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ADMISSIBLE FUNCTIONS AND FRAMES FOR THE METAPLECTIC REPRESENTATION

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ADMISSIBLE FUNCTIONS AND FRAMES FOR THE METAPLECTIC REPRESENTATION

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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วิทยานิพนธ์ฉบับนี้ ศึกษาการสมมูลของตัวแทนเมตาเพลกทิกกับผลรวมของตัวแทนสัมพรรคสำหรับชั้นของกลุ่มย่อยของกลุ่มซิมเพลกทิก โดยเริ่มต้นศึกษาจากผลรวมของการแปลงเวฟเลท พบว่าเงื่อนไขการยอมรับสำหรับการแปลงโดยตัวแทนสัมพรรค สามารถนำไปใช้กับการแปลงโดยผลรวมของตัวแทนเวฟเลทได้ และแสดงการสร้างเวกเตอร์ยอมรับ เวกเตอร์ยอมรับแบบจำกัดแถบ และกรอบสำหรับผลรวมของตัวแทนสัมพรรคโดยการใช้เซตตัดขวาง นอกจากนี้ ระบุชั้นของกลุ่มย่อยของกลุ่มซิมเพลกทิกซึ่งสมสัณฐานกับกลุ่มสัมพรรค ซึ่งแสดงให้เห็นว่าตัวแทนย่อยของตัวแทนเมตาเพลกทิกสามารถสมมูลกับตัวแทนสัมพรรคได้ และนำเสนอตัวอย่างเพื่อให้เห็นการสมมูลนี้ด้วย ตลอดจน จำแนกภาคขยายของกลุ่มไฮเซนเบิร์กแบบหลายมิติโดยวงศ์แบบหนึ่งพารามิเตอร์ของเมทริกซ์แบบเปลี่ยนขนาด โดยขึ้นกับการสมสัณฐาน พบว่าภาคขยายนี้เป็นกลุ่มย่อยของกลุ่มซิมเพลกทิก และตัวแทนเมตาเพลกทิกสมมูลกับผลรวมของตัวแทนเวฟเลท

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SUMS OF WAVELET TRANSFORMS / ADMISSIBLE VECTOR / METAPLECTIC REPRESENTATION / EXTENSION OF HEISENBERG GROUP

This thesis investigates equivalence of the metaplectic representation with sums of affine representations for a class of subgroups of the symplectic group. Beginning with the study of sums of wavelet transforms, it is shown that the usual admissibility conditions for the transform by affine representations apply to transforms by sums of affine representations as well. A construction of admissible vectors, bandlimited admissible vectors and frames for sums of affine representations by means of transversals is given. A class of subgroups of the symplectic group which are isomorphic to affine groups are identified, and it is shown how subrepresentations of the metaplectic representation can be equivalent to affine representations. This equivalence is illuminated by numerous examples. Finally, extensions of the multidimensional Heisenberg group by a one-parameter family of matrix dilations are classified up to isomorphism. These extensions are shown to be subgroups of the symplectic group, and their metaplectic representation is equivalent to a sum of wavelet representations.

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CHAPTER I

INTRODUCTION

A recurrent theme in analysis is the decomposition of a vector or function into basic building blocks, also known as spectral analysis. For example, the Fourier series is the decomposition of a periodic function into an infinite sum of trigonometric functions of fixed frequency. Thus, if one knows the frequency response of a linear system, one can deduce the response of the system to any periodic signal. Similarly, the Fourier transform can be considered as the decomposition of a non-periodic function f into a continuous spectrum of basic frequencies. Under some mild assumptions, the function f can be reconstructed from its Fourier transform by means of the inverse Fourier transform.

In general, it is difficult to recognize local properties of f from its Fourier transform. In applications such as image processing for example, one encounters functions which possess steep but well localized gradients. The Fourier transform is poorly suited to the analysis of such functions. First of all, the steep gradients lead to Fourier transforms which decay only slowly at infinity. In addition, the locations of the gradients cannot be detected easily from the Fourier transform.

1.1 The Wavelet Transform

The wavelet transform was introduced in Grossmann, Morlet and Paul (1985, 1986) as one tool to overcome these difficulties. Here the given function

$f \in L^2(\mathbb{R})$ is sampled in a window of variable location and scale,

$$Wf(a,x) = a^{-1/2} \int_{\mathbb{R}} f(y)\, \overline{\psi(\, a^{-1}(y-x)\,)}\, dy \qquad (a > 0, x \in \mathbb{R})$$

for some fixed window function $\psi \in L^2(\mathbb{R})$. Thus, the wavelet transform of f is a function depending on the two parameters location and scale, yielding information on both location and size of steep gradients. The question of reconstructing a given function from its wavelet transform was solved by Grossmann et al. (1985) using the theory of square integrable representations by Duflo and Moore (1976). ψ is called *admissible*, or a *wavelet*, if such a reconstruction is possible for every f, and it turns out that the set of admissible functions is dense in $L^2(\mathbb{R})$.

The wavelet transform can be naturally extended to n-dimensional Euclidean space,

$$Wf(a,\vec{x}) = |\det a|^{-1/2} \int_{\mathbb{R}^n} f(\vec{y})\, \overline{\psi(\, a^{-1}(\vec{y}-\vec{x})\,)}\, d\vec{y} \qquad (\vec{x} \in \mathbb{R}^n,\ f, \psi \in L^2(\mathbb{R}^n)\,) \tag{1.1}$$

where a is now an element of a closed subgroup H of $GL_n(\mathbb{R})$. Bernier and Taylor (1996) obtained results on existence and characterization of admissible functions in case that H possess open, free orbits in $\mathbb{R}^n$, extending those of Grossmann et al. (1985, 1986), still exploiting the results from the theory of square integrable representations.

Laugesen, Weaver, Weiss and Wilson (2002) finally succeeded to give general sufficient and necessary conditions on the matrix group H for the existence of admissible functions, as well as a characterization of these functions.

1.2 Transforms from Group Representations

The wavelet transform is only one example of decompositions arising from group representations. Let G be a locally compact group with Haar measure ν

and π a unitary representation of G on some Hilbert space $\mathcal{H}$, and fix $\phi \in \mathcal{H}$. The *voice transform* of an element f of $\mathcal{H}$ is the function Vf defined on G by

$$(Vf)(g) = \langle f, \pi(g)\phi \rangle . \tag{1.2}$$

The vector ϕ is called *admissible*, if the linear map $f \mapsto V_\phi f$ is a multiple of an isometry, that is if

$$\|V_\phi f\|^2_{L^2(G)} = c_\phi \|f\|^2_{\mathcal{H}} \qquad \forall f \in \mathcal{H} \tag{1.3}$$

for some constant $c_\phi > 0$. In this case, an application of the polarization identity leads to the reproducing formula

$$f = \frac{1}{c_\phi} \int_G \langle f, \pi(g)\phi \rangle \, \pi(g)\phi \, d\nu(g) \tag{1.4}$$

as a weak integral in $\mathcal{H}$. If such a ϕ exists, then the group G is called *admissible*, ϕ is called an *admissible vector* and $\{\pi(g)\phi\}_{g\in G}$ is called a *resolution of the identity.*

Having the form of a weak integral, the reproducing formula (1.4) is in general difficult to compute. It is preferable to find a discrete subset I of G which gives a basis-like reconstruction of the form

$$f = \sum_{i\in I} \langle f, \pi(g_i)\phi \rangle \; \pi(g_i)\phi; \tag{1.5}$$

which is the concept of a frame.

The groups considered in the wavelet transform (1.1) are affine groups, that is semi-direct products $G = H \rtimes \mathbb{R}^n$ that may be represented as matrix groups

$$\left\{ (h, \vec{x}) = \begin{bmatrix} h & \vec{x} \\ 0 & 1 \end{bmatrix} \; : \; h \in H, \; \vec{x} \in \mathbb{R}^n \right\}$$

for some closed subgroup H of $GL_n(\mathbb{R})$, acting by translations and dilations on $L^2(\mathbb{R}^n)$,

$$\pi(h, \vec{x})\phi = T_{\vec{x}} D_h \phi \qquad\qquad (h, \vec{x}) \in G, \; \phi \in L^2(\mathbb{R}^n). \tag{1.6}$$

Constructions which yield frames from this representation were initially obtained by Heil and Walnut (1989), Bernier and Taylor (1996) and Heinlein (2003) and others, and frame construction continues to be an active field of research in wavelet theory.

1.3 The Objectives of This Thesis

In a recent series of papers Cordero, De Mari, Nowak and Tabacco (2006a, 2006b, 2010) studied admissibility for the voice transform associated with the metaplectic representation of subgroups of the metaplectic group $Sp(n, \mathbb{R})$ on $L^2(\mathbb{R}^n)$. By employing the Wigner distribution, the authors obtained conditions for a function $\psi \in L^2(\mathbb{R}^n)$ to be admissible, mirroring those given by Laugesen et al. (2002) for the wavelet transform. The Wigner distribution is, however, difficult to work with; as a consequence, almost all examples in these papers constructed admissible functions for a special class of groups only; these groups are isomorphic to affine groups and through an ad-hoc process, admissibility conditions could be derived from those for the wavelet transform. A recent thesis of King (2009) continued this ad-hoc construction.

In this thesis, we build on these examples, and investigate the underlying mechanism in detail. Thus, we study subgroups of the symplectic group $Sp(n, \mathbb{R})$ which take the form of semidirect products $K = D \rtimes M$, where M is an n-dimensional vector group and D a closed subgroup of $GL_n(\mathbb{R})$ which acts linearly on M. These groups are isomorphic to subgroups, or finite extensions of subgroups of the affine group, and thus possess a wavelet representation as well. It turns out that in many cases, the metaplectic representation of K decomposes into a finite sum of subrepresentations, each of which is equivalent to a sum of modulated wavelet representations.

We therefore begin by introducing modulated wavelet representations, and by studying their sums in a systematic way. In particular, we derive conditions for a vector to be admissible, present a concrete method for the construction of admissible and bandlimited admissible vectors, and discuss the construction of frames.

We then make use of these results to show how to obtain admissibility conditions for the metaplectic representation of the groups K from those for the corresponding sum of wavelet representations. In particular, this process will allow us to introduce frames for the metaplectic representation. We present a series of examples which illuminate this process, one of which will show that the concept of admissibility of Cordero et al. (2006a) through the Wigner distribution is more narrow than the usual one.

It turns out that the groups involved in the examples given in Cordero et al. (2006a, 2010), King (2009) and Czaja and King (preprint) all belong to the same class of groups, namely are extensions of the Heisenberg group by a one-parameter family of matrix groups. We introduce these extensions, classify them up to isomorphism and show that they can indeed be presented as subgroups of $Sp(n,\mathbb{R})$ of the form $D \rtimes M$ as discussed in this thesis. Our admissibility results coincide with those for the groups given in Cordero et al. (2006a, 2010), King (2009) and Czaja and King (preprint).

