.51}$$

From the gauge constraint,

$$A_a^3 \;=\; -\frac{\partial_i}{\partial_3}\,A_a^i \quad , \quad \delta A^k \;=\; \left(\delta^{ki} - \delta^{k3}\frac{\partial_i}{\partial_3}\right)\delta A^i\,, \tag{6.3.52}$$

we may then write

$$\left[A_a^i(x), \pi_b^l(x')\right] \;=\; \mathrm{i}\delta_{ab}g^{li}\,\delta^3(\mathbf{x} - \mathbf{x}')\,, \tag{6.3.53}$$

$$\left[-\frac{\partial_i}{\partial_3}A_a^i(x), \pi_b^l(x')\right] \;=\; -\mathrm{i}\delta_{ab}g^{li}\frac{\partial_i}{\partial_3}\,\delta^3(\mathbf{x} - \mathbf{x}')\,, \tag{6.3.54}$$

$$\delta(x^0 - x'^0)\left[-\frac{\partial_i}{\partial_3}A_a^i(x), \pi_b^l(x')\right] \;=\; -\mathrm{i}\delta_{ab}g^{li}\frac{\partial_i}{\partial_3}\,\delta^4(x - x')\,. \tag{6.3.55}$$

We may combine Eqs. (6.3.51) and (6.3.55), in the form

$$\delta(x^0 - x'^0)\left[A_a^i(x) + A_a^3(x), \pi_b^l(x')\right] = \mathrm{i}\delta_{ab}g^{li}\delta^4(x - x') - \mathrm{i}\delta_{ab}g^{li}\frac{\partial_i}{\partial_3}\,\delta^4(x - x')\,, \tag{6.3.56}$$

or

$$\delta(x^0 - x'^0)\left[A_a^k(x), \pi_b^l(x')\right] = \mathrm{i}\delta_{ab}\left(g^{li}\delta^{ki} - \delta^{k3}\frac{\partial^l}{\partial_3}\right)\delta^4(x - x') \,. \tag{6.3.57}$$

This allows us to obtain the expression

$$\delta(x^0 - x'^0)\left[A_a^k(x), \pi_b^l(x')\right] \;=\; \mathrm{i}\delta_{ab}\left[\delta^{kl} - \delta^{k3}\partial_3^{-1}\partial^l\right]\delta^4(x - x') \,, \tag{6.3.58}$$

with now $k, l = 1, 2, 3$.

Equations (6.3.44) and (6.3.58), will allow to derive the commutation relation in Eq. (6.3.59) below

$$\begin{aligned}
&[1{-}4\lambda\bar{\psi}(x)\psi(x)]\,G_a^{k0} \\
&\quad= \pi_a^k(x) - \partial_k \int (\mathrm{d}x') D_{ab}(x, x')\left[J_b^0(x') + g_0\bar{\psi}(x')\gamma^0 t_b \psi(x') + \nabla_j'^{bc}\pi_c^j(x')\right] , \\
&[1{-}4\lambda\bar{\psi}(x)\psi(x)]\left[A_a^k(x'), G_a^{k0}(x)\right]\delta(x^0 - x'^0) \\
&\quad= \left[A_a^k(x), \pi_a^k(x')\right]\delta(x^0 - x'^0) \\
&\qquad - \partial_k \int (\mathrm{d}x'')\, D_{ab}(x, x'')\nabla_j''^{bc}\left[A_k^a(x'), \pi_c^j(x'')\right]\delta(x^0 - x'^0) \\
&\quad= \mathrm{i}\delta_{aa}\left[\delta^{kk} - \delta^{k3}\partial_3^{-1}\partial^k\right]\delta^4(x - x') \\
&\qquad - \partial_k \int (\mathrm{d}x'')\, D_{ab}(x, x'')\nabla_j''^{bc}\left[A_k^a(x'), \pi_c^j(x'')\right]\delta(x^0 - x'^0) \\
&\quad= 2\mathrm{i}\delta_{aa}\delta^4(x - x') - \partial_k \int (\mathrm{d}x'')\, D_{ab}(x, x'')\nabla_j''^{bc}\left[A_k^a(x'), \pi_c^j(x'')\right]\delta(x^0 - x'^0) \,,
\end{aligned} \tag{6.3.59}$$

where we recall that $D_{ab}(x,x'')$ involves the factor $\delta(x^0-x'^0)$. The latter then implies that the last term in Eq. (6.3.59) is given by

$$
\begin{aligned}
& -\partial_k \int (\mathrm{d}x'')\, D_{ab}(x,x'')\nabla_j''^{bc}\left[A_k^a(x'),\pi_c^j(x'')\right]\delta(x^0-x'^0) \\
= & -\partial_k \int (\mathrm{d}x'')\, D_{ab}(x,x'')\nabla_j''^{bc}\mathrm{i}\delta_{ac}\big[\delta^{kj}-\delta^{k3}\partial_3'^{-1}\partial'^j\big]\delta^4(x-x'') \\
= & -\mathrm{i}\partial_k \int (\mathrm{d}x'')\, D_{ab}(x,x'')\nabla_j''^{ba}\big[\delta^{kj}-\delta^{k3}\partial_3'^{-1}\partial'^j\big]\delta^3(\mathbf{x}'-\mathbf{x}'')\delta(x^0-x'^0)\,. \qquad (6.3.60)
\end{aligned}
$$

Now we take the limit $\mathbf{x}'\to\mathbf{x}$ in the latter and integrate over $\mathrm{d}^3\mathbf{x}$ to obtain

$$
\begin{aligned}
& -\mathrm{i}\int(\mathrm{d}x'')\int \mathrm{d}^3\mathbf{x}\,\partial_k\big[\delta^{kj}-\delta^{k3}\partial_3'^{-1}\partial'^j\big]D_{ab}(x,x'')\nabla_j''^{ba}\delta^3(\mathbf{x}'-\mathbf{x}'')\delta(x^0-x'^0) \\
= & -\mathrm{i}\int(\mathrm{d}x'')\int \mathrm{d}^3\mathbf{x}\,[\partial^j-\partial^j]D_{ab}(x,x'')\nabla_j''^{ba}\delta^3(\mathbf{x}'-\mathbf{x}'')\delta(x^0-x'^0) \\
= & \quad 0\,. \qquad (6.3.61)
\end{aligned}
$$

This result will be used later in deriving the modification of the FP factor.

The action principle gives

$$
\begin{aligned}
\frac{\partial}{\partial\lambda}\langle 0_+|0_-\rangle &= \mathrm{i}\left\langle 0_+\middle|\int(\mathrm{d}x)\,\hat{\mathscr{L}}_{1T}\middle|0_-\right\rangle \\
&= \mathrm{i}\left\langle 0_+\middle|\int(\mathrm{d}x)\frac{\partial}{\partial\lambda}\mathscr{L}_{1T}\middle|0_-\right\rangle \\
&= \mathrm{i}\left\langle 0_+\middle|\int(\mathrm{d}x)\,\bar\psi\psi G_{\mu\nu}^a G_a^{\mu\nu}\middle|0_-\right\rangle \\
\frac{\partial}{\partial\lambda}\langle 0_+|0_-\rangle &= \mathrm{i}\int(\mathrm{d}x)\,\langle 0_+|\bar\psi(x)\psi(x)G_{\mu\nu}^a(x)G_a^{\mu\nu}(x)|0_-\rangle\ , \qquad (6.3.62)
\end{aligned}
$$

where we refer to Eq. (6.3.1) for the appearance of the coupling parameter λ.

We note that

$$\begin{aligned} G_{\mu\nu}G^{\mu\nu} &= G_{0\nu}G^{0\nu} + G_{i\nu}G^{i\nu} \\ &= \underbrace{G_{00}G^{00}}_{=0} + G_{0j}G^{0j} + G_{i0}G^{i0} + G_{ij}G^{ij} \ . \end{aligned}$$

That is

$$G_{\mu\nu}G^{\mu\nu} = 2\, G_{0i}G^{0i} + G_{ij}G^{ij} \ . \tag{6.3.63}$$

Consider the matrix element

$$\begin{aligned} &\left\langle 0_+ \middle| \left(G^a_{\mu\nu}(x) G^{\mu\nu}_a(x') \right)_+ \middle| 0_- \right\rangle \\ &= 2 \left\langle 0_+ \middle| \left(G^a_{k0}(x) G^{k0}_a(x') \right)_+ \middle| 0_- \right\rangle + \left\langle 0_+ \middle| \left(G^a_{kl}(x) G^{kl}_a(x') \right)_+ \middle| 0_- \right\rangle . \end{aligned} \tag{6.3.64}$$

The second term is simply equal to

$$G'^a_{kl}(x) G'^{kl}_a(x') \left\langle 0_+ \,|\, 0_- \right\rangle \ , \tag{6.3.65}$$

expressed in terms of functional derivatives using our notation below Eq. (6.2.62). From Eq. (6.1.6), we may rewrite

$$\begin{aligned} G^a_{k0}(x) &= \partial_k A^a_0 - \partial_0 A^a_k + g_0 f^{abc} A^b_k A^c_0 \\ &= \left(\partial_k A^a_0 + g_0 f^{abc} A^b_k A^c_0 \right) - \partial_0 A^a_k \\ &= \left(\partial_k \delta^{ac} A^c_0 + g_0 f^{abc} A^b_k A^c_0 \right) - \partial_0 A^a_k \\ &= \left(\delta^{ac} \partial_k + g_0 f^{abc} A^b_k \right) A^c_0 - \partial_0 A^a_k \ , \end{aligned}$$

$$G^a_{k0}(x) \;=\; \nabla^{ac}_k A^c_0 - \partial_0 A^a_k \;, \tag{6.3.66}$$

and subsequently use the integral

$$f(x) \;=\; \int (\mathrm{d}z)\; \delta^4(x-z) f(z)\;. \tag{6.3.67}$$

To determine the first term on the right-hand side of Eq. (6.3.64), we rewrite

$$G^a_{k0}(x) \;=\; \int (\mathrm{d}z)\; \delta^4(x-z)\nabla^{ac}_k(z) A^c_0(z) - \int (\mathrm{d}z)\; \delta^4(x-z)\partial^z_0 A^a_k(z)\;. \tag{6.3.68}$$

We then have the useful identity:

$$\begin{aligned}
&\Big\langle 0_+ \Big| \big(G^a_{k0}(x) G^{k0}_a(x')\big)_+ \Big| 0_- \Big\rangle \\
&\quad=\; G'^a_{k0}(x)\; G'^{k0}_a(x')\; \langle 0_+ \,|\, 0_- \rangle \\
&\qquad + \int (\mathrm{d}z)\; \delta^4(x-z)\delta(z^0 - x'^0)\; \big\langle 0_+ \big| \big[A^a_k(z), G^{k0}_a(x')\big] \big| 0_- \big\rangle \\
&\qquad - \mathrm{i} \int (\mathrm{d}z)\; \delta^4(x-z)\nabla'^{ac}_k(z) \left\langle 0_+ \left| \frac{\delta}{\delta J^0_c} G^{k0}_a(x') \right| 0_- \right\rangle \;,
\end{aligned} \tag{6.3.69}$$

where the second term comes from the non-commutativity of the time derivative and the time ordering operation as resulting from the last term in Eq. (6.3.68), and the third term follows from the rule of functional differentiation in Eq. (6.2.63) as resulting from the first integral in Eq. (6.3.68).

From Eqs. (6.3.44), (6.3.59) and (6.3.61), the right-hand side of Eq. (6.3.69) simplifies for $x' \to x$ to

$$\Big[G'^a_{k0}(x)\; G'^{k0}_a(x) + \Delta'(x)\Big]\; \langle 0_+ \,|\, 0_- \rangle \;, \tag{6.3.70}$$

where

$$\Delta'(x) = 2\int(\mathrm{d}z)\frac{\delta^4(z-x)}{\Big[1-4\lambda\bar{\psi}'(x)\psi'(x)\Big]}K'(x,z)\,, \tag{6.3.71}$$

$$K'(x,z) = \mathrm{i}\Big[\delta_{aa}\delta^4(0)+\frac{1}{2}\,\partial^x_k\nabla'^{ac}_k(z)D'_{ac}(x,z)\Big]\,, \tag{6.3.72}$$

involving a familiar $\delta^4(0)$ term.

All told, the expression Eq. (6.3.62) becomes

$$\begin{aligned}\frac{\partial}{\partial\lambda}\langle 0_+\,|\,0_-\rangle &= \mathrm{i}\int(\mathrm{d}x)\bar{\psi}'(x)\psi'(x)G'^a_{\mu\nu}G'^{\mu\nu}_a(x)\,\langle 0_+\,|\,0_-\rangle \\ &\quad +2\mathrm{i}\int(\mathrm{d}x)\bar{\psi}'(x)\psi'(x)\Delta'(x)\,\langle 0_+\,|\,0_-\rangle\,,\end{aligned} \tag{6.3.73}$$

which upon an elementary integration over λ leads to

$$\langle 0_+\,|\,0_-\rangle = \mathrm{e}^{\mathrm{i}M'}\exp\Big[\mathrm{i}\lambda\int(\mathrm{d}x)\bar{\psi}'(x)\psi'(x)G'^a_{\mu\nu}(x)G'^{\mu\nu}_a(x)\Big]\,\langle 0_+\,|\,0_-\rangle_{\lambda=0}\,, \tag{6.3.74}$$

where

$$M' = -\int(\mathrm{d}x)(\mathrm{d}z)\,\delta^4(x-z)\ln\big[1-4\lambda\bar{\psi}'(x)\psi'(x)\big]\,K'(x,z)\,, \tag{6.3.75}$$

and $\langle 0_+\,|\,0_-\rangle_{\lambda=0}$ is the vacuum-to-vacuum amplitude corresponding to the Lagrangian density $\mathscr{L}_T$ in Eq. (6.1.1) involving the FP factor in Eq. (6.2.82). That is, the familiar FP factor gets modified by a multiplicative factor $\exp[\mathrm{i}M']$ for the gauge invariant Lagrangian density $\mathscr{L}_1$ in Eq. (6.3.1).

6.4 Gauge Breaking Interactions

In the present section we consider the addition of a gauge breaking term to the Lagrangian density $\mathscr{L}$ in Eq. (6.1.2). It is well known that even the addition of the simple source term $\mathscr{L}_S$ in Eq. (6.1.3) to $\mathscr{L}$ causes difficulties (cf. Rivers, 1987, p.204) in the quantization problem in the path integral formalism as the action $\int(\mathrm{d}x)\ \mathscr{L}_T(x)$, with $\mathscr{L}_T(x)$ defined in Eq. (6.1.1), is not gauge invariant. We will see how easy it is to handle the addition of a gauge breaking term to $\mathscr{L}_T$ in our formalism in the functional differential approach to quantum field theory.

Consider the Lagrangian density

$$\mathscr{L}_{2T} \;=\; \mathscr{L}_T + \frac{\lambda}{2} A^a_\mu A^\mu_a \bar{\psi}\psi\ , \tag{6.4.1}$$

where from Eq. (6.1.1),

$$\begin{aligned}\mathscr{L}_{2T} \;=\;& -\frac{1}{4}\, G^a_{\mu\nu} G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}\left[(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\right] - m_0\bar{\psi}\psi \\ & + g_0\bar{\psi}\gamma_\mu A^\mu\psi + \bar{\eta}\psi + \bar{\psi}\eta + J^\mu_a A^a_\mu + \frac{\lambda}{2} A^a_\mu A^\mu_a \bar{\psi}\psi\ .\end{aligned} \tag{6.4.2}$$

The action is

$$\begin{aligned}\mathcal{W}_{2T} \;=\;& \int(\mathrm{d}x)\ \mathscr{L}_{2T} \\ =\;& \int(\mathrm{d}x)\bigg\{ -\frac{1}{4}\, G^a_{\mu\nu} G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\big] \\ & \qquad -m_0\bar{\psi}\psi + g_0\bar{\psi}\gamma_\mu A^\mu\psi + \bar{\eta}\psi + \bar{\psi}\eta \\ & \qquad + J^\mu_a A^a_\mu + \frac{\lambda}{2} A^a_\mu A^\mu_a \bar{\psi}\psi \bigg\}\ ,\end{aligned} \tag{6.4.3}$$

with the variation defined by

$$\delta\mathcal{W}_{2T} = \int(\mathrm{d}x)\bigg\{ -\frac{1}{4}\,\delta\big(G^a_{\mu\nu}G^{\mu\nu}_a\big) + \frac{1}{2\mathrm{i}}\,\delta\big[(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\big]$$

$$-m_0\delta(\bar{\psi}\psi) + g_0\delta\big(\bar{\psi}\gamma_\mu A^\mu\psi\big) + \bar{\eta}\delta\psi + (\delta\bar{\psi})\eta$$

$$+J^\mu_a\delta A^a_\mu + \frac{\lambda}{2}\,\delta\big(A^a_\mu A^\mu_a\bar{\psi}\psi\big)\bigg\}\,. \tag{6.4.4}$$

Consider the last term in the above equation:

$$\frac{\lambda}{2}\,\delta\big(A^a_\mu A^\mu_a\bar{\psi}\psi\big)$$

$$= \frac{\lambda}{2}\Big[A^a_\mu A^\mu_a\,\delta(\bar{\psi}\psi) + \delta\big(A^a_\mu A^\mu_a\big)\bar{\psi}\psi\Big]$$

$$= \frac{\lambda}{2}\Big\{A^a_\mu A^\mu_a\big[\bar{\psi}\delta\psi + (\delta\bar{\psi})\psi\big] + \big[A^a_\mu\delta A^\mu_a + (\delta A^a_\mu)A^\mu_a\big]\bar{\psi}\psi\Big\}$$

$$= \frac{\lambda}{2}\Big[A^a_\mu A^\mu_a\bar{\psi}\delta\psi + A^a_\mu A^\mu_a(\delta\bar{\psi})\psi + A^a_\mu(\delta A^\mu_a)\bar{\psi}\psi + (\delta A^a_\mu)A^\mu_a\bar{\psi}\psi\Big]\,. \tag{6.4.5}$$

From Eqs. (6.2.11) - (6.2.14), we can rewrite Eq. (6.4.4) as

$$\delta\mathcal{W}_{2T} = \int(\mathrm{d}x)\bigg\{\big(\partial_\mu G^{\mu\nu}_a\big)\delta A^a_\nu + g_0 f^{acb}A^\mu_c G^b_{\mu\nu}\delta A^\nu_a - m_0\bar{\psi}\delta\psi - m_0(\delta\bar{\psi})\psi$$

$$+\frac{1}{2\mathrm{i}}\Big[(\partial_\mu\bar{\psi})\gamma^\mu\delta\psi + \delta(\partial\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\delta(\partial_\mu\psi) - \delta\bar{\psi}\gamma^\mu(\partial_\mu\psi)\Big]$$

$$+g_0\bar{\psi}\gamma_\mu A^\mu\delta\psi + g_0\bar{\psi}\gamma_\mu(\delta A^\mu)\psi + g_0(\delta\bar{\psi})\gamma_\mu A^\mu\psi$$

$$+\bar{\eta}\delta\psi + (\delta\bar{\psi})\eta + J^\mu_a\delta A^a_\mu$$

$$+\frac{\lambda}{2}\Big[A^a_\mu A^\mu_a\bar{\psi}\delta\psi + A^a_\mu A^\mu_a(\delta\bar{\psi})\psi + A^a_\mu(\delta A^\mu_a)\bar{\psi}\psi + (\delta A^a_\mu)A^\mu_a\bar{\psi}\psi\Big]\,,$$

$$(6.4.6)$$

$$\frac{\delta \mathcal{W}_{2T}}{\delta \psi} = \frac{1}{2\mathrm{i}}(\partial_\mu \bar{\psi})\gamma^\mu + \frac{1}{2\mathrm{i}}(\partial_\mu \bar{\psi})\gamma^\mu + g_0 \bar{\psi}\gamma_\mu A^\mu - m_0 \bar{\psi} + \bar{\eta} + \frac{\lambda}{2} A^a_\mu A^\mu_a \bar{\psi}$$

$$= 0\,. \tag{6.4.7}$$

Or

$$(\partial_\mu \bar{\psi})\frac{\gamma^\mu}{\mathrm{i}} + g_0 \bar{\psi}\gamma_\mu A^\mu + \frac{\lambda}{2} A^a_\mu A^\mu_a \bar{\psi} - m_0 \bar{\psi} = -\bar{\eta}\,, \tag{6.4.8}$$

$$\bar{\psi}\left(\gamma^\mu \frac{\partial_\mu}{\mathrm{i}} + g_0 \gamma_\mu A^\mu + \frac{\lambda}{2} A^a_\mu A^\mu_a - m_0\right) = -\bar{\eta}\,, \tag{6.4.9}$$

$$\bar{\psi}\left(\gamma^\mu \frac{\nabla^*_\mu}{\mathrm{i}} + \frac{\lambda}{2} A^a_\mu A^\mu_a - m_0\right) = -\bar{\eta}\,. \tag{6.4.10}$$

On the other hand,

$$\frac{\delta \mathcal{W}_{2T}}{\delta \bar{\psi}} = -\frac{1}{2\mathrm{i}}(\partial_\mu \psi)\gamma^\mu - \frac{1}{2\mathrm{i}}(\partial_\mu \psi)\gamma^\mu + g_0 \gamma_\mu A^\mu \psi - m_0 \psi + \eta + \frac{\lambda}{2} A^a_\mu A^\mu_a \psi$$

$$= 0\,, \tag{6.4.11}$$

or

$$(\partial_\mu \psi)\frac{\gamma^\mu}{\mathrm{i}} - g_0 \gamma_\mu A^\mu \psi - \frac{\lambda}{2} A^a_\mu A^\mu_a \psi + m_0 \psi = \eta\,, \tag{6.4.12}$$

$$\left(\frac{\gamma^\mu}{\mathrm{i}}\,\partial_\mu - g_0 \gamma_\mu A^\mu - \frac{\lambda}{2} A^a_\mu A^\mu_a + m_0\right)\psi = \eta\,, \tag{6.4.13}$$

$$\left(\gamma^\mu \frac{\nabla_\mu}{\mathrm{i}} - \frac{\lambda}{2} A^a_\mu A^\mu_a + m_0\right)\psi = \eta\,, \tag{6.4.14}$$

with $\mu, \nu = 0, 1, 2, 3$ and where ∇_μ is defined in Eq. (6.1.11).

Consider only terms involving the variation δA^a_μ as appearing in the last term of the right-hand side of Eq. (6.4.6) given by

$$\begin{aligned}
\frac{\lambda}{2}\Big[A^a_\mu\big(\delta A^\mu_a\big)\bar{\psi}\psi + \big(\delta A^a_\mu\big)A^\mu_a\bar{\psi}\psi\Big] &= \frac{\lambda}{2}\Big[A^\mu_a\big(\delta A^a_\mu\big)\bar{\psi}\psi + \big(\delta A^a_\mu\big)A^\mu_a\bar{\psi}\psi\Big] \\
&= \lambda\bar{\psi}\psi A^\mu_a\delta A^a_\mu \\
&= \lambda\bar{\psi}\psi\big(A^0_a\delta A^a_0 + A^k_a\delta A^a_k\big) \\
&= \lambda\bar{\psi}\psi\left[A^0_a\delta A^a_0 + A^k_a\left(\delta^{ki} - \delta^{k3}\frac{\partial_i}{\partial_3}\right)\delta A^a_i\right] \\
&= \lambda\bar{\psi}\psi\left(A^0_a\delta A^a_0 + A^i_a\delta A^a_i - \frac{\partial_i}{\partial_3}A^3_a\delta A^a_i\right) \\
&= \lambda\bar{\psi}\psi\left[A^0_a\delta A^a_0 + A^i_a\delta A^a_i - \frac{\partial_i}{\partial_3}\left(-\frac{\partial^i}{\partial_3}A^i_a\right)\delta A^a_i\right] \\
&= \lambda\bar{\psi}\psi\left(A^0_a\delta A^a_o + A^i_a\delta A^a_i + \frac{\boldsymbol{\partial}^2}{\partial_3^2}A^i_a\delta A^a_i\right), \\
\frac{\lambda}{2}\Big[A^a_\mu\big(\delta A^\mu_a\big)\bar{\psi}\psi + \big(\delta A^a_\mu\big)A^\mu_a\bar{\psi}\psi\Big] &= \lambda\bar{\psi}\psi A^0_a\delta A^a_o + \lambda\bar{\psi}\psi A^i_a\delta A^a_i + \lambda\bar{\psi}\psi\frac{\boldsymbol{\partial}^2}{\partial_3^2}A^i_a\delta A^a_i\,.
\end{aligned} \tag{6.4.15}$$

From Eqs. (6.2.24) - (6.2.27), we can rewrite Eq. (6.4.6) as

$$\begin{aligned}
\delta\mathcal{W}_{2T} &= \int(\mathrm{d}x)\bigg\{\delta^{ab}\partial_k G^{k0}_b\delta A^a_0 + \delta^{ab}\partial_\mu G^{\mu i}_b\delta A^a_i - \frac{\partial^i}{\partial_3}\,\delta^{ab}\partial_\mu G^{\mu 3}_b\delta A^a_i \\
&\quad + g_0 f^{acb}A^c_k G^{k0}_b\delta A^a_0 + g_0 f^{acb}A^c_\mu G^{\mu i}_b\delta A^a_i - \frac{\partial^i}{\partial_3}\,g_0 f^{acb}A^c_\mu G^{\mu 3}_b\delta A^a_i \\
&\quad + \frac{1}{2\mathrm{i}}\Big[(\partial_\mu\bar{\psi})\gamma^\mu\delta\psi + \delta(\partial\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\delta(\partial_\mu\psi) - \delta\bar{\psi}\gamma^\mu(\partial_\mu\psi)\Big] \\
&\quad + g_0\bar{\psi}\gamma_\mu A^\mu\delta\psi + g_0(\delta\bar{\psi})\gamma_\mu A^\mu\psi - m_0\bar{\psi}\delta\psi - m_0(\delta\bar{\psi})\psi
\end{aligned}$$

$$+g_0\bar{\psi}\gamma_0 t^a(\delta A_a^0)\psi + g_0\bar{\psi}\gamma^i t^a(\delta A_a^i)\psi - g_0\bar{\psi}\gamma^3 t^a\frac{\partial_i}{\partial_3}(\delta A_a^i)\psi$$

$$+\bar{\eta}\delta\psi + (\delta\bar{\psi})\eta + J_a^0\delta A_0^a + J_a^i\delta A_i^a - J_a^3\frac{\partial^i}{\partial_3}\,\delta A_i^a$$

$$+\lambda\bar{\psi}\psi A_a^0\delta A_o^a + \lambda\bar{\psi}\psi A_a^i\delta A_i^a + \lambda\bar{\psi}\psi\frac{\boldsymbol{\partial}^2}{\partial_3^2}A_a^i\delta A_i^a$$

$$+\frac{\lambda}{2}\Big[A_\mu^a A_a^\mu\bar{\psi}\delta\psi + A_\mu^a A_a^\mu(\delta\bar{\psi})\psi\Big]\Bigg\}\,, \qquad (6.4.16)$$

$$\frac{\delta\mathcal{W}_{2T}}{\delta A_0^a(x)} = J_a^0 + g_0\bar{\psi}\gamma^0 t_a\psi + \delta^{ab}\partial_k G_b^{k0} + g_0 f^{acb}A_k^c G_b^{k0} + \lambda\bar{\psi}\psi A_a^0\,,$$

$$= 0\,, \qquad (6.4.17)$$

or

$$\delta^{ab}\partial_k G_b^{k0} + g_0 f^{acb}A_k^c G_b^{k0} = -J_a^0 - \lambda A_a^0\bar{\psi}\psi - g_0\bar{\psi}\gamma^0 t_a\psi\,, \qquad (6.4.18)$$

$$\left(\delta^{ab}\partial_k + g_0 f^{acb}A_k^c\right)G_b^{k0} = -J_a^0 - \lambda A_a^0\bar{\psi}\psi - g_0\bar{\psi}\gamma^0 t_a\psi\,, \qquad (6.4.19)$$

$$\nabla_k^{ab}G_b^{k0} = -J_a^0 - \lambda A_a^0\bar{\psi}\psi - g_0\bar{\psi}\gamma^0 t_a\psi\,. \qquad (6.4.20)$$