This thesis is organized as follows. Chapter II gives a short introduction into the concepts and a review of the mathematical tools required. Chapter III begins with a quick summary of the usual wavelet transform, and presents our results on sums of modulated wavelet transforms, including the characterization and construction of admissible vectors and of frames. The relationship between the metaplectic representation and sums of wavelet representations for subgroups of the symplectic group arising as semi-direct products is explored in Chapter IV, and illustrated by numerous examples. Chapter V is devoted to a discussion and classification of extensions of the Heisenberg group by one-parameter groups of matrices. Finally, these results are all briefly summarized in Chapter VI.

CHAPTER II

BASIC BACKGROUND

In this chapter, we review the mathematical concepts and theorems from the literature that are used in this thesis. We begin with the basics of locally compact groups, followed by matrix groups and their Lie algebras. We also review the fundamental concepts in time-frequency analysis, such as the Fourier transform and its properties, frames in Hilbert space, the Wigner distribution and the metaplectic representation. Throughout, it is assumed that the reader is familiar with the foundations of real analysis, such as measure theory and function spaces. Details and proofs of the material presented here can be found in Folland (1989, 1999), Baker (2001), Knapp (1996) and Gröchenig (2000).

2.1 Locally Compact Groups

Definition 2.1. *A topological group is a group G endowed with a topology such that the group operations $(h, k) \mapsto hk$ and $h \mapsto h^{-1}$ are continuous from $G \times G$ and G to G.*

Simple examples include topological vector spaces (the group operation being addition), groups of invertible $n \times n$ real matrices (with the relative topology induced from $\mathbb{R}^{n\times n}$) and all groups equipped with the discrete topology. If G is a topological group, we denote the identity element of G by e, and for $A, B \subset G$ and $h \in G$ we define $hA = \{hk : k \in A\}$, $Ah = \{kh : k \in A\}$, $A^{-1} = \{k^{-1} : k \in A\}$ and $AB = \{kh : k \in A, h \in B\}$. We say that A is symmetric if $A = A^{-1}$.

Here are some of the basic properties of topological groups:

Proposition 2.2. *Let G be a topological group.*

(a) *The maps $g \in G \mapsto hg$ ($h \in G$ fixed) and $g \in G \mapsto g^{-1}$ are homeomorphisms. In particular, the topology of G is translation and inversion invariant: If U is an open subset of G and $h \in G$, then Uh, hU and U^{-1} are open.*

(b) *For every neighborhood U of e there exists a symmetric open neighborhood V of e with $V \subset U$.*

(c) *For every open neighborhood U of e there exists an open neighborhood V of e with $VV \subset U$.*

(d) *If H is a subgroup of G then so is $\overline{H}$.*

(e) *Every open subgroup of G is also closed.*

(f) *If K_1, K_2 are compact subsets of G then so is K_1K_2.*

If f is a function on the topological group G and $k \in G$, the left and the right translates of f through k are defined by

$$L_kf(h) \;=\; f(k^{-1}h), \qquad R_kf(h) \;=\; f(hk).$$

(The point of using k^{-1} on the left and k on the right is to obtain homomorphism properties: $L_{kl} = L_kL_l$ and $R_{kl} = R_kR_l$.) f is called *left* (resp. *right*) *uniformly continuous* if for every $\varepsilon > 0$ there is a neighborhood V of e such that $\|L_kf - f\|_\infty < \varepsilon$ (resp. $\|R_kf - f\|_\infty < \varepsilon$) for $k \in V$.

We write $f \in C_c(G)$, if f is continuous and there exists a compact subset K of G outside of which f vanishes. The smallest such subset K is called the *support of f*, denoted $\operatorname{supp}(f)$.

Proposition 2.3. *If $f \in C_c(G)$, then f is left and right uniformly continuous.*

A locally compact group is a topological group whose topology is locally compact and Hausdorff. The next proposition shows that closed and open subgroups of a locally compact group are again locally compact.

Proposition 2.4. *Let G be a locally compact Hausdorff space.*

(a) If $H \subset G$ is closed, then H is also locally compact Hausdorff.

(b) If $K \subset G$ is open, then K is also locally compact Hausdorff.

2.1.1 Haar Measure

Locally compact groups are of interest because they carry a translation invariant measure: If G is a locally compact group, being a topological space, G has a measurable structure, namely the σ-algebra generated by the open sets, called the *Borel σ-algebra*. A measure μ on the Borel sets is called a *Borel measure*, and it is called a *Radon measure* if

(a) $\mu(K)$ is finite for every compact set K;

(b) every Borel set E is outer regular: $\mu(E) = \inf\{\mu(U) : E \subset U, U \text{ open}\}$; and

(c) every open set E is inner regular: $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$.

These conditions assure that every $f \in C_c(G)$ is integrable, and $C_c(G)$ is dense in $L^p(G)$ for every $1 \leqslant p < \infty$.

A Borel measure μ on G is called *left-invariant* (resp. *right-invariant*) if $\mu(hE) = \mu(E)$ (resp. $\mu(Eh) = \mu(E)$) for all $h \in G$ and all Borel subsets E of G. If in addition, μ is a nonzero Radon measure, then it is called a *left* (resp. *right*) *Haar measure*. For example, the Lebesgue measure is a (left and right) Haar measure on $\mathbb{R}^n$. The following proposition summarizes some elementary properties of Haar measures; in it, we employ the notation

$$C_c^+ = \{f \in C_c(G) \;:\; f \geqslant 0 \text{ and } \|f\|_\infty > 0\}\,.$$

Proposition 2.5. *Let G be a locally compact group, and μ a nonzero Radon measure on G*

(a) μ is a left Haar measure if and only if the measure $\tilde{\mu}$ defined by $\tilde{\mu}(E) = \mu(E^{-1})$ is a right Haar measure.

(b) μ is a left Haar measure if and only if $\int_G f(k^{-1}h)d\mu(h) = \int_G f(h)d\mu(h)$ for all $f \in C_c^+$, $k \in G$ if and only if $\int_G f(k^{-1}h)d\mu(h) = \int_G f(h)d\mu(h)$ for all $f \in L^1(G)$, $k \in G$.

(c) If μ is a left Haar measure on G, then $\mu(U) > 0$ for every nonempty open $U \subset G$ and $\int_G f(h)d\mu(h) > 0$ for all $f \in C_c^+$.

(d) If μ is a left Haar measure on G, then $\mu(G) < \infty$ if and only if G compact.

Theorem 2.6. *Every locally compact group G possesses a left Haar measure. The left Haar measure is essentially unique, that is, if μ and ν are left Haar measures on G, there exists $c > 0$ such that $\mu = c\nu$. By symmetry, similar statements hold for a right Haar measure.*

If μ is a left Haar measure on G and $h \in G$, the measure $\mu_h(E) = \mu(Eh)$ is again a left Haar measure, because of the commutativity of left and right translations which results from the associative law. Hence, by Theorem 2.6, there is a positive number $\Delta(h)$ such that $\mu_h = \Delta(h)\mu$. The function $\Delta : G \to (0, \infty)$ thus defined is independent of the choice of μ by Theorem 2.6, and is called the *modular function* of G.

Proposition 2.7. *Δ is a continuous homomorphism from G into the multiplicative group of positive real numbers. Moreover, if μ is a left Haar measure on G, then for any $f \in L^1(G)$ and k in G we have*

$$\int_G f(hk)d\mu(h) = \Delta(k^{-1}) \int_G f(h)d\mu(h). \tag{2.1}$$

Proof. For every $h, k \in G$ and Borel subset E of G of positive measure,

$$\Delta(hk)\mu(E) = \mu(Ehk) = \Delta(k)\mu(Eh) = \Delta(k)\Delta(h)\mu(E),$$

and as μ is nonzero, Δ is a homomorphism. Also, since $\mathbf{1}_E(hk) = \mathbf{1}_{Ek^{-1}}(h)$,

$$\int_G \mathbf{1}_E(hk)d\mu(h) = \mu(Ek^{-1}) = \Delta(k^{-1})\mu(E) = \Delta(k^{-1}) \int_G \mathbf{1}_E(h)d\mu(h).$$

This proves (2.1) when $f = \mathbf{1}_E$ is the characteristic function of a Borel set E, and the general case follows by the definition of the integral. Finally, using Proposition 2.3 and Radon condition (a) one easily shows that the map $k \mapsto \int_G f(hk)d\mu(h)$ is continuous for any $f \in C_c(G)$, so the continuity of Δ follows from (2.1). $\square$

Evidently, a left Haar measure on G is also a right Haar measures precisely when Δ is identically 1, in which case G is called *unimodular*. Of course, every Abelian group is unimodular.

Proposition 2.8. *If G is compact, then G is unimodular.*

Proof. For any $h \in G$, obviously $G = Gh$. Hence if μ is a right Haar measure, we have $\mu(G) = \mu(Gh) = \Delta(h)\mu(G)$, and since $0 < \mu(G) < \infty$, by compactness we conclude that $\Delta(h) = 1$. $\square$

We observed above that if μ is a left Haar measure, then $\tilde{\mu}(E) = \mu(E^{-1})$ is a right Haar measure. We now show how to compute it in terms of μ and Δ.

Proposition 2.9. $d\tilde{\mu}(h) = \Delta(h)^{-1}d\mu(h)$.

Proof. By (2.1), for $f \in C_c(G)$,

$$\int_G f(h)\Delta(h)^{-1}d\mu(h) = \Delta(k)\int_G f(hk)\Delta(hk)^{-1}d\mu(h) = \int_G f(hk)\Delta(h)^{-1}d\mu(h).$$

Thus the Radon measure determined by $\Delta^{-1}d\mu$ is right-invariant, so by theorem 2.6, $\Delta^{-1}d\mu = cd\tilde{\mu}$ for some $c > 0$. If $c \neq 1$, we can pick a symmetric neighborhood U of e in G such that $|\Delta(h)^{-1} - 1| < \frac{1}{2}|c-1|$ for $h \in U$. But $\tilde{\mu}(U) = \mu(U^{-1}) = \mu(U)$, so

$$|c-1|\mu(U) = |c\tilde{\mu}(U) - \mu(U)| = \left|\int_U (\Delta(h)^{-1} - 1)d\mu(h)\right| < \frac{1}{2}|c-1|\mu(U),$$

a contradiction. Hence $c = 1$ and $d\tilde{\mu} = \Delta^{-1}d\mu$. $\square$

This proposition shows that left and right Haar measures are mutually absolutely continuous.

2.1.2 Continuous Group Actions

Definition 2.10. *Let X be a set, G a group. By a (left)action of G on X, we mean a map*

$$\alpha : G \times X \to X$$

satisfying

(a) $\alpha(e, x) = x \quad \forall x \in X$ *where e denotes the identity of G,*

(b) $\alpha(h, \alpha(h', x)) = \alpha(hh', x) \quad \forall h, h' \in G,\ x \in X.$

The triple (X, G, α) is also called a transformation group, and X is called a G-set.