Similarly,

$$\frac{\delta\mathcal{W}_{2T}}{\delta A_i^a(x)} = J_a^i - J_a^3\frac{\partial_i}{\partial_3} + g_0\bar{\psi}\gamma^i t_a\psi - g_0\bar{\psi}\gamma^3 t_a\frac{\partial_i}{\partial_3}\,\psi + \delta^{ab}\partial_\mu G_b^{\mu i}$$

$$-\frac{\partial_i}{\partial_3}\,\delta^{ab}\partial_\mu G_b^{\mu 3} + g_0 f^{acb}A_\mu^c G_b^{\mu i} - \frac{\partial_i}{\partial_3}g_0 f^{acb}A_\mu^c G_b^{\mu 3}$$

$$+\lambda\bar{\psi}\psi A_a^i + \lambda\bar{\psi}\psi\frac{\boldsymbol{\partial}^2}{\partial_3^2}A_a^i$$

$$= 0\,, \qquad (6.4.21)$$

or

$$\left(\delta^{ab}\partial_\mu + g_0 f^{acb} A^c_\mu\right) G^{\mu i}_b - \frac{\partial_i}{\partial_3}\left(\delta^{ab}\partial_\mu G^{\mu 3}_b + g_0 f^{acb} A^c_\mu G^{\mu 3}_b + g_0\bar{\psi}\gamma^3 t_a\psi + J^3_a\right)$$

$$= - g_0\bar{\psi}\gamma^i t_a\psi - J^i_a - \lambda\bar{\psi}\psi A^i_a - \lambda\bar{\psi}\psi\frac{\boldsymbol{\partial}^2}{\partial_3^2}A^i_a \,,$$

$$\left(\delta^{ab}\partial_\mu + g_0 f^{acb} A^c_\mu\right) G^{\mu i}_b - \frac{\partial_i}{\partial_3}\left[\left(\delta^{ab}\partial_\mu + g_0 f^{acb} A^c_\mu\right) G^{\mu 3}_b + g_0\bar{\psi}\gamma^3 t_a\psi + J^3_a\right]$$

$$= - g_0\bar{\psi}\bar{\psi}\gamma^i t_a\psi - J^i_a - \left(1 + \frac{\boldsymbol{\partial}^2}{\partial_3^2}\right)\lambda\bar{\psi}\psi A^i_a \,, \qquad (6.4.22)$$

$$\nabla^{ab}_\mu G^{\mu i}_b - \frac{\partial_i}{\partial_3}\left(\nabla^{ab}_\mu G^{\mu 3}_b + g_0\bar{\psi}\gamma^3 t_a\psi + J^3_a\right)$$

$$= - g_0\bar{\psi}\bar{\psi}\gamma^i t_a\psi - J^i_a - \left(1 + \frac{\boldsymbol{\partial}^2}{\partial_3^2}\right)\lambda\bar{\psi}\psi A^i_a \,. \qquad (6.4.23)$$

We make use of the field equations, to write

$$G^{k0}_a = \pi^k_a - \partial_k D_{ab}\left[J^0_b + \lambda A^0_b\bar{\psi}\psi + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla^{bc}_\nu \pi^\nu_c\right] . \qquad (6.4.24)$$

Also from Eq. (6.1.6), we have

$$G^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g_0 f^{abc} A^\mu_b A^\nu_c \,,$$

$$G^{k0}_a = \partial^k A^0_a - \partial^0 A^k_a + g_0 f^{abc} A^k_b A^0_c \,, \qquad (6.4.25)$$

thus

$$\partial_k G^{k0}_a = \partial_k\partial^k A^0_a - \partial^0 \underbrace{\left(\partial_k A^k_a\right)}_{=0} + g_0 f^{acb}\partial_k A^k_c A^0_b$$

$$
\begin{aligned}
&= \delta^{ab}\partial_k\partial^k A_b^0 + g_0 f^{acb}\partial_k A_c^k A_b^0 \\
&= \left(\delta^{ab}\partial^k + g_0 f^{acb} A_c^k\right)\partial_k A_b^0 \ , \\
\partial_k G_a^{k0} &= \nabla_k^{ab}\partial_k A_b^0 \ . \qquad (6.4.26)
\end{aligned}
$$

Upon multiplying Eq. (6.4.24) by

$$
\nabla_l^{ca}\partial^l \frac{1}{\boldsymbol{\partial}^2}\,\partial_k \ ,
$$

and using Eqs. (6.2.50) and (6.4.26), we get

$$
\begin{aligned}
\nabla_l^{ca}\partial^l \frac{1}{\boldsymbol{\partial}^2}\,\partial_k G_a^{k0} &= \left(\nabla_l^{ca}\partial^l \frac{1}{\boldsymbol{\partial}^2}\nabla_k^{ab}\partial_k\right) A_b^0 \\
&= \nabla_l^{ca}\partial^l \frac{1}{\boldsymbol{\partial}^2}\,\partial_k \pi_a^k \\
&\quad -\nabla_l^{ca}\frac{\partial^l}{\boldsymbol{\partial}^2}\partial_k\partial^k D_{ab}\big[J_b^0 + \lambda A_b^0\bar{\psi}\psi + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla_\nu^{bc}\pi_c^\nu\big] \\
&= \nabla_l^{ca}\partial^l \frac{1}{\boldsymbol{\partial}^2}\,\partial_k \pi_a^k \\
&\quad -\delta^4(x,x')\delta^{cb}\big[J_b^0 + \lambda A_b^0\bar{\psi}\psi + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla_\nu^{bc}\pi_c^\nu\big] \ , \\
\nabla_l^{ca}\partial^l \frac{1}{\boldsymbol{\partial}^2}\,\partial_k G_a^{k0} &= \nabla_l^{ca}\partial^l \frac{1}{\boldsymbol{\partial}^2}\,\partial_k \pi_a^k \\
&\quad -\delta^4(x,x')\big[J_c^0 + \lambda A_c^0\bar{\psi}\psi + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla_\nu^{bc}\pi_c^\nu\big] \ , \qquad (6.4.27)
\end{aligned}
$$

or

$$
\left(\nabla_l^{ca}\partial^l \frac{1}{\boldsymbol{\partial}^2}\nabla_k^{ab}\partial_k\right) A_b^0 = -J_c^0 - \lambda A_c^0\bar{\psi}\psi + \dots \ , \qquad (6.4.28)
$$

where the dots correspond to terms *independent* of J_b^0 and A_b^0. We introduce the Green operator $N^{be}(x,x')$ satisfying

$$\left[\nabla_l^{ca}\partial^l \frac{1}{\boldsymbol{\partial}^2}\nabla_k^{ab}\partial_k + \lambda\delta^{cb}\bar{\psi}(x)\psi(x)\right] N^{be}(x,x') \;=\; \delta^{ce}\delta^4(x-x')\,, \quad (6.4.29)$$

to obtain from Eq. (6.4.28)

$$\left[\nabla_l^{ca}\partial^l \frac{1}{\boldsymbol{\partial}^2}\nabla_k^{ab}\partial_k + \lambda\delta^{cb}\bar{\psi}(x)\psi(x)\right] A_b^0 \;=\; -J_c^0 + \dots\,, \quad (6.4.30)$$

$$A_b^0 \;=\; N^{be}(x,x')\big[-J_c^0 + \dots\big]\,, \quad (6.4.31)$$

$$\frac{\delta}{\delta J_b^0(x)} A_b^0(x) \;=\; -N^{bb}(x,x)\,. \quad (6.4.32)$$

Hence the action principle and Eq. (6.4.32) give

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\langle 0_+\,|\,0_-\rangle &= \mathrm{i}\left\langle 0_+\left|\int(\mathrm{d}x)\frac{\partial}{\partial\lambda}\,\mathscr{L}_{2T}\right|0_-\right\rangle \\
&= \frac{\mathrm{i}}{2}\left\langle 0_+\left|\int(\mathrm{d}x)A_\mu^a A_a^\mu\bar{\psi}\psi\right|0_-\right\rangle \\
&= \frac{\mathrm{i}}{2}\int(\mathrm{d}x)\,\langle 0_+\,|A_\mu^a A_a^\mu\bar{\psi}\psi|0_-\rangle \\
&= \frac{\mathrm{i}}{2}\int(\mathrm{d}x)\Bigg[(-\mathrm{i})\frac{\delta}{\delta J_a^\mu(x')}\,\langle 0_+\,|A_a^\mu\bar{\psi}\psi|0_-\rangle \\
&\qquad +\mathrm{i}\left\langle 0_+\left|\frac{\delta}{\delta J_a^\mu(x')}A_a^\mu\bar{\psi}\psi\right|0_-\right\rangle\Bigg] \\
&= \frac{\mathrm{i}}{2}\int(\mathrm{d}x)\left[A_\mu'^a A_a'^\mu\bar{\psi}'\psi'\,\langle 0_+\,|\,0_-\rangle + \mathrm{i}\bar{\psi}'\psi'\left\langle 0_+\left|\frac{\delta}{\delta J_a^0(x')}A_a^0\right|0_-\right\rangle\right] \\
&= \frac{\mathrm{i}}{2}\int(\mathrm{d}x)\Big[A_\mu'^a A_a'^\mu\bar{\psi}'\psi'\,\langle 0_+\,|\,0_-\rangle - \mathrm{i}\bar{\psi}'\psi' N'^{bb}(x,x)\,\langle 0_+\,|\,0_-\rangle\Big]\,,
\end{aligned}$$

$$\frac{\partial}{\partial\lambda}\langle 0_+|0_-\rangle = \frac{\mathrm{i}}{2}\int(\mathrm{d}x)A'^{a}_{\mu}(x)A'^{\mu}_{a}(x)\bar{\psi}'(x)\psi'(x)\langle 0_+|0_-\rangle$$

$$+\frac{1}{2}\int(\mathrm{d}x)\bar{\psi}'(x)\psi'(x)N'^{bb}(x,x)\langle 0_+|0_-\rangle\ . \tag{6.4.33}$$

Upon integrating the latter over λ, by using in the process Eq. (6.4.29), we obtain

$$\int\delta\langle 0_+|0_-\rangle = \frac{\mathrm{i}}{2}\int\delta\lambda\Big[\int(\mathrm{d}x)A'^{a}_{\mu}A'^{\mu}_{a}\bar{\psi}'\psi' + \frac{1}{2}\int(\mathrm{d}x)\bar{\psi}'\psi'N'^{bb}\Big]\langle 0_+|0_-\rangle\,, \tag{6.4.34}$$

which integrates out to

$$\langle 0_+|0_-\rangle_\lambda = \exp\left[\frac{\mathrm{i}\lambda}{2}\int(\mathrm{d}x)A'^{a}_{\mu}A'^{\mu}_{a}\bar{\psi}'\psi' + \frac{1}{2}\int(\mathrm{d}x)\lambda\bar{\psi}'\psi'N'^{bb}\right]\langle 0_+|0_-\rangle_{\lambda=0}\ . \tag{6.4.35}$$

Consider the second term in the exponential on the right-hand side of Eq. (6.4.35), and using Eqs. (6.4.29) and (6.2.74). This may be expressed as

$$\frac{1}{2}\int(\mathrm{d}x)\lambda\bar{\psi}'\psi'N'^{bb} = \frac{1}{2}\int(\mathrm{d}x)\bar{\psi}'\psi'\frac{1}{\left[\nabla'^{ca}_{l}\partial^{l}\dfrac{1}{\boldsymbol{\partial}^2}\nabla'^{ab}_{k}\partial_k + \lambda\bar{\psi}'\psi'\right]}\lambda$$

$$= \frac{1}{2}\,\mathrm{Tr}\left[\frac{\lambda\bar{\psi}'\psi'}{\nabla'^{ca}_{l}\partial^{l}\dfrac{1}{\boldsymbol{\partial}^2}\nabla'^{ab}_{k}\partial_k + \lambda\bar{\psi}'\psi'}\right]. \tag{6.4.36}$$

Thus we obtain

$$\langle 0_+|0_-\rangle_\lambda = \exp\left\{\frac{\mathrm{i}\lambda}{2}\int(\mathrm{d}x)A'^{a}_{\mu}A'^{\mu}_{a}\bar{\psi}'\psi' + \frac{1}{2}\,\mathrm{Tr}\left[\frac{\lambda\bar{\psi}'\psi'}{\nabla'^{ca}_{l}\partial^{l}\dfrac{1}{\boldsymbol{\partial}^2}\nabla'^{ab}_{k}\partial_k + \lambda\bar{\psi}'\psi'}\right]\right\}$$

$$\times\langle 0_+|0_-\rangle_{\lambda=0}\ , \tag{6.4.37}$$

$$\langle 0_+ | 0_- \rangle_\lambda = \exp\left[-\frac{1}{2} \operatorname{Tr} \ln \left(1 + \frac{\lambda}{\nabla'_l \partial^l (\boldsymbol{\partial}^2)^{-1} \nabla'_k \partial_k} \bar{\psi}' \psi' \right) \right]$$
$$\times \exp\left[\frac{\mathrm{i}\lambda}{2} \int (\mathrm{d}x) A'^a_\mu A'^\mu_a \bar{\psi}' \psi' \right] \langle 0_+ | 0_- \rangle_{\lambda=0} \ , \qquad (6.4.38)$$

showing an obvious modification of the FP factor with latter occurring in $\langle 0_+ | 0_- \rangle_{\lambda=0}$. The quantum action (dynamical) principle leads systematically to the FP of non-abelian gauge theories with no much effort. It is emphasized, in the process of the analysis, that no restrictions may be set on the external current J^a_μ, coupled to the gauge field A^μ_a (such as $\partial^\mu J^a_\mu = 0$), until all functional differentiations with respect to it are taken so that all of its components may be varied independently. We have considered gauge invariant as well as gauge non-invariant interactions and have shown that the FP factor needs to be modified in more general cases and expressions for these modifications were derived. [It is well known that even the simple gauge breaking source term $\mathscr{L}_S$ in Eq. (6.1.3) causes complications in the path integral formalism. The path integral may, of course, be readily derived from the action principle.] The presence of the source term $\mathscr{L}_S$ in the Lagrangian density is essential in order to generate the Green functions of the theory from the vacuum-to-vacuum transition amplitude, as a generating functional, by functional differentiations. We have also show, in particular, that a gauge invariant theory does not necessarily imply the familiar FP factor for proper quantization. Finally we note that even for abelian gauge theories, as obtained from the bulk of this chapter by taking the limit of f^{abc} to zero and replacing t^a by the identity, may lead to modifications, as multiplicative factors in $\langle 0_+ | 0_- \rangle$, as clearly seen from the expressions in Eqs. (6.3.75) and (6.4.38).

CHAPTER VII

QUADRATIC ACTIONS IN DEPENDENT FIELDS AND THE ACTION PRINCIPLE: A THEOREM

7.1 Introduction

The purpose of this chapter is to investigate systematically, in a unified manner, within the functional *differential* formalism of quantum field theory (Schwinger, 1951, 1953, 1954; Manoukian, 1985, 1986, 1987, 2006; Limboonsong and Manoukian, 2006; Manoukian, Sukkhasena and Siranan, 2007), field theories with interaction Lagrangian densities $\mathcal{L}_I(x;\lambda)$, with λ a generic coupling constant, such that $\partial\mathcal{L}_I(x;\lambda)/\partial\lambda$ may be expressed as quadratic functions in dependent fields and, in general, as arbitrary functions of independent fields. These include, as special cases, present renormalizable gauge field theories(see, Chapter 1). For example, the non-abelian ones, such as QCD, are quadratic, while QED is linear in dependent fields. The functional differential treatment necessitates the introduction of *external sources* in order to generate the vacuum-to-vacuum transition amplitude, as a generating functional, from which amplitudes for basic processes may be extracted. The novelty of this work is that we show that for all the general Lagrangians, mentioned above, the vacuum-to-vacuum transition amplitude may be explicitly derived in functional *differential* form, in a unified manner, leading to modifications of computational rules by including such factors as Faddeev-Popov ones (Faddeev and Popov, 1967; Fradkin and Tyutin, 1970) and *modifications* thereof. The derivation is given in the *presence* of external sources, without recourse to path integrals, and without relying on any symmetry and invariance arguments. There has also been a renewed interest in Schwinger's action principle recently (see, e.g., Das and Scherer, 2005; Kawai, 2005; Iliev, 2003) emphasizing, in general, however, operator aspects of a theory, as deriving, for example, commutation relations, rather than dealing with computational ones related directly to generating functionals as done here in our work.

7.2 General Class of Lagrangians

Consider Lagrangian densities which may depend on one or more coupling constants. We scale these couplings by a parameter λ which is eventually set equal to one. The resulting Lagrangian densities will be denoted by $\underline{\mathcal{L}}(x;\lambda)$. The class of Lagrangian densities considered are of the following types

$$\underline{\mathcal{L}}(x;\lambda) = \mathcal{L}(x;0) + \mathcal{L}_I(x;\lambda) + J_1^i(x)\,\chi_i(x) + J_2^j(x)\,\eta_j(x)\,, \tag{7.2.1}$$

where $\chi_i(x)$ and $\eta_j(x)$ are independent and dependent fields, respectively. $J_1^i(x)$, $J_2^j(x)$ are external sources coupled to these respective fields. The interaction Lagrangian densities sought are of the following forms

$$\mathcal{L}_I(x;\lambda) = B(x;\lambda) + B^j(x;\lambda)\,\eta_j(x) + \frac{1}{2}\,B^{jk}(x;\lambda)\,\eta_j(x)\,\eta_k(x)\,, \tag{7.2.2}$$

with $\mathcal{L}_I(x;0) = 0$, where

$$\frac{\partial B(x;\lambda)}{\partial\lambda}\,,\ \frac{\partial B^j(x;\lambda)}{\partial\lambda}\,,\ \frac{\partial B^{jk}(x;\lambda)}{\partial\lambda} \;=\; \frac{\partial B^{kj}(x;\lambda)}{\partial\lambda}\,, \tag{7.2.3}$$

may be expressed in terms of the independent fields, and the latter two may involve space derivatives applied to the dependent fields $\eta_j(x)$. By definition, the canonical conjugate momenta of the fields $\eta_j(x)$ vanish. That is, formally,

$$\frac{\partial\underline{\mathcal{L}}(x;\lambda)}{\partial(\partial_0\eta_j(x))} \;=\; 0\,. \tag{7.2.4}$$

Let

$$\frac{\partial \mathcal{L}(x;0)}{\partial \eta_j(x)} = A^{jk}(x)\eta_k(x) \; . \tag{7.2.5}$$

The *constraint* equation of the dependent fields $\eta_k(x)$ follow from Eqs. (7.2.1) and (7.2.2) are obtained as follows. We rewrite Eq. (7.2.1) in detail as

$$\begin{aligned} \underline{\mathcal{L}}(x;\lambda) &= \mathcal{L}(x;0) + B(x;\lambda) + B^j(x;\lambda)\,\eta_j(x) \\ &+\frac{1}{2}\,B^{jk}(x;\lambda)\,\eta_j(x)\,\eta_k(x) \\ &+J_1^i(x)\,\chi_i(x) + J_2^j(x)\,\eta_j(x)\;, \end{aligned} \tag{7.2.6}$$

with

$$\begin{aligned} \mathcal{L}(x;0) &= \underline{\mathcal{L}}(x;\lambda) - B(x;\lambda) - B^j(x;\lambda)\,\eta_j(x) \\ &-\frac{1}{2}\,B^{jk}(x;\lambda)\,\eta_j(x)\,\eta_k(x) \\ &-J_1^i(x)\,\chi_i(x) - J_2^j(x)\,\eta_j(x)\;, \\ \frac{\partial \mathcal{L}(x;0)}{\partial \eta_j(x)} &= -B^j(x;\lambda) - J_2^j(x) \\ &-B^{jk}(x;\lambda)\,\eta_k(x)\;, \end{aligned} \tag{7.2.7}$$

$$\frac{\partial \mathcal{L}(x;0)}{\partial \eta_j(x)} + B^{jk}(x;\lambda)\,\eta_k(x) = -[B^j(x;\lambda) + J_2^j(x)]\;, \tag{7.2.8}$$

$$A^{jk}(x)\,\eta_k(x) + B^{jk}(x;\lambda)\,\eta_k(x) = -[B^j(x;\lambda) + J_2^j(x)]\;, \tag{7.2.9}$$

$$[A^{jk}(x) + B^{jk}(x;\lambda)]\,\eta_k(x) = -[B^j(x;\lambda) + J_2^j(x)]\;, \tag{7.2.10}$$

leading to the important equation:

$$M^{jk}(x;\lambda)\,\eta_k(x) \;=\; -[B^j(x;\lambda) + J_2^j(x)]\;, \tag{7.2.11}$$

where

$$M^{jk}(x;\lambda) = A^{jk}(x) + B^{jk}(x;\lambda)\;. \tag{7.2.12}$$

Let $D_{jk}(x, x'; \lambda)$ denote the Green operator function satisfying

$$M^{ij}(x;\lambda)\; D_{jk}(x, x';\lambda) = \delta^i{}_k\; \delta^4(x, x')\;. \tag{7.2.13}$$

We use the notation

$$M^{jk}(x;\lambda) \equiv \mathscr{O}^{jk}(x) \tag{7.2.14}$$

and denote the right-hand side of Eq. (7.2.11) is $J^j(x)$, that is

$$J^j(x) \;=\; -[B^j(x,\lambda) + J_2^j(x)]\;. \tag{7.2.15}$$

From Eqs. (7.2.14) and (7.2.13), we note that

$$\mathscr{O}^{jk}(x)\,\eta^k(x) \;=\; J^j(x)\;, \tag{7.2.16}$$

$$\mathscr{O}^{jk}(x)\; D^{kl}(x, x') \;=\; \delta^4(x, x')\delta^{jl}\;. \tag{7.2.17}$$

Hence

$$\eta^j(x) \;=\; \int (\mathrm{d}x)\; D^{jk}(x, x')\; J^k(x')\;, \tag{7.2.18}$$

and we may write

$$\eta_j(x) = -\int (\mathrm{d}x')\; D_{jk}(x, x';\lambda)\; [B^k(x';\lambda) + J_2^k(x')]\;, \tag{7.2.19}$$

giving a constraint which is explicit source J_2^k-dependent, and is also a function of the independent fields.

7.3 The Quantum Dynamical Principle at Work and Explicit Expression for the Vacuum-to-Vacuum Transition Amplitude

Let $|0_\mp\rangle$ denote the vacuum states of a theory before/after the external are switched on/off, respectively. We are interested in the variation of the vacuum-to-vacuum transition amplitude $\langle 0_+ \,|\, 0_- \rangle$, governed by the Lagrangian density $\underline{\mathcal{L}}(x;\lambda)$ in Eq. (7.2.1), with respect to the parameter λ as well as with respect the external sources $J_1^i(x)$, $J_2^j(x)$. To this end, we invoke the quantum dynamical principle which states (see, e.g., Manoukian, 1985, 1986, 1987, 2006; Manoukian, Sukkhasena and Siranan, 2007)

$$\frac{\partial}{\partial\lambda}\langle 0_+ \,|\, 0_- \rangle = \mathrm{i}\left\langle 0_+ \left| \int (\mathrm{d}x) \frac{\partial}{\partial\lambda} \mathcal{L}_I(x;\lambda) \right| 0_- \right\rangle , \tag{7.3.1}$$

$$(-\mathrm{i})\frac{\delta}{\delta J_1^i(x)}\langle 0_+ \,|\, 0_- \rangle = \langle 0_+ \,|\chi_i(x)| 0_- \rangle , \tag{7.3.2}$$

$$(-\mathrm{i})\frac{\delta}{\delta J_2^j(x)}\langle 0_+ \,|\, 0_- \rangle = \langle 0_+ \,|\eta_j(x)| 0_- \rangle . \tag{7.3.3}$$

Consider the matrix element $\langle 0_+ \,| F(x;\lambda, J_1, J_2 | 0_- \rangle$ of an operator which is not only a function of the independent fields but which may also have an explicit dependence on λ and the external sources J_1^i, J_2^j. An explicit λ, J_2^j dependence may occur, for example, when the dependent fields $\eta_j(x)$ are expressed in terms of the independent fields and J_2^j as given in Eq. (7.2.19).

The quantum dynamical principle, in particular, then states (see, Limboonsong and Manoukian, 2006; Manoukian, 2006; Manoukian, Sukkhasena and Siranan, 2007) that

$$(-\mathrm{i})\frac{\delta}{\delta J_2^j(x')}\,\langle 0_+\,|F(x;\lambda,J_1,J_2|0_-\rangle$$

$$=\langle 0_+\,|(F(x;\lambda,J_1,J_2)\,\eta_j(x'))_+|0_-\rangle-\mathrm{i}\left\langle 0_+\left|\frac{\delta}{\delta J_2^j(x')}F(x;\lambda,J_1,J_2)\right|0_-\right\rangle\ , \quad (7.3.4)$$

where $(\ldots)_+$ denotes the time-ordered product, and the functional derivative, with respect to $J_2^j(x')$, in the second term on the right-hand side of Eq. (7.3.4), is applied to the explicit J_2-dependent term (if any) that occurs in F.

Let $\partial B'^j(x;\lambda)/\partial\lambda$ denote $\partial B^j(x;\lambda)/\partial\lambda$ with the fields $\chi_i(x)$ in the latter replaced by the functional derivatives $(-\mathrm{i})\ \delta/\delta J_1^i(x)$. From Eq. (7.3.4),

$$(-\mathrm{i})\frac{\delta}{\delta J_2^k(x')}\left\langle 0_+\left|\frac{\partial}{\partial\lambda}\,B^j(x;\lambda)\right|0_-\right\rangle$$

$$=\left\langle 0_+\left|(\frac{\partial}{\partial\lambda}\,B^j(x;\lambda)\,\eta_k(x'))_+\right|0_-\right\rangle-\mathrm{i}\left\langle 0_+\left|\frac{\delta}{\delta J_2^k(x')}\frac{\partial}{\partial\lambda}\,B^j(x;\lambda)\right|0_-\right\rangle\ , \quad (7.3.5)$$

where we have used the fact that $\partial B^j(x;\lambda)/\partial\lambda$ is expressed in terms of the independent fields and has no explicit J_2^k-dependence, and hence the second term on the right-hand side of Eq. (7.3.4) is zero for this corresponding case.

Then from above equation we have

$$(-\mathrm{i})\frac{\delta}{\delta J_2^k(x')}\,\frac{\partial}{\partial\lambda}\,B'^j(x;\lambda)\,\langle 0_+\,|\,0_-\rangle=\left\langle 0_+\left|\left(\frac{\partial}{\partial\lambda}\,B^j(x;\lambda)\,\eta_k(x')\right)_+\right|0_-\right\rangle\ . \quad (7.3.6)$$

On the other hand, let $F(x;\lambda,J_1,J_2)\equiv\partial B^{jk}(x;\lambda)\,\eta_k(x')/\partial\lambda$ and replace $\eta_k(x')$ by $(-\mathrm{i})\ \delta/\delta J_2^k(x')$, Eq. (7.3.4) also give

$$(-\mathrm{i})\frac{\delta}{\delta J_2^j(x'')}\left\langle 0_+\left|\frac{\partial}{\partial\lambda}\,B^{jk}(x;\lambda)\,\eta_k(x')\right|0_-\right\rangle$$

$$=\left\langle 0_+\left|\left(\frac{\partial}{\partial\lambda}\,B^{jk}(x;\lambda)\,\eta_k(x')\,\eta_j(x'')\right)_+\right|0_-\right\rangle$$

$$-\,\mathrm{i}\left\langle 0_+\left|\left(\frac{\delta}{\delta J_2^j(x'')}\,\frac{\partial}{\partial\lambda}\,B^{jk}(x;\lambda)\,\eta_k(x')\right)_+\right|0_-\right\rangle\;, \qquad (7.3.7)$$

$$(-\mathrm{i})\frac{\delta}{\delta J_2^j(x'')}(-\mathrm{i})\frac{\delta}{\delta J_2^k(x')}\,\frac{\partial}{\partial\lambda}\,B'^{jk}(x;\lambda)\,\langle 0_+\,|\,0_-\rangle$$

$$=\left\langle 0_+\left|\left(\frac{\partial}{\partial\lambda}\,B^{jk}(x;\lambda)\,\eta_k(x')\,\eta_j(x'')\right)_+\right|0_-\right\rangle$$

$$-\,\mathrm{i}\left\langle 0_+\left|\left(\frac{\partial}{\partial\lambda}\,B^{jk}(x;\lambda)\,\frac{\delta}{\delta J_2^j(x'')}\,\eta_k(x')\right)_+\right|0_-\right\rangle\;, \qquad (7.3.8)$$

where from Eq. (7.2.19),

$$\eta_j(x) \;=\; -\int(\mathrm{d}x')\;D_{jk}(x,x';\lambda)[B^k(x';\lambda)+J_2^k(x')]\;,$$

$$\eta_k(x') \;=\; -\int(\mathrm{d}x'')\;D_{kj}(x',x'';\lambda)[B^j(x'';\lambda)+J_2^j(x'')]\;, \qquad (7.3.9)$$

and hence

$$\frac{\delta}{\delta J_2^j(x'')}\,\eta_k(x') \;=\; -D_{kj}(x',x'';\lambda)\;. \qquad (7.3.10)$$

Therefore the second term on the right-hand side of Eq. (7.3.8) is simply

$$-\mathrm{i}\left\langle 0_+\left|\left(\frac{\partial}{\partial\lambda}\,B^{jk}(x;\lambda)\,\frac{\delta}{\delta J_2^j(x'')}\,\eta_k(x')\right)_+\right|0_-\right\rangle$$

$$= \mathrm{i}\left\langle 0_+\left|\frac{\partial}{\partial\lambda}\,B^{jk}(x;\lambda)\,D_{kj}(x',x'';\lambda)\right|0_-\right\rangle$$

$$= \mathrm{i}\frac{\partial}{\partial\lambda}\,B'^{jk}(x;\lambda)\,D'_{kj}(x',x'';\lambda)\,\langle 0_+\,|\,0_-\rangle\ , \qquad (7.3.11)$$

with $D'_{kj}(x',x'';\lambda)$ denoting $D_{kj}(x',x'';\lambda)$ with the fields $\chi_i(x)$ replaced by $(-\mathrm{i})\,\delta/\delta J_1^j(x)$.

All told, we may solve for $\langle 0_+\,|\partial\mathcal{L}_I(x;\lambda)/\partial\lambda|0_-\rangle$ in terms of functional derivatives, with respect to the external sources, as applied to $\langle 0_+\,|\,0_-\rangle$ directly from Eqs. (7.2.2), (7.3.1) and (7.3.5) - (7.3.11) by considering Eq. (7.2.2),

$$\mathcal{L}_I(x;\lambda) \;=\; B(x;\lambda) + B^j(x;\lambda)\,\eta_j(x) + \frac{1}{2}\,B^{jk}(x;\lambda)\,\eta_j(x)\,\eta_k(x)\ ,$$

and

$$\frac{\partial}{\partial\lambda}\,\mathcal{L}_I(x;\lambda) \;=\; \frac{\partial}{\partial\lambda}\,B(x;\lambda) + \left(\frac{\partial}{\partial\lambda}\,B^j(x;\lambda)\right)\eta_j(x)$$

$$+\frac{1}{2}\left(\frac{\partial}{\partial\lambda}\,B^{jk}(x;\lambda)\right)\eta_j(x)\,\eta_k(x)\ , \qquad (7.3.12)$$

and by substituting the above equation into the right-hand side of Eq. (7.3.1), to get

$$\frac{\partial}{\partial\lambda}\,\langle 0_+\,|\,0_-\rangle = \mathrm{i}\left\langle 0_+\left|\int(\mathrm{d}x)\frac{\partial}{\partial\lambda}\,\mathcal{L}_I(x;\lambda)\right|0_-\right\rangle$$

$$= \mathrm{i}\left\langle 0_+\right|\int(\mathrm{d}x)\left[\frac{\partial}{\partial\lambda}\,B(x;\lambda) + \left(\frac{\partial}{\partial\lambda}\,B^j(x;\lambda)\right)\eta_j(x)\right.$$

$$\left. + \,\frac{1}{2}\left(\frac{\partial}{\partial\lambda}\,B^{jk}(x;\lambda\right)\eta_j(x)\,\eta_k(x)\right]\left|0_-\right\rangle\ . \qquad (7.3.13)$$

To find the second term on the right-hand side of the above equation we use Eq. (7.3.6), and note that

$$\begin{aligned}
&\left\langle 0_+ \left| \int (\mathrm{d}x) \left(\frac{\partial}{\partial \lambda} B^j(x;\lambda) \right) \eta_j(x) \right| 0_- \right\rangle \\
&\quad = \int (\mathrm{d}x) \left\langle 0_+ \left| \left(\frac{\partial}{\partial \lambda} B^j(x;\lambda)\, \eta_j(x) \right)_+ \right| 0_- \right\rangle \\
&\quad = \int (\mathrm{d}x)(-\mathrm{i}) \frac{\delta}{\delta J_2^j(x)} \frac{\partial}{\partial \lambda} B'^j(x;\lambda) \langle 0_+ | 0_- \rangle \ , \qquad (7.3.14)
\end{aligned}$$

and find out that the last term on the right-hand side of the Eq. (7.3.13), by using Eq. (7.3.8), is

$$\begin{aligned}
&\left\langle 0_+ \left| \int (\mathrm{d}x) \frac{1}{2} \left(\frac{\partial}{\partial \lambda} B^{jk}(x;\lambda) \right) \eta_j(x)\, \eta_k(x) \right| 0_- \right\rangle \\
&\quad = \frac{1}{2} \int (\mathrm{d}x) \left\langle 0_+ \left| \left(\frac{\partial}{\partial \lambda} B^{jk}(x;\lambda)\, \eta_j(x)\, \eta_k(x) \right)_+ \right| 0_- \right\rangle \\
&\quad = \frac{1}{2} \int (\mathrm{d}x) \Bigg[(-\mathrm{i}) \frac{\delta}{\delta J_2^j(x)} (-\mathrm{i}) \frac{\delta}{\delta J_2^k(x)} \frac{\partial}{\partial \lambda} B'^{jk}(x;\lambda) \langle 0_+ | 0_- \rangle \\
&\qquad + \mathrm{i} \left\langle 0_+ \left| \left(\frac{\partial}{\partial \lambda} B^{jk}(x;\lambda) \frac{\delta}{\delta J_2^j(x)} \eta_k(x) \right)_+ \right| 0_- \right\rangle \Bigg] . \qquad (7.3.15)
\end{aligned}$$

From Eq. (7.3.10), we also have

$$\begin{aligned}
\frac{\delta}{\delta J_2^j(x'')} \eta_k(x') &= -D_{kj}(x', x''; \lambda) \ , \\
\frac{\delta}{\delta J_2^j(x)} \eta_k(x) &= -D_{kj}(x, x; \lambda) \ , \qquad (7.3.16)
\end{aligned}$$

and we get

$$\left\langle 0_+ \left| \int (\mathrm{d}x)\, \frac{1}{2} \left(\frac{\partial}{\partial\lambda}\, B^{jk}(x;\lambda) \right) \eta_j(x)\, \eta_k(x) \right| 0_- \right\rangle$$

$$= \frac{1}{2} \int (\mathrm{d}x) \Bigg[(-\mathrm{i}) \frac{\delta}{\delta J_2^j(x)} (-\mathrm{i}) \frac{\delta}{\delta J_2^k(x)}\, \frac{\partial}{\partial\lambda}\, B'^{jk}(x;\lambda)\, \langle 0_+ \,|\, 0_- \rangle$$

$$- \mathrm{i} \left\langle 0_+ \left| \frac{\partial}{\partial\lambda}\, B^{jk}(x;\lambda)\, D_{kj}(x,x;\lambda) \right| 0_- \right\rangle \Bigg]$$

$$= \frac{1}{2} \int (\mathrm{d}x) \Bigg[(-\mathrm{i}) \frac{\delta}{\delta J_2^j(x)} (-\mathrm{i}) \frac{\delta}{\delta J_2^k(x)}\, \frac{\partial}{\partial\lambda}\, B'^{jk}(x;\lambda)\, \langle 0_+ \,|\, 0_- \rangle$$

$$- \mathrm{i}\, \frac{\partial}{\partial\lambda}\, B'^{jk}(x;\lambda)\, D'_{jk}(x,x;\lambda)\, \langle 0_+ \,|\, 0_- \rangle \Bigg] . \qquad (7.3.17)$$

Upon substituting Eqs. (7.3.14) and (7.3.17) into Eq. (7.3.13), we obtain the chain of equalities:

$$\frac{\partial}{\partial\lambda}\, \langle 0_+ \,|\, 0_- \rangle = \mathrm{i} \int (\mathrm{d}x) \Bigg\{ \frac{\partial}{\partial\lambda}\, B'(x;\lambda) + (\mathrm{i}) \frac{\delta}{\delta J_2^j(x)}\, \frac{\partial}{\partial\lambda}\, B'^{j}(x;\lambda)$$

$$+ \frac{1}{2} \Bigg[(-\mathrm{i}) \frac{\delta}{\delta J_2^j(x)} (-\mathrm{i}) \frac{\delta}{\delta J_2^k(x)}\, \frac{\partial}{\partial\lambda}\, B'^{jk}(x;\lambda)$$

$$- \mathrm{i}\, \frac{\partial}{\partial\lambda}\, B'^{jk}(x;\lambda)\, D'_{kj}(x,x;\lambda) \Bigg] \Bigg\} \langle 0_+ \,|\, 0_- \rangle$$

$$= \mathrm{i}\int (\mathrm{d}x)\left\{\frac{\partial}{\partial\lambda}\,B'(x;\lambda) + \eta_j(x)\frac{\partial}{\partial\lambda}\,B'^{\,j}(x;\lambda)\right.$$

$$+\,\frac{1}{2}\Bigg[\eta_j(x)\;\eta_k(x)\frac{\partial}{\partial\lambda}\,B'^{\,jk}(x;\lambda)$$

$$\left.-\,\mathrm{i}\,\frac{\partial}{\partial\lambda}\,B'^{\,jk}(x;\lambda)\;D'_{kj}(x,x;\lambda)\Bigg]\right\}\langle 0_+\,|\,0_-\rangle$$

$$= \mathrm{i}\int (\mathrm{d}x)\left\{\Bigg[\frac{\partial}{\partial\lambda}\,B'(x;\lambda) + \eta_j(x)\frac{\partial}{\partial\lambda}\,B'^{\,j}(x;\lambda)\right.$$

$$+\,\frac{1}{2}\,\eta_j(x)\;\eta_k(x)\frac{\partial}{\partial\lambda}\,B'^{\,jk}(x;\lambda)\Bigg]$$

$$\left.-\,\frac{\mathrm{i}}{2}\,\frac{\partial}{\partial\lambda}\,B'^{\,jk}(x;\lambda)\;D'_{kj}(x,x;\lambda)\right\}\langle 0_+\,|\,0_-\rangle$$

$$= \Bigg[\mathrm{i}\int (\mathrm{d}x)\;\frac{\partial}{\partial\lambda}\Big(B'(x;\lambda) + \eta_j(x)\;B'^{\,j}(x;\lambda)$$

$$+\,\frac{1}{2}\,\eta_j(x)\;\eta_k(x)\;B'^{\,jk}(x;\lambda)\Big)$$

$$+\,\frac{1}{2}\int (\mathrm{d}x)\left(\frac{\partial}{\partial\lambda}\,B'^{\,jk}(x;\lambda)\right)D'_{kj}(x,x;\lambda)\Bigg]\,\langle 0_+\,|\,0_-\rangle\;,$$

$$\frac{\partial}{\partial\lambda}\,\langle 0_+\,|\,0_-\rangle = \Bigg[\mathrm{i}\int (\mathrm{d}x)\frac{\partial}{\partial\lambda}\,\mathcal{L}'_I(x;\lambda)$$

$$+\,\frac{1}{2}\int (\mathrm{d}x)\left(\frac{\partial}{\partial\lambda}\,B'^{\,jk}(x;\lambda)\right)D'_{kj}(x,x;\lambda)\Bigg]\,\langle 0_+\,|\,0_-\rangle\;,\tag{7.3.18}$$

where $\mathcal{L}'_I$ denotes $\mathcal{L}_I(x;\lambda)$ with $\chi_i(x)$, $\eta_j(x)$ replaced in the latter by $(-\mathrm{i})\delta/\delta J_1^i(x)$,

$(-\mathrm{i})\delta/\delta J_2^j(x)$, respectively.

Upon integrating Eq. (7.3.18) over λ from 0 to 1, give through the initial equation

$$\frac{\partial}{\partial\lambda}\langle 0_+|0_-\rangle = \left[\mathrm{i}\int(\mathrm{d}x)\frac{\partial}{\partial\lambda}\,\mathcal{L}_I'(x;\lambda)\right.$$

$$\left. + \frac{1}{2}\int(\mathrm{d}x)\left(\frac{\partial}{\partial\lambda}\,B'^{jk}(x;\lambda)\right)D_{kj}'(x,x;\lambda)\right]\langle 0_+|0_-\rangle\ , \tag{7.3.19}$$

the integral expression

$$\int\delta\,\langle 0_+|0_-\rangle = \int_0^1\delta\lambda\left[\mathrm{i}\int(\mathrm{d}x)\frac{\partial}{\partial\lambda}\,\mathcal{L}_I'(x;\lambda)\right.$$

$$\left. + \frac{1}{2}\int(\mathrm{d}x)\left(\frac{\partial}{\partial\lambda}\,B'^{jk}(x;\lambda)\right)D_{kj}'(x,x;\lambda)\right]\langle 0_+|0_-\rangle\ , \tag{7.3.20}$$

or,

$$\langle 0_+|0_-\rangle = \exp\left\{\int_0^1\mathrm{d}\lambda\left[\mathrm{i}\int(\mathrm{d}x)\frac{\partial}{\partial\lambda}\,\mathcal{L}_I'(x;\lambda)\right]\right.$$

$$\left. + \int_0^1\mathrm{d}\lambda\left[\frac{1}{2}\int(\mathrm{d}x)\left(\frac{\partial}{\partial\lambda}\,B'^{jk}(x;\lambda)\right)D_{kj}'(x,x;\lambda)\right]\right\}\langle 0_+|0_-\rangle_0$$

$$= \exp\left[\mathrm{i}\int(\mathrm{d}x)\,\mathcal{L}_I'(x)\right.$$

$$\left. + \frac{1}{2}\int(\mathrm{d}x)\int_0^1\mathrm{d}\lambda\left(\frac{\partial}{\partial\lambda}\,B'^{jk}(x;\lambda)\right)D_{kj}'(x,x;\lambda)\right]\langle 0_+|0_-\rangle_0\ . \tag{7.3.21}$$

That is, we have

$$\langle 0_+ \,|\, 0_- \rangle = \exp\left[\mathrm{i} \int (\mathrm{d}x)\, \mathcal{L}'_I(x) \right.$$

$$\left. + \frac{1}{2} \int (\mathrm{d}x) \int_0^1 \mathrm{d}\lambda \left(\frac{\partial}{\partial \lambda}\, B'^{jk}(x;\lambda) \right) D'_{kj}(x,x;\lambda) \right] \langle 0_+ \,|\, 0_- \rangle_0 \ , \tag{7.3.22}$$

where $\langle 0_+ \,|\, 0_- \rangle_0$ is governed by the Lagrangian density $[\mathcal{L}(x;0) + J_1^i(x)\ \chi_i(x) + J_2^j(x)\ \eta_j(x)]$ in Eq. (7.2.1), and $\mathcal{L}'_I(x) \equiv \mathcal{L}'_I(x;0)$, with the latter defined below Eq. (7.3.18).

Equation (7.3.22) provides the solution for the generating functional $\langle 0_+ \,|\, 0_- \rangle$ in the presence of external sources. We thus see that for interaction Lagrangian densities such that $\partial \mathcal{L}_I(x;\lambda)/\partial\lambda$ are quadratic in dependent fields $(\partial B^{jk}(x;\lambda)/\partial\lambda \neq 0)$, as described above, the rules for computations, via the generating functional $\langle 0_+ \,|\, 0_- \rangle$ are modified by the presence of the multiplicative functional differential operator factor

$$\exp\left[\frac{1}{2} \int (\mathrm{d}x) \int_0^1 \mathrm{d}\lambda \left(\frac{\partial}{\partial \lambda}\, B'^{jk}(x;\lambda) \right) D'_{kj}(x,x;\lambda) \right] . \tag{7.3.23}$$

7.4 Applications to Abelian and Non-Abelian Gauge Theories

As special cases of the general Lagrangians described through Eq. (7.2.1) and developed above, consider non-abelian gauge theories with Lagrangian densities

$$\underline{\mathcal{L}} = \mathcal{L} + \mathcal{L}_S \ , \tag{7.4.1}$$

$$\mathcal{L} = -\frac{1}{4}\, G^a_{\mu\nu} G_a^{\mu\nu} + \frac{1}{2\mathrm{i}}\, [\partial_\mu \bar{\psi} \gamma^\mu \psi - \bar{\psi}\gamma^\mu \partial_\mu \psi] - m_0 \bar{\psi}\psi + g_0 \bar{\psi} \gamma_\mu A^\mu \psi \ , \tag{7.4.2}$$

$$\mathcal{L}_S = \bar{\rho}\psi + \bar{\psi}\rho + J^\mu_a A^a_\mu \ , \tag{7.4.3}$$

$$A_\mu = A^a_\mu t_a, \qquad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g_0[A_\mu, A_\nu] \, , \tag{7.4.4}$$

$$G_{\mu\nu} = G^a_{\mu\nu} t_a \, , \tag{7.4.5}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 f^{abc} A^b_\mu A^c_\nu \, . \tag{7.4.6}$$

The t^a matrices are generators of the underlying algebra, and the f^{abc}, totally anti-symmetric, are the structure constants satisfying the Jacobi identity

$$[t^a, t^b] \; = \; \mathrm{i} f^{abc} t^c \, . \tag{7.4.7}$$

$\mathcal{L}_S$ is the source term with the J^μ_a classical functions, while ρ, $\bar{\rho}$ are so-called anti-commuting Grassmann variables.

Upon setting

$$\nabla^{ab}_\mu = \delta^{ab}\partial_\mu + g_0 f^{acb} A^c_\mu \, , \tag{7.4.8}$$

working in the Coulomb gauge $\partial_i A^i_a = 0$, $i = 1, 2, 3$, and introducing the Green operator function $D^{cd}(x, x'; g_0)$, satisfying

$$[\delta^{ac}\boldsymbol{\partial}^2 + g_0 f^{abc} A^b_k \partial_k] \, D^{cd}(x, x'; g_0) = \delta^4(x, x') \, \delta^{ad} \, , \tag{7.4.9}$$

$k = 1, 2, 3$, one may solve for G^{k0}_a (see, Limboonsong and Manoukian, 2006) as follows. First we note that in detail

$$\begin{aligned} \underline{\mathcal{L}} \;&=\; \mathcal{L} + \mathcal{L}_S \, , \\ \underline{\mathcal{L}} \;&=\; -\frac{1}{4}\, G^a_{\mu\nu} G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}\, [\partial_\mu \bar{\psi}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi] - m_0\bar{\psi}\psi + g_0\bar{\psi}\gamma_\mu A^\mu\psi \\ &\quad + \bar{\rho}\psi + \bar{\psi}\rho + J^\mu_a A^a_\mu \, , \end{aligned} \tag{7.4.10}$$

and the action is given by

$$
\begin{aligned}
W &= \int (\mathrm{d}x)\, \underline{\mathcal{L}}(x) \\
&= \int (\mathrm{d}x) \bigg\{ -\frac{1}{4}\, G^a_{\mu\nu} G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}\, [\partial_\mu \bar{\psi} \gamma^\mu \psi - \bar{\psi} \gamma^\mu \partial_\mu \psi] - m_0 \bar{\psi}\psi \\
&\qquad + g_0 \bar{\psi} \gamma_\mu A^\mu \psi + \bar{\rho}\psi + \bar{\psi}\rho + J^\mu_a A^a_\mu \bigg\}\,, \qquad (7.4.11)
\end{aligned}
$$

with its variation having the general form:

$$
\begin{aligned}
\delta W &= \int (\mathrm{d}x) \bigg\{ -\frac{1}{4}\, \delta(G^a_{\mu\nu} G^{\mu\nu}_a) + \frac{1}{2\mathrm{i}}\, \delta[\partial_\mu \bar{\psi} \gamma^\mu \psi - \bar{\psi} \gamma^\mu \partial_\mu \psi] - m_0 \delta(\bar{\psi}\psi) \\
&\qquad + g_0\, \delta(\bar{\psi} \gamma_\mu A^\mu \psi) + \bar{\rho}\, \delta\psi + \delta(\bar{\psi})\rho + J^\mu_a \delta A^a_\mu \bigg\}\,. \qquad (7.4.12)
\end{aligned}
$$

For the first term on the right-hand side of the above equation we have

$$
\begin{aligned}
-\frac{1}{4}\, \delta(G^a_{\mu\nu} G^{\mu\nu}_a) &= -\frac{1}{4} \left(G^a_{\mu\nu}\, \delta G^{\mu\nu}_a + G^{\mu\nu}_a \delta G^a_{\mu\nu} \right) \\
&= -\frac{1}{4} \big[G^a_{\mu\nu}\, \delta \left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g_0 f^{abc} A^\mu_b A^\nu_c \right) \\
&\qquad + G^{\mu\nu}_a \delta \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 f^{abc} A^b_\mu A^c_\nu \right) \big] \\
&= -\frac{1}{4} [G^a_{\mu\nu} \partial^\mu \delta A^\nu_a - G^a_{\mu\nu} \partial^\nu \delta A^\mu_a + g_0 f^{abc} G^a_{\mu\nu} \delta (A^\mu_b A^\nu_c) \\
&\qquad + G^a_{\mu\nu} \partial_\mu \delta A^a_\nu - G^{\mu\nu}_a \partial_\nu \delta A^a_\mu + g_0 f^{abc} G^{\mu\nu}_a \delta (A^b_\mu A^c_\nu)] \\
&= -\frac{1}{4} [G^{\mu\nu}_a \partial_\mu \delta A^a_\nu + G^{\mu\nu}_a \partial_\mu \delta A^a_\nu + g_0 f^{abc} G^a_{\mu\nu} \delta (A^\mu_b A^\nu_c) \\
&\qquad + G^{\mu\nu}_a \partial_\mu \delta A^a_\nu + G^{\mu\nu}_a \partial_\mu \delta A^a_\nu + g_0 f^{abc} G^{\mu\nu}_a \delta (A^b_\mu A^c_\nu)]\,.
\end{aligned}
$$

That is,

$$-\frac{1}{4}\,\delta(G^a_{\mu\nu}G^{\mu\nu}_a) = -\frac{1}{4}\left[4G^{\mu\nu}_a\partial_\mu\delta A^a_\nu + g_0 f^{abc}G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c) + g_0 f^{abc}G^{\mu\nu}_a\delta(A^b_\mu A^c_\nu)\right]. \tag{7.4.13}$$

By using the identity $A\delta B = \delta(AB) - (\delta A)B$ we have

$$G^{\mu\nu}_a\partial_\mu\delta A^a_\nu = \partial_\mu(G^{\mu\nu}_a\delta A^a_\nu) - (\partial_\mu G^{\mu\nu}_a)\delta A^a_\nu\,, \tag{7.4.14}$$

and

$$\int(\mathrm{d}x)\,\partial_\mu(G^{\mu\nu}_a\delta A^a_\nu) = \oint \mathrm{d}\Sigma_\mu\, G^{\mu\nu}_a\delta A^a_\nu = 0\,, \tag{7.4.15}$$

leading to

$$G^{\mu\nu}_a\partial_\mu\delta A^a_\nu = -(\partial_\mu G^{\mu\nu}_a)\,\delta A^a_\nu\,. \tag{7.4.16}$$

Upon substituting Eq. (7.4.16) into the first term on the right-hand side of Eq. (7.4.13) gives

$$\begin{aligned}
-\frac{1}{4}\,\delta(G^a_{\mu\nu}G^{\mu\nu}_a) &= -\frac{1}{4}\left[-4(\partial_\mu G^{\mu\nu}_a)\,\delta A^a_\nu + g_0 f^{abc}\,G^a_{\mu\nu}\,\delta(A^\mu_b A^\nu_c) + g_0 f^{abc}\,G^a_{\mu\nu}\delta(A^\mu_b A^\nu_c)\right] \\
&= -\frac{1}{4}\left[-4(\partial_\mu G^{\mu\nu}_a)\,\delta A^a_\nu + 2g_0 f^{abc}\,G^a_{\mu\nu}\,\delta(A^\mu_b A^\nu_c)\right] \\
&= (\partial_\mu G^{\mu\nu}_a)\,\delta A^a_\nu - \frac{1}{2}\,g_0 f^{abc}\,G^a_{\mu\nu}\,\delta(A^\mu_b A^\nu_c) \\
&= (\partial_\mu G^{\mu\nu}_a)\,\delta A^a_\nu - \frac{1}{2}\,g_0 f^{abc}\,G^a_{\mu\nu}[A^\mu_b\delta A^\nu_c + (\delta A^\mu_b)A^\nu_c]
\end{aligned}$$

$$= (\partial_\mu G_a^{\mu\nu})\, \delta A_\nu^a - \frac{1}{2}\, g_0 f^{abc}\, G_{\mu\nu}^a A_b^\mu\, \delta A_c^\nu - \frac{1}{2}\, g_0 f^{abc}\, G_{\mu\nu}^a (\delta A_b^\mu) A_c^\nu$$

$$= (\partial_\mu G_a^{\mu\nu})\, \delta A_\nu^a - \frac{1}{2}\, g_0 f^{abc} A_b^\mu\, G_{\mu\nu}^a\, \delta A_c^\nu - \frac{1}{2}\, g_0 f^{abc} (\delta A_b^\mu)\, G_{\mu\nu}^a A_c^\nu$$

$$= (\partial_\mu G_a^{\mu\nu})\, \delta A_\nu^a - \frac{1}{2}\, g_0 f^{bca} A_c^\mu\, G_{\mu\nu}^b\, \delta A_a^\nu - \frac{1}{2}\, g_0 f^{bac} (\delta A_a^\mu)\, G_{\mu\nu}^b A_c^\nu$$

$$= (\partial_\mu G_a^{\mu\nu})\, \delta A_\nu^a + \frac{1}{2}\, g_0 f^{acb} A_c^\mu\, G_{\mu\nu}^b\, \delta A_a^\nu - \frac{1}{2}\, g_0 f^{acb} (\delta A_a^\mu)\, G_{\mu\nu}^b A_c^\nu$$

$$= (\partial_\mu G_a^{\mu\nu})\, \delta A_\nu^a + \frac{1}{2}\, g_0 f^{acb} A_c^\mu\, G_{\mu\nu}^b\, \delta A_a^\nu + \frac{1}{2}\, g_0 f^{acb} (\delta A_a^\nu)\, G_{\mu\nu}^b A_c^\mu\ ,$$

or

$$-\frac{1}{4}\, \delta(G_{\mu\nu}^a G_a^{\mu\nu}) = (\partial_\mu G_a^{\mu\nu})\, \delta A_\nu^a + g_0 f^{acb} A_c^\mu\, G_{\mu\nu}^b\, \delta A_a^\nu\ . \qquad (7.4.17)$$

For the second and the third terms on the right-hand side of Eq. (7.4.12), we get

$$+\frac{1}{2\mathrm{i}}\, \delta[(\partial_\mu \bar{\psi}\, \gamma^\mu \psi) - \bar{\psi}\, \gamma^\mu \partial_\mu \psi] - m_0\, \delta(\bar{\psi}\, \psi)$$

$$= \frac{1}{2\mathrm{i}} \left\{ \delta[(\partial_\mu \bar{\psi})\, \gamma^\mu \psi] - \delta(\bar{\psi}\, \gamma^\mu \partial_\mu \psi) \right\} - m_0 [\bar{\psi}\, \delta\psi + (\delta\bar{\psi})\psi]$$

$$= \frac{1}{2\mathrm{i}} \left[(\partial_\mu \bar{\psi})\, \gamma^\mu\, \delta\psi + \delta(\partial_\mu \bar{\psi})\, \gamma^\mu \psi - \bar{\psi}\, \gamma^\mu\, \delta(\partial_\mu \psi) - \delta\bar{\psi}\, \gamma^\mu (\partial_\mu \psi) \right]$$

$$-m_0 \bar{\psi}\, \delta\psi - m_0 (\delta\bar{\psi})\, \psi\ , \qquad (7.4.18)$$

Finally for the fourth term on the right-hand side of Eq. (7.4.12), we derive

$$g_0\, \delta(\bar{\psi}\, \gamma_\mu A^\mu \psi) = g_0 [\bar{\psi}\, \gamma_\mu A^\mu \delta\psi + \delta(\bar{\psi}\, \gamma_\mu A^\mu)\psi]$$

$$= g_0 \{\bar{\psi}\, \gamma_\mu A^\mu \delta\psi + [\bar{\psi}\, \gamma_\mu (\delta A^\mu) + (\delta\bar{\psi})\, \gamma_\mu A^\mu]\psi\}$$

$$= \quad g_0[\bar{\psi}\,\gamma_\mu A^\mu \delta\psi + \bar{\psi}\,\gamma_\mu(\delta A^\mu)\psi + (\delta\bar{\psi})\,\gamma_\mu A^\mu\psi]\,,$$

$$g_0\,\delta(\bar{\psi}\,\gamma_\mu A^\mu\psi) \quad = \quad g_0\bar{\psi}\,\gamma_\mu A^\mu\delta\psi + g_0\bar{\psi}\,\gamma_\mu(\delta A^\mu)\psi + g_0(\delta\bar{\psi})\,\gamma_\mu A^\mu\psi\,. \tag{7.4.19}$$

Upon substituting the above equalities into Eq. (7.4.12), we obtain

$$\begin{aligned}
\delta W \quad = \quad & \int(\mathrm{d}x)\Big\{(\partial_\mu G_a^{\mu\nu})\,\delta A_\nu^a + g_0 f^{acb} A_c^\mu G_{\mu\nu}^b\,\delta A_a^\nu - m_0\bar{\psi}\delta\psi - m_0(\delta\bar{\psi})\psi \\
& +\frac{1}{2\mathrm{i}}\big[(\partial_\mu\bar{\psi})\gamma^\mu\delta\psi + \delta(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\delta(\partial_\mu\bar{\psi}) - \delta\bar{\psi}\,\gamma^\mu(\partial_\mu\psi)\big] \\
& +g_0\bar{\psi}\gamma_\mu A^\mu\delta\psi + g_0\bar{\psi}\gamma_\mu(\delta A^\mu)\psi + g_0(\delta\bar{\psi})\gamma_\mu A^\mu\psi \\
& +\bar{\rho}\,\delta\psi + (\delta\bar{\psi})\rho + J_a^\mu\delta A_\mu^a\Big\}\,.
\end{aligned} \tag{7.4.20}$$