It is often convenient to denote $\alpha(h, x)$ by $h \cdot x$ (or $\alpha_h(x)$). Then (a) and (b) become

(a′) $e \cdot x = x$ (or $\alpha_e(x) = x$) $\quad \forall x \in X$, (b′) $h \cdot (h' \cdot x) = (hh') \cdot x \quad (\text{or } \alpha_h(\alpha_{h'}(x)) = \alpha_{hh'}(x)) \;\; \forall h, h' \in G,\, x \in X.$

If X is a topological space and G a topological group, then one also requires that the map α be continuous, and calls X a G-space. In this case,

(a) α is an open map,

(b) for fixed $h \in G$, the map $x \mapsto h \cdot x$ is a homeomorphism of X onto X, since it has a continuous inverse, namely $x \mapsto h^{-1} \cdot x$.

Given $x \in X$, the set $\mathcal{O}(x) = G \cdot x = \{h \cdot x \,:\, h \in G\}$ is called the *orbit* of x. The *stabilizer* of $x \in X$ is the set $G_x = \{h \in G \,:\, h \cdot x = x\}$. It is a closed subgroup of G provided that X is a T_1-space. The orbit $\mathcal{O}(x)$ is called *free* if $G_x = \{e\}$. The *global stabilizer* is $G_0 = \bigcap\limits_{x \in X} G_x$, and the action is called *effective* if $G_0 = \{e\}$.

For example, let $X = \mathbb{R}^n$ and $G = GL_n(\mathbb{R})$. There is a natural action of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$ given by multiplication of a matrix with vector,

$$a \cdot \vec{x} = \alpha_a(\vec{x}) = a\vec{x}$$

for $a \in GL_n(\mathbb{R})$ and $\vec{x} \in \mathbb{R}^n$. The stabilizer of $\vec{x} = 0$ is $GL_n(\mathbb{R})$ itself, and each $\vec{x} \neq 0$ has a nontrivial stabilizer: if $\vec{x} = (x_1, x_2, ..., x_n)^T$, $(n \geqslant 2)$, then $\alpha_a(\vec{x}) = \vec{x}$ where α_a denotes reflection along the line through $\vec{x}$.

Now if a is a diagonal matrix, $a = \mathrm{diag}(a_1, a_2, ..., a_n)$ with $a_k > 0$ for all k, let $D = \{a^t \,:\, t \in \mathbb{R}\}$. D is called the one-parameter subgroup of $GL_n(\mathbb{R})$ generated by a. The orbit of $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq 0$, is free, as whenever $x_k \neq 0$, and $t \neq 0$ then $a_k^t x_k \neq x_k$. This also shows that the action of D on $\mathbb{R}^n$ is effective.

Similarly, if $\widehat{\mathbb{R}^n}$ denotes Euclidean space whose elements are written as row vectors, then

$$a \cdot \vec{\gamma} = \vec{\gamma} a^{-1}$$

for $a \in GL_n(\mathbb{R})$ and $\vec{\gamma} \in \widehat{\mathbb{R}^n}$, defines an action of $GL_n(\mathbb{R})$ on $\widehat{\mathbb{R}^n}$.

2.1.3 Semi-Direct Products

Let D and N be locally compact groups and let α be an action of D on N by group automorphisms. We can form a new group by setting

$$G = \{(a, n) \,:\, a \in D,\, n \in N\}$$

endowed with the product topology and the group law

$$(a, n)(a', n') = (aa', n\alpha_a(n')) = (aa', \alpha_a(\alpha_{a^{-1}}(n)n'))$$

for $(a, n), (a', n') \in G$. It is not difficult to verify that G is a locally compact group with identity (e_D, e_N) where e_D is the identity of D, and e_N is the identity of N, and that the inverse of element (a, n) is $(a^{-1}, \alpha_{a^{-1}}(n)^{-1})$.

Furthermore, identifying N with the closed subset $\{(e_D, n) \,:\, n \in N\}$ of G, then N is a normal subgroup of G, and G/N is naturally isomorphic to D. We call G *the semi-direct product of D and N with respect to* α and write $G = D \rtimes N$.

Now let μ_D and μ_N denote the left Haar measures on D and N respectively. As α_a is a homeomorphism for each $a \in D$, it is also a Borel isomorphism and hence $\mu_a(E) := \mu_N(\alpha_a(E))$ is a left Haar measure on N. By uniqueness of the Haar measure, there is a number $J(a) > 0$, such that $\mu_N(\alpha_a(E)) = J(a)\mu_N(E)$ for all Borel subset E of N. Since α is a group automorphism, then $J : D \to (0, \infty)$ is a homomorphism. Equivalently by the definition of the integral,

$$J(a)\int_N f(\alpha_a(n))d\mu_N(n) = \int_N f(n)d\mu_N(n) \tag{2.2}$$

first for all characteristic functions $f = \mathbf{1}_E$ (E Borel) and hence for all $f \in L^1(N)$. The Haar measure ν on G is determined by

$$d\nu(a, n) = J(a^{-1})d\mu_D(a)\, d\mu_N(n). \tag{2.3}$$

To verify this, note that for all $(a', n') \in G$ and $f \in C_c(G)$, we have by left invariant of μ_N and μ_D,

$$\begin{aligned}
\int_G f\left((a',n')(a,n)\right) d\nu(a,n) &= \int_D \int_N f\left(a'a, \alpha_{a'}(\alpha_{a'^{-1}}(n')n)\right) \frac{d\mu_N(n) d\mu_D(a)}{J(a)} \\
&= \int_D \int_N f\left(a'a, \alpha_{a'}(n)\right) \frac{d\mu_N(n) d\mu_D(a)}{J(a)} = \int_D \int_N f\left(a'a, n\right) \frac{d\mu_N(n) d\mu_D(a)}{J(a')J(a)} \\
&= \int_D \int_N f\left(a, n\right) \frac{d\mu_N(n) d\mu_D(a)}{J(a')J(a'^{-1}a)} = \int_D \int_N f\left(a, n\right) \frac{d\mu_N(n) d\mu_D(a)}{J(a)} \\
&= \int_G f\left(a,n\right) d\nu(a,n).
\end{aligned}$$

2.1.4 Group Representations

Definition 2.11. *Let G be a locally compact group and $\mathcal{H}$ be a Hilbert space. A (unitary) representation π of G on $\mathcal{H}$ is a mapping satisfying:*

(a) $\pi : G \to \mathcal{U}(\mathcal{H})$. ($\mathcal{U}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$.)

(b) π is a homomorphism: $\pi(hk) = \pi(h)\pi(k)$ for all $h, k \in G$.

(c) π is continuous with respect to the strong operator topology of $\mathcal{U}(\mathcal{H})$, that is $h \mapsto \pi(h)\psi$ is continuous for each $\psi \in \mathcal{H}$.

If $\mathcal{H} = \mathbb{C}$, then π is called a character of G.

A closed subspace $\mathcal{K}$ of $\mathcal{H}$ is called *π-invariant*, if $\pi(h)\psi \in \mathcal{K}$ for all $\psi \in \mathcal{K}$, $h \in G$. The orthogonal complement $\mathcal{K}^\perp$ is also π-invariant.

Now let $\{\pi_j\}_{j \in J}$ be a collection of representations of G on Hilbert spaces $\mathcal{H}_j$, and let $\mathcal{H} = \bigoplus_{j \in J} \mathcal{H}_j$. Since each $\pi_j(h)$ is a unitary operator, then the operator $\pi(h)$ on $\mathcal{H}$ defined by

$$\pi(h)\psi = \sum_{j \in J} \pi_j(h)\psi_j$$

for $\psi = \sum_{j \in J} \psi_j \in \mathcal{H}$, $\psi_j \in \mathcal{H}_j$ is a well defined unitary operator on $\mathcal{H}$. One can verify that π is a representation of G on $\mathcal{H}$, called *the sum of the representations*

$\{\pi_j\}_{j\in J}$, and we write

$$\pi = \bigoplus_{j\in J} \pi_j.$$

If π is a representation of $\mathcal{H}$ and $\mathcal{K}$ a π-invariant subspace, then π restricts to a representation $\pi\big|_{\mathcal{K}}$ of G on $\mathcal{K}$. Since $\mathcal{K}^\perp$ is also π-invariant, then

$$\pi = \pi\big|_{\mathcal{K}} \oplus \pi\big|_{\mathcal{K}^\perp}.$$

Definition 2.12. *A representation π is called irreducible if $\{0\}$ and $\mathcal{H}$ are the only π-invariant closed subspaces of $\mathcal{H}$.*

For example, every character is irreducible.

Theorem 2.13. (Schur's Lemma) *Let π be a unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(a) π is irreducible.

(b) For every $\psi \in \mathcal{H}\setminus\{0\}$ the subspace spanned by the finite linear combinations of $\pi(g)\psi$, $g \in G$, is dense in $\mathcal{H}$.

(c) If a bounded operator $S : \mathcal{H} \to \mathcal{H}$ satisfies $\pi(g)S = S\pi(g)$ for all $g \in G$, then $S = \lambda I_{\mathcal{H}}$ for some $\lambda \in \mathbb{C}$.

2.2 Matrix Groups

In this section, $\Bbbk$ will denote the fields $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$.

Let $M_{n,m}(\Bbbk)$ be the set of $m \times n$ matrices whose entries are in $\Bbbk$. We denote the (i, j)-th entry of an $m \times n$ matrix a by a_{ij} and also write $a = [a_{ij}]$. If $m = n$, then we write $M_n(\Bbbk)$ for this set.

$M_{n,m}(\Bbbk)$ is a $\Bbbk$-vector space under the operations of matrix addition and scalar multiplication. The zero vector is the $m \times n$ zero matrix $0_{m,n}$ which we will often denote by 0 when the size is clear from the context. As a vector space $M_{n,m}(\mathbb{k})$ is isomorphic to $\mathbb{k}^{nm}$ and thus inherits the topology of $\mathbb{k}^{nm}$. $M_n(\mathbb{k})$ is also a ring with the usual addition and multiplication of square matrices, with zero 0_n and the $n \times n$ identity matrix I_n as its unity; $M_n(\mathbb{k})$ is not commutative except when $n = 1$. We usually give $M_n(\mathbb{k})$ the operator norm, but as all norms on a finite dimensional vector space are equivalent, then $a_n \to a$ in $M_n(\mathbb{k})$ if and only if the sequences of corresponding entries all converge: $(a_n)_{ij} \to a_{ij}$ for all i, j. The conjugate transpose of a matrix will be denoted by a^*, and the transpose by a^T.