We recall that the canonical momenta conjugate to A^μ defined by

$$\pi^\mu \equiv \pi[A^\mu] \;=\; \frac{\delta W}{\delta\dot{A}_\mu} \;=\; \frac{\delta W}{\delta(\partial_0 A_\mu)}\,. \tag{7.4.21}$$

Consider the first term on the right-hand side of Eq. (7.4.20). To this end we have

$$\begin{aligned}
\int(\mathrm{d}x)(\partial_\mu G_a^{\mu\nu})\,\delta A_\nu^a \;&=\; \int(\mathrm{d}x)\big[(\partial_\mu G_a^{\mu 0})\,\delta A_0^a + (\partial_\mu G_a^{\mu k})\,\delta A_k^a\big], \quad k = 1,2,3 \\
&= \int(\mathrm{d}x)\big[(\partial_\mu G_a^{\mu 0})\,\delta A_0^a + (\partial_\mu G_a^{\mu k})(\delta^{ki} - \delta^{k3}\frac{\partial^i}{\partial_3})\,\delta A_i^a \\
&= \int(\mathrm{d}x)\big[(\partial_\mu G_a^{\mu 0})\,\delta A_0^a + (\partial_\mu G_a^{\mu i} - \frac{\partial^i}{\partial_3}\,\partial_\mu G_a^{\mu 3})\,\delta A_i^a\,, \\
\int(\mathrm{d}x)(\partial_\mu G_a^{\mu\nu})\,\delta A_\nu^a \;&=\; \int(\mathrm{d}x)\big[-G_a^{\mu 0}\delta(\partial_\mu A_0^a) - (G_a^{\mu i} - \frac{\partial^i}{\partial_3}G_a^{\mu 3})\,\delta(\partial_\mu A_i^a)\big]\,.
\end{aligned} \tag{7.4.22}$$

The canonical conjugate momenta, π_a^i are given by the equations

$$\begin{aligned}\pi_a^i &= \frac{\delta W}{\delta(\partial_0 A_i^a)} \\ &= -G_a^{0i} + \frac{\partial^i}{\partial_3}\, G_a^{03} \\ &= G_a^{i0} - \frac{\partial^i}{\partial_3}\, g_a^{30}\ ,\end{aligned} \tag{7.4.23}$$

and, for the dependent fields A_a^0, A_a^3:

$$\pi_a^0 \ = \ 0 \qquad , \qquad \pi_a^3 \ = \ 0\ . \tag{7.4.24}$$

That is, we may write

$$\pi_a^\mu \ = \ G_a^{\mu 0} - \partial_3^{-1} g^{\mu k}\, \partial_k G_a^{30}\ . \tag{7.4.25}$$

Upon multiplying Eq. (7.4.25) by ∇_μ^{ba} gives:

$$\nabla_\mu^{ba} \pi_a^\mu \ = \ \nabla_\mu^{ba} G_a^{\mu 0} - \nabla_\mu^{ba} \partial_3^{-1} g^{\mu k} \partial_k G_a^{30}\ . \tag{7.4.26}$$

From the field equations,

$$\nabla_\mu^{ba} G_b^{\mu\nu} \ = \ -(\delta^\nu{}_\sigma \delta^{ac} - g^{\nu k} \partial_k D^{ab} \nabla_\sigma^{bc})[J_c^\sigma + g_0 \bar{\psi}\gamma^\sigma t_c \psi]\ , \tag{7.4.27}$$

$$\nabla_\mu^{ba} G_a^{\mu 0} \ = \ -(\delta^0{}_\sigma \delta^{bc} - \underbrace{g^{0k}}_{=0} \partial_k D^{ba} \nabla_\sigma^{ac})[J_c^\sigma + g_0 \bar{\psi}\gamma^\sigma t_c \psi]\ , \tag{7.4.28}$$

$$\nabla_\mu^{ba} G_a^{\mu 0} \ = \ -J_b^0 - g_0 \bar{\psi}\gamma^0 t_b \psi\ , \tag{7.4.29}$$

we may rewrite Eq. (7.4.26) as,

$$\nabla_\mu^{ba} \pi_a^\mu \ = \ -J_b^0 - g_0 \bar{\psi}\gamma^0 t_b \psi - \nabla_\mu^{ba} \partial_3^{-1} g^{\mu k} \partial_k G_a^{30}\ , \tag{7.4.30}$$

and also

$$\nabla_\mu^{ba}\partial_3^{-1}g^{\mu k}\partial_k G_a^{30} \;=\; -J_b^0 - g_0\bar{\psi}\gamma^0 t_b\psi - \nabla_\mu^{ba}\pi_a^\mu \;, \tag{7.4.31}$$

$$\nabla_k^{ba}\partial_3^{-1}g^{kk}\partial_k G_a^{30} \;=\; -J_b^0 - g_0\bar{\psi}\gamma^0 t_b\psi - \nabla_k^{ba}\pi_a^k \;, \tag{7.4.32}$$

$$\nabla_k^{ba}\partial_k\,\partial_3^{-1}G_a^{30} \;=\; -J_b^0 - g_0\bar{\psi}\gamma^0 t_b\psi - \nabla_k^{ba}\pi_a^k \;. \tag{7.4.33}$$

Using Eqs. (7.4.9) and (7.2.16) - (7.2.18), we derive that

$$\partial_3^{-1}G_a^{30} \;=\; -D_{ab}[J_b^0 + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla_k^{ba}\pi_a^k] \;. \tag{7.4.34}$$

Substituting the above equation into Eq. (7.4.25), gives

$$\pi_a^\mu \;=\; G_a^{\mu 0} + g^{\mu k}\partial_k D_{ab}[J_b^0 + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla_k^{ba}\pi_a^k] \;, \tag{7.4.35}$$

$$G_a^{\mu 0} \;=\; \pi_a^\mu - g^{\mu k}\partial_k D_{ab}[J_b^0 + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla_k^{ba}\pi_a^k] \;, \tag{7.4.36}$$

$$G_a^{k0} \;=\; \pi_a^k - \partial_k D_{ab}[J_b^0 + g_0\bar{\psi}\gamma^0 t_b\psi + \nabla_k^{ba}\pi_a^k] \;, \tag{7.4.37}$$

$$G_a^{k0} \;=\; -\partial_k D_{ab}J_b^0 + F_a^k \;. \tag{7.4.38}$$

written in a matrix notation, where F_a^k is not an explicit function of the dependent fields and of the external sources. From the very definition of G_a^{k0} in Eq. (7.4.6), we also have

$$G_{\mu\nu}^a \;=\; \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_0 f^{abc}A_\mu^b A_\nu^c \;,$$

$$G_a^{k0} \;=\; \partial^k A_a^0 - \partial^0 A_a^k + g_0 f^{abc}A_b^k A_c^0 \;, \tag{7.4.39}$$

$$\begin{aligned} \partial_k G_a^{k0} &= \partial_k \partial^k A_a^0 - \partial^0 \partial_k A_a^k + g_0 f^{abc} A_b^k \partial_k A_c^0 \\ &= \delta^{ab} \boldsymbol{\partial}^2 A_b^0 + g_0 f^{acb} A_c^k \partial_k A_b^0 \\ &= [\delta^{ab} \boldsymbol{\partial}^2 + g_0 f^{acb} A_k^c \partial_k] A_b^0 \; , \\ \partial_k G_a^{k0} &= \nabla_k^{ab} \partial_k A_b^0 \; . \end{aligned} \tag{7.4.40}$$

Hence we may solve for A_b^0 to obtain

$$A_b^0 \; = \; \frac{1}{[\delta^{ab} \boldsymbol{\partial}^2 + g_0 f^{acb} A_k^c \partial_k]} \, \partial_k G_a^{k0} \; . \tag{7.4.41}$$

Now we substitute Eq. (7.4.38) into above equation to obtain

$$\begin{aligned} A_b^0 &= \frac{1}{[\delta^{ab} \boldsymbol{\partial}^2 + g_0 f^{acb} A_k^c \partial_k]} \, \partial_k (-\partial^k D_{ab} J_b^0 + F_a^k) \\ &= \frac{1}{[\delta^{ab} \boldsymbol{\partial}^2 + g_0 f^{acb} A_k^c \partial_k]} (-\boldsymbol{\partial}^2 D_{ab} J_b^0 + \partial_k F_a^k) \; , \\ A_b^0 &= -D_{bc} \boldsymbol{\partial}^2 D_{ca} J_a^0 + K_b \; , \end{aligned} \tag{7.4.42}$$

where K_b is not an explicit function of the external sources.

Now we show that the time derivative $\partial_0 A_b^k$ may be solved in terms of A_c^0 and the independent fields themselves. To this end, we note that

$$\begin{aligned} G_a^{\mu\nu} &= \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g_0 f^{abc} A_b^\mu A_c^\nu \; , \\ \partial^\nu A_a^\mu &= \partial^\mu A_a^\nu + g_0 f^{abc} A_b^\mu A_c^\nu - G_a^{\mu\nu} \; , \end{aligned} \tag{7.4.43}$$

$$\begin{aligned}
\partial^0 A_a^k &= \partial^k A_a^0 + g_0 f^{abc} A_b^k A_c^0 - G_a^{k0} \\
&= \delta_{ab}\partial_k A_b^0 + g_0 f^{acb} A_c^k A_b^0 - G_a^{k0} \\
&= (\delta_{ab}\partial_k + g_0 f^{acb} A_c^k)\, A_b^0 - G_a^{k0} \,, \\
\partial^0 A_a^k &= \nabla_{ab}^k A_b^0 - G_a^{k0} \,, \qquad (7.4.44)
\end{aligned}$$

and from Eq. (7.4.42) we also have

$$D_{bc}\boldsymbol{\partial}^2 D_{ca} J_a^0 = -(A_b^0 - K_b) \,, \qquad (7.4.45)$$

$$\frac{1}{[\delta_{bc}\boldsymbol{\partial}^2 + g_0 f^{cdb} A_k^d \partial_k]}\, \boldsymbol{\partial}^2 D_{ca} J_a^0 = -(A_b^0 - K_b) \,, \qquad (7.4.46)$$

$$\boldsymbol{\partial}^2 D_{ca} J_a^0 = -[\delta_{bc}\boldsymbol{\partial}^2 + g_0 f^{cdb} A_k^d \partial_k](A_b^0 - K_b) \,, \qquad (7.4.47)$$

$$\begin{aligned}
D_{ca} J_a^0 &= -[\delta_{bc} + \frac{g_0}{\boldsymbol{\partial}^2} f^{cdb} A_k^d \partial_k](A_b^0 - K_b) \\
&= -[\delta_{ca}\delta^{ab} + \frac{g_0}{\boldsymbol{\partial}^2}\, \delta^{ab} f^{cdb} A_k^d \partial_k](A_b^0 - K_b) \\
&= -[\delta_{ca} + \frac{g_0}{\boldsymbol{\partial}^2} f^{cdb} A_k^d \partial_k]\delta^{ab}(A_b^0 - K_b) \,, \\
D_{ca} J_a^0 &= -[\delta_{ca} + \frac{g_0}{\boldsymbol{\partial}^2} f^{cdb} A_k^d \partial_k](A_a^0 - K_a) \,. \qquad (7.4.48)
\end{aligned}$$

Accordingly, Eqs. (7.4.44), (7.4.48) and (7.4.38) lead to the following expression for $\partial^0 A_a^k$:

$$\begin{aligned}
\partial^0 A_a^k &= \nabla_{ab}^k A_b^0 - G_a^{k0} \\
&= \nabla_{ab}^k A_b^0 - [-\partial^k D_{ab} J_b^0 + F_a^k]
\end{aligned}$$

$$
\begin{aligned}
&= \nabla^k_{ab} A^0_b + \partial^k D_{ab} J^0_b - F^k_a] \\
&= \nabla^k_{ab} A^0_b - \partial^k [\delta_{ab} + \frac{1}{\boldsymbol{\partial}^2}\, g_0 f^{acb} A^c_k \partial_k](A^0_a - K_a) - F^k_a \\
&= (\delta_{ab}\partial^k + g_0 f^{acb} A^k_c) A^0_b - \partial^k [\delta_{ab} + \frac{1}{\boldsymbol{\partial}^2}\, g_0 f^{acb} A^c_k \partial_k](A^0_a - K_a) - F^k_a \\
&= \delta_{ab}\partial^k A^0_b + g_0 f^{acb} A^k_c A^0_b - \partial^k \delta_{ab} A^0_a + \partial^k \delta_{ab} K_a \\
&\quad - \frac{\partial^k}{\boldsymbol{\partial}^2}\, g_0 f^{acb} A^c_k \partial_k A^0_a + \frac{\partial^k}{\boldsymbol{\partial}^2}\, g_0 f^{acb} A^c_k \partial_k K_a - F^k_a \\
&= g_0 f^{acb} A^k_c A^0_b - \frac{\partial^k}{\boldsymbol{\partial}^2}\, g_0 f^{acb} A^k_c \partial_k A^0_a + L^k_a \\
&= g_0 f^{acb} A^k_c \frac{\partial^l}{\boldsymbol{\partial}^2}\, A^0_b - g_0 f^{acb} \frac{\partial^k}{\boldsymbol{\partial}^2}\, A^l_c \partial_l A^0_b + L^k_a\,, \\
\partial^0 A^k_a &= g_0 f^{acb} [A^k_c \,\frac{\partial^l}{\boldsymbol{\partial}^2} - \frac{\partial^k}{\boldsymbol{\partial}^2}\, A^l_c]\, \partial_l A^0_b + L^k_a\,, \qquad (7.4.49)
\end{aligned}
$$

where L^k_a has no explicit dependence on the external sources and on the dependent fields A^0_b.

Now we substitute Eq. (7.4.6) into Eq. (7.4.2). For the first term of Eq. (7.4.2), we have

$$
\begin{aligned}
-\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a &= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 f^{abc} A^b_\mu A^c_\nu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g_0 f^{abc} A^\mu_b A^\nu_c) \\
&= -\frac{1}{4}\partial_\mu A^a_\nu \partial^\mu A^\nu_a + \frac{1}{4}\partial_\mu A^a_\nu \partial^\nu A^\mu_a - \frac{1}{4}(\partial_\mu A^a_\nu) g_0 f^{abc} A^\mu_b A^\nu_c \\
&\quad + \frac{1}{4}\partial_\nu A^a_\mu \partial^\mu A^\nu_a - \frac{1}{4}\partial_\nu A^a_\mu \partial^\nu A^\mu_a + \frac{1}{4}(\partial_\nu A^a_\mu) g_0 f^{abc} A^\mu_b A^\nu_c \\
&\quad - \frac{1}{4} g_0 f^{abc} A^b_\mu A^c_\nu (\partial^\mu A^\nu_a) + \frac{1}{4} g_0 f^{abc} A^b_\mu A^c_\nu (\partial^\nu A^\mu_a) \\
&\quad - \frac{1}{4}(g_0 f^{abc})^2 A^b_\mu A^c_\nu A^\mu_b A^\nu_c
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{4}\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{4}\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{4}(\partial^\mu A^\nu_a) g_0 f^{abc} A^b_\mu A^c_\nu \\
&\quad -\frac{1}{4}\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{4}\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{4}(\partial^\mu A^\nu_a) g_0 f^{abc} A^b_\mu A^c_\nu \\
&\quad +\frac{1}{4} g_0 f^{abc} A^b_\mu A^c_\nu (\partial^\nu A^\mu_a) + \frac{1}{4} g_0 f^{abc} A^b_\mu A^c_\nu (\partial^\nu A^\mu_a) \\
&\quad -\frac{1}{4}(g_0 f^{abc})^2 A^b_\mu A^c_\nu A^\mu_b A^\nu_c \,,
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a &= -\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{2}(\partial^\mu A^\nu_a) g_0 f^{abc} A^b_\mu A^c_\nu \\
&\quad +\frac{1}{2} g_0 f^{abc} A^b_\mu A^c_\nu (\partial^\nu A^\mu_a) - \frac{1}{4}(g_0 f^{abc})^2 A^b_\mu A^c_\nu A^\mu_b A^\nu_c \,. \qquad (7.4.50)
\end{aligned}
$$

That is, we may write $\mathcal{L}$ as

$$
\begin{aligned}
\mathcal{L} &= -\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{2}(\partial^\mu A^\nu_a) g_0 f^{abc} A^b_\mu A^c_\nu + \frac{1}{2} g_0 f^{abc} A^b_\mu A^c_\nu (\partial^\nu A^\mu_a) \\
&\quad -\frac{1}{4}(g_0 f^{abc})^2 A^b_\mu A^c_\nu A^\mu_b A^\nu_c + \frac{1}{2\mathrm{i}}[(\partial_\mu \bar{\psi})\gamma^\mu \psi - \bar{\psi}\gamma^\mu \partial_\mu \psi] \\
&\quad -m_0 \bar{\psi}\psi + g_0 \bar{\psi}\gamma_\mu A^\mu \psi \,, \qquad (7.4.51)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial g_0}\mathcal{L} &= \frac{\partial}{\partial g_0}\Big\{ -\partial_\mu A^a_\nu \partial^\mu A^\nu_a - \frac{1}{2}(\partial^\mu A^\nu_a) g_0 f^{abc} A^b_\mu A^c_\nu + \frac{1}{2} g_0 f^{abc} A^b_\mu A^c_\nu (\partial^\nu A^\mu_a) \\
&\quad -\frac{1}{4}(g_0 f^{abc})^2 A^b_\mu A^c_\nu A^\mu_b A^\nu_c + \frac{1}{2\mathrm{i}}[(\partial_\mu \bar{\psi})\gamma^\mu \psi - \bar{\psi}\gamma^\mu \partial_\mu \psi] \\
&\quad -m_0 \bar{\psi}\psi + g_0 \bar{\psi}\gamma_\mu A^\mu \psi \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}(\partial^\mu A_a^\nu) f^{abc} A_\mu^b A_\nu^c + \frac{1}{2} f^{abc} A_\mu^b A_\nu^c (\partial^\nu A_a^\mu) \\
&\quad -\frac{2}{4} g_0 f^{abc} f^{abc} A_\mu^b A_\nu^c A_b^\mu A_c^\nu + \bar{\psi}\gamma_\mu A^\mu \psi \\
&= -\frac{1}{2} f^{abc} A_\mu^b A_\nu^c (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g_0 f^{abc} A_b^\mu A_c^\nu) + \bar{\psi}\gamma_\mu A^\mu \psi \\
&= -\frac{1}{2} f^{abc} A_\mu^b A_\nu^c G_a^{\mu\nu} + \bar{\psi}\gamma_\mu A^\mu \psi \\
&= -\frac{1}{2}\left(f^{abc} A_0^b A_\nu^c G_a^{0\nu} + f^{abc} A_k^b A_\nu^c G_a^{k\nu} \right) + \bar{\psi}\gamma_\mu A^\mu \psi
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\Bigg(f^{abc} A_0^b A_0^c \underbrace{G_a^{00}}_{=0} + f^{abc} A_0^b A_l^c G_a^{0l} + f^{abc} A_k^b A_0^c G_a^{k0} \\
&\quad + f^{abc} A_k^b A_l^c G_a^{kl} \Bigg) + \bar{\psi}\gamma_\mu A^\mu \psi \\
&= -\frac{1}{2}\left(f^{abc} A_0^b A_l^c G_a^{0l} + f^{abc} A_k^b A_0^c G_a^{k0} + f^{abc} A_k^b A_l^c G_a^{kl} \right) \\
&\quad + \bar{\psi}\gamma_\mu A^\mu \psi \\
&= -\frac{1}{2}\left(f^{abc} A_l^b A_0^c G_a^{l0} + f^{abc} A_k^b A_0^c G_a^{k0} + f^{abc} A_k^b A_l^c G_a^{kl} \right) \\
&\quad + \bar{\psi}\gamma_\mu A^\mu \psi \\
&= -\frac{1}{2}\left(2 f^{abc} A_k^b A_0^c G_a^{k0} + f^{abc} A_k^b A_l^c G_a^{kl} \right) + \bar{\psi}\gamma_\mu A^\mu \psi \,, \\
\frac{\partial}{\partial g_0}\mathcal{L} &= -f^{abc} A_k^b A_0^c G_a^{k0} - \frac{1}{2} f^{abc} A_k^b A_l^c G_a^{kl} + \bar{\psi}\gamma_\mu A^\mu \psi \,. \qquad (7.4.52)
\end{aligned}
$$

Finally we record the above expression in the form

$$\frac{\partial}{\partial g_0}\,\mathcal{L} \;=\; -f^{abc}A^b_k(A^c_0G^{k0}_a + \frac{1}{2}A^c_lG^{kl}_a) + \bar{\psi}\gamma_\mu A^\mu\psi\;. \tag{7.4.53}$$

From the definition of G^{k0}_a in Eq. (7.4.6), and the fact that $\partial^0 A^k_a$ may be expressed in terms of the A^0_b, as shown in Eq. (7.4.49), and the independent fields themselves, we see that Eq. (7.4.53) is quadratic in the dependent fields A^0_b.

The structure G^{k0}_a in Eq. (7.4.6) may be expressed, from Eq. (7.4.49), as a linear function of the dependent fields A^0_a, and directly from Eq. (7.4.38) we have

$$\begin{aligned} G^{k0}_a &= -\partial^k D_{ab}J^0_b + F^k_a \\ &= -\delta^0{}_\nu\,\partial_k D_{ab}J^\nu_b + F^k_a\;, \end{aligned} \tag{7.4.54}$$

$$\frac{\delta}{\delta J^\nu_b}\,G^{k0}_a \;=\; -\delta^0{}_\nu\,\partial_k D_{ab}(x,x';g_0)\;. \tag{7.4.55}$$

Hence Eqs. (7.3.1) - (7.3.4), (7.3.6), (7.3.8), (7.4.53) and (7.4.55) give

$$\begin{aligned} \frac{\partial}{\partial g_0}\,\langle 0_+|\,0_-\rangle &= \mathrm{i}\left\langle 0_+\middle|\int(\mathrm{d}x)\frac{\partial}{\partial g_0}\,\mathcal{L}\middle|0_-\right\rangle \\ &= \mathrm{i}\Big\langle 0_+\Big|\int(\mathrm{d}x)\Big[-f^{abc}A^b_k(A^c_0G^{k0}_a + \frac{1}{2}A^c_lG^{kl}_a) \\ &\qquad + \bar{\psi}\gamma_\mu A^\mu\psi\Big]\Big|0_-\Big\rangle\;, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial g_0}\,\langle 0_+|\,0_-\rangle &= \mathrm{i}\Big\langle 0_+\Big|\int(\mathrm{d}x)\Big(-f^{abc}A^b_kA^c_0G^{k0}_a \\ &\qquad -\frac{1}{2}f^{abc}A^b_kA^c_lG^{kl}_a + \bar{\psi}\gamma_\mu A^\mu\psi\Big)\Big|0_-\Big\rangle. \end{aligned} \tag{7.4.56}$$

The first term on the right-hand of Eq. (7.4.56) may be rewritten as

$$\mathrm{i}\left\langle 0_+\left|\int(\mathrm{d}x)\Big(-f^{abc}A_k^b A_0^c G_a^{k0}\Big)\right|0_-\right\rangle$$

$$= -\mathrm{i}\int(\mathrm{d}x) f^{abc} A_k^{\prime b}\left\langle 0_+\left|(A_0^c\, G_a^{k0})_+\right|0_-\right\rangle \,. \qquad (7.4.57)$$

Using the quantum dynamical principle, we have from Eq. (7.3.4) that

$$\left\langle 0_+\left|(A_0^c\, G_a^{k0})_+\right|0_-\right\rangle = (-\mathrm{i})\frac{\delta}{\delta J_c^0}\left\langle 0_+\left|G_a^{k0}\right|0_-\right\rangle + \mathrm{i}\left\langle 0_+\left|\frac{\delta}{\delta J_c^0}\, G_a^{k0}\right|0_-\right\rangle \,. \qquad (7.4.58)$$

On the other hand, from Eq. (7.4.55) we also obtain

$$\begin{aligned}\frac{\delta}{\delta J_b^\nu}\, G_a^{k0} &= -\delta^0{}_\nu\, \partial_k D_{ab}(x,x';g_0)\,,\\ \frac{\delta}{\delta J_c^0}\, G_a^{k0} &= -\delta^0{}_0\, \partial_k D_{ac}(x,x';g_0)\\ &= -\partial_k D_{ac}\,. \end{aligned} \qquad (7.4.59)$$

That is, we have the equality

$$\left\langle 0_+\left|(A_0^c\, G_a^{k0})_+\right|0_-\right\rangle = A_0^{\prime c} G_a^{\prime k0}\left\langle 0_+\middle|\,0_-\right\rangle - \mathrm{i}\,\partial_k D'_{ac}(x,x;g_0)\left\langle 0_+\middle|\,0_-\right\rangle \,. \qquad (7.4.60)$$

We note that

$$\mathrm{i}\left\langle 0_+\left|\int(\mathrm{d}x)(-f^{abc}A_k^b A_0^c G_a^{k0})\right|0_-\right\rangle$$

$$= -\mathrm{i}\int(\mathrm{d}x) f^{abc} A_k^{\prime b}\left\langle 0_+\left|A_0^c\, G_a^{k0}\right|0_-\right\rangle$$

$$= -\mathrm{i}\int(\mathrm{d}x) f^{abc} A_k^{\prime b}\Big[A_0^{\prime c} G_a^{\prime k0}\left\langle 0_+\middle|\,0_-\right\rangle - \mathrm{i}\partial_k D'_{ac}(x,x;g_0)\left\langle 0_+\middle|\,0_-\right\rangle\Big]$$

$$= \int (\mathrm{d}x)\Big[-\mathrm{i} f^{abc} A_k^{\prime b} A_0^{\prime c} G_a^{\prime k0} \langle 0_+ \,|\, 0_- \rangle - f^{bca} A_k^{\prime b} \partial_k D'_{ac}(x,x;g_0) \langle 0_+ \,|\, 0_- \rangle \Big] \,. \tag{7.4.61}$$

The second term on the right-hand side of Eq. (7.4.56) may be also rewritten in the form:

$$\mathrm{i} \left\langle 0_+ \middle| \int (\mathrm{d}x) \Big(-\frac{1}{2} f^{abc} A_k^b A_l^c G_a^{kl} \Big) \middle| 0_- \right\rangle$$

$$= -\frac{\mathrm{i}}{2} \int (\mathrm{d}x) f^{abc} A_k^{\prime b} \left\langle 0_+ \middle| \Big(A_l^c \, G_a^{kl} \Big)_+ \middle| 0_- \right\rangle \,. \tag{7.4.62}$$

From Eq. (7.3.6), we then have

$$\left\langle 0_+ \middle| \Big(A_l^c \, G_a^{kl} \Big)_+ \middle| 0_- \right\rangle = (-\mathrm{i}) \frac{\delta}{\delta J_c^l} \, G_a^{\prime kl} \langle 0_+ \,|\, 0_- \rangle \,. \tag{7.4.63}$$

Upon substituting the above equation into Eq. (7.4.62), gives

$$\mathrm{i} \left\langle 0_+ \middle| \int (\mathrm{d}x) \left(-\frac{1}{2} f^{abc} A_k^b A_l^c G_a^{kl} \right) \middle| 0_- \right\rangle$$

$$= -\frac{\mathrm{i}}{2} \int (\mathrm{d}x) f^{abc} A_k^{\prime b} \left\langle 0_+ \middle| \Big(A_l^c \, G_a^{kl} \Big)_+ \middle| 0_- \right\rangle$$

$$= -\frac{\mathrm{i}}{2} \int (\mathrm{d}x) f^{abc} A_k^{\prime b} A_l^{\prime c} G_a^{\prime kl} \langle 0_+ \,|\, 0_- \rangle$$

$$= \int (\mathrm{d}x) \left[-\frac{\mathrm{i}}{2} f^{abc} A_k^{\prime b} A_l^{\prime c} G_a^{\prime kl} \langle 0_+ \,|\, 0_- \rangle \right] . \tag{7.4.64}$$

Finally for the third term on the right-hand side of Eq. (7.4.56) we have:

$$\mathrm{i} \left\langle 0_+ \middle| \int (\mathrm{d}x) \bar{\psi} \gamma^\mu A_\mu \psi \middle| 0_- \right\rangle = \mathrm{i} \int (\mathrm{d}x) \left\langle 0_+ \middle| \bar{\psi} \gamma^\mu A_\mu \psi \middle| 0_- \right\rangle$$

$$= \mathrm{i}\int(\mathrm{d}x)\bigg[(-\mathrm{i})\frac{\delta}{\delta J_a^\mu}\left\langle 0_+\left|\bar\psi\gamma^\mu t_a\psi\right|0_-\right\rangle + \mathrm{i}\left\langle 0_+\left|\underbrace{\frac{\delta}{\delta J_a^\mu}\,(\bar\psi\gamma^\mu t_a\psi)}_{=0}\right|0_-\right\rangle\bigg]\,,$$

or

$$\mathrm{i}\left\langle 0_+\left|\int(\mathrm{d}x)\bar\psi\gamma^\mu A_\mu\psi\right|0_-\right\rangle \;=\; \int(\mathrm{d}x)\mathrm{i}\bar\psi'\gamma^\mu A'_\mu\psi'\,\langle 0_+\,|\,0_-\rangle\ . \tag{7.4.65}$$

Equations (7.4.61), (7.4.64) and (7.4.65), allow us to rewrite Eq. (7.4.56) as

$$\begin{aligned}\frac{\partial}{\partial g_0}\langle 0_+\,|\,0_-\rangle \;&=\; \int(\mathrm{d}x)\bigg[-\mathrm{i}f^{abc}A_k'^{b}\left(A_0'^{c}G_a'^{k0}+\frac{1}{2}A_l'^{c}G_a'^{kl}\right)\\ &\quad +\mathrm{i}\bar\psi'\gamma^\mu A'_\mu\psi' - f^{bca}A_k'^{b}\partial_k D'_{ac}(x,x;g_0)\bigg]\,\langle 0_+\,|\,0_-\rangle\\ &=\;\bigg[\mathrm{i}\int(\mathrm{d}x)\,\frac{\partial}{\partial g_0}\,\mathcal{L}'(x;g_0)\\ &\quad -\int(\mathrm{d}x)f^{bca}A_k'^{b}(x)\partial_k D'^{ac}(x,x;g_0)\bigg]\,\langle 0_+\,|\,0_-\rangle\ ,\end{aligned} \tag{7.4.66}$$

where $A_k'^{b} = (-\mathrm{i})\delta/\delta J_b^k$. Upon using the definition of $D^{ac}(x,x';g_0)$ in Eq. (7.4.9), and integrating Eq. (7.4.66) over g_0, and using a matrix notation

$$D^{ab}(x,x';g_0) \;=\; \left[\left\langle 0_+\left|\left(\frac{1}{\boldsymbol{\partial}^2 - \mathrm{i}g_0 A_k\partial_k}\right)\right|0_-\right\rangle\right]^{ab}\ , \tag{7.4.67}$$

the notation

$$\mathrm{Tr}\,[f] \;=\; \int(\mathrm{d}x)f^{aa}(x,x)\,, \tag{7.4.68}$$

and the fact that $f^{bca}A^b_k = \mathrm{i}(A_k)^{ca}$, we have from Eq. (7.4.66) the important result:

$$\begin{aligned}
\frac{\partial}{\partial g_0}\langle 0_+ | 0_- \rangle &= \left[\mathrm{i}\int(\mathrm{d}x)\frac{\partial}{\partial g_0}\,\mathcal{L}'(x;g_0) - \int(\mathrm{d}x)\frac{\mathrm{i}A'_k\partial_k}{\boldsymbol{\partial}^2 - \mathrm{i}g_0A_l\partial_l}\right]\langle 0_+ | 0_- \rangle \\
&= \left\{\mathrm{i}\int(\mathrm{d}x)\frac{\partial}{\partial g_0}\,\mathcal{L}'(x;g_0) + \mathrm{Tr}\left[\frac{-\mathrm{i}A'_k\partial_k}{\boldsymbol{\partial}^2 - \mathrm{i}g_0A_l\partial_l}\right]\right\}\langle 0_+ | 0_- \rangle\ ,
\end{aligned} \tag{7.4.69}$$

$$\begin{aligned}
\int\delta\langle 0_+ | 0_- \rangle &= \exp\left[\int\delta g_0\left\{\mathrm{i}\int(\mathrm{d}x)\frac{\partial}{\partial g_0}\,\mathcal{L}'(x;g_0) + \mathrm{Tr}\left[\frac{-\mathrm{i}A'_k\partial_k}{\boldsymbol{\partial}^2 - \mathrm{i}g_0A_l\partial_l}\right]\right\}\right] \\
&\quad \times\langle 0_+ | 0_- \rangle_0
\end{aligned} \tag{7.4.70}$$

which integrates out to

$$\begin{aligned}
\langle 0_+ | 0_- \rangle &= \exp\left[\mathrm{i}\int(\mathrm{d}x)\,\mathcal{L}'(x;g_0)\right] \\
&\quad \times\exp\mathrm{Tr}\ln\left[1 - \mathrm{i}\frac{g_0}{\boldsymbol{\partial}^2}A'_k\partial_k\right]\langle 0_+ | 0_- \rangle_0\ ,
\end{aligned} \tag{7.4.71}$$

we obtain the modifying Faddeev-Popov multiplicative factor

$$\exp\mathrm{Tr}\ln\left[1 - \mathrm{i}\frac{g_0}{\boldsymbol{\partial}^2}A'_k\partial_k\right]. \tag{7.4.72}$$

The general derivation given above for interaction Lagrangian densities such that $\partial\mathcal{L}_I(x;\lambda)/\partial\lambda$ may be expressed as quadratic functions in dependent fields involves no symmetry arguments. As a matter of fact, we may consider the addition of a gauge-invariant breaking term $g_1A^\mu_aA^a_\mu\bar{\psi}\psi/2$ to the Lagrangian density in Eq. (7.4.2) which is again quadratic in A^0_a and presumably contributes to the generation of masses to the vector fields through a non-vanishing expectation value of $\bar{\psi}\psi$. A detailed analysis shows (see, Limboonsong and Manoukian, 2006) that the modifying extra multiplicative factor to $\exp(\mathrm{i}\int(\mathrm{d}x)\,\mathcal{L}'_I(x))\,\langle 0_+\,|\,0_-\rangle\,|_0$ occurring in $\langle 0_+\,|\,0_-\rangle$ is given by

$$\exp\left[-\frac{1}{2}\mathrm{Tr}\ln\left(1+\frac{g_1}{\nabla'_l\partial_l(\boldsymbol{\partial}^2)^{-1}\nabla'_k\partial_k}\bar{\psi}'\psi'\right)\right]$$

$$\times\exp\mathrm{Tr}\ln\left(1-\mathrm{i}g_0\frac{1}{\boldsymbol{\partial}^2}A'_k\partial_k\right), \tag{7.4.73}$$

where $\bar{\psi}'=(-\mathrm{i})\delta/\delta\rho$, $\psi'=(-\mathrm{i})\delta/\delta\bar{\rho}$, and $\mathcal{L}'_I$ is the new interaction Lagrangian density functional differential operator expressed in terms of functional derivatives with respect to the external sources.

We have seen, within the functional differential formalism of quantum field theory in the presence of external sources, that interaction Lagrangian densities $\mathcal{L}_I(x;\lambda)$ such that $\partial\mathcal{L}_I(x;\lambda)/\partial\lambda$ may be expressed as quadratic functions of dependent fields (i.e., $\partial B^{jk}(x;\lambda)/\partial\lambda\neq 0$ in Eq. (7.2.2)) and *arbitrary* functions of independent fields, necessarily lead to modifications of the rules for computations, via the generating functional $\langle 0_+\,|\,0_-\rangle$ as a functional of the external sources which are coupled to the fields, and no appeal was made, through the analysis, to path integrals. The general expression for such a modification is given in Eq. (7.3.22) as a functional differential operator occurring as a multiplicative factor in $\langle 0_+\,|\,0_-\rangle$. Such Lagrangians play *central* roles in fundamental physics and present renormalizable gauge theories fall into this category. It is important, however, to emphasize that such modifications are not tied up to non-abelian gauge theories, through the emergence of so-called Faddeev-Popov factors, as one might naively expect, but apply to theories which, in general, are quadratic functions in dependent fields as described above. As a matter of fact the addition of a gauge term breaking term in the form $(g_1/2)A^\mu A_\mu\bar{\psi}\psi$ to the interaction Lagrangian density of QED (abelian gauge theory), which is again quadratic in A^0, leads, according to Eq. (7.4.73), the following extra functional differential multiplicative factor

$$\exp\left[-\frac{1}{2}\mathrm{Tr}\ln\left(1+\frac{g_0}{\boldsymbol{\partial}^2}\bar{\psi}'\psi'\right)\right], \tag{7.4.74}$$

multiplying $\exp[\mathrm{i}\int(\mathrm{d}x)\ \mathcal{L}'_I(x)]\,\langle 0_+\,|\,0_-\rangle_0$, where $\mathcal{L}'_I(x)$ is the *new* interaction Lagrangian density functional differential operator including the additional term just mentioned as a simplified version of Eq. (7.4.73). That is, *a non-trivial modification arises even for such an abelian gauge theory*. The technical question now arises as to what happens to model Lagrangian densities that one may set up which are cubic or of higher order in dependent fields in the sense investigated above. The main complication with such theories becomes obvious by noting that the corresponding Green function operator function to the one in Eq. (7.2.13) will now *depend* on dependent fields themselves. Accordingly, when we apply the corresponding rule in Eq. (7.3.8) for finally expressing the matrix element $\langle 0_+\,|(\partial\mathcal{L}_I/\partial\lambda)|0_-\rangle$, as a functional differential operator, with respect to the external sources, to be eventually applied to $\langle 0_+\,|\,0_-\rangle$, the expression $\delta\eta_k(x')/\delta J_2^j(x'')$ will again depend, rather non-trivially, on the dependent fields $\eta_j(x)$. This makes the procedure of expressing the matrix element just mentioned as a functional differential operation to be applied to $\langle 0_+\,|\,0_-\rangle$ quite unmanageable. Such field theories require very special tools and will not be considered here.

CHAPTER VIII

CONCLUSION

In this final chapter, we summarize the main results developed, proved and obtained in the thesis. We were guided by the structures of *all* present Lagrangians in quantum field theory describing the dynamics of elementary particles in High-Energy Physics to systematically study, prove and establish rules for the construction of the vacuum-to-vacuum transition amplitude in the functional *differential* formalism of quantum via the application of the Quantum Dynamical Principle in the presence of *constraints* and in the *presence* of external sources, where necessarily and *a priori* no conservation laws are imposed on these sources so that all of their components may be varied independently. In particular we note that the entire analysis and all the general Lagrangians considered in this work are *physically* relevant. The reason why constraints are considered here is that all of the Lagrangian describing the dynamics of elementary particles are gauge theories. In gauge theories, a gauge constraint then necessarily arises. Throughout, we work in the so-called Coulomb gauge for the massless spin 1 gauge field to ensure the positivity of the underlying Hilbert space and that only two polarization states consistly exist for the gauge field. Before carrying out systematically constraints in the differential formalism of quantum field theory, we have also developed rules for computations in quantum physics with constraints, again in the differential formalism, and explicit expressions for the transformations were developed. The rules developed are as follows: [1] In developing the rules to follow, we were inspired by the situation occurring in quantum electrodynamics in the Coulomb gauge. Suppose we are given a Hamiltonian $H(\mathbf{q},\mathbf{p})$ as a function of independent pairs of canonical conjugate variables $\{q_i, p_i, i = 1,\ldots,n\} \equiv \{\mathbf{q},\mathbf{p}\}$, that is, it is defined in a phase space of dimensionality equal to $2n$. We are also given a set of pairwise commuting operator functions $\{G_j(\mathbf{q}(t),\mathbf{p}(t)), j = 1,\ldots,k\}$ of these variables. These allow us to describe the dynamics of any Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})$ in, *a priori*, $2(n+k)$ dimensional phase space in which constraints are imposed given by

$$Q_j(\tau) - G_j(\mathbf{q},(\tau),\mathbf{p}(\tau)) \;=\; 0, \quad j = 1,\ldots,k\,, \tag{8.0.1}$$

for all τ in an interval $[t',t]$, with $\mathbf{Q} = (Q_1,\ldots,Q_k)$, for which $\mathbf{P} = 0$, *such that*

$$\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{G}(\mathbf{q},\mathbf{p}),0) \;=\; H(\mathbf{q},\mathbf{p})\,. \tag{8.0.2}$$

The transformation functions $\langle \mathbf{q},\mathbf{Q},t\,|\,\mathbf{q}',\mathbf{Q}',t'\rangle$ of the constrained dynamics is given by

$$\langle \mathbf{qQ}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle \;=\; \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\;\tilde{H}'(\tau)\right)\langle \mathbf{qQ}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle_0^{\wedge}\Big|\,, \tag{8.0.3}$$

where

$$\langle \mathbf{qQ}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle_0^{\wedge} = \delta^{(k)}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\cdot)} - \mathbf{G}'(\cdot)\right)\delta^{(k)}\left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)}\frac{\delta}{\delta\mathbf{s}(\cdot)}\right)\langle \mathbf{q}\,t\,|\,\mathbf{q}'t'\rangle_0\; A, \tag{8.0.4}$$

$$A = \delta^k\left(\mathbf{Q} - \mathbf{Q}' - \int_{t'}^{t}\mathrm{d}\tau\;\mathbf{s}(\tau)\right)\exp\left(\frac{\mathrm{i}}{\hbar}\,\mathbf{Q}\cdot\int_{t'}^{t}\mathrm{d}\tau\;\mathbf{f}(\tau)\right)$$

$$\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{\tau}\mathrm{d}\tau'\;\mathbf{s}(\tau)\cdot\mathbf{f}(\tau)\right), \tag{8.0.5}$$

$$\langle \mathbf{q}\,t\,|\,\mathbf{q}'\,t'\rangle_0 = \delta^n\left(\mathbf{q} - \mathbf{q}' - \int_{t'}^{t}\mathrm{d}\tau\;\mathbf{S}(\tau)\right)\exp\left(\frac{\mathrm{i}}{\hbar}\,\mathbf{q}\cdot\int_{t'}^{t}\mathrm{d}\tau\;\mathbf{F}(\tau)\right)$$

$$\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{\tau}\mathrm{d}\tau'\;\mathbf{S}(\tau)\cdot\mathbf{F}(\tau')\right), \tag{8.0.6}$$

and the vertical bar $|$ in Eq. (8.0.3) refers to the fact that all the external sources are to be set to zero after all the relevant functional differentiations have been carried out.

$\delta^{(k)}(-\mathrm{i}\hbar\delta/\delta\mathbf{f}(\cdot) - \mathbf{G}'(\cdot))$ and $\delta^{(k)}(\mathrm{i}\hbar\delta/(2\pi\hbar)\delta\mathbf{s}(\cdot))$ in Eq. (8.0.4), as arising from the conditions in Eqs. (3.2.44) and (3.2.45), refer, each, to the product of k-dimensional deltas with τ running over all points in the interval $[t', t]$, i.e.,

$$\delta^{(k)}(D(\cdot)) \; = \; \prod_{t' \leq \tau \leq t} \delta^{k}(D(\tau)). \tag{8.0.7}$$

The numericals $\mathbf{Q}$, $\mathbf{Q}'$ are defined as follows:

$$\begin{aligned} \mathbf{Q} \;&=\; \mathbf{Q}^{\mathrm{c}}(\tau)\Big|_{\tau \to t}, \\ \mathbf{Q}' \;&=\; \mathbf{Q}^{\mathrm{c}}(\tau)\Big|_{\tau \to t'}, \end{aligned}$$

where $\mathbf{Q}^{\mathrm{c}}(\tau)$ is the *classical* function

$$\mathbf{Q}^{\mathrm{c}}(\tau) \;=\; \frac{\langle \mathbf{q}t\,|\mathbf{G}(\mathbf{q}(\tau), \mathbf{p}(\tau))|\mathbf{q}'t'\rangle}{\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle}\,\Bigg|,$$

and in detail

$$\mathbf{Q}^{\mathrm{c}}(\tau) \;=\; \frac{1}{\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle}\, G'(\tau) \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t} \mathrm{d}\tau'\; H'(\tau')\right) \langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle_0 \,\Bigg|,$$

where

$$H'(\tau') \;=\; H\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau')}, \mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau')}\right), \tag{8.0.8}$$

obtained from $H(\mathbf{q}, \mathbf{p})$ by replacing $\mathbf{q}$, $\mathbf{p}$, respectively, by $-\mathrm{i}\hbar\delta/\delta\mathbf{F}(\tau')$, $\mathrm{i}\hbar\delta/\delta\mathbf{S}(\tau')$, while

$$\tilde{H}'(\tau') \;=\; \tilde{H}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau')}, \mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau')}, -\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\tau')}, \mathrm{i}\hbar\frac{\delta}{\delta\mathbf{s}(\tau')}\right),$$

as similarly obtained from a Hamiltonian $\tilde{H}(\mathbf{q}, \mathbf{p}, \mathbf{Q}, \mathbf{P})$. We note that because of the equality in Eq. (8.0.2), we may replace $\tilde{H}'(\tau)$ in Eq. (8.0.3) by $H'(\tau)$ as a consequence of the constraints imposed by the delta functionals:

$$\delta^{(k)}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{f}(\cdot)}-\mathbf{G}'(\cdot)\right)\delta^{(k)}\left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)}\frac{\delta}{\delta\mathbf{s}(\cdot)}\right), \tag{8.0.9}$$

in Eq. (8.0.4).

The procedure for describing the dynamics of a Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})$ with constraints may be summarized through the following:

$$\underbrace{H(\mathbf{q},\mathbf{p})}_{\substack{\text{Phase Space of}\\ \dim(2n)}} \quad\longrightarrow\quad \underbrace{\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})}_{\substack{\text{Phase Space of}\\ \dim(2n+2k)}} \quad\longrightarrow\quad \tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})\Big|_{\text{constraints}}, \tag{8.0.10}$$

with the transformations functions of the constrained dynamics with Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p},\mathbf{Q},\mathbf{P})$ given in Eq. (8.0.3). [2] On the other hand, suppose, we are given a Hamiltonian $\tilde{H}(\mathbf{q},\mathbf{p})$ as a function of independent variables $\mathbf{q}=(q_1,\ldots,q_n)$ and their canonical conjugate momenta $\mathbf{p}=(p_1,\ldots,p_n)$, we consider a new system by defining constraint operator functions

$$\mathbf{G}(\mathbf{q}(\tau),\mathbf{p}(\tau)) \;=\; \left\{G_1(\mathbf{q}(\tau),\mathbf{p}(\tau),\ldots,G_k(\mathbf{q}(\tau),\mathbf{p}(\tau))\right\}, \tag{8.0.11}$$

as of pairwise commuting operator functions $G_j(\mathbf{q}(\tau),\mathbf{p}(\tau))$, which together we introduce canonical conjugate momenta for them

$$\hat{\mathbf{G}}(\mathbf{q}(\tau),\mathbf{p}(\tau)) \;=\; \left\{\hat{G}_1(\mathbf{q}(\tau),\mathbf{p}(\tau)),\ldots,\hat{G}_k(\mathbf{q}(\tau),\mathbf{p}(\tau))\right\}, \tag{8.0.12}$$

such that

$$\mathbf{G}(\mathbf{q}(\tau),\mathbf{p}(\tau)) \;=\; \mathbf{0}\,, \tag{8.0.13}$$

$$\hat{\mathbf{G}}(\mathbf{q}(\tau), \mathbf{p}(\tau)) \ = \ \mathbf{0} \,, \tag{8.0.14}$$

for *all* τ in the interval $[t', t]$.

The new Hamiltonian of the constrained dynamics is then defined by

$$H(\mathbf{q}^*, \mathbf{p}^*) \ = \ \tilde{H}(\mathbf{q}, \mathbf{p})\Big|_{\mathbf{G}=\mathbf{0}, \hat{\mathbf{G}}=\mathbf{0}} \,, \tag{8.0.15}$$

and Eqs. (8.0.13) and (8.0.14) define the constraints, and with $(\mathbf{q}, \mathbf{p}) \to (\mathbf{q}^*, \mathbf{p}^*, \mathbf{G}, \hat{\mathbf{G}})$ defining a canonical transformation, i.e., the Jacobian of the transformation is unity:

$$J \ = \ \left| \frac{\partial(\mathbf{q}, \mathbf{p})}{\partial(\mathbf{q}^*, \mathbf{p}^*, \mathbf{G}, \hat{\mathbf{G}})} \right| \ = \ 1 \,, \tag{8.0.16}$$

as obtained within a classical context.

The procedure for describing the dynamics of the new constrained dynamics with Hamiltonian $H(\mathbf{q}^*, \mathbf{p}^*)$ may be then summarized through the following:

$$\underbrace{\tilde{H}(\mathbf{q}, \mathbf{p})}_{\substack{\text{Phase Space of} \\ \text{dim } (2n)}} \qquad \longrightarrow \qquad \underbrace{\tilde{H}(\mathbf{q}, \mathbf{p})\Big|_{\mathbf{G}=\mathbf{0}, \hat{\mathbf{G}}=\mathbf{0}} \equiv H(\mathbf{q}^*, \mathbf{p}^*)}_{\substack{\text{Phase Space of} \\ \text{dim } (2(n-k))}} \,. \tag{8.0.17}$$

Given the Hamiltonian $\tilde{H}(\mathbf{q}, \mathbf{p})$ with the constraints Eqs. (8.0.13) and (8.0.14) now imposed, the transformation function $\langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_{\mathrm{C}}$, with the $\mathbf{q}$ (and similarly the $\mathbf{q}'$) not necessarily independent variables is then given by

$$\begin{aligned} \langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_{\mathrm{C}} \ = \ & \delta^{(k)} \left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{f}(\cdot)} - \mathbf{G}'(\cdot) \right) \delta^{(k)} \left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{f}(\cdot)} \right) \\ & \times \delta^{(k)} \left(\frac{1}{(2\pi\hbar)} \left(\mathrm{i}\hbar \, \frac{\delta}{\delta \mathbf{s}(\cdot)} - \hat{\mathbf{G}}'(\cdot) \right) \right) \delta^{(k)} \left(\frac{\mathrm{i}\hbar}{(2\pi\hbar)} \frac{\delta}{\delta \mathbf{s}(\cdot)} \right) \\ & \times \exp \left(-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \ \tilde{H}'(\tau) \right) \langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle \, \langle \mathbf{Q}t \,|\, \mathbf{Q}'t' \rangle \,, \end{aligned} \tag{8.0.18}$$

where

$$\tilde{H}'(\tau) \; = \tilde{H}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}, \mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}}\right), \tag{8.0.19}$$

$$\begin{aligned}\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle \; &= \delta^k\left(\mathbf{q}-\mathbf{q}'-\int_{t'}^{t}\mathrm{d}\tau\;\mathbf{S}(\tau)\right)\exp\left(\frac{\mathrm{i}}{\hbar}\mathbf{q}\cdot\int_{t'}^{t}\mathrm{d}\tau\;\mathbf{F}(\tau)\right)\\ &\quad\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\;\mathbf{S}(\tau)\cdot\Theta(\tau-\tau')\mathbf{F}(\tau')\right),\end{aligned} \tag{8.0.20}$$

$$\begin{aligned}\langle \mathbf{Q}t\,|\,\mathbf{Q}'t'\rangle \; &= \delta^k\left(\mathbf{Q}-\mathbf{Q}'-\int_{t'}^{t}\mathrm{d}\tau\;\mathbf{s}(\tau)\right)\exp\left(\frac{\mathrm{i}}{\hbar}\mathbf{Q}\cdot\int_{t'}^{t}\mathrm{d}\tau\;\mathbf{f}(\tau)\right)\\ &\quad\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\;\mathbf{s}(\tau)\cdot\Theta(\tau-\tau')\mathbf{f}(\tau')\right),\end{aligned} \tag{8.0.21}$$

and the numericals $\mathbf{Q}$, $\mathbf{Q}'$ are defined as follows:

$$\mathbf{Q} \;=\; \mathbf{Q}^{\mathrm{c}}(\tau)\Big|_{\tau\to t} \;=\; \mathbf{0}\,, \tag{8.0.22}$$

$$\mathbf{Q}' \;=\; \mathbf{Q}^{\mathrm{c}}(\tau)\Big|_{\tau\to t'} \;=\; \mathbf{0}\,, \tag{8.0.23}$$

where $\mathbf{Q}^{\mathrm{c}}(\tau)$ is the classical function having the expression

$$\mathbf{Q}^{\mathrm{c}}(\tau) \;=\; \frac{\langle \mathbf{q}t\,|\mathbf{G}(\mathbf{q}(\tau),\mathbf{p}(\tau))|\mathbf{q}'t'\rangle}{\langle \mathbf{q}t\,|\,\mathbf{q}'t'\rangle} \;=\; 0\,. \tag{8.0.24}$$

The analysis carried out in quantum field theory, is in the functional differential formalism as well. We have first investigated the origin of the classic Faddeev-Popov factor. To this end, we consider first the generic Lagrangian density:

$$\mathscr{L}_T = \mathscr{L} + \mathscr{L}_S \quad (8.0.25)$$

$$\mathscr{L} = -\frac{1}{4}\, G^a_{\mu\nu} G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}[(\partial_\mu \bar{\psi})\gamma^\mu \psi - \bar{\psi}\gamma^\mu \partial_\mu \psi] - m_0 \bar{\psi}\psi$$

$$+ g_0 \bar{\psi}\gamma_\mu A^\mu \psi \,, \quad (8.0.26)$$

$$\mathscr{L}_S = \bar{\eta}\psi + \bar{\psi}\eta + J^\mu_a A^a_\mu \,, \quad (8.0.27)$$

$$A_\mu = A^a_\mu t_a \quad , \quad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i} g_0 [A_\mu, A_\nu] \,, \quad (8.0.28)$$

$$G_{\mu\nu} = G^a_{\mu\nu} t_a \,, \quad (8.0.29)$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 f^{abc} A^b_\mu A^c_\nu \,. \quad (8.0.30)$$

The t^a are generators of the underlying algebra, and the f^{abc}, totally antisymmetric, are the structure constants satisfying the Jacobi identity, $[t^a, t^b] = \mathrm{i} f^{abc} t^c$. Note that A_μ is a matrix. $\mathscr{L}_S$ is the source term with the J^a_μ classical functions, while η, $\bar{\eta}$ are so-called anti-commuting Grassmann variables. We have worked in the Coulomb gauge $\partial_k A^a_k = 0$, $k = 1, 2, 3$. The multiplicative factor in the exact vacuum-to-vacuum transition amplitude was then derived in the functional differential formalism to be $\det\left[1 - \mathrm{i} g_0 \,\frac{1}{\boldsymbol{\partial}^2}\, A'_k \partial^k\right]$, where $A'^a_k(x) = -\mathrm{i}\delta/\delta J^a_k(x)$, and $A'_k = A'^a_k t^a$. We have then considered a generalization of the Lagrangian density in Eq. (8.0.26) which is still gauge *invariant* and requires a modification of the Faddeev-Popov factor just given. The Lagrangian density is given through the following: We consider the modification of the Lagrangian density $\mathscr{L}$ in Eq. (8.0.26):

$$\mathscr{L} \to \mathscr{L} + \lambda \bar{\psi}\psi G^a_{\mu\nu} G^{\mu\nu}_a \equiv \mathscr{L}_1 \,, \quad (8.0.31)$$

which is obviously gauge invariant under the simultaneous local gauge transformations in Eqs. (6.1.7) - (6.1.9).

The Lagrangian density $\mathscr{L}_{1T} = \mathscr{L}_1 + \mathscr{L}_S$, where $\mathscr{L}_S$ is defined in Eq. (8.0.27) , are given by

$$\begin{aligned}
\mathscr{L}_{1T} &= \mathscr{L}_1 + \mathscr{L}_S \\
&= -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}\left[(\partial_\mu \bar{\psi})\gamma^\mu \psi - \bar{\psi}\gamma^\mu \partial_\mu \psi\right] - m_0 \bar{\psi}\psi + g_0 \bar{\psi}\gamma_\mu A^\mu \psi \\
&\quad + \lambda \bar{\psi}\psi G^a_{\mu\nu} G^{\mu\nu}_a + \bar{\eta}\psi + \bar{\psi}\eta + J^\mu_a A^a_\mu \, . \qquad (8.0.32)
\end{aligned}$$

The corresponding vacuum-to-vacuum transition amplitude $\langle 0_+ \,|\, 0_- \rangle$ was then derived to be

$$\langle 0_+ \,|\, 0_- \rangle = \mathrm{e}^{\mathrm{i}M'} \exp\left[\mathrm{i}\lambda \int (\mathrm{d}x) \bar{\psi}'(x)\psi'(x) G'^a_{\mu\nu}(x) G'^{\mu\nu}_a(x)\right] \langle 0_+ \,|\, 0_- \rangle_{\lambda=0} \; , \qquad (8.0.33)$$

where

$$M' = -\int (\mathrm{d}x)(\mathrm{d}z)\; \delta^4(x-z) \ln\left[1 - 4\lambda \bar{\psi}'(x)\psi'(x)\right] K'(x,z) \; , \qquad (8.0.34)$$

and $\langle 0_+ \,|\, 0_- \rangle_{\lambda=0}$ is the vacuum-to-vacuum amplitude corresponding to the Lagrangian density $\mathscr{L}_T$ in Eq. (8.0.25) involving the FP factor $\det\left[1 - \mathrm{i}g_0 \dfrac{1}{\boldsymbol{\partial}^2} A'_k \partial^k\right]$. That is, the familiar FP factor gets *modified by a multiplicative factor* $\exp[\mathrm{i}M']$ for the gauge invariant Lagrangian density $\mathscr{L}_1$ in Eq. (8.0.31).