Proposition 2.14. *The determinant* $\det : M_n(\mathbb{k}) \to \mathbb{k}$ *is a continuous function.*

Proof. The determinant is obtained by composing the continuous function $M_n(\mathbb{k}) \to \mathbb{k}^{n^2}$ identifying $M_n(\mathbb{k})$ with $\mathbb{k}^{n^2}$ with a polynomial function $\mathbb{k}^{n^2} \to \mathbb{k}$. □

Next, we consider two particular subsets of $M_n(\mathbb{k})$:

$$GL_n(\mathbb{k}) = \{a \in M_n(\mathbb{k}) \,:\, \det a \neq 0\} \quad \text{and} \quad SL_n(\mathbb{k}) = \{a \in M_n(\mathbb{k}) \,:\, \det a = 1\}$$

which are both groups under matrix multiplication. Furthermore, $SL_n(\mathbb{k})$ is a subgroup of $GL_n(\mathbb{k})$. By Proposition 2.14, then $GL_n(\mathbb{k}) = M_n(\mathbb{k}) \setminus \det^{-1}\{0\}$ is an open subset of $M_n(\mathbb{k})$, similarly, $SL_n(\mathbb{k}) = \det^{-1}\{1\} \subseteq GL_n(\mathbb{k})$ is closed in $M_n(\mathbb{k})$ and $GL_n(\mathbb{k})$. By Proposition 2.4, then $GL_n(\mathbb{k})$ and $SL_n(\mathbb{k})$ are a locally compact groups.

Definition 2.15. *A subgroup G of $GL_n(\mathbb{k})$ which is closed in $GL_n(\mathbb{k})$ is called a matrix group.*

Proposition 2.16. *Let G be a matrix group. Then every closed subgroup H of G is also a matrix group (called a matrix subgroup of G).*

Proof. Since H is closed in G and G is closed in $GL_n(\mathbb{k})$, then H is closed in $GL_n(\mathbb{k})$. □

Since a matrix group H is a closed subgroup of $GL_n(\mathbb{k})$, then by Proposition 2.4, its topology is locally compact. Hence H has a left Haar measure.

Proposition 2.17. *Let G be a matrix group. If H is a matrix subgroup of G and K is a matrix subgroup of H, then K is a matrix subgroup of G.*

Proof. This is a straightforward generalization of Proposition 2.16. □

Definition 2.18. *Let G and H be two matrix groups. A group homomorphism $\varphi : G \to H$ is called a matrix group homomorphism if it is continuous and its image $\varphi(G)$ is closed in H, that is, is a matrix subgroup of H. In addition, if φ^{-1} exists and is continuous, then φ is called a matrix group isomorphism.*

Remark 2.1. It is important to require that $\varphi(G)$ be closed in H. For example, let r be an irrational number. Now $G = \{e^n : n \in \mathbb{Z}\}$ is a closed subgroup of $GL_1(\mathbb{R}) \equiv \mathbb{R}^+$. Define $\varphi : G \to GL_2(\mathbb{R})$ by $\varphi(e^n) = R_{nr} \in GL_2(\mathbb{R})$ where $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix. Then φ is a continuous homomorphism, but $\overline{\varphi(G)} = \{R_\theta : 0 \leqslant \theta < 2\pi\} \neq \varphi(G)$. Hence $\varphi(G)$ is not a matrix group.

Remark 2.2. Let $\varphi : G \to H$ be a matrix group homomorphism. Since φ is continuous, then $\ker\varphi = \varphi^{-1}(\{e\})$ is a closed subset of G hence is a matrix group. The quotient group $G/\ker\varphi$ can be identified with the matrix group $\varphi(G)$ by the usual quotient isomorphism $\overline{\varphi} : G/\ker\varphi \to \varphi(G)$ (which need not be a homomorphism of matrix groups since $G/\ker\varphi$ need not be a matrix group).

2.3 Lie Algebras

The theory of Lie algebra is a rich and well developed field. As we will use this theory for the classification of matrix groups, we focus here on the connection between matrix groups and Lie algebras.

Definition 2.19. *Let $\mathfrak{g}$ be a vector space over $\mathbb{k}$. $\mathfrak{g}$ is called a Lie algebra over $\mathbb{k}$, if there exists a $\mathbb{k}$-bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, satisfying:*

(a) skew-symmetry:

$$[x, y] = -[y, x]$$

(b) Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all $x, y, z \in \mathfrak{g}$.

For example, $M_n(\mathbb{k})$ is a Lie algebra over $\mathbb{k}$ with the Lie bracket

$$[A, B] = AB - BA$$

for $A, B \in M_n(\mathbb{k})$.

If $\mathfrak{a}$ and $\mathfrak{b}$ are subsets of a Lie algebra $\mathfrak{g}$, we write

$$[\mathfrak{a}, \mathfrak{b}] = \operatorname{span}\{[x, y] \,:\, x \in \mathfrak{a},\, y \in \mathfrak{b}\}.$$

Definition 2.20. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{k}$.*

(a) The center of $\mathfrak{g}$ is

$$z(\mathfrak{g}) = \{z \in \mathfrak{g} \,:\, [z, x] = 0 \;\; \text{for all} \;\; x \in \mathfrak{g}\}.$$

(b) $[\mathfrak{g}, \mathfrak{g}]$ is called the derived algebra of $\mathfrak{g}$.

Definition 2.21. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{k}$ and $\mathfrak{h}$ a vector subspace of $\mathfrak{g}$. Then*

(a) $\mathfrak{h}$ is called a Lie subalgebra of $\mathfrak{g}$, if $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.

(b) $\mathfrak{h}$ is called a Lie ideal of $\mathfrak{g}$, if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Proposition 2.22. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{k}$. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathfrak{g}$, then so are $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$, and $[\mathfrak{a}, \mathfrak{b}]$.*

Proof. The conclusions for $\mathfrak{a}+\mathfrak{b}$ and $\mathfrak{a}\cap\mathfrak{b}$ are obvious. In the case of $[\mathfrak{a},\mathfrak{b}]$, by the Jacobi identity and skew-symmetry, we have

$$[\mathfrak{g},[\mathfrak{a},\mathfrak{b}]] = [[\mathfrak{g},\mathfrak{a}],\mathfrak{b}] + [\mathfrak{a},[\mathfrak{g},\mathfrak{b}]] \subset [\mathfrak{a},\mathfrak{b}] + [\mathfrak{a},\mathfrak{b}] \subset [\mathfrak{a},\mathfrak{b}].$$

□

Definition 2.23. *Let $\mathfrak{g},\mathfrak{h}$ be Lie algebras over $\mathbb{k}$. A $\mathbb{k}$-linear transformation $\Phi : \mathfrak{g}\to\mathfrak{h}$ is called a Lie algebra homomorphism, if for all $x, y\in\mathfrak{g}$,*

$$\Phi([x,y]) = [\Phi(x),\Phi(y)].$$

Such a homomorphism is called a Lie algebra isomorphism, if it is also a $\mathbb{k}$-linear isomorphism.

Let $\mathfrak{a}$ be an ideal of the Lie algebra $\mathfrak{g}$. The quotient vector space $\mathfrak{g}/\mathfrak{a}$ inherits a bracket operation defined by

$$[x+\mathfrak{a}, y+\mathfrak{a}] = [x,y]+\mathfrak{a}$$

for $x, y\in\mathfrak{g}$. This is easily seen to make $\mathfrak{g}/\mathfrak{a}$ into a Lie algebra so that the quotient linear transformation $\mathfrak{g}\mapsto\mathfrak{g}/\mathfrak{a}$ is a homomorphism of Lie algebras. $\mathfrak{g}/\mathfrak{a}$ is referred to as the *quotient Lie algebra* of $\mathfrak{g}$ with respect to the ideal $\mathfrak{a}$. The usual isomorphism results apply:

Proposition 2.24. *Let $\Phi : \mathfrak{g}\to\mathfrak{h}$ be a homomorphism of $\mathbb{k}$-Lie algebras. Then*

(a) $\ker\Phi$ *is an ideal of* $\mathfrak{g}$.

(b) If $\mathfrak{g}_0$ and $\mathfrak{h}_0$ are ideals in $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and $\Phi(\mathfrak{g}_0)\subset\mathfrak{h}_0$, then Φ induces a Lie algebra homomorphism $\overline{\Phi} : \mathfrak{g}/\mathfrak{g}_0\to\mathfrak{h}/\mathfrak{h}_0$ given by

$$\overline{\Phi}(x+\mathfrak{g}_0) = \Phi(x)+\mathfrak{h}_0 \qquad\qquad (x\in\mathfrak{g}).$$

Furthermore, if Φ is an isomorphism mapping $\mathfrak{g}_0$ onto $\mathfrak{h}_0$, then $\overline{\Phi}$ will also be an isomorphism.

Remark 2.3. Let $\Phi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of $\mathbb{k}$-Lie algebras. By Proposition 2.24, the quotient Lie algebra $\mathfrak{g}/\ker\Phi$ can be identified with $\Phi(\mathfrak{g})$ by the quotient isomorphism $\overline{\Phi} : \mathfrak{g}/\ker\Phi \to \Phi(\mathfrak{g})$.

Every element $u \in \mathfrak{g}$ defines a $\mathbb{k}$-linear automorphism ad_u of $\mathfrak{g}$, called the *adjoint action* of u on $\mathfrak{g}$ by

$$\mathrm{ad}_u(x) = [u, x]$$

for $x \in \mathfrak{g}$. Once a basis of $\mathfrak{g}$ is chosen, $\mathrm{ad} : \mathfrak{g} \to M_n(\mathbb{k})$ where $n = \dim(\mathfrak{g})$.

Definition 2.25. *Let $\mathfrak{g}$ be a finite dimensional $\mathbb{k}$-Lie algebra. The derived algebras $\mathfrak{g}^j$ of $\mathfrak{g}$ are defined recursively*

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \quad ..., \quad \mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j].$$

Then the decreasing sequence

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \cdots$$

is called the derived series of $\mathfrak{g}$. If $\mathfrak{g}^j = 0$ for some j, we say that $\mathfrak{g}$ is solvable. The maximal solvable ideal of $\mathfrak{g}$ is called its radical.

Definition 2.26. *Let $\mathfrak{g}$ be a finite dimensional $\mathbb{k}$-Lie algebra. The ideals $\mathfrak{g}_j$ of $\mathfrak{g}$ are defined recursively*

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \quad ..., \quad \mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j].$$

Then the decreasing sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$$

is called the lower central series of $\mathfrak{g}$. If $\mathfrak{g}_j = 0$ for some j, we say that $\mathfrak{g}$ is nilpotent. The smallest j with $\mathfrak{g}_j = 0$ is called the degree of nilpotency. The maximal nilpotent ideal of $\mathfrak{g}$ is called its nilradical.

Examples.

(1) Every nilpotent Lie algebra is solvable. The Lie algebra $\mathfrak{g}$ consisting of matrices of the form $\begin{bmatrix} a_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & a_n \end{bmatrix}$ is solvable, but not nilpotent if $a_k \neq 0$ for some k.