On the other hand for an interaction with a gauge breaking term:

$$\mathscr{L}_{2T} = \mathscr{L}_T + \frac{\lambda}{2} A^a_\mu A^\mu_a \bar{\psi}\psi \; , \qquad (8.0.35)$$

where from Eq. (8.0.25),

$$\mathscr{L}_{2T} = -\frac{1}{4}\, G^{a}_{\mu\nu}G^{\mu\nu}_{a} + \frac{1}{2\mathrm{i}}\left[(\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\right] - m_0\bar{\psi}\psi$$

$$+ g_0\bar{\psi}\gamma_\mu A^\mu\psi + \bar{\eta}\psi + \bar{\psi}\eta + J^\mu_a A^a_\mu + \frac{\lambda}{2}A^a_\mu A^\mu_a\bar{\psi}\psi\,. \qquad (8.0.36)$$

we derive for $\langle 0_+\,|\,0_-\rangle$, the expression

$$\langle 0_+\,|\,0_-\rangle_\lambda = \exp\left[-\frac{1}{2}\,\mathrm{Tr}\ln\left(1+\frac{\lambda}{\nabla'_l\partial^l(\boldsymbol{\partial}^2)^{-1}\nabla'_k\partial_k}\bar{\psi}'\psi'\right)\right]$$

$$\times\exp\left[\frac{\mathrm{i}\lambda}{2}\int(\mathrm{d}x)A'^a_\mu A'^\mu_a\bar{\psi}'\psi'\right]\langle 0_+\,|\,0_-\rangle_{\lambda=0}\ , \qquad (8.0.37)$$

showing an obvious modification of the FP factor with latter occurring in $\langle 0_+\,|\,0_-\rangle_{\lambda=0}$. It is most interesting to note that even in *abelian* gauge theory, that is of QED, the addition of a gauge breaking term $\lambda A_\mu A^\mu\bar{\psi}\psi/2$ to $\mathscr{L}_{QED}$ generates a modifying factor. From Eq. (8.0.37) this may be read to lead to

$$\langle 0_+\,|\,0_-\rangle = \exp\left[-\frac{1}{2}\,\mathrm{Tr}\ln\left(1+\frac{\lambda}{\boldsymbol{\partial}^2}\,\bar{\psi}'\psi'\right)\right]$$

$$\times\exp\left(\mathrm{i}\int(\mathrm{d}x)\left[\mathscr{L}'_{QED}(x)+\frac{\lambda}{2}\,A'_\mu A'^\mu\bar{\psi}'\psi'\right]\right)$$

$$\times\,\langle 0_+\,|\,0_-\rangle_0 \qquad (8.0.38)$$

where $\langle 0_+\,|\,0_-\rangle_0$ is the vacuum-to-vacuum transition amplitude of electron free electrons and photons.

Finally, we have realized that all the Lagrangians density in quantum field theory of elementary particles are at most *quadratic* in the dependent fields. This is true even for the *generalizations* of such interactions as we have discussed above and accordingly we have established and proved the following theorem: Consider Lagrangian densities which may depend on one or more coupling constants. We scale these couplings by a parameter λ which is eventually set equal to one. The resulting Lagrangian densities will be denoted by $\underline{\mathcal{L}}(x;\lambda)$. The class of Lagrangian densities considered are of the following types

$$\underline{\mathcal{L}}(x;\lambda) = \mathcal{L}(x;0) + \mathcal{L}_I(x;\lambda) + J_1^i(x)\,\chi_i(x) + J_2^j(x)\,\eta_j(x)\;, \tag{8.0.39}$$

where $\chi_i(x)$ and $\eta_j(x)$ are independent and dependent fields, respectively. $J_1^i(x)$, $J_2^j(x)$ are external sources coupled to these respective fields. The interaction Lagrangian densities sought are of the following forms

$$\mathcal{L}_I(x;\lambda) = B(x;\lambda) + B^j(x;\lambda)\,\eta_j(x) + \frac{1}{2}\,B^{jk}(x;\lambda)\,\eta_j(x)\,\eta_k(x)\;, \tag{8.0.40}$$

with $\mathcal{L}_I(x;0) = 0$, where

$$\frac{\partial B(x;\lambda)}{\partial\lambda}\;,\;\frac{\partial B^j(x;\lambda)}{\partial\lambda}\;,\;\frac{\partial B^{jk}(x;\lambda)}{\partial\lambda}\;=\;\frac{\partial B^{kj}(x;\lambda)}{\partial\lambda}\;, \tag{8.0.41}$$

may be expressed in terms of the independent fields, and the latter two may involve space derivatives applied to the dependent fields $\eta_j(x)$. By definition, the canonical conjugate momenta of the fields $\eta_j(x)$ vanish. That is, formally,

$$\frac{\partial\underline{\mathcal{L}}(x;\lambda)}{\partial(\partial_0\eta_j(x))}\;=\;0\;. \tag{8.0.42}$$

Then we have the explicit solution:

$$\langle 0_+\,|\,0_-\rangle = \exp\Bigg[\mathrm{i}\int(\mathrm{d}x)\;\mathcal{L}_I'(x) + \frac{1}{2}\int(\mathrm{d}x)\int_0^1\mathrm{d}\lambda\left(\frac{\partial}{\partial\lambda}\,B'^{jk}(x;\lambda)\right)D'_{kj}(x,x;\lambda)\Bigg]\,\langle 0_+\,|\,0_-\rangle_0\;, \tag{8.0.43}$$

where $\langle 0_+\,|\,0_-\rangle_0$ is governed by the Lagrangian density $[\mathcal{L}(x;0) + J_1^i(x)\;\chi_i(x)\;+$

$J_2^j(x)\,\eta_j(x)]$ in Eq. (8.0.39), and $\mathcal{L}'_I(x) \equiv \mathcal{L}'_I(x;0)$.

Equation (8.0.43) provides the solution for the generating functional $\langle 0_+\,|\,0_-\rangle$ in the presence of external sources. We thus see that for interaction Lagrangian densities such that $\partial\mathcal{L}_I(x;\lambda)/\partial\lambda$ are quadratic in dependent fields $(\partial B^{jk}(x;\lambda)/\partial\lambda \neq 0)$, as described above, the rules for computations, via the generating functional $\langle 0_+\,|\,0_-\rangle$ are modified by the presence of the multiplicative functional differential operator factor

$$\exp\left[\frac{1}{2}\int(\mathrm{d}x)\int_0^1 \mathrm{d}\lambda\left(\frac{\partial}{\partial\lambda}\,B'^{jk}(x;\lambda)\right)D'_{kj}(x,x;\lambda)\right]. \tag{8.0.44}$$

All the present Lagrangians in gauge theories of elementary particles are special cases of this theorem *including* the more *generalized* theories we have developed in the bulk of the thesis. Special applications of Eq. (8.0.44) were given in Sect. 7.4. We close this chapter by pointing out that one of the greatest challenge is to handle rigorously, at the level of the present work, quantum theory of gravitation where the Lagrangian density in the dependent field (in a Coulomb-like gauge) is not quadratic not even cubic but of infinite order!

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APPENDIX

PUBLICATIONS

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Number of Eigenvalues of a Given Potential
—— *Explicit Functional Expressions* ——

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Explicit functional expressions are derived for the number of eigenvalues of a given potential as well as of their sum. These functional forms involve direct functional differentiations of a given functional written in closed form. The expressions are obtained from trace functionals which allow, in the process, of a direct Fourier analysis.

Over the years, upper bounds have been derived for the number of eigenvalues, falling within specific ranges, for given potentials. The first bound was due to Bargmann[1] who worked with spherically symmetric potentials and, in the process, obtained a bound depending on the orbital angular momentum. This was then extended by Schwinger[2] for more general potentials, not necessarily spherically symmetric, and a similar result was obtained by Birman.[3] Related upper bounds have been also derived by others, cf. Ghirardi and Rimini.[4] The most significant application of the Schwinger bound for the number of eigenvalues of a given potential, or more precisely of the negative of the sum of the negative eigenvalues, was carried out in the problem of the stability of matter,[5],[6] and, in particular, in deriving a lower bound to the expectation value of the kinetic energy operator. The purpose of this communication is to derive an explicit functional expression for the number of eigenvalues as well as for their sum. Our strategy of attack is the following. We first obtain expressions for the quantities we are seeking in terms of the spectral measure of the underlying Hamiltonian H in the problem. We relate these expressions to corresponding integrals involving Green functions. We then recast the derived results, by using in the process the quantum dynamical (action) principle[7)–9)] in terms of trace functionals of the transformation function $\langle \boldsymbol{x}\, T \,|\, \boldsymbol{x}\, 0 \rangle$ and we finally carry out a Fourier decomposition[7] of the latter.

For a given Hamiltonian H, its spectral decomposition may be written as

$$H = \int_{-\infty}^{\infty} \lambda \, dP_H(\lambda). \tag{1}$$

The number of eigenvalues $< \xi$, counting degeneracy, may be simply written in the form

$$N(\xi) = \int d^\nu \boldsymbol{x} \int_{-\infty}^{\infty} \Theta(\xi - \lambda) \, d \left\langle \boldsymbol{x} \,|\, P_H(\lambda) \,|\, \boldsymbol{x} \right\rangle, \tag{2}$$

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where ν denotes the dimensionality of space, and Θ is the step function. ξ may be taken to fall between eigenvalues. We may introduce an integral representation for Θ, to rewrite ($\epsilon \to +0$)

$$N(\xi) = \frac{1}{2\pi i} \int d^\nu \boldsymbol{x} \int_{-\infty}^{\infty} \frac{dT}{T - i\epsilon}\, e^{i\xi T/\hbar} \int_{-\infty}^{\infty} e^{-i\lambda T/\hbar}\, d\left\langle \boldsymbol{x} \middle| P_H(\lambda) \middle| \boldsymbol{x} \right\rangle . \tag{3}$$

On the other hand, the Green (transformation) function $\langle \boldsymbol{x}\, T \,|\, \boldsymbol{x}'\, 0\rangle$ is given by

$$\left\langle \boldsymbol{x}\, T \middle| \boldsymbol{x}'\, 0 \right\rangle = \int_{-\infty}^{\infty} e^{-i\lambda T/\hbar}\, d\left\langle \boldsymbol{x} \middle| P_H(\lambda) \middle| \boldsymbol{x}' \right\rangle \tag{4}$$

from the time evolution of the problem. Accordingly, (3) becomes

$$N(\xi) = \frac{1}{2\pi i} \int d^\nu \boldsymbol{x} \int_{-\infty}^{\infty} \frac{dT}{T - i\epsilon}\, e^{i\xi T/\hbar} \left\langle \boldsymbol{x}\, T \middle| \boldsymbol{x}\, 0 \right\rangle . \tag{5}$$

For the *sum* of eigenvalues $N[\xi]$ having values $< \xi$, we have to multiply the integrand in (2) by λ. From (4), (5), we then have

$$N[\xi] = \frac{1}{2\pi i} \int d^\nu \boldsymbol{x} \int_{-\infty}^{\infty} \frac{dT}{T - i\epsilon}\, e^{i\xi T/\hbar}\, i\hbar \frac{d}{dT} \left\langle \boldsymbol{x}\, T \middle| \boldsymbol{x}\, 0 \right\rangle . \tag{6}$$

Given a Hamiltonian $H(\boldsymbol{x}, \boldsymbol{p})$, we may couple $\boldsymbol{x}$ and $\boldsymbol{p}$ linearly to external c-number sources $\boldsymbol{F}(\tau)$, $\boldsymbol{S}(\tau)$ and define the new Hamiltonian:

$$H'(\tau) = H - \boldsymbol{x} \cdot \boldsymbol{F}(\tau) + \boldsymbol{p} \cdot \boldsymbol{S}(\tau). \tag{7}$$

We may now use the quantum dynamical (action) principle,[7)–9)] expression

$$\left\langle \boldsymbol{x}\, T \middle| \boldsymbol{x}\, 0 \right\rangle = \exp\left[-\frac{i}{\hbar} \int_0^T d\tau\, H\left(-i\hbar \frac{\delta}{\delta \boldsymbol{F}(\tau)} \,,\, i\hbar \frac{\delta}{\delta \boldsymbol{S}(\tau)} \right) \right] \left\langle \boldsymbol{x}\, T \middle| \boldsymbol{x}\, 0 \right\rangle_0 , \tag{8}$$

where

$$\begin{aligned} \left\langle \boldsymbol{x}\, T \middle| \boldsymbol{x}\, 0 \right\rangle_0 = \delta^\nu &\left(\int_0^T d\tau\, \boldsymbol{S}(\tau) \right) \exp\left[\frac{i}{\hbar} \boldsymbol{x} \cdot \int_0^T d\tau\, \boldsymbol{F}(\tau) \right] \\ &\times \exp\left[-\frac{i}{\hbar} \int_0^T d\tau \int_0^T d\tau'\, \boldsymbol{S}(\tau) \cdot \Theta(\tau - \tau')\, \boldsymbol{F}(\tau') \right] , \end{aligned} \tag{9}$$

to obtain, upon integration over $\boldsymbol{x}$, the explicit functional expressions for $N(\xi)$, $N[\xi]$:

$$N(\xi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dT}{T - i\epsilon}\, e^{i\xi T/\hbar}\, K(T), \tag{10}$$

$$N[\xi] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dT}{T - i\epsilon}\, e^{i\xi T/\hbar}\, i\hbar \frac{d}{dT} K(T), \tag{11}$$

where

$$K(T) = \exp\left[-\frac{i}{\hbar}\int_0^T d\tau\ H\left(-i\hbar\frac{\delta}{\delta \boldsymbol{F}(\tau)}\ ,\ i\hbar\frac{\delta}{\delta \boldsymbol{S}(\tau)}\right)\right]$$
$$\times\ (2\pi\hbar)^\nu \delta^\nu\left(\int_0^T d\tau\ \boldsymbol{S}(\tau)\right)\delta^\nu\left(\int_0^T d\tau\ \boldsymbol{F}(\tau)\right)$$
$$\times \exp\left[-\frac{i}{\hbar}\int_0^T d\tau \int_0^T d\tau'\ \boldsymbol{S}(\tau)\cdot\Theta(\tau-\tau')\ \boldsymbol{F}(\tau')\right]\Bigg| \tag{12}$$

and the bar $|$ corresponds to taking the limits $\boldsymbol{S}, \boldsymbol{F} \to \boldsymbol{0}$, *after* the functional differentiations are carried out.

Since in (5) and (6), we are considering the trace operation, we may carry out Fourier decompositions[8)] as follows:

$$\boldsymbol{F}(\tau) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \boldsymbol{F}_n\ e^{-i2\pi n\tau/T}, \tag{13}$$

$$\boldsymbol{S}(\tau) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \boldsymbol{S}_n\ e^{i2\pi n\tau/T}, \tag{14}$$

$$\frac{\delta}{\delta\boldsymbol{F}(\tau)} = \sum_{n=-\infty}^{\infty} e^{i2\pi n\tau/T}\frac{\partial}{\partial \boldsymbol{F}_n}, \tag{15}$$

$$\frac{\delta}{\delta\boldsymbol{S}(\tau)} = \sum_{n=-\infty}^{\infty} e^{-i2\pi n\tau/T}\frac{\partial}{\partial \boldsymbol{S}_n}, \tag{16}$$

where $\partial F^i_n/\partial F^j_m = \delta^{ij}\delta_{nm}$ and so on. These spectral decompositions of the auxiliary c-fields correspond to projections on subspaces, labeled by n, with which eigenvectors of the Hamiltonian in question will be associated.

$K(T)$ in (12) then simplifies to

$$K(T) = \exp\left[-\frac{i}{\hbar}\int_0^T d\tau\ H\left(-i\hbar\frac{\delta}{\delta \boldsymbol{F}(\tau)}\ ,\ i\hbar\frac{\delta}{\delta \boldsymbol{S}(\tau)}\right)\right]$$
$$\times\ (2\pi\hbar)^\nu \delta^\nu(\boldsymbol{S}_0)\,\delta^\nu(\boldsymbol{F}_0)\ \exp\left(\frac{1}{\hbar}\sum_{n\neq 0}\frac{\boldsymbol{S}_n\cdot\boldsymbol{F}_n}{2\pi n}\right)\Bigg|. \tag{17}$$

In evaluating the latter, we may use (15), (16) in H. The expressions in (10), (11) together with (17) are the main results of the paper. The expression for $K(T)$ in (17) is directly related to the spectral resolution of the time evolution operator expressed in terms of c-functional methods involving no quantum operators.

To verify the consistency of the formulation, consider the harmonic oscillator problem with $\nu = 1$, $H = p^2/2m + m\omega^2 x^2/2$. Then from (15) and (16), we have

$$\int_0^T d\tau\ H\left(-i\hbar\frac{\delta}{\delta F(\tau)}\ ,\ i\hbar\frac{\delta}{\delta S(\tau)}\right) = -\hbar^2 T\sum_{n=-\infty}^{\infty}\left[\frac{1}{2m}\frac{\partial}{\partial S_n}\ \frac{\partial}{\partial S_{-n}}\right.$$

$$+\frac{m\omega^2}{2}\frac{\partial}{\partial F_n}\frac{\partial}{\partial F_{-n}}\Bigg] . \tag{18}$$

Carrying out the differentiations in (17) with respect to F_n, we obtain

$$K(T)=\prod_{n=1}^{\infty}\exp\left(\frac{i\hbar T}{m}\frac{\partial}{\partial S_n}\frac{\partial}{\partial S_{-n}}\right)\,\exp\left(-\frac{iTm\omega^2}{\hbar(2\pi n)^2}\,S_n\,S_{-n}\right)\Bigg| . \tag{19}$$

To carry out the differentiations with respect to S_n, we may use the convenient representation

$$e^{-i\beta S_n S_{-n}}=\int_{-\infty}^{\infty}d\lambda_2\int_{-\infty}^{\infty}\frac{d\lambda_1}{2\pi}\,e^{i\lambda_1\lambda_2}\,e^{-i\lambda_2 S_{-n}}\,e^{-i\beta\lambda_1 S_n}, \tag{20}$$

which gives for $K(T)$ the final result

$$K(T)=\sum_{n=0}^{\infty}\exp\left[-\frac{iT}{\hbar}\hbar\omega\left(n+\frac{1}{2}\right)\right] \tag{21}$$

upon recognizing the infinite product representation of $(T\omega/2)/\sin(T\omega/2)$. All told, (10) and (11) lead to

$$N(\xi)=\sum_{n=0}^{\infty}\Theta\Bigg(\xi-\hbar\omega\left(n+\frac{1}{2}\right)\Bigg), \tag{22}$$

$$N[\xi]=\sum_{n=0}^{\infty}\hbar\omega\left(n+\frac{1}{2}\right)\Theta\Bigg(\xi-\hbar\omega\left(n+\frac{1}{2}\right)\Bigg), \tag{23}$$

as expected, where we have used the integral representation of the step function after carrying out the differentiation with respect to T in (11) to obtain (23).

The explicit expressions for $N(\xi)$ and $N[\xi]$ in (5), (10) and (6),(11) are exact ones in contrast to earlier ones[1)–4)] which are formulated in terms of bounds. Our expressions are expressed in terms of c-functional methods thus avoiding quantum operator techniques. There are some advantages using our formalism. For example, (17) may be efficient in carrying out a perturbation expansion. For example for the anharmonic oscillator Hamiltonian $H=p^2/2m+m\omega^2x^2/2+\lambda x^3$, we readily obtain

$$\begin{aligned}&\int_0^T d\tau\;H\Bigg(-i\hbar\frac{\delta}{\delta F(\tau)}\,,\,i\hbar\frac{\delta}{\delta S(\tau)}\Bigg)\\&\quad=-\hbar^2T\sum_{n=-\infty}^{\infty}\left[\frac{1}{2m}\frac{\partial}{\partial S_n}\frac{\partial}{\partial S_{-n}}+\frac{m\omega^2}{2}\frac{\partial}{\partial F_n}\frac{\partial}{\partial F_{-n}}\right]\\&\qquad+i\lambda\hbar^3T\sum_{n_1+n_2+n_3\neq 0}\frac{\partial}{\partial F_{n_1}}\frac{\partial}{\partial F_{n_2}}\frac{\partial}{\partial F_{n_3}}\end{aligned} \tag{24}$$

and from (17) to

$$K(T) = \sqrt{\frac{2\pi\hbar m}{iT}}\ \exp\left(\frac{i\hbar T m\omega^2}{2}\sum_{n=-\infty}^{\infty}\frac{\partial}{\partial F_n}\frac{\partial}{\partial F_{-n}}\right)$$
$$\times\exp\left(\hbar^2 T\lambda \sum_{n_1+n_2+n_3\neq 0}\frac{\partial}{\partial F_{n_1}}\frac{\partial}{\partial F_{n_2}}\frac{\partial}{\partial F_{n_3}}\right)$$
$$\times\,\delta(F_0)\left[\prod_{n=1}^{\infty}\exp\left(-\frac{iT}{(2\pi n)^2}\frac{F_n F_{-n}}{m\hbar}\right)\right]. \qquad (25)$$

A similar calculation as worked out through (19) and (20) gives

$$K(T) = \sum_{n=0}^{\infty}\exp\left\{-\frac{iT}{\hbar}\left[\hbar\omega\left(n+\frac{1}{2}\right)\right.\right.$$
$$\left.\left.-\lambda^2\left(\frac{\hbar}{2m\omega}\right)^3\frac{(30n^2+30n+11)}{\hbar\omega}+\dots\right]\right\} \qquad (26)$$

up to second order in λ in the exponential from which $N(\xi)$ and $N[\xi]$ are directly determined.

Our expressions given in (5) and (6) are also useful in obtaining the so-called first quantum correction [cf. Ref. 10)] to the semi-classical one with $\langle \boldsymbol{x}\, T \,|\, \boldsymbol{x}\, 0\rangle$ given by

$$\big\langle \boldsymbol{x}\, T \,\big|\, \boldsymbol{x}\, 0\big\rangle = \left(\frac{m}{2\pi i\hbar^2\tau}\right)^{\nu/2} e^{-i\tau V(\boldsymbol{x})}$$
$$\times\exp\left[\frac{\hbar^2\tau^2}{12m}\boldsymbol{\nabla}^2 V(\boldsymbol{x}) - \frac{i\hbar^2\tau^3}{24m}\left(\boldsymbol{\nabla}V(\boldsymbol{x})\right)^2\right] \qquad (27)$$

written in terms of the parameter $\tau \equiv T/\hbar$ for a given potential $V(\boldsymbol{x})$.

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Action Principle and Modification of the Faddeev–Popov Factor in Gauge Theories

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The quantum action (dynamical) principle is exploited to investigate the nature and origin of the Faddeev–Popov (FP) factor in gauge theories without recourse to path integrals. Gauge invariant as well as gauge non-invariant interactions are considered to show that the FP factor needs to be modified in more general cases and expressions for these modifications are derived. In particular we show that a gauge invariant theory does not necessarily imply the familiar FP factor for proper quantization.

KEY WORDS: action principle; gauge theories; Faddeev–Popov factor; quantization rules.

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1. INTRODUCTION

In earlier communications (Manoukian, 1986, 1987; Manoukian and Siranan, 2005), we have seen that the quantum action (dynamical) principle (Schwinger, 1951a,b, 1953a,b, 1954, 1972, 1973; Lam, 1965; Manoukian, 1985) may be used to quantize gauge theories in constructing the vacuum-to-vacuum transition amplitude and the Faddeev–Popov (FP) factor (Faddeev and Popov, 1967), encountered in non-abelian gauge theories (e.g., (Abers and Lee, 1973; Rivers, 1987; 't Hooft, 2000; Veltman, 2000; Gross, 2005; Politzer, 2005; Wilczek, 2005)), may be obtained *directly* from the action principle without much effort. No appeal was made to path integrals, and there was not even the need to go into the well-known complicated structure of the Hamiltonian (Fradkin and Tyutin, 1970) in non-abelian gauge theories. For extensive references on the gauge problem in gauge theories

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see Manoukian and Siranan (2005). The latter reference traces its historical development from early papers to most recent ones.

In the present investigation, we consider the generic non-abelian gauge theory Lagrangian density

$$\mathscr{L}_{\mathrm{T}} = \mathscr{L} + \mathscr{L}_{\mathrm{S}} \tag{1}$$

and modifications thereof, where

$$\mathscr{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \frac{1}{2i} [(\partial_\mu \overline{\psi}) \gamma^\mu \psi - \overline{\psi} \gamma^\mu \partial_\mu \psi] - m_0 \overline{\psi} \psi + g_0 \overline{\psi} \gamma_\mu A^\mu \psi \tag{2}$$

$$\mathscr{L}_{\mathrm{S}} = \overline{\eta} \psi + \overline{\psi} \eta + J^\mu_a A^a_\mu \tag{3}$$

$$A_\mu = A^a_\mu t_a, \qquad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_0 [A_\mu A_\nu] \tag{4}$$

$$G_{\mu\nu} = G^a_{\mu\nu} t_a \tag{5}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 f^{abc} A^b_\mu A^c_\nu. \tag{6}$$

The t^a are generators of the underlying algebra, and the f^{abc}, totally antisymmetric, are the structure constants satisfying the Jacobi identity, $[t^a, t^b] = i f^{abc} t^c$. Note that A_μ is a matrix. $\mathscr{L}_{\mathrm{S}}$ is the source term with the J^a_μ classical functions, while η, $\overline{\eta}$ are so-called anti-commuting Grassmann variables.

The Lagrangian density $\mathscr{L}$ in (2) is invariant under simultaneous local gauge transformations:

$$\psi \longrightarrow U\psi, \qquad \overline{\psi} \longrightarrow \overline{\psi} U^{-1}, \tag{7}$$

$$A_\mu \longrightarrow U A_\mu U^{-1} + \frac{i}{g_0} U \partial_\mu U^{-1} \tag{8}$$

$$G_{\mu\nu} \longrightarrow U G_{\mu\nu} U^{-1} \tag{9}$$

where $U = U(\theta) = \exp[i g_0 \theta^a t^a]$, $\theta = \theta^a t^a$, $\theta = \theta(x)$.

Upon setting

$$\nabla_\mu = \partial_\mu - i g_0 A_\mu \tag{10}$$

with

$$\nabla^{ab}_\mu = \delta^{ab} \partial_\mu + g_0 f^{acb} A^c_\mu \tag{11}$$

we have the basic commutator

$$[\nabla_\mu, \nabla_\nu] = -i g_0 G_{\mu\nu} \tag{12}$$

and the identity

$$\nabla_\mu^{ab}\nabla_\nu^{bc}G_c^{\mu\nu}=0. \tag{13}$$

[The latter generalizes the elementary identity $\partial_\mu\partial_\nu F^{\mu\nu}=0$, in abelian gauge theory, to non-abelian ones, where $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$.]

We consider gauge invariant (Section 3.) as well as gauge non-invariant (Section 4.) modifications of the Lagrangian density and show by a systematic use of the quantum action principle that the familiar FP factor needs to be modified in more general cases and explicit expressions for these modifications are derived. In particular, we show that a gauge invariant theory does *not* necessarily imply the familiar FP factor for proper quantization, as may be perhaps expected (cf. Rivers (1987, p. 204), and modifications thereof may be necessary. Before doing so, however, we use the action principle to derive, in Section 2., the FP factor and investigate its origin for the classic Lagrangian density $\mathscr{L}$, without recourse to path integrals, as an anticipation of what to expect in more general cases. Throughout, we work in the celebrated Coulomb gauge $\partial_k A_a^k=0$, $k=1,2,3$.

2. ACTION PRINCIPLE AND THE ORIGIN OF THE FP FACTOR

To obtain the expression for the vacuum-to-vacuum transition amplitude $\langle 0_+\,|\,0_-\rangle$, in the presence of external sources J_μ^a, η^a, $\overline{\eta}^a$, as the generator of all the Green functions of the theory, *no* restrictions may be set, in particular, on the external current J_μ^a, coupled to the gauge fields A_a^μ, such as $\partial^\mu J_\mu^a=0$, so that *variations of the components of* J_μ^a *may be carried out independently*, until the entire analysis is completed, and all functional differentiations are carried out to generate Green functions. This point cannot be overemphasized. As we will see, the *generality* condition that must be adopted on the external current J_μ^a together with the presence of *dependent* gauge field components in (A_a^μ), as a result of the structure of the Lagrangian density $\mathscr{L}$ in (2) and the gauge constraint, are responsible for the *origin* and the presence of the FP factor in the theory for a proper quantization in the realm of the quantum action principle.

We define the Green operator $D^{ab}(x,x')$ satisfying the differential equation

$$\left[\delta^{ac}\vec{\partial}^2+g_0 f^{abc}A_k^b\partial_k\right]D^{cd}(x,x')=\delta^4(x,x')\delta^{ad}. \tag{14}$$

Since the differential operator on the left-hand side of $D^{cd}(x,x')$ is independent of the time derivative, $D^{cd}(x,x')$ involves a $\delta(x^0-x'^0)$ factor. Using the gauge constraint, one may, for example, eliminate A_a^3 in favor of A_a^1, A_a^2. That is, we may treat the A_a^3 as dependent fields.

The field equations are given by

$$\nabla_\mu^{ab}G_b^{\mu\nu}=-\left(\delta^\nu{}_\sigma\,\delta^{ac}-g^{\nu k}\partial_k D^{ab}\nabla_\sigma^{bc}\right)\left[J_c^\sigma+g_0\overline{\psi}\gamma^\sigma t_c\psi\right] \tag{15}$$

with $\mu, \nu = 0, 1, 2, 3$, $k = 1, 2, 3$, and

$$\left[\gamma^\mu \frac{\nabla_\mu}{i} + m_0\right]\psi \;\; = \eta \tag{16}$$

$$\overline{\psi}\left[\gamma^\mu \frac{\overleftarrow{\nabla}^*_\mu}{i} - m_0\right] = -\overline{\eta} \tag{17}$$

where ∇_μ is defined in (10).

The canonical conjugate variables to A^1_a, A^2_a, are given by

$$\pi^i_a = G^{i0}_a - \partial_3^{-1}\partial^i G^{30}_a, \qquad i = 1, 2. \tag{18}$$

With $\pi^0_a = 0$, $\pi^3_a = 0$, we may rewrite (18) as

$$\pi^\mu_a = G^{\mu 0}_a - \partial_3^{-1} g^{\nu k}\partial_k G^{30}_a \tag{19}$$

$k = 1, 2, 3$. One may then readily express $G^{\mu 0}_a$ as follows:

$$G^{\mu 0}_a = \pi^\mu_a - g^{\mu k}\partial_k D_{ab}\left[J^0_b + g_0\overline{\psi}\gamma^0 t_b\psi + \nabla^{bc}_\nu \pi^\nu_c\right]. \tag{20}$$

We note that the right-hand side of (20) is expressed in terms of the independent fields A^1_a, A^2_a, their canonical conjugate momenta and involves no time derivatives. Here we recall that A^3_a is expressed in terms of A^1_a, A^2_a with no time derivative. Accordingly, with the (independent) fields and their canonical conjugate momenta kept *fixed*, we obtain the following functional derivative

$$\frac{\delta}{\delta J^\nu_b(x')} G^{\mu 0}_a(x) = -g^{\mu k}\delta^0{}_\nu\, \partial_k D_{ab}(x, x') \tag{21}$$

$\mu, \nu = 0, 1, 2, 3$, $k = 1, 2, 3$. On the other hand, $G^{kl}_a = \partial^k A^l_a - \partial^l A^k_a$, $k, l = 1, 2, 3$, may be expressed in terms of the independent fields A^1_a, A^2_a and involves no time derivatives. Accordingly with A^1_a, A^2_a and their canonical conjugate variables kept fixed, we also have

$$\frac{\delta}{\delta J^\nu_b(x')} G^{kl}_a(x) = 0. \tag{22}$$

Similarly, with ψ and $\overline{\psi}$ kept fixed, we have the obvious functional derivative expression

$$\frac{\delta}{\delta J^\nu_b(x')}[\overline{\psi}(x)\gamma^\mu t^a \psi(x)] = 0. \tag{23}$$

The action principle gives

$$\frac{\partial}{\partial g_0}\langle 0_+ \,|\, 0_- \rangle = i\left\langle 0_+ \left| \int (dx)\, \hat{\mathscr{L}}_1 \right| 0_- \right\rangle \tag{24}$$

where

$$\hat{\mathscr{L}}_1 = \frac{\partial}{\partial g_0}\mathscr{L} = -\frac{1}{2} f^{abc} A^b_\mu A^c_\nu G^{\mu\nu}_a + \overline{\psi}\gamma^\mu A_\mu \psi. \tag{25}$$

We may also write

$$f^{abc} A^b_\mu A^c_\nu G^{\mu\nu}_a = 2 f^{abc} A^b_k A^c_0 G^{k0}_a + f^{abc} A^b_k A^c_l G^{kl}_a \tag{26}$$

and set $(-i)\delta/\delta J^\mu_a = A'^a_\mu$, $(-i)\delta/\delta\overline{\eta} = \psi'$, $(-i)\delta/\delta\eta = \overline{\psi}'$. [Here we note that $G'^a_{\mu\nu}$ on the right-hand side of (5.30) of Manoukian (1986) should be replaced by $F'^a_{\mu\nu} = \partial_\mu A'^a_\nu - \partial_\nu A'^a_\mu$.]

Now we use the rule of functional differentiations (cf. Manoukian (2006), Ch. 11) that for an operator $\mathcal{O}(x)$

$$(-i)\frac{\delta}{\delta J^\mu_a(x')}\langle 0_+ | \mathcal{O}(x) | 0_- \rangle = \left\langle 0_+ \left| \left(A^a_\mu(x')\mathcal{O}(x)\right)_+ \right| 0_- \right\rangle - i\left\langle 0_+ \left| \frac{\delta}{\delta J^\mu_a(x')}\mathcal{O}(x) \right| 0_- \right\rangle \tag{27}$$

where $(\ldots)_+$ denotes the time-ordered product, and the functional derivative of $\mathcal{O}(x)$ in the second term on the right-hand of (27) is taken as in (21)–(23) with the (independent) fields and their canonical conjugate momenta kept fixed. Here we recall that A^3_a may be expressed in terms of A^1_a, A^2_a and involves no time derivatives.

From (24)–(27), together with (21)–(23), we obtain

$$\frac{\partial}{\partial g_0}\langle 0_+ | 0_- \rangle = \int (dx)\left[i\hat{\mathscr{L}}'_1(x) - f^{bca} A'^b_k \partial^k D'^{ac}(x,x)\right]\langle 0_+ | 0_- \rangle . \tag{28}$$

Using a matrix notation

$$D^{ab}(x,x') = \left[\left\langle x \left| \left(\frac{1}{\vec{\partial}^2 - i g_0 A_k \partial_k} \right) \right| x' \right\rangle\right]^{ab}, \tag{29}$$

the notation

$$Tr[f] = \int (dx)\, f^{aa}(x,x), \tag{30}$$

and the fact that $f^{bca} A^b_k = i(A_k)^{ca}$, we may rewrite the second factor within the square brackets in (28) as

$$Tr\left\{ -iA'_k\, \partial^k \frac{1}{[\vec{\partial}^2 - i g_0 A'_l\, \partial^l]} \right\}. \tag{31}$$

An elementary integration over g_0 from 0 to some g_0 value then gives the familiar FP factor for $\langle 0_+ \,|\, 0_- \rangle$ in (28)

$$\det\left[1 - i g_0 \frac{1}{\vec{\partial}^2} A'_k\, \partial^k\right]. \tag{32}$$

3. GAUGE INVARIANCE AND MODIFICATION OF THE FP FACTOR

Now consider the modification of the Lagrangian density $\mathscr{L}$ in (2):

$$\mathscr{L} \longrightarrow \mathscr{L} + \lambda \overline{\psi}\psi G^a_{\mu\nu} G^{\mu\nu}_a \equiv \mathscr{L}_1 \tag{33}$$

which is obviously gauge invariant under the simultaneous local gauge transformations in (7)–(9).

The field equations corresponding to the Lagrangian density $\mathscr{L}_{1\mathrm{T}} = \mathscr{L}_1 + \mathscr{L}_{\mathrm{S}}$, where $\mathscr{L}_{\mathrm{S}}$ is defined in (3), are given by

$$\nabla^{ab}_\mu \left(\left[1 - 4\lambda\overline{\psi}\psi\right] G^{\mu\nu}_b\right) = -\left(\delta^\nu{}_\sigma\, \delta^{ac} - g^{\nu k}\partial_k D^{ab} \nabla^{bc}_\sigma\right) \times \left[J^\sigma_c + g_0 \overline{\psi}\gamma^\sigma t_c \psi\right] \tag{34}$$

$$\left[\gamma^\mu \frac{\nabla_\mu}{i} - \lambda G^a_{\mu\nu} G^{\mu\nu}_a + m_0\right]\psi = \eta \tag{35}$$

$$\overline{\psi}\left[\gamma^\mu \frac{\overleftarrow{\nabla}^*_\mu}{i} + \lambda G^a_{\mu\nu} G^{\mu\nu}_a - m_0\right] = -\overline{\eta}. \tag{36}$$

The canonical conjugate momenta to A^1_a, A^2_a are given by

$$\pi^i_a = [1 - 4\lambda\overline{\psi}\psi] G^{i0}_a - \partial_3^{-1}\partial^i [1 - 4\lambda\overline{\psi}\psi] G^{30}_a \tag{37}$$

$i = 1, 2$. One may then express G^{k0}_a as follows:

$$\left[1 - 4\lambda\overline{\psi}(x)\psi(x)\right] G^{k0}_a(x) = \pi^k_a(x) - \partial_k \int (dx')\, D_{ab}(x, x') \left[J^0_b(x') + g_0 \overline{\psi}(x')\gamma^0 t_b \psi(x') + \nabla'^{bc}_j \pi^j_c(x')\right] \tag{38}$$

$k = 1, 2, 3$, with π^3_a set equal to zero.

With the (independent) fields and their canonical conjugate momenta kept fixed, we then have

$$\left[1 - 4\lambda\overline{\psi}(x)\psi(x)\right] \frac{\delta}{\delta J^\nu_b(x')} G^{k0}_a(x) = -\partial_k D_{ab}(x, x')\, \delta^0{}_\nu. \tag{39}$$

The equal time commutation relations of the independent fields $A_a^1(x)$, $A_a^2(x)$ are given by

$$\delta(x^0 - x'^0)\big[A_a^i(x), \pi_b^j(x')\big] = i\delta_{ab}\delta^{ij}\delta^4(x - x') \tag{40}$$

with $i, j = 1, 2$. From the gauge constraint, we may then write

$$\delta(x^0 - x'^0)\left[A_a^k(x), \pi_b^l(x')\right] = i\delta_{ab}\left[\delta^{kl} - \delta^{k3}\partial_3^{-1}\partial^l\right]\delta^4(x - x') \tag{41}$$

with now $k, l = 1, 2, 3$.

From (38), (41), we then obtain the commutation relation

$$\begin{aligned}\left[1 - 4\lambda\overline{\psi}(x)\psi(x)\right]&\left[A_{ka}(x'), G_a^{k0}(x)\right]\delta(x^0 - x'^0)\\ &= 2i\delta_{aa}\delta^4(x - x') - \partial_k\int(dx'')\,D_{ab}(x, x'')\nabla_j''^{bc}\\ &\quad\times\left[A_{ka}(x'), \pi_c^j(x'')\right]\delta(x^0 - x'^0),\end{aligned} \tag{42}$$

where we recall that $D_{ab}(x, x'')$ involves the factor $\delta(x^0 - x''^0)$. The latter then implies that the last term in (42) is given by

$$-i\partial_k\int(dx'')\,D_{ab}(x, x'')\nabla_j''^{ba}[\delta^{kj} - \delta^{k3}\partial_3'^{-1}\partial'^j]\delta^3(\mathbf{x}' - \mathbf{x}'')\,\delta(x^0 - x'^0). \tag{43}$$

Now we take the limit $\mathbf{x}' \to \mathbf{x}$ in the latter and integrate over $d^3\mathbf{x}$ to obtain

$$-i\int(dx'')\int d^3\mathbf{x}\left[\partial_j - \partial_j\right]D_{ab}(x, x'')\nabla_j''^{ba}\delta^3(\mathbf{x} - \mathbf{x}'')\,\delta(x^0 - x'^0) = 0. \tag{44}$$

This result will be used later in deriving the modification of the FP factor.

The action principle gives

$$\frac{\partial}{\partial\lambda}\langle 0_+\,|\,0_-\rangle = i\int(dx)\left\langle 0_+\left|\overline{\psi}(x)\psi(x)G_{\mu\nu}^a(x)G_a^{\mu\nu}(x)\right|0_-\right\rangle. \tag{45}$$

Consider the matrix element

$$\begin{aligned}\left\langle 0_+\left|\left(G_{\mu\nu}^a(x)G_a^{\mu\nu}(x')\right)_+\right|0_-\right\rangle &= 2\left\langle 0_+\left|\left(G_{k0}^a(x)G_a^{k0}(x')\right)_+\right|0_-\right\rangle\\ &\quad + \left\langle 0_+\left|\left(G_{kl}^a(x)G_a^{kl}(x')\right)_+\right|0_-\right\rangle.\end{aligned} \tag{46}$$

The second term is simply equal to

$$G_{kl}'^a(x)G_a'^{kl}(x')\,\langle 0_+\,|\,0_-\rangle \tag{47}$$

expressed in terms of functional derivatives using our notation below Eq. (26). While to determine the first term, we rewrite

$$G_{k0}^a(x) = \int(dz)\,\delta^4(x - z)\nabla_k^{ac}(z)A_0^c(z) - \int(dz)\,\delta^4(x - z)\,\partial_0^z A_k^a(z). \tag{48}$$

We then have

$$\left\langle 0_+ \left| \left(G^a_{k0}(x) G^{k0}_a(x')\right)_+ \right| 0_- \right\rangle = G'^a_{k0}(x) G'^{k0}_a(x') \langle 0_+ | 0_- \rangle$$
$$+ \int (dz)\, \delta^4(x-z)\, \delta(z^0 - x'^0) \langle 0_+ | [A^a_k(z), G^{k0}_a(x')] | 0_- \rangle$$
$$-i \int (dz)\, \delta^4(x-z) \nabla'^{ac}_k(z) \left\langle 0_+ \left| \frac{\delta}{\delta J^0_c(z)} G^{k0}_a(x') \right| 0_- \right\rangle \qquad (49)$$

where the second term comes from the non-commutativity of the time derivative and the time ordering operation as resulting from the last term in (48), and the third term follows from the rule of functional differentiation in (27) as resulting from the first integral in (48).

From (38), (42), (44), the right-hand side of (49) simplifies for $x' \to x$ to

$$\left[G'^a_{k0}(x) G'^{k0}_a(x) + \Delta'(x)\right] \langle 0_+ | 0_- \rangle \qquad (50)$$

where

$$\Delta'(x) = 2 \int (dz) \frac{\delta^4(z-x)}{[1 - 4\lambda \overline{\psi}'(x) \psi'(x)]} K'(x,z) \qquad (51)$$

$$K'(x,z) = i \left[\delta_{aa} \delta^4(0) + \frac{1}{2} \partial^x_k \nabla'^{ac}_k(z) D'_{ac}(x,z)\right] \qquad (52)$$

involving a familiar $\delta^4(0)$ term.

All told, the expression (45) becomes

$$\frac{\partial}{\partial \lambda} \langle 0_+ | 0_- \rangle = i \int (dx)\, \overline{\psi}'(x) \psi'(x) G'^a_{\mu\nu}(x) G'^{\mu\nu}_a(x) \langle 0_+ | 0_- \rangle$$
$$+ 2i \int (dx)\, \overline{\psi}'(x) \psi'(x) \Delta'(x) \langle 0_+ | 0_- \rangle \qquad (53)$$

which upon an elementary integration over λ leads to

$$\langle 0_+ | 0_- \rangle = e^{iM'} \exp\left[i\lambda \int (dx)\, \overline{\psi}'(x) \psi'(x) G'^a_{\mu\nu}(x) G'^{\mu\nu}_a(x)\right] \langle 0_+ | 0_- \rangle_{\lambda=0} \qquad (54)$$

where

$$M' = -\int (dx)(dz)\, \delta^4(x-z) \ln[1 - 4\lambda \overline{\psi}'(x) \psi'(x)] K'(x,z) \qquad (55)$$

and $\langle 0_+ | 0_- \rangle_{\lambda=0}$ is the vacuum-to-vacuum amplitude corresponding to the Lagrangian density $\mathscr{L}_T$ in (1) involving the FP factor in (32). That is, the familiar FP factor gets modified by a multiplicative factor $\exp[iM']$ for the gauge invariant Lagrangian density $\mathscr{L}_1$ in (33).

4. GAUGE BREAKING INTERACTIONS

In the present section we consider the addition of a gauge breaking term to the Lagrangian density $\mathscr{L}$ in (2). It is well known that even the addition of the simple source term $\mathscr{L}_S$ in (3) to $\mathscr{L}$ causes difficulties (cf. Rivers (1987), p. 204) in the quantization problem in the path integral formalism as the action $\int(dx)\mathscr{L}_T(x)$, with $\mathscr{L}_T(x)$ defined in (1), is not gauge invariant. We will see how easy it is to handle the addition of a gauge breaking term to $\mathscr{L}_T$.

Consider the Lagrangian density

$$\mathscr{L}_{2T} = \mathscr{L}_T + \frac{\lambda}{2} A^a_\mu A^\mu_a \overline{\psi}\psi. \tag{56}$$

Then an analysis similar to the one in Section 3. shows that

$$G^{k0}_a = \pi^k_a - \partial_k D_{ab}\left[J^0_b + \lambda A^0_b \overline{\psi}\psi + g_0 \overline{\psi}\gamma^0 t_b \psi + \nabla^{bc}_\nu \pi^\nu_c\right]. \tag{57}$$

Using the fact that

$$\partial_k G^{k0}_a = \nabla^{ab}_k \partial_k A^0_b \tag{58}$$

we obtain upon multiplying (57) by

$$\nabla^{ca}_l \partial^l \frac{1}{\vec{\partial}^2} \partial_k$$

and using (14), we obtain

$$\left(\nabla^{ca}_l \partial^l \frac{1}{\vec{\partial}^2} \nabla^{ab}_k \partial_k\right) A^0_b = -J^0_c - \lambda A^0_c \overline{\psi}\psi + \ldots \tag{59}$$

where the dots correspond to terms *independent* of J^0_b and A^0_b. We introduce the Green operator $N^{be}(x, x')$ satisfying

$$\left[\nabla^{ca}_l \partial^l \frac{1}{\vec{\partial}^2} \nabla^{ab}_k \partial_k + \lambda \delta^{cb} \overline{\psi}(x)\psi(x)\right] N^{be}(x, x') = \delta^{ce}\delta^4(x - x') \tag{60}$$

to obtain from (59)

$$\frac{\delta}{\delta J^0_b(x)} A^0_b(x) = -N^{bb}(x, x). \tag{61}$$

Hence the action principle and (61) give

$$\begin{aligned} \frac{\partial}{\partial\lambda}\langle 0_+ | 0_-\rangle &= \tfrac{i}{2}\int(dx)\, A'^a_\mu(x) A'^\mu_a(x) \overline{\psi}'(x)\psi'(x)\, \langle 0_+ | 0_-\rangle \\ &\quad - \tfrac{1}{2}\int(dx)\, \overline{\psi}'(x)\psi'(x) N'^{bb}(x, x)\, \langle 0_+ | 0_-\rangle\,. \end{aligned} \tag{62}$$

Upon integrating the latter over λ, by using in the process (60), we obtain

$$\langle 0_+ | 0_- \rangle = \exp\left[-\frac{1}{2} Tr \ln \left(1 + \frac{\lambda}{\nabla_l' \partial^l (\vec{\partial}^2)^{-1} \nabla_k' \partial_k} \overline{\psi}' \psi' \right) \right] \times \exp\left[i \frac{\lambda}{2} \int (dx)\, A'^{a}_{\mu}(x) A'^{\mu}_{a}(x) \overline{\psi}'(x) \psi'(x) \right] \langle 0_+ | 0_- \rangle_{\lambda=0} \quad (63)$$

showing an obvious modification of the FP factor with the latter occurring in $\langle 0_+ | 0_- \rangle_{\lambda=0}$.

5. CONCLUSION

The quantum action (dynamical) principle leads systematically to the FP of non-abelian gauge theories with no much effort. It is emphasized, in the process of the analysis, that no restrictions may be set on the external current J^a_μ, coupled to the gauge field A^μ_a (such as $\partial^\mu J^a_\mu = 0$), until all functional differentiations with respect to it are taken so that all of its components may be varied independently. We have considered gauge invariant as well as gauge non-invariant interactions and have shown that the FP factor needs to be modified in more general cases and expressions for these modifications were derived. [It is well known that even the simple gauge breaking source term $\mathscr{L}_S$ in (3) causes complications in the path integral formalism. The path integral may, of course, be readily derived from the action principle.] The presence of the source term $\mathscr{L}_S$ in the Lagrangian density is essential in order to generate the Green functions of the theory from the vacuum-to-vacuum transition amplitude, as a generating functional, by functional differentiations. We have also shown, in particular, that a gauge invariant theory does not necessarily imply the familiar FP factor for proper quantization. Finally we note that even for abelian gauge theories, as obtained from the bulk of the paper by taking the limit of f^{abc} to zero and replacing t^a by the identity, may lead to modifications, as multiplicative factors in $\langle 0_+ | 0_- \rangle$, as clearly seen from the expressions in (55) and (63).

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Quadratic Actions in Dependent Fields and the Action Principle

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Abstract General field theories are considered, within the functional *differential* formalism of quantum field theory, with interaction Lagrangian densities $\mathcal{L}_I(x;\lambda)$, with λ a generic coupling constant, such that the following expression $\partial\mathcal{L}_I(x;\lambda)/\partial\lambda$ may be expressed as quadratic functions in *dependent* fields but may, in general, be arbitrary functions of *independent* fields. These necessarily include, as special cases, present renormalizable gauge theories. It is shown, in a unified manner, that the vacuum-to-vacuum transition amplitude (the generating functional) may be explicitly derived in functional differential form which, in general, leads to modifications to computational rules by including such factors as Faddeev–Popov ones and *modifications* thereof which are explicitly obtained. The derivation is given in the *presence* of external sources and does not rely on any symmetry and invariance arguments as is often done in gauge theories and no appeal is made to path integrals.

Keywords Functional differential formalism of quantum field theory · Dependent fields · Action principle · Quantization rules · Gauge theories

1 Introduction

The purpose of this communication is to investigate systematically, in a unified manner, within the functional *differential* formalism of quantum field theory [1–13], field theories with interaction Lagrangian densities $\mathcal{L}_I(x;\lambda)$, with λ a generic coupling constant, such that $\partial\mathcal{L}_I(x;\lambda)/\partial\lambda$ may be expressed as quadratic functions in dependent fields and, in general, as arbitrary functions of independent fields. These include, as special cases, present renormalizable gauge field theories. For example, the non-abelian ones, such as QCD, are quadratic, while QED is linear in dependent fields. The functional differential treatment necessitates the introduction of *external sources* in order to generate the vacuum-to-vacuum transition amplitude, as a generating functional, from which amplitudes for basic processes

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may be extracted. The novelty of this work is that we show that for all the general Lagrangians, mentioned above, the vacuum-to-vacuum transition amplitude may be explicitly derived in functional *differential* form, in a unified manner, leading to modifications of computational rules by including such factors as Faddeev–Popov ones [14, 15] and *modifications* thereof. The derivation is given in the *presence* of external sources, without recourse to path integrals, and without relying on any symmetry and invariance arguments. There has also been a renewed interest in Schwinger's action principle recently (see, e.g., [16–18]) emphasizing, in general, however, operator aspects of a theory, as deriving, for example, commutation relations, rather than dealing with computational ones related directly to generating functionals as done here.

2 General Class of Lagrangians

Consider Lagrangian densities which may depend on one or more coupling constants. We scale these couplings by a parameter λ which is eventually set equal to one. The resulting Lagrangian densities will be denoted by $\underline{\mathcal{L}}(x;\lambda)$. The class of Lagrangian densities considered are of the following types

$$\underline{\mathcal{L}}(x;\lambda) = \mathcal{L}(x;0) + \mathcal{L}_I(x;\lambda) + J_1^i(x)\chi_i(x) + J_2^j(x)\eta_j(x) \tag{1}$$

where $\chi_i(x)$ and $\eta_j(x)$ are independent and dependent fields, respectively. $J_1^i(x)$, $J_2^j(x)$ are external sources coupled to these respective fields. The interaction Lagrangian densities sought are of the following forms

$$\mathcal{L}_I(x;\lambda) = B(x;\lambda) + B^j(x;\lambda)\eta_j(x) + \frac{1}{2}B^{jk}(x;\lambda)\eta_j(x)\eta_k(x) \tag{2}$$

with $\mathcal{L}_I(x;0) = 0$, where $\partial B(x;\lambda)/\partial\lambda$, $\partial B^j(x;\lambda)/\partial\lambda$, $\partial B^{jk}(x;\lambda)/\partial\lambda = \partial B^{kj}(x;\lambda)/\partial\lambda$ may be expressed in terms of the independent fields, and the latter two may involve space derivatives applied to the dependent fields $\eta_j(x)$. By definition, the canonical conjugate momenta of the fields $\eta_j(x)$ vanish. That is, formally, $\partial\underline{\mathcal{L}}(x;\lambda)/\partial(\partial_0\eta_j(x)) = 0$. Let $\partial\mathcal{L}(x;0)/\partial\eta_j(x) = A^{jk}(x)\eta_k(x)$. The *constraint* equation of the dependent fields $\eta_k(x)$ follow from (1, 2) to be

$$M^{jk}(x;\lambda)\eta_k(x) = -[B^j(x;\lambda) + J_2^j(x)] \tag{3}$$

where

$$M^{jk}(x;\lambda) = A^{jk}(x) + B^{jk}(x;\lambda) \tag{4}$$

Let $D_{jk}(x,x';\lambda)$ denote the Green operator function satisfying

$$M^{ij}(x;\lambda)D_{jk}(x,x';\lambda) = \delta^i{}_k\delta^4(x,x') \tag{5}$$

From (3), this leads to

$$\eta_j(x) = -\int (\mathrm{d}x')D_{jk}(x,x';\lambda)[B^k(x';\lambda) + J_2^k(x')] \tag{6}$$

giving a constraint which is *explicit* source J_2^k-dependent, and is also a function of the independent fields.

Let $|0_{\mp}\rangle$ denote the vacuum states of a theory before/after the external are switched on/off, respectively. We are interested in the variation of the vacuum-to-vacuum transition amplitude $\langle 0_+ | 0_- \rangle$, governed by the Lagrangian density $\underline{\mathcal{L}}(x;\lambda)$ in (1), with respect to the parameter λ as well as the external sources $J_1^i(x)$, $J_2^j(x)$. To this end, we invoke the quantum dynamical principle which states (see, e.g., [7–10, 12, 13])

$$\frac{\partial}{\partial\lambda}\langle 0_+ | 0_- \rangle = \mathrm{i}\,\langle 0_+ | \int (\mathrm{d}x) \frac{\partial}{\partial\lambda} \mathcal{L}_I(x;\lambda)\,|0_-\rangle \tag{7}$$

$$(-\mathrm{i})\frac{\delta}{\delta J_1^i(x)}\langle 0_+ | 0_- \rangle = \langle 0_+ |\, \chi_i(x)\,|0_-\rangle \tag{8}$$

$$(-\mathrm{i})\frac{\delta}{\delta J_2^j(x)}\langle 0_+ | 0_- \rangle = \langle 0_+ |\, \eta_j(x)\,|0_-\rangle \tag{9}$$

Consider the matrix element $\langle 0_+ |\, F(x;\lambda, J_1, J_2)\,|0_-\rangle$ of an operator which is not only a function of the independent fields but which may also have an explicit dependence on λ and the external sources J_1^i, J_2^j. An explicit λ, J_2^j dependence may occur, for example, when the dependent fields $\eta_j(x)$ are expressed in terms of the independent fields and J_2^j as given in (6).

The quantum dynamical principle, in particular, then states (see [11–13]) that

$$\begin{aligned}(-\mathrm{i})&\frac{\delta}{\delta J_2^j(x')}\langle 0_+ |\, F(x;\lambda,J_1,J_2)\,|0_-\rangle \\ &= \langle 0_+ |\,(F(x;\lambda,J_1,J_2)\eta_j(x'))_+\,|0_-\rangle - \mathrm{i}\,\langle 0_+ | \frac{\delta}{\delta J_2^j(x')} F(x;\lambda,J_1,J_2)\,|0_-\rangle\end{aligned} \tag{10}$$

where $(\ldots)_+$ denotes the time-ordered product, and the functional derivative, with respect to $J_2^j(x')$, in the second term on the right-hand side of (10), is applied to the explicit J_2-dependent term (if any) that occurs in F.

Let $\partial B'^j(x;\lambda)/\partial\lambda$ denote $\partial B^j(x;\lambda)/\partial\lambda$ with the fields $\chi_i(x)$ in the latter replaced by the functional derivatives $(-\mathrm{i})\delta/\delta J^i(x)$. From (8–10), we then have

$$(-\mathrm{i})\frac{\delta}{\delta J_2^k(x')}\frac{\partial}{\partial\lambda} B'^j(x;\lambda)\,\langle 0_+ | 0_- \rangle = \langle 0_+ | \left(\frac{\partial}{\partial\lambda} B^j(x;\lambda)\eta_k(x')\right)_+ |0_-\rangle \tag{11}$$

where we have used the fact that $\partial B^j(x;\lambda)/\partial\lambda$ is expressed in terms of the independent fields and has no explicit J_2^k-dependence, and hence the second term on the right-hand side of (10) is zero for this corresponding case.