(2) The Lie algebra $\mathfrak{g}$ consisting of matrices of the form $\begin{bmatrix} 0 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 0 \end{bmatrix}$ is nilpotent.

Remark 2.4. If $\Phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism of $\mathfrak{g}$ onto the Lie algebra $\mathfrak{h}$, then Φ sends center to center, nilradical to nilradical, the derived series to the derived series, etc.

2.3.1 Tangent Space of Matrix Groups as Lie Algebras

Definition 2.27. *Let G be a subset of $M_n(\mathbb{k})$. A differentiable curve in G is a function*

$$\gamma : (a, b) \to G$$

for which the derivative $\gamma'(t)$ exists for each $t \in (a, b)$. Here $\gamma'(t)$ is defined as an element of $M_n(\mathbb{k})$ by

$$\gamma'(t) = \lim_{s \to t} \frac{1}{s-t} \left(\gamma(t) - \gamma(s)\right),$$

provided this limit exists.

The usual product rule applies. Suppose $\alpha, \beta : (a, b) \to M_n(\mathbb{k})$ are differentiable curves. Then

$$\gamma := \alpha\beta : (a, b) \to M_n(\mathbb{k})$$

is a differentiable curve, and

$$\gamma'(t) = \frac{d}{dt}(\alpha\beta)(t) = \alpha'(t)\beta(t) + \alpha(t)\beta'(t) \qquad (t \in (a, b)).$$

Similarly, if $\alpha : (a,b) \to GL_n(\mathbb{R})$ is a differentiable curve, then $\alpha^{-1} : (a,b) \to GL_n(\Bbbk)$ is differentiable and

$$\frac{d}{dt}\alpha(t)^{-1} = -\alpha(t)^{-1}\alpha'(t)\alpha(t)^{-1} \qquad (t \in (a,b)). \tag{2.4}$$

Definition 2.28. *Let G be a matrix group. The tangent space to G at $P \in G$ is*

$$T_P G = \{\gamma'(0) \in M_n(\Bbbk) : \gamma \textit{ is a differentiable curve in } G \textit{ with } \gamma(0) = P\}.$$

Proposition 2.29. *$T_P G$ is a real vector subspace of $M_n(\Bbbk)$.*

Proof. Suppose that α, β are differentiable curves in G for which $\alpha(0) = \beta(0) = P$. Then by the product rule

$$\gamma : dom(\alpha) \cap dom(\beta) \to G; \quad \gamma(t) = \alpha(t)P^{-1}\beta(t),$$

is also a differentiable curve, and

$$\gamma'(t) = \alpha'(t)P^{-1}\beta(t) + \alpha(t)P^{-1}\beta'(t),$$

hence

$$\gamma'(0) = \alpha'(0)P^{-1}\beta(0) + \alpha(0)P^{-1}\beta'(0) = \alpha'(0) + \beta'(0),$$

which shows that T_P is closed under addition.

Similarly, if $r \in \mathbb{R}$ and α is a differentiable curve in G with $\alpha(0) = P$, then $\eta(t) = \alpha(rt)$ defines another such curve. Since $\eta'(0) = r\alpha'(0)$, we see that $T_P G$ is closed under real scalar multiplication. $\square$

We will use the notation $\mathfrak{g} = T_I G$ for this real vector subspace of $M_n(\Bbbk)$ when $P = I$. $\mathfrak{g}$ is called the Lie algebra of G, due to the following theorem:

Theorem 2.30. *Let G be a matrix group. Then $\mathfrak{g}$ is an $\mathbb{R}$-Lie subalgebra of $M_n(\Bbbk)$.*

Proof. By Proposition 2.29, it suffices to show that for two differentiable curves α and β in G which satisfy $\alpha(0) = \beta(0) = I$, we have $[\alpha'(0), \beta'(0)] \in \mathfrak{g}$.

Consider the function

$$F : dom(\alpha) \times dom(\beta) \to G; \quad F(s,t) = \alpha(s)\beta(t)\alpha(s)^{-1}.$$

This is clearly continuous and differentiable with respect to each of the variables s, t. For each $s \in dom(\alpha)$, the function $F(s, \cdot) : dom(\beta) \to G$ is a differentiable curve in G with $F(s, 0) = I$, and

$$\left.\frac{d}{dt}F(s,t)\right|_{t=0} = \alpha(s)\beta'(0)\alpha(s)^{-1},$$

so that

$$\alpha(s)\beta'(0)\alpha(s)^{-1} \in \mathfrak{g}.$$

Since $\mathfrak{g}$ is a linear subspace of $M_n(\Bbbk)$, it is closed in $M_n(\Bbbk)$, hence we also have

$$\lim_{s\to 0} \frac{1}{s}\left(\alpha(s)\beta'(0)\alpha(s)^{-1} - \beta'(0)\right) \in \mathfrak{g}.$$

This limit exists, as by the product rule and (2.4)

$$\begin{aligned}\lim_{s\to 0}\frac{1}{s}\left(\alpha(s)\beta'(0)\alpha(s)^{-1} - \beta'(0)\right) &= \left.\frac{d}{ds}\alpha(s)\beta'(0)\alpha(s)^{-1}\right|_{s=0} \\ &= \alpha'(0)\beta'(0)\alpha(0)^{-1} - \alpha(0)\beta'(0)\alpha(0)^{-1}\alpha'(0)\alpha(0)^{-1} \\ &= \alpha'(0)\beta'(0) - \beta'(0)\alpha'(0) = [\alpha'(0), \beta'(0)]\end{aligned}$$

which shows that $[\alpha'(0), \beta'(0)] \in \mathfrak{g}$. □

Definition 2.31. *Let G and H be matrix groups and $\varphi : G \to H$ be a continuous map. Then φ is said to be a differentiable map if it satisfies:*

(a) for every differentiable curve $\gamma : (a,b) \to G$, the curve $\varphi \circ \gamma : (a,b) \to H$ is differentiable

(b) if two differentiable curves $\alpha, \beta : (a, b) \to G$ *satisfy*

$$\alpha(0) = \beta(0), \qquad \alpha'(0) = \beta'(0),$$

then

$$(\varphi \circ \alpha)'(0) = (\varphi \circ \beta)'(0).$$

A continuous homomorphism of matrix groups that is also a differentiable map is called a *Lie homomorphism.*

Theorem 2.32. *Let* $\varphi : G \to H$ *be a matrix group homomorphism. Then* φ *is differentiable, that is, is a Lie homomorphism.*

Let $\varphi : G \to H$ be a matrix group homomorphism. If $\gamma : (a, b) \to G$ is a differentiable curve through the identity I of G, then by Theorem 2.32, $\varphi \circ \gamma : (a, b) \to H$ is a differentiable curve through the identity of H. Define a map $\mathrm{d}\varphi : \mathfrak{g} \to \mathfrak{h}$ by

$$\mathrm{d}\varphi(\gamma'(0)) = (\varphi \circ \gamma)'(0).$$

Theorem 2.33. *Let* G *and* H *be matrix groups and* $\varphi : G \to H$ *be a differentiable homomorphism. Then the derivative* $d\varphi : \mathfrak{g} \to \mathfrak{h}$ *is a Lie algebra homomorphism.*

2.3.2 Exponentials and One-Parameter Subgroups

The matrix exponential of $A \in M_n(\Bbbk)$ is defined by the matrix-valued series

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

which converges for all $A \in M_n(\Bbbk) \in M_n(\Bbbk)$. The logarithm of A is defined by the the matrix-valued series

$$\log(A) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (A - I)^n$$

which converges for $\|A - I\| < 1$.

Proposition 2.34. *Let* $A, B \in M_n(\Bbbk)$.

(a) The maps $A \mapsto e^A$ *and* $A \mapsto \log(A)$ *are continuous.*

(b) If A, B *commute, then* $e^{(A+B)} = e^A e^B$.

(c) $e^A \in GL_n(\Bbbk)$ *and* $(e^A)^{-1} = e^{-A}$.

(d) $\det e^A = e^{tr(A)}$.

(e) If $\|A - I\| < 1$*, then* $e^{\log(A)} = A$.

(f) If $\|e^B - I\| < 1$*, then* $\log e^B = B$.

Definition 2.35. *Let* G *be a matrix group. A one-parameter group in* G *is a continuous function* $\gamma : \mathbb{R} \to G$ *which is differentiable and also satisfies*

$$\gamma(s+t) = \gamma(s)\gamma(t)$$

for all $s, t \in \mathbb{R}$.

Proposition 2.36. *Let* G *be a matrix group with Lie algebra* $\mathfrak{g}$*. Then*

(a) $e^A \in G$ *for all* $A \in \mathfrak{g}$.

(b) The exponential map $\exp : A \mapsto e^A$ *maps an open neighborhood of 0 in* $\mathfrak{g}$ *homeomorphically onto an open neighborhood of* I *in* G.

Remark 2.5. Let $A \in M_n(\Bbbk)$ be such that $e^{tA} \in G$ for all $t \in \mathbb{R}$. Then $\gamma(t) = e^{tA}$ is a one-parameter subgroup in G and $A = \gamma'(0) \in \mathfrak{g}$. In fact,

$$\frac{d}{dt}e^{tA} = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{1}{n!}(tA)^n = \sum_{n=0}^{\infty}\frac{d}{dt}\frac{1}{n!}(tA)^n = \sum_{n=1}^{\infty}\frac{t^{n-1}}{(n-1)!}A^n = A\sum_{n=0}^{\infty}\frac{1}{n!}(tA)^n = Ae^{tA}.$$

Here we have used the fact that limits and derivatives can be taken entrywise. Conversely, if $A \in \mathfrak{g}$, then $tA \in \mathfrak{g}$ for all $t \in \mathbb{R}$, and by Proposition 2.36, $e^{tA} \in G$ for all $t \in \mathbb{R}$. We thus have a characterization of $\mathfrak{g}$:

$$\mathfrak{g} = \left\{A \in M_n(\Bbbk) \,:\, e^{tA} \in G \text{ for all } t \in \mathbb{R}\right\}.$$

Remark 2.6. Let $\gamma : (a, b) \to G$ be a differentiable curve in G through the identity. Since γ is continuous, then $\gamma(a, b)$ is a connected set lying in the connectedness component of the identity of G. Thus the Lie algebra $\mathfrak{g}$ of G is determined by the connectedness component of the identity of G.

As a consequence of Theorem 2.33, isomorphic matrix groups have isomorphic Lie algebra. For the converse one has:

Proposition 2.37. *Let G and H be simply connected matrix groups with Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Then G and H are isomorphic if and only if $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic.*

2.4 The Fourier Transform

Throughout, $\mathbb{R}^n$ will denote Euclidean space with elements written as column vectors. We usually use symbols $\vec{x}$, $\vec{y}$, etc. to denote these column vectors. $\widehat{\mathbb{R}^n}$ will denote Euclidean space with elements written as row vectors, which we denote by Greek symbols $\vec{\gamma}$, $\vec{\omega}$ etc.

The inner product in a Hilbert space will be denoted by $\langle \cdot, \cdot \rangle$. In case of Euclidean space if one of the vectors is written as row vector, we can also write the inner product as, $\vec{\gamma}\vec{x}$. We will often deal with column vectors as arguments of functions. In this case and whenever convenient we will write $\vec{x} = (x_1, x_2, ..., x_n)$ even though $\vec{x}$ is a column vector.

The idea of the Fourier transform on $L^2\left([0, 1]^n\right)$ is simply a restatement of the concept of Hilbert space basis. Since $L^2\left([0, 1]^n\right)$ is naturally identified with $L^2(\Pi^n)$, where $\Pi = \{z \in \mathbb{C} : |z| = 1\}$ denotes the complex unit circle, one may work in the latter space as well.

Theorem 2.38. *The collection of functions* $\{e_{\vec{m}}(\vec{x}) = e^{2\pi i\langle \vec{m},\vec{x}\rangle}\}_{\vec{m}\in\mathbb{Z}^n}$ *is an orthonormal basis of* $L^2([0,1]^n)$.

Proof. Orthogonality follows from the fact that $\int_0^1 e^{2\pi imt}dt$ equals 1 if $m = 0$ and equals 0 otherwise. Next, since $e_{\vec{m}}e_{\vec{l}} = e_{\vec{m}+\vec{l}}$, the set of finite linear combinations of the $e_{\vec{m}}$'s is an algebra. It clearly separates points on Π^n; also, $e_0 = 1$ and $\overline{e}_{\vec{m}} = e_{-\vec{m}}$. Since Π^n is compact, the Stone-Weierstrass theorem implies that this algebra is dense in $C(\Pi^n)$ in the uniform norm and hence in the $L^2(\Pi^n)$ norm. But $C(\Pi^n)$ is itself dense in $L^2(\Pi^n)$, hence this algebra is dense in $L^2(\Pi^n)$. It follows that $\{e_{\vec{m}}\}_{\vec{m}\in\mathbb{Z}^n}$ is a Hilbert space basis of $L^2(\Pi^n)$. $\square$

If $f \in L^2([0,1]^n)$, we define its *Fourier transform* $\widehat{f}$, a function on $\mathbb{Z}^n$, by

$$\widehat{f}(\vec{m}) = \langle f, e_{\vec{m}}\rangle = \int_{[0,1]^n} f(\vec{x})e^{-2\pi i\langle \vec{m},\vec{x}\rangle}d\vec{x} \tag{2.5}$$

for $\vec{m} \in \mathbb{Z}^n$, and we call the series

$$\sum_{\vec{m}\in\mathbb{Z}^n} \widehat{f}(\vec{m})e_{\vec{m}},$$

the *Fourier series* of f.

The term *Fourier transform* is also used to denote the map $f \mapsto \widehat{f}$. Theorem 2.38 then implies that the Fourier transform maps $L^2([0,1]^n)$ onto $l^2(\mathbb{Z}^n)$, that $\|\widehat{f}\|_2 = \|f\|_2$ (Parseval's identity) and that the Fourier series of f converges to f in the $L^2([0,1]^n)$ norm.

The integral (2.7) makes sense if f is merely in $L^1([0,1]^n)$, and as $|\widehat{f}(\vec{m})| \leqslant \|f\|_1$, the Fourier transform extends to a norm-decreasing map from $L^1([0,1]^n)$ to $l^\infty(\mathbb{Z}^n)$.

The situation on $\mathbb{R}^n$ is more complicated. The formal analogue of Theorem 2.38 should be

$$f(\vec{x}) = \int_{\widehat{\mathbb{R}^n}} \widehat{f}(\vec{\gamma})e^{2\pi i\vec{\gamma}\vec{x}}d\vec{\gamma}, \qquad \text{where } \widehat{f}(\vec{\gamma}) = \int_{\mathbb{R}^n} f(\vec{x})e^{-2\pi i\vec{\gamma}\vec{x}}d\vec{x}.$$

These relations turn out to be valid when suitably interpreted, but some care is needed. In the first place, the integral defining $\widehat{f}(\vec{\gamma})$ is only assured to exist if $f \in L^1(\mathbb{R}^n)$. We therefore begin by defining the *Fourier transform* of $f \in L^1(\mathbb{R}^n)$ by

$$\mathcal{F}f(\vec{\gamma}) = \widehat{f}(\vec{\gamma}) = \int_{\mathbb{R}^n} f(\vec{x})e^{-2\pi i\vec{\gamma}\vec{x}}d\vec{x}$$

for $\vec{\gamma} \in \widehat{\mathbb{R}^n}$. (We use the notation $\mathcal{F}$ for the Fourier transform only where it is needed for clarity. Also, the argument of $\widehat{f}$ is usually written as a row vector.) Clearly the operator $\mathcal{F}$ is linear and $\|\widehat{f}\|_\infty \leqslant \|f\|_1$, and from the theorem below,

$$\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\widehat{\mathbb{R}^n}).$$

We summarize the elementary properties of $\mathcal{F}$ in a theorem.

Theorem 2.39. *Suppose $f, g \in L^1(\mathbb{R}^n)$.*

(a) If $\vec{x}^\alpha f \in L^1(\mathbb{R}^n)$ for a multi-index $|\alpha| \leqslant k$, then $\widehat{f} \in C^k(\widehat{\mathbb{R}^n})$ and $\partial^\alpha \widehat{f} = (\widehat{-2\pi i\vec{x})^\alpha f}$.

(b) If $f \in C^k(\mathbb{R}^n)$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$ for $|\alpha| \leqslant k$, and $\partial^\alpha f \in C_0(\mathbb{R}^n)$ for $|\alpha| \leqslant k-1$, then $(\widehat{\partial^\alpha f})(\vec{\gamma}) = (2\pi i\vec{\gamma})^\alpha \hat{f}(\vec{\gamma})$.

(c) (Riemann-Lebesgue Lemma): $\widehat{f} \in C_0(\widehat{\mathbb{R}^n})$.

Parts (a) and (b) of Theorem 2.39 point to a fundamental property of the Fourier transform: Smoothness properties of f are reflected in the rate of decay of $\widehat{f}$ at infinity, and vice versa.

We are now ready to invert the Fourier transform. If $f \in L^1(\mathbb{R}^n)$, we define the *inverse Fourier transform* by

$$\check{f}(\vec{x}) = \widehat{f}(-\vec{x}) = \int_{\widehat{\mathbb{R}^n}} f(\vec{\gamma})e^{2\pi i\vec{\gamma}\vec{x}}d\vec{\gamma}$$

for $\vec{x} \in \mathbb{R}^n$. Note that $\widehat{f}$ need not be integrable in general, but if it is, then by the next theorem we can reconstruct f from $\widehat{f}$ by

$$f(\vec{x}) = (\widehat{f})^\vee(\vec{x}) = \int_{\widehat{\mathbb{R}^n}} \widehat{f}(\vec{\gamma})e^{2\pi i \vec{\gamma}\vec{x}} d\vec{\gamma} \tag{2.6}$$

for a.e. $\vec{x} \in \mathbb{R}^n$.

Theorem 2.40. (The Fourier Inversion Theorem) *If $f \in L^1(\mathbb{R}^n)$ and $\widehat{f} \in L^1(\widehat{\mathbb{R}^n})$, then f agrees almost everywhere with a continuous function f_0, and $(\widehat{f})^\vee = (\check{f})^\wedge = f_0$.*

Corollary 2.41. *If $f \in L^1(\mathbb{R}^n)$ and $\widehat{f} = 0$, then $f = 0$ a.e. That is, the Fourier transform $\mathcal{F}$ is a one-to-one mapping.*

The following is the analogue of theorem 2.38.

Theorem 2.42. (The Plancherel Theorem) *If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\widehat{f} \in L^2(\widehat{\mathbb{R}^n})$, $\|\widehat{f}\|_2 = \|f\|_2$, and the restriction of $\mathcal{F}$ to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ extends uniquely to a unitary isomorphism of $L^2(\mathbb{R}^n)$ onto $L^2(\widehat{\mathbb{R}^n})$, which we also denote by $\mathcal{F}$.*

Since $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\widehat{\mathbb{R}^n})$ is unitary, then its inverse certainly exists. However, only when $\widehat{f} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ can it be computed by formula (2.6). We use the symbol $\check{f}$ to denote the inverse Fourier transform of a function $f \in L^2(\widehat{\mathbb{R}^n})$ as well.

Next we define three types of unitary operators on $L^2(\mathbb{R}^n)$. These operators are fundamental in the wavelet transform, and will be used throughout. Given $h \in GL_n(\mathbb{R})$, $c \in M_n(\mathbb{R})$ with $c = c^T$, and $\vec{x} \in \mathbb{R}^n$, the dilation operator D_h, the translation operator $T_{\vec{x}}$, the modulation operator $E_{\vec{x}}$ and the chirp operator N_c are defined by

$$(D_h f)(\vec{y}) = |\det h|^{-1/2} f(h^{-1}\vec{y}), \qquad (T_{\vec{x}} f)(\vec{y}) = f(\vec{y} - \vec{x}),$$

$$(E_{\vec{x}} f)(\vec{y}) = e^{2\pi i \langle \vec{y}, \vec{x} \rangle} f(\vec{y}), \qquad (N_c f)(\vec{y}) = e^{-i\pi \langle c\vec{y}, \vec{y} \rangle} f(\vec{y})$$

for $f \in L^2(\mathbb{R}^n)$ and $\vec{y} \in \mathbb{R}^n$. The corresponding dilation and modulation operators on $\text{Ł}^2(\widehat{\mathbb{R}^n})$ are

$$(D_h f)(\vec{\gamma}) = |\det h|^{1/2} f(\vec{\gamma}h) \qquad \text{and} \qquad (E_{\vec{x}} f)(\vec{\gamma}) = e^{2\pi i \vec{\gamma}\vec{x}} f(\vec{\gamma})$$

for $f \in L^2(\widehat{\mathbb{R}^n})$ and $\vec{\gamma} \in \widehat{\mathbb{R}^n}$, respectively. These are easily seen to be unitary operator on $L^2(\mathbb{R}^n)$. For example,

$$\|D_h f\|_2^2 = \int_{\mathbb{R}^n} \left|\det h^{-1}\right| \left|f(h^{-1}\vec{y})\right|^2 d\vec{y} = \int_{\mathbb{R}^n} |f(\vec{y})|^2 \, d\vec{y} = \|f\|_2^2$$

for all $f \in L^2(\mathbb{R}^n)$, so that D_h is an isometry. Here we have used the fact that

$$|\det a| \int_{\mathbb{R}^n} f(a\vec{x}) d\vec{x} = \int_{\mathbb{R}^n} f(\vec{x}) d\vec{x}$$

for all $a \in GL_n(\mathbb{R})$ and $f \in L^1(\mathbb{R}^n)$. In addition, $D_h D_k = D_{hk}$ for all $h, k \in GL_n(\mathbb{R})$. Using techniques from group representations (see Folland (1999), for example), one shows that the mappings $h \mapsto D_h$, $\vec{x} \mapsto T_{\vec{x}}$, $\vec{x} \mapsto E_{\vec{x}}$ and $c \mapsto N_c$ are strongly continuous homomorphisms of the respective groups into the group of unitary operators on $L^2(\mathbb{R}^n)$ (respectively $L^2(\widehat{\mathbb{R}^n})$), that is, they are group representations.

Proposition 2.43. *For $h \in GL_n(\mathbb{R})$ and $\vec{x} \in \mathbb{R}^n$,*

(a) $\mathcal{F} D_h = D_h \mathcal{F}$

(b) $\mathcal{F} T_{\vec{x}} = E_{-\vec{x}} \mathcal{F}$.

Proof. For $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\begin{aligned}(\mathcal{F} D_h f)(\vec{\gamma}) &= \int_{\mathbb{R}^n} (D_h f)(\vec{y}) e^{-2\pi i \vec{\gamma}\vec{y}} d\vec{y} = \int_{\mathbb{R}^n} |\det h|^{-1/2} f(h^{-1}\vec{y}) e^{-2\pi i \vec{\gamma}\vec{y}} d\vec{y} \\ &= \int_{\mathbb{R}^n} |\det h|^{1/2} f(\vec{y}) e^{-2\pi i \vec{\gamma} h \vec{y}} d\vec{y} = |\det h|^{1/2} (\mathcal{F} f)(\vec{\gamma} h) = (D_h \mathcal{F} f)(\vec{\gamma})\end{aligned}$$

and also

$$(\mathcal{F}T_{\vec{x}}f)(\vec{\gamma}) = \int_{\mathbb{R}^n} (T_{\vec{x}}f)(\vec{y})e^{-2\pi i\vec{\gamma}\vec{y}}d\vec{y} = \int_{\mathbb{R}^n} f(\vec{y}-\vec{x})e^{-2\pi i\vec{\gamma}\vec{y}}d\vec{y} = \int_{\mathbb{R}^n} f(\vec{y})e^{-2\pi i\vec{\gamma}(\vec{y}+\vec{x})}d\vec{y}$$
$$= e^{-2\pi i\vec{\gamma}\vec{x}}\int_{\mathbb{R}^n} f(\vec{y})e^{-2\pi i\vec{\gamma}\vec{y}}d\vec{y} = e^{-2\pi i\vec{\gamma}\vec{x}}(\mathcal{F}f)(\vec{\gamma}) = (E_{-\vec{x}}\mathcal{F}f)(\vec{\gamma}).$$

The assertion follows from density of $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ and continuity of all operators involved. $\square$

2.5 The Affine Group

As already shown in the section on group actions, the group $D = GL_n(\mathbb{R})$ acts naturally on $N = \mathbb{R}^n$ by matrix multiplication,

$$h \cdot \vec{x} = \alpha_h(\vec{x}) = h\vec{x}$$

for $h \in GL_n(\mathbb{R})$ and $\vec{x} \in \mathbb{R}^n$. The corresponding semi-direct product is called *the n-dimensional affine group*,

$$\mathrm{Aff}_n(\mathbb{R}) := GL_n(\mathbb{R}) \rtimes \mathbb{R}^n,$$

with the group law

$$(h,\vec{x})(k,\vec{y}) = (hk, \vec{x}+h\vec{y})$$

for $(h,\vec{x}),(k,\vec{y}) \in \mathrm{Aff}_n(\mathbb{R})$.

Elements of $\mathrm{Aff}_n(\mathbb{R})$ are best represented in matrix form. It is easy to verify that $(h,\vec{x}) \mapsto \begin{bmatrix} h & \vec{x} \\ 0 & 1 \end{bmatrix}$ is an isomorphism and homeomorphism of $\mathrm{Aff}_n(\mathbb{R})$ onto a closed subgroup of $GL_{n+1}(\mathbb{R})$; one therefore often identifies $\mathrm{Aff}_n(\mathbb{R})$ with the group considering these matrices,

$$\mathrm{Aff}_n(\mathbb{R}) = \left\{ (h,\vec{x}) := \begin{bmatrix} h & \vec{x} \\ 0 & 1 \end{bmatrix} \; : \; h \in GL_n(\mathbb{R}),\; \vec{x} \in \mathbb{R}^n \right\}.$$

Now given $\vec{z} \in \mathbb{R}^n$, we have

$$\begin{bmatrix} h & \vec{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{z} \\ 1 \end{bmatrix} = \begin{bmatrix} h\vec{z} + \vec{x} \\ 1 \end{bmatrix}$$

or in short $(h, \vec{x}) \cdot \vec{z} = h\vec{z} + \vec{x}$. Recall that a map $\vec{z} \mapsto h\vec{z} + \vec{x}$ is called an *affine map*, therefore the name affine group. If $(h, \vec{x}), (k, \vec{y}) \in \mathrm{Aff}_n(\mathbb{R})$, then

$$\begin{aligned} ((h, \vec{x})(k, \vec{y})) \cdot \vec{z} &= (hk, \vec{x} + h\vec{y}) \cdot \vec{z} = hk\vec{z} + (h\vec{y} + \vec{x}) = h(k\vec{z} + \vec{y}) + \vec{x} \\ &= (h, \vec{x}) \cdot (k\vec{z} + \vec{y}) = (h, \vec{x}) \cdot ((k, \vec{y}) \cdot \vec{z}) \,. \end{aligned}$$

That is $(h, \vec{x}) \cdot \vec{z}$ is a (clearly continuous) action of $\mathrm{Aff}_n(\mathbb{R})$ on $\mathbb{R}^n$, called the *affine action.*

Since the affine action involves dilations and translations of vectors, it induces a representation π of $\mathrm{Aff}_n(\mathbb{R})$ on $\mathcal{H} = L^2(\mathbb{R}^n)$. In fact, given $\psi \in L^2(\mathbb{R}^n)$ and $(h, \vec{x}) \in \mathrm{Aff}_n(\mathbb{R})$, we have

$$\begin{aligned} |\det h|^{-1/2} \psi((h, \vec{x})^{-1} \cdot \vec{z}) &= |\det h|^{-1/2} \, \psi \left((h^{-1}, -h^{-1}\vec{x}) \cdot \vec{z} \right) \\ &= |\det h|^{-1/2} \, \psi(h^{-1}\vec{z} - h^{-1}\vec{x}) = |\det h|^{-1/2} \, \psi(h^{-1}(\vec{z} - \vec{x})) = (T_{\vec{x}} D_h \psi)(\vec{z}). \end{aligned} \tag{2.7}$$

Now for each $(h, \vec{x}) \in \mathrm{Aff}_n(\mathbb{R})$, $\pi(h, \vec{x})$ defined by

$$\pi(h, \vec{x}) = T_{\vec{x}} D_h$$

is a unitary operator on $L^2(\mathbb{R}^n)$. Furthermore, π is a homomorphism as

$$\begin{aligned} (\pi((h, \vec{x})(k, \vec{y}))\psi)(\vec{z}) &= |\det hk|^{-1/2} \, \psi(((h, \vec{x})(k, \vec{y}))^{-1} \cdot \vec{z}) \\ &= |\det h|^{-1/2} \, |\det k|^{-1/2} \, \psi(((k, \vec{y})^{-1}(h, \vec{x})^{-1}) \cdot \vec{z}) \\ &= |\det h|^{-1/2} \, |\det k|^{-1/2} \, \psi((k, \vec{y})^{-1} \cdot ((h, \vec{x})^{-1} \cdot \vec{z})) \\ &= |\det h|^{-1/2} \, (\pi(k, \vec{y})\psi)((h, \vec{x})^{-1} \cdot \vec{z}) = (\pi(h, \vec{x})\pi(k, \vec{y})\psi)(\vec{z}). \end{aligned}$$

Since the mappings $h \mapsto D_h$ and $\vec{x} \mapsto T_{\vec{x}}$ are strongly continuous maps into the set of unitary operators on $L^2(\mathbb{R}^n)$, so is the composition $\pi : (h, \vec{x}) \mapsto T_{\vec{x}} D_h$. Hence π is a representation of $\mathrm{Aff}_n(\mathbb{R})$ on $L^2(\mathbb{R}^n)$. As shown in (2.7), given $\psi \in L^2(\mathbb{R}^n)$ and $(h, \vec{x}) \in \mathrm{Aff}_n(\mathbb{R})$, then

$$(\pi(h, \vec{x})\psi)(\vec{y}) = (T_{\vec{x}} D_h \psi)(\vec{y}) = |\det h|^{-1/2} \psi(h^{-1}(\vec{y} - \vec{x}))$$

for all $\vec{y} \in \mathbb{R}^n$.

Recall that the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\widehat{\mathbb{R}^n})$ is an isomorphism of Hilbert spaces. Thus it induces a representation δ of $\mathrm{Aff}_n(\mathbb{R})$ on the phase space $L^2(\widehat{\mathbb{R}^n})$ defined by

$$\delta = \mathcal{F} \circ \pi \circ \mathcal{F}^{-1}.$$

Computing, we obtain by Proposition 2.43 for $(h, \vec{x}) \in G$,

$$\begin{aligned}\delta(h, \vec{x}) &= \mathcal{F} \circ \pi(h, \vec{x}) \circ \mathcal{F}^{-1} = \mathcal{F} T_{\vec{x}} D_h \mathcal{F}^{-1} = E_{-\vec{x}} \mathcal{F} D_h \mathcal{F}^{-1} \\ &= E_{-\vec{x}} D_h \mathcal{F} \mathcal{F}^{-1} = E_{-\vec{x}} D_h\end{aligned}$$

that is

$$(\delta(h, \vec{x})\widehat{\psi})(\vec{\gamma}) = |\det h|^{1/2} e^{-2\pi i \vec{\gamma}\vec{x}} \widehat{\psi}(\vec{\gamma} h)$$

for $\psi \in L^2(\mathbb{R}^n)$ and $\vec{\gamma} \in \widehat{\mathbb{R}^n}$.

We will also consider subgroups of $\mathrm{Aff}_n(\mathbb{R})$ arising from subgroups of $GL_n(\mathbb{R})$. Given a closed subgroup H of $GL_n(\mathbb{R})$, we consider the corresponding subgroup G of $\mathrm{Aff}_n(\mathbb{R})$,

$$G = \{(h, \vec{x}) \in \mathrm{Aff}_n(\mathbb{R}) \ : \ h \in H, \vec{x} \in \mathbb{R}^n\}.$$

We identify H with the subgroup $\{(h, \vec{x}) \in G \ : \ h \in H, \vec{x} = 0\}$ and refer to it as the *dilation subgroup* of G, and $\mathbb{R}^n$ with the subgroup $\{(h, \vec{x}) \in G \ : \ h = e, \, \vec{x} \in \mathbb{R}^n\}$, and call it the *translation subgroup* of G. Thus $G = H \rtimes \mathbb{R}^n$ is a closed subgroup of $\mathrm{Aff}_n(\mathbb{R})$. By (2.2),

$$|\det h| \int_{\mathbb{R}^n} f(\alpha_h(\vec{x}))d\vec{x} = |\det h| \int_{\mathbb{R}^n} f(h\vec{x})d\vec{x} = \int_{\mathbb{R}^n} f(\vec{x})d\vec{x}$$

for all $f \in C_c(\mathbb{R}^n)$, it follows from (2.2) that

$$d\nu(h, \vec{x}) = \frac{d\mu(h)\, d\lambda(\vec{x})}{|\det h|} \tag{2.8}$$

is a left Haar measure for G, where μ is a left Haar measure on H and λ the Lebesgue measure on $\mathbb{R}^n$.

In addition, the restriction of the representation π of $\mathrm{Aff}_n(\mathbb{R})$ to G is called *the affine representation* of G, or *the wavelet representation.*

2.6 Frame Theory

Definition 2.44. *A sequence $\{e_j : j \in J\}$ in a Hilbert space $\mathcal{H}$ is called a frame if there exist positive constants $A, B > 0$ such that for all $f \in \mathcal{H}$*

$$A \|f\|^2 \leq \sum_{j \in J} |\langle f, e_j \rangle|^2 \leq B \|f\|^2 . \tag{2.9}$$

The constants A, B are called frame bounds. If $A = B$, then $\{e_j : j \in J\}$ is called a tight frame. If $A = B = 1$, then it is called a Parseval frame, as (2.9) reduces to Parseval's identity.

Thus a frame is a natural generalization of the concept of Hilbert space basis. One now wants to reconstruct f from its frame coefficients $\langle f, e_j \rangle$. Note that the vectors e_j need not be linearly independent. However, by (2.9) the set $\{e_j : j \in J\}$ is bounded.

Definition 2.45. *For any subset $\{e_j : j \in J\} \subset \mathcal{H}$, the coefficient operator or analysis operator C is given by*

$$Cf = \{\langle f, e_j \rangle : j \in J\} .$$

The synthesis operator or reconstruction operator D is defined for a finite sequence $c = (c_j)_{j\in J}$ by

$$Dc = \sum_{j\in J} c_j e_j \in \mathcal{H},$$

and the frame operator S is defined by

$$Sf = \sum_{j\in J} \langle f, e_j\rangle\, e_j.$$

The following proposition guarantees that the operators D and S are well defined for a sequence $c = (c_j)_{j\in J}$ with infinitely many nonzero terms.

Proposition 2.46. *Suppose that $\{e_j : j \in J\}$ is a frame for $\mathcal{H}$.*

(a) C is a bounded operator from $\mathcal{H}$ into $\ell^2(J)$ of norm less than or equal to $B^{1/2}$ with closed range.

(b) The operators C and D are adjoint to each other; that is, $D = C^*$. Consequently, D extends to a bounded linear operator from $\ell^2(J)$ into $\mathcal{H}$ and satisfies $\|D\| \leqslant B^{1/2}$.

(c) The frame operator $S = C^*C = DD^*$ maps $\mathcal{H}$ onto $\mathcal{H}$ and is a positive invertible operator satisfying $AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}$ and $B^{-1}I_{\mathcal{H}} \leq S^{-1} \leq A^{-1}I_{\mathcal{H}}$. In particular, $\{e_j : j \in J\}$ is a tight frame if and only if $S = AI_{\mathcal{H}}$.

(d) The optimal frame bounds are $B_{opt} = \|S\|$ and $A_{opt} = \|S^{-1}\|^{-1}$, where $\|\cdot\|$ is the usual operator norm of S.

Proof. (a) The statement follows directly from the frame inequalities (2.9). (b) Let $c = (c_j)_{j\in J}$ be a finite sequence. Since C is a bounded linear operator of norm $\|C\| \leqslant B^{1/2}$, then its adjoint C^* is also linear and bounded, and for every finite sequence $\{c_j\}_{j\in J}$,

$$\langle C^*c, f\rangle = \langle c, Cf\rangle = \sum_{j\in J} c_j \overline{\langle f, e_j\rangle} = \left\langle \sum_{j\in J} c_j e_j, f \right\rangle = \langle Dc, f\rangle\,.$$

It follows that C^* is the extension of D to a bounded linear operator $C^* : \ell^2(J) \to \mathcal{H}$ with the same operator norm, $\|D\| = \|C^*\| = \|C\|$. Thus (b) follows. We now formally write

$$Dc = \sum_{j \in J} c_j e_j$$

for $c \in \ell^2(J)$. (c) Obviously the frame operator can be expressed as $S = C^*C = DD^*$ and consequently S is defined, self-adjoint and positive. Since

$$\langle Sf, f \rangle = \langle Cf, Cf \rangle = \|Cf\|^2 = \sum_{j \in J} |\langle f, e_j \rangle|^2 \tag{2.10}$$

the operator inequality $AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}$ is just (2.9) rewritten. S is invertible on $\mathcal{H}$ because $A > 0$. Inequalities are preserved under multiplication with positive commuting operator, therefore $AS^{-1} \leq SS^{-1} \leq BS^{-1}$, as desired. (d) follows from the frame inequalities (2.9) and the fact that the operator norm of a positive operator is determined by $\|S\| = \sup\{\langle Sf, f \rangle : \|f\| \leq 1\}$. The argument for A_{opt} is similar. $\square$

Statement (b) shows that $\sum_{j \in J} c_j e_j$ and $\sum_{j \in J} \langle f, e_j \rangle e_j$ are well defined for an arbitrary ℓ^2-sequence by means of the adjoint operator, even though the frame vectors e_j are not orthogonal in general. Convergence of this series is to be understood as follows.

Corollary 2.47. *Let $\{e_j : j \in J\}$ be a frame for $\mathcal{H}$. If $f = \sum_{j \in J} c_j e_j$ for some $c \in \ell^2(J)$, then for every $\varepsilon > 0$ there exists a finite subset $F_0 = F_0(\varepsilon) \subset J$ such that*

$$\left\| f - \sum_{j \in F} c_j e_j \right\| < \varepsilon$$

for all finite subsets $F \supset F_0$. We say that the series $\sum_{j \in J} c_j e_j$ converges unconditionally to $f \in \mathcal{H}$.

Proof. Choose $F_0 \subset J$ such that $\sum_{n \notin F_0} |c_j|^2 < \varepsilon / B^{1/2}$. Given a finite subset $F \supset F_0$ of J, let $c_F = c \cdot \mathbf{1}_F \in \ell^2(J)$ be the finite sequence with terms $c_{F,j} = c_j$ if $j \in F$ and $c_{F,j} = 0$ if $j \notin F$. Then $\sum_{j \in F} c_j e_j = Dc_F$ and by Proposition 2.46 (b) we obtain

$$\left\| f - \sum_{j \in F} c_j e_j \right\| = \|Dc - Dc_F\| = \|D(c - c_F)\| \leq B^{1/2} \|c - c_F\|_2 < \varepsilon.$$

$\square$

As another consequence of Proposition 2.46 we obtain a first reconstruction formula for f from the frame coefficients $\langle f, e_j \rangle$.

Corollary 2.48. *If $\{e_j : j \in J\}$ is a frame with frame bounds $A, B > 0$, then $\{S^{-1}e_j : j \in J\}$ is a frame with frame bounds $B^{-1}, A^{-1} > 0$, called the dual frame. Every $f \in \mathcal{H}$ has non-orthogonal expansions*

$$f = \sum_{j \in J} \left\langle f, S^{-1} e_j \right\rangle e_j$$

and

$$f = \sum_{j \in J} \langle f, e_j \rangle S^{-1} e_j$$

where both sums converge unconditionally in $\mathcal{H}$.

Proof. First observe that by (2.10),

$$\sum_{j \in J} \left| \left\langle f, S^{-1} e_j \right\rangle \right|^2 = \sum_{j \in J} \left| \left\langle S^{-1} f, e_j \right\rangle \right|^2 = \left\langle S(S^{-1} f), S^{-1} f \right\rangle = \left\langle S^{-1} f, f \right\rangle.$$

Therefore Proposition 2.46 (c) implies that

$$B^{-1} \|f\|^2 \leq \left\langle S^{-1} f, f \right\rangle = \sum_{j \in J} \left| \left\langle f, S^{-1} e_j \right\rangle \right|^2 \leq A^{-1} \|f\|^2$$

which shows that the collection $\{S^{-1} e_j : j \in J\}$ is a frame with frame bounds B^{-1} and A^{-1}.

Using the factorizations $I_{\mathcal{H}} = S^{-1}S = SS^{-1}$, we obtain the series expansions

$$f = S(S^{-1}f) = \sum_{j\in J} \left\langle S^{-1}f, e_j \right\rangle e_j = \sum_{j\in J} \left\langle f, S^{-1}e_j \right\rangle e_j$$

and

$$f = S^{-1}Sf = \sum_{j\in J} \langle f, e_j \rangle S^{-1}e_j.$$

Because both $\{\langle f, e_j\rangle\}_{j\in J}$ and $\{\langle f, S^{-1}e_j\rangle\}_{j\in J}$ are in $\ell^2(J)$, both of the above series converge unconditionally by Corollary 2.47. $\square$

The above corollary shows that f can be reconstructed from its sequence of frame coefficients by means of the dual frame. In some particular cases, reconstruction is easier:

Proposition 2.49.

(a) If $\{e_j : j \in J\}$ is a Parseval frame of $\mathcal{H}$, then

$$f = \sum_{j\in J} \langle f, e_j \rangle e_j.$$

(b) If $\{e_j : j \in J\}$ is a Parseval frame of $\mathcal{H}$ and if $\|e_j\| = 1$ for all $j \in J$, then $\{e_j\}$ is an orthonormal basis.

Proof. (a) Since $A = B = 1$, then $S = I_{\mathcal{H}}$. (b) By (2.9) we have

$$1 = \|e_m\|^2 = \sum_{j\in J} |\langle e_m, e_j\rangle|^2 = 1 + \sum_{j\neq m} |\langle e_m, e_j\rangle|^2,$$

and consequently $\langle e_m, e_j\rangle = \delta_{j,m}$. $\square$

2.7 The Wigner Distribution

In time-frequency analysis, one studies a function and its Fourier transform simultaneously, as one wishes to understand its behavior with respect to time and frequency. The Wigner distribution was introduced for this purpose.

Definition 2.50. *The Wigner distribution $\mathcal{W}_f$ of a function $f \in L^2(\mathbb{R}^n)$ is defined on $\mathbb{R}^{2n}$ to be*

$$\mathcal{W}_f(\vec{x}, \vec{w}) = \int_{\mathbb