On the other hand, (10) also gives

$$\begin{aligned}(-\mathrm{i})&\frac{\delta}{\delta J_2^j(x'')}(-\mathrm{i})\frac{\delta}{\delta J_2^k(x')}\frac{\partial}{\partial\lambda} B'^{jk}(x;\lambda)\,\langle 0_+ | 0_- \rangle \\ &= \langle 0_+ | \left(\frac{\partial}{\partial\lambda} B^{jk}(x;\lambda)\eta_j(x'')\eta_k(x')\right)_+ |0_-\rangle - \mathrm{i}\,\langle 0_+ | \left(\frac{\partial}{\partial\lambda} B^{jk}(x;\lambda)\frac{\delta}{\delta J_2^j(x'')}\eta_k(x')\right)_+ |0_-\rangle\end{aligned} \tag{12}$$

where from (6),

$$\frac{\delta}{\delta J_2^j(x'')}\eta_k(x') = -D_{kj}(x',x'';\lambda) \tag{13}$$

Hence the second term on the right-hand side of (12) is simply

$$+\mathrm{i}\frac{\partial}{\partial\lambda}B'^{jk}(x;\lambda)\ D'_{kj}(x',x'';\lambda)\ \langle 0_+|0_-\rangle \tag{14}$$

with $D'_{kj}(x',x'';\lambda)$ denoting $D_{kj}(x',x'';\lambda)$ with the fields $\chi_i(x)$ replaced by $(-\mathrm{i})\delta/\delta J_1^i(x)$.

All told, we may *solve* for $\langle 0_+|\,\partial\mathcal{L}_I(x;\lambda)/\partial\lambda\,|0_-\rangle$ in terms of functional derivatives, with respect to the external sources, as *applied to* $\langle 0_+|0_-\rangle$ directly from (2, 7, 11–14) to obtain

$$\frac{\partial}{\partial\lambda}\langle 0_+|0_-\rangle = \left[\mathrm{i}\int(\mathrm{d}x)\frac{\partial}{\partial\lambda}\mathcal{L}'(x;\lambda) + \frac{1}{2}\int(\mathrm{d}x)\left(\frac{\partial}{\partial\lambda}B'^{jk}(x;\lambda)\right)D'_{kj}(x,x;\lambda)\right]\langle 0_+|0_-\rangle \tag{15}$$

where $\mathcal{L}'_I(x;\lambda)$ denotes $\mathcal{L}_I(x;\lambda)$ with $\chi_i(x)$, $\eta_j(x)$ replaced in the latter by $(-\mathrm{i})\delta/\delta J_1^i(x)$, $(-\mathrm{i})\delta/\delta J_2^j(x)$, respectively.

Upon integrating (15) over λ from 0 to 1, gives

$$\langle 0_+|0_-\rangle = \exp\left(\mathrm{i}\int(\mathrm{d}x)\mathcal{L}'_I(x)\right) \times\exp\left[\frac{1}{2}\int(\mathrm{d}x)\int_0^1\mathrm{d}\lambda\left(\frac{\partial}{\partial\lambda}B'^{jk}(x;\lambda)\right)D'_{kj}(x,x;\lambda)\right]\langle 0_+|0_-\rangle_0 \tag{16}$$

where $\langle 0_+|0_-\rangle_0$ is governed by the Lagrangian density $[\mathcal{L}(x;0)+J_1^i(x)\chi_i(x)+J_2^j(x)\eta_j(x)]$ in (1), and $\mathcal{L}'_I(x)\equiv\mathcal{L}'_I(x;0)$, with the latter defined below (15).

Equation (16) provides the solution for the generating functional $\langle 0_+|0_-\rangle$ in the *presence* of external sources. We thus see that *for interaction Lagrangian densities such that* $\partial\mathcal{L}_I(x;\lambda)/\partial\lambda$ *are quadratic in dependent fields* $(\partial B^{jk}(x;\lambda)/\partial\lambda\neq 0)$, *as described above, the rules for computations, via the generating functional* $\langle 0_+|0_-\rangle$ *are modified by the presence of the multiplicative functional differential operator factor*

$$\exp\left[\frac{1}{2}\int(\mathrm{d}x)\int_0^1\mathrm{d}\lambda\left(\frac{\partial}{\partial\lambda}B'^{jk}(x;\lambda)\right)D'_{kj}(x,x;\lambda)\right] \tag{17}$$

As special cases of the general Lagrangians described through (1) and developed above, consider *non-abelian* gauge theories with Lagrangian densities

$$\underline{\mathcal{L}} = \mathcal{L} + \mathcal{L}_S \tag{18}$$

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + \frac{1}{2\mathrm{i}}\left[\partial_\mu\bar{\psi}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\right] - m_o\bar{\psi}\psi + g_o\bar{\psi}\gamma_\mu A^\mu\psi \tag{19}$$

$$\mathcal{L}_S = \bar{\rho}\psi + \bar{\psi}\rho + J^\mu_a A^a_\mu \tag{20}$$

$$A_\mu = A^a_\mu t_a, \qquad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g_o[A_\mu, A_\nu] \tag{21}$$

$$G_{\mu\nu} = G^a_{\mu\nu}t_a \tag{22}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_o f^{abc}A^b_\mu A^c_\nu \tag{23}$$

The t^a matrices are generators of the underlying algebra, and the f^{abc}, totally anti-symmetric, are the structure constants satisfying the Jacobi identity $[t^a, t^b] = \mathrm{i} f^{abc} t^c$. $\mathcal{L}_S$ is the source term with the J^μ_a classical functions, while ρ, $\bar{\rho}$ are so-called anti-commuting Grassmann variables.

Upon setting

$$\nabla^{ab}_\mu = \delta^{ab}\partial_\mu + g_o f^{acb} A^c_\mu \tag{24}$$

working in the Coulomb gauge $\partial_i A^i_a = 0$, $i = 1, 2, 3$, and introducing the Green operator function $D^{cd}(x, x'; g_o)$, satisfying

$$[\delta^{ac}\boldsymbol{\partial}^2 + g_o f^{abc} A^b_k \partial_k] D^{cd}(x, x'; g_o) = \delta^4(x, x')\delta^{ad} \tag{25}$$

$k = 1, 2, 3$, one may solve for G^{k0}_a (see [11]) as follows

$$G^{k0}_a = -\partial^k D_{ab} J^0_b + F^k_a \tag{26}$$

in a matrix notation, where F^k_a is *not an explicit* function of the dependent fields and of the external sources. From the very definition of G^{k0}_a in (23), we also have

$$\begin{aligned} \partial_k G^{k0}_a &= \nabla^{ab}_k \partial_k A^0_b \\ &= [\delta^{ab}\boldsymbol{\partial}^2 + g_o f^{acb} A^c_k \partial_k] A^0_b \end{aligned} \tag{27}$$

Hence we may solve for A^0_b to obtain

$$A^0_b = -D_{bc}\boldsymbol{\partial}^2 D_{ca} J^0_a + K_b \tag{28}$$

where K_b is not an explicit function of the external sources.

Now we show that the time derivative $\partial_0 A^k_b$ may be solved in terms of A^0_c and the independent fields themselves. To this end, we note that

$$\partial^0 A^k_a = \nabla^k_{ab} A^0_b - G^{k0}_a \tag{29}$$

and from (28)

$$D_{ca} J^0_a = -\left[\delta_{ca} + \frac{1}{\boldsymbol{\partial}^2} f^{cda} A^d_k \partial_k\right](A^0_a - K_a) \tag{30}$$

Accordingly, (29, 30) and (26) lead to

$$\partial^0 A^k_a = g_o f^{acb}\left[A^k_c \frac{\partial^l}{\boldsymbol{\partial}^2} - \frac{\partial^k}{\boldsymbol{\partial}^2} A^l_c\right]\partial_l A^0_b + L^k_a \tag{31}$$

where L^k_a has no explicit dependence on the external sources and on the dependent fields A^0_b.

Finally we note that

$$\frac{\partial}{\partial g_o}\mathcal{L}_I = -f^{abc} A^b_k \left(A^c_0 G^{k0}_a + \frac{1}{2} A^c_l G^{kl}_a\right) + \bar{\psi}\gamma^\mu A_\mu \psi \tag{32}$$

From the definition of G^{k0}_a in (23), *and the fact that* $\partial^0 A^k_a$ may be expressed in terms of the A^0_b, as shown in (31), and the independent fields themselves, we see that (32) is *quadratic* in the dependent fields A^0_b.

The structure G_a^{k0} in (23) may be expressed, from (31), as a linear function of the dependent fields A_a^0, and directly from (26) we have

$$\frac{\delta}{\delta J_b^\nu} G_a^{k0}(x) = -\delta^0{}_\nu\, \partial_k D_{ab}(x, x'; g_o) \tag{33}$$

Hence (7–12, 32, 33) give

$$\frac{\partial}{\partial g_o}\langle 0_+ | 0_-\rangle = \left[\mathrm{i} \int (\mathrm{d}x) \frac{\partial}{\partial g_o} \mathcal{L}_I'(x; g_o) - \int (\mathrm{d}x) f^{bca} A_k^b(x) D'^{ac}(x, x; g_o) \right] \langle 0_+ | 0_-\rangle \tag{34}$$

where $A_k'^b(x) = (-\mathrm{i})\delta/\delta J_b^k(x)$. Upon using the definition of $D^{ac}(x, x', g_o)$ in (25), and integrating (34) over g_o, we obtain the modifying Faddeev–Popov multiplicative factor

$$\exp \mathrm{Tr} \ln \left[1 - \mathrm{i} g_o \frac{1}{\boldsymbol{\partial}^2} A_k' \partial^k \right] \tag{35}$$

as a *special case* of (17), where

$$\mathrm{Tr}[f] \equiv \int (\mathrm{d}x)\, f^{aa}(x, x) \tag{36}$$

The general derivation given above for interaction Lagrangian densities such that $\partial \mathcal{L}_I(x; \lambda)/\partial\lambda$ may be expressed as quadratic functions in dependent fields involves no symmetry arguments. As a matter of fact, we may consider the addition of a gauge-invariant breaking term $(g_1/2) A_a^\mu A_\mu^a \bar{\psi}\psi$ to the Lagrangian density in (19) which is again quadratic in A_a^0 and presumably contributes to the generation of masses to the vector fields through a non-vanishing expectation value of $\bar{\psi}\psi$. A detailed analysis shows (see [11]) that the modifying *extra multiplicative factor* to $\exp(\mathrm{i}\int(\mathrm{d}x)\mathcal{L}_I'(x))\,\langle 0_+ | 0_-\rangle\,|_0$ occurring in $\langle 0_+ | 0_-\rangle$ is given by

$$\exp\left[-\frac{1}{2} \mathrm{Tr} \ln \left(1 + \frac{g_1}{\nabla_l' \partial_l (\boldsymbol{\partial}^2)^{-1} \nabla_k' \partial_k} \bar{\psi}' \psi' \right) \right] \times \exp \mathrm{Tr} \ln \left(1 - \mathrm{i} g_o \frac{1}{\boldsymbol{\partial}^2} A_k' \partial_k \right) \tag{37}$$

where $\bar{\psi}' = (-\mathrm{i})\delta/\delta\rho$, $\psi' = (-\mathrm{i})\delta/\delta\bar{\rho}$, and $\mathcal{L}_I'$ is the *new* interaction Lagrangian density functional differential operator expressed in terms of functional derivatives with respect to the external sources.

3 Conclusion

We have seen, within the functional differential formalism of quantum field theory in the presence of external sources, that interaction Lagrangian densities $\mathcal{L}_I(x; \lambda)$ such that $\partial \mathcal{L}_I(x; \lambda)/\partial\lambda$ may be expressed as quadratic functions of dependent fields (i.e., $\partial B^{jk}(x; \lambda)/\partial\lambda \neq 0$ in (2)) and *arbitrary* functions of independent fields, necessarily lead to modifications of the rules for computations, via the generating functional $\langle 0_+|0_-\rangle$ as a functional of the external sources which are coupled to the fields, and no appeal was made, through the analysis, to path integrals. The general expression for such a modification is given in (17) as a functional differential operator occurring as a multiplicative factor in $\langle 0_+|0_-\rangle$. Such Lagrangians play *central* roles in fundamental physics and present renormalizable gauge theories fall into this category. It is important, however, to emphasize that such modifications are not tied up to non-abelian gauge theories, through the emergence of so-called Faddeev–Popov factors, as one might naively expect, but apply to theories which, in general, are quadratic functions in dependent fields as described above. As a matter of fact the addition of a gauge term breaking term in the form $(g_1/2)A^\mu A_\mu \bar{\psi}\psi$ to the interaction Lagrangian density of QED (abelian gauge theory), which is again quadratic in A^0, leads, according to (37), the following extra functional differential multiplicative factor

$$\exp\left[-\frac{1}{2}\operatorname{Tr}\ln\left(1+\frac{g_1}{\boldsymbol{\partial}^2}\bar{\psi}'\psi'\right)\right] \tag{38}$$

multiplying $\exp[\mathrm{i}\int(\mathrm{d}x)\mathcal{L}'_I(x)]\,\langle 0_+|0_-\rangle_0$, where $\mathcal{L}'_I(x)$ is the *new* interaction Lagrangian density functional differential operator including the additional term just mentioned as a simplified version of (37). That is, *a non-trivial modification arises even for such an abelian gauge theory*. The technical question now arises as to what happens to model Lagrangian densities that one may set up which are cubic or of higher order in dependent fields in the sense investigated above. The main complication with such theories becomes obvious by noting that the corresponding Green function operator function to the one in (5) will now *depend* on dependent fields themselves. Accordingly, when we apply the corresponding rule in (12) for finally expressing the matrix element $\langle 0_+|\,(\partial\mathcal{L}_I/\partial\lambda)\,|0_-\rangle$, as a functional differential operator, with respect to the external sources, to be eventually applied to $\langle 0_+|0_-\rangle$, the expression $\delta\eta_k(x')/\delta J_2^j(x'')$ will again depend, rather non-trivially, on the dependent fields $\eta_j(x)$. This makes the procedure of expressing the matrix element just mentioned as a functional differential operation to be applied to $\langle 0_+|0_-\rangle$ quite unmanageable. Such field theories require very special tools and will be investigated, within the functional differential formalism, in a subsequent report.

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Constraints, dependent fields and the quantum dynamical principle

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Abstract

Within the functional *differential* formalism of quantum systems, referred to as the quantum dynamical principle, given independent pairs of canonical conjugate variables $\{q_i(t), p_i(t), i = 1, \ldots, n\} \equiv \{\mathbf{q}(t), \mathbf{p}(t)\}$, and a set of pairwise commuting operator functions $\{G_j(\mathbf{q}(t), \mathbf{p}(t)), j = 1, \ldots, k\}$ of these variables defined, transformation functions are explicitly given, for constrained dynamical systems, expressed as functional differential operations applied to a given functional written in closed form. In the functional differential treatment *external sources* are, *a priori*, necessarily introduced in the theory. The connection of this work to the so-called Faddeev–Popov technique in path integrals is pointed out.

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1. Introduction

The purpose of this paper is to show within the functional *differential* treatment of quantum systems (e.g. [1–9]), also known as the quantum dynamical principle (QDP), given independent pairs of canonical conjugate variables $\{q_i(t), p_i(t), i = 1, \ldots, n\} \equiv \{\mathbf{q}(t), \mathbf{p}(t)\}$ and a set of pairwise commuting operator functions $\{G_j(\mathbf{q}(t), \mathbf{p}(t)), j = 1, \ldots, k\}$ of these variables defined, *transformation* functions may be then explicitly given for constrained dynamical systems. These transformation functions are expressed as functional differential operations, involving functional differentiations with respect to external sources, applied to a given functional of these sources written in closed form. The very elegant QDP has been indisputably recognized as a powerful tool over the years. There has been a renewed interest recently in Schwinger's action principle (see, e.g. [10–15]) emphasizing generally, however, operator aspects, as deriving, for example, commutation relations, rather than dealing with computational ones related directly to transformation functions as done here. We note that in the functional differential formalism external sources are, *a priori*, necessarily introduced to generate transformation functions and matrix elements of various operators. It will be understood throughout the bulk of this paper that all these sources will eventually be set equal to zero after all the relevant functional differentiations with respect to them have been carried out. The connection of this work to the so-called Faddeev–Popov technique in path integrals will be pointed out.

2. Functional differentiations and constraints

Consider a Hamiltonian $H(\mathbf{q}, \mathbf{p})$ as a function of independent pairs of canonical conjugate variables $\{q_i, p_i, i = 1, \ldots n\} \equiv \{\mathbf{q}, \mathbf{p}\}$. We introduce external sources $\{F_i(\tau), S_i(\tau), i = 1, \ldots, n\} \equiv \{\mathbf{F}(\tau), \mathbf{S}(\tau)\}$ and define the extended Hamiltonian $\underline{H}(\tau)$, in the presence of these sources, by

$$\underline{H}(\tau) = H(\mathbf{q}, \mathbf{p}) - \mathbf{q} \cdot \mathbf{F}(\tau) + \mathbf{p} \cdot \mathbf{S}(\tau) \quad (1)$$

with $\mathbf{F}(\tau)$, $\mathbf{S}(\tau)$ vanishing outside an interval $[t', t]$, with $t' < t$. Of physical interest are the transformation functions $\langle \mathbf{q}t | \mathbf{q}'t' \rangle$, in particular, in the limit of vanishing external sources. The explicit functional expression for the latter is well known (see, e.g. [4, section 11.2]) and is given by

$$\langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle \Big| = \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t} \mathrm{d}\tau \, H'(\tau) \right) \langle \mathbf{q}t \,|\, \mathbf{q}'t' \rangle_0 \Big|, \quad (2)$$

where $H'(\tau)$ is the functional differential operator

$$H'(\tau) = H\left(-\mathrm{i}\hbar \frac{\delta}{\delta \mathbf{F}(\tau)}, \mathrm{i}\hbar \frac{\delta}{\delta \mathbf{S}(\tau)} \right) \quad (3)$$

obtained from $H(\mathbf{q},\mathbf{p})$ in (1), by replacing $\mathbf{q}$,$\mathbf{p}$, respectively, by $-\mathrm{i}\hbar\delta/\delta\mathbf{F}(\tau)$, $\mathrm{i}\hbar\delta/\delta\mathbf{S}(\tau)$. Here $\langle\mathbf{q}t|\mathbf{q}'t'\rangle_0$ is given by

$$\langle\mathbf{q}t\,|\,\mathbf{q}'t'\rangle_0=\delta^n\left(\mathbf{q}-\mathbf{q}'-\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{S}(\tau)\right)\times\exp\left(\frac{\mathrm{i}}{\hbar}\mathbf{q}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{F}(\tau)\right)\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{\tau}\mathrm{d}\tau'\,\mathbf{S}(\tau)\cdot\mathbf{F}(\tau')\right)\quad(4)$$

defining a functional of $\mathbf{F}(\tau)$, $\mathbf{S}(\tau)$. The vertical bar | in (2) means to set these external sources equal to zero after the functional differentiations with respect to them, as defined in the exponential expression in (2), are carried out.

Given a set of operator functions $\{G_j(\mathbf{q}(\tau),\mathbf{p}(\tau)),\ j=1,\ldots,k\}$, as mentioned above, with time development given by the Hamiltonian $H(\mathbf{q},\mathbf{p})$, we may introduce the following c-functions:

$$Q_j^c(\tau)=\frac{\langle\mathbf{q}t\,|G_j(\mathbf{q}(\tau),\mathbf{p}(\tau))|\,\mathbf{q}'t'\rangle|}{\langle\mathbf{q}t\,|\,\mathbf{q}'t'\rangle|}\quad(5)$$

for τ in the interval $[t',t]$. We may promote the $Q_j^c(\tau)$ to quantum variables $Q_j(\tau)$ by noting: (A) The canonical conjugate momenta $P_j(\tau)$ of dependent fields $Q_j(\tau)$ must vanish, by definition. (B) We may introduce external sources $\mathbf{f}(\tau)$, $\mathbf{s}(\tau)$ to generate functionals of the latter fields as done for the $q_j(\tau)$, $p_j(\tau)$ fields and, in the process, make use of (4). (C) $H(\mathbf{q},\mathbf{p})$ in (1) is a function of independent pairs of the canonical conjugate variables in $\{q_i,p_i,i=1,\ldots,n\}$, and hence no explicit functional differentiation operations with respect to the sources $\mathbf{f}(\tau)$, $\mathbf{s}(\tau)$ appear in $H'(\tau)$.

To the above end, we note that (5) may be rewritten as

$$\left[Q_j^c(\tau)\langle\mathbf{q}t\,|\,\mathbf{q}'t'\rangle-\langle\mathbf{q}t\,|G_j(\mathbf{q}(\tau),\mathbf{p}(\tau))|\,\mathbf{q}'t'\rangle\right]|=0\quad(6)$$

or from which we have

$$\left[Q_j^c(\tau)-G_j'(\tau)\right]\langle\mathbf{q}t\,|\,\mathbf{q}'t'\rangle|=0\quad(7)$$

with

$$G_j'(\tau)=G_j\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\tau)},\ \mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)\quad(8)$$

by an immediate application of the QDP. By promoting the $Q_j^c(\tau)$ to quantum variables, with $\langle\mathbf{q}t|\mathbf{q}'t'\rangle$ generalized to $\langle\mathbf{qQ}t|\mathbf{q}'\mathbf{Q}'t'\rangle^\wedge$, we must have from (A), (B), (C), above *and* (7)

$$\left[-\mathrm{i}\hbar\frac{\delta}{\delta f_j(\tau)}-G_j'(\tau)\right]\langle\mathbf{qQ}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle^\wedge|=\langle\mathbf{qQ}t\,|\left[Q_j(\tau)-G_j(\mathbf{q}(\tau),\mathbf{p}(\tau))\right]|\,\mathbf{q}'\mathbf{Q}'t'\rangle^\wedge|=0,\quad(9)$$

$$\mathrm{i}\hbar\frac{\delta}{\delta s_j(\tau)}\langle\mathbf{qQ}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle^\wedge|=\langle\mathbf{qQ}t\,|P_j(\tau)|\,\mathbf{q}'\mathbf{Q}'t'\rangle^\wedge|=0\quad(10)$$

for all $t'\leqslant\tau\leqslant t$. Since a relation $xg(x)=0$, implies that $g(x)$ involves the *factor* $\delta(x)$, we note from (9), (10), (4) and finally from (2) that

$$\langle\mathbf{qQ}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle^\wedge|=\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,H'(\tau)\right)\langle\mathbf{qQ}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle_0^\wedge|,\quad(11)$$

where

$$\langle\mathbf{qQ}t\,|\,\mathbf{q}'\mathbf{Q}'t'\rangle_0^\wedge=\delta^{(k)}\left(-\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{F}(\cdot)}-G'(\cdot)\right)\times\delta^{(k)}\left(\mathrm{i}\hbar\frac{\delta}{\delta\mathbf{S}(\cdot)}\right)\langle\mathbf{q}t\,|\,\mathbf{q}'t'\rangle_0A,\quad(12)$$

$$A=\delta^k\left(\mathbf{Q}-\mathbf{Q}'-\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{s}(\tau)\right)\exp\left(\frac{\mathrm{i}}{\hbar}\mathbf{Q}\cdot\int_{t'}^{t}\mathrm{d}\tau\,\mathbf{f}(\tau)\right)\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{\tau}\mathrm{d}\tau'\,\mathbf{s}(\tau)\cdot\mathbf{f}(\tau')\right)\quad(13)$$

and the vertical bar | in (11) refers to the fact that all the external sources are to be set to zero after all the relevant functional differentiations have been carried out. $\delta^{(k)}(-\mathrm{i}\hbar\delta/\delta\mathbf{f}(\cdot)-G'(\cdot))$ and $\delta^{(k)}(\mathrm{i}\hbar\delta/\delta\mathbf{s}(\cdot))$ in (12), as arising *from* the conditions in (9) and (10), refer, each, to the product of k-dimensional deltas with τ running over all points in the interval $[t',t]$, i.e. $\delta^{(k)}(D(\cdot))=\Pi_{t'\leqslant\tau\leqslant t}\delta^k(D(\tau))$. We also note that functional differentiation operations with respect to external sources commute, unlike quantum operators, showing the power of the underlying formalism.

3. Conclusion

Equation (11) for $\langle\mathbf{qQ}t|\mathbf{q}'\mathbf{Q}'t'\rangle^\wedge|$ gives the expression for the transformation functions with the constraints given in (9) for all τ in $[t',t]$. They involve functional differential operations, with respect to external sources, to be applied to the functional $[\langle\mathbf{q}t|\mathbf{q}'t'\rangle_0A]$ with the latter given in closed form in (4) and (13).

Finally, we may make contact with the Faddeev–Popov technique, in the path integral formalism, by noting that the path integral representation for $[\langle\mathbf{q}t|\mathbf{q}'t'\rangle_0\,A]$ (see, [4, section 11.4]) on the extreme right-hand side of (12) is given by

$$\left[\langle\mathbf{q}t\,|\,\mathbf{q}'t'\rangle_0A\right]=\int_{\mathbf{q}(t')=\mathbf{q}',\mathbf{Q}(t')=\mathbf{Q}'}^{\mathbf{q}(t)=\mathbf{q},\mathbf{Q}(t)=\mathbf{Q}}\mathcal{D}(\mathbf{q}(\cdot),\mathbf{p}(\cdot))\,\mathcal{D}(\mathbf{Q}(\cdot),\mathbf{P}(\cdot))\times\exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\,v(\tau)\right),\quad(14)$$

where

$$v(\tau)=[\mathbf{p}(\tau)\cdot\dot{\mathbf{q}}(\tau)+\mathbf{P}(\tau)\cdot\dot{\mathbf{Q}}(\tau)+\mathbf{q}(\tau)\cdot\mathbf{F}(\tau)-\mathbf{p}(\tau)\cdot\mathbf{S}(\tau)+\mathbf{Q}(\tau)\cdot\mathbf{f}(\tau)-\mathbf{P}(\tau)\cdot\mathbf{s}(\tau)]\quad(15)$$

and by carrying out the explicit functional differentiations with respect to the external sources in (11) and (12) and by finally setting the sources equal to zero. Here, we note that the Hamiltonian of the system describing its time evolution *appears* in the first factor on the right-hand side of (11) as a functional differential operator with respect to external sources as defined in (3).

As an explicit illustration following the procedure developed through (5)–(13), consider a Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}^2) \tag{16}$$

in three-dimensional (3D) which is obviously rotationally invariant with the dynamics occurring in the 3D space, where V is arbitrary. Now suppose one is interested in developing the dynamics to be constrained to a fixed 2D plane making a given angle α with the (q_1, q_3)-plane. To do this, we introduce canonical conjugate operators

$$G_1 = q_1 \sin\alpha - q_2 \cos\alpha, \tag{17}$$

$$\hat{G}_1 = p_1 \sin\alpha - p_2 \cos\alpha. \tag{18}$$

The Hamiltonian of the dynamical system restricted to the 2D plane described above is then given by

$$H(\mathbf{q}*, \mathbf{p}*) = \frac{\mathbf{p}*^2}{2m} + V(\mathbf{q}*^2), \tag{19}$$

where $\mathbf{p}* = (p*_1, p*_2)$, $\mathbf{q}* = (q*_1, q*_2)$ with

$$p*_1 = p_1 \cos\alpha + p_2 \sin\alpha, \quad p*_2 = p_3, \tag{20}$$

$$q*_1 = q_1 \cos\alpha + q_2 \sin\alpha, \quad q*_2 = q_3, \tag{21}$$

and hence with the $(q*_1, q*_2)$-plane *making* an angle α with the (q_1, q_3)-plane corresponding to a rotation about the q_3-axis by the angle α. By carrying out the functional differential operations in (11) and (12) and using the expression in (14), it is readily seen that one obtains the Faddeev–Popov form in (11), upon including an extra multiplicative factor appearing in (12) given by $\delta(\mathrm{i}\hbar\delta/\delta s_1 \cdot -\hat{G}'_1(\cdot))\delta(-\mathrm{i}\hbar\delta/\delta f_1(\cdot))$, for the dynamical system described by the Hamiltonian in (16) to satisfy *both* constraints $G_1 = 0$, $\hat{G}_1 = 0$ by noting, in the process, that under such constraints the Hamiltonian in (16) reduces to the one in (19), and finally setting the external sources equal to zero.

The power and simplicity of the functional *differential* formalism via the QDP are evident. The constraints are implemented by functional differentiations, with respect to additional external sources, of transformation functions generalized to dynamical systems, in the presence of dependent degrees of freedom, which are readily spelled out from the corresponding systems with no constraints. Contact with the path integral formalism is also directly made from the explicit expression one has of transformation functions for simplified 'Hamiltonians' involving only dynamical variables coupled to external sources as given in (14) and (15). We emphasize that in the functional *differential* treatment, via the action principle, external sources must, *a priori*, be introduced.

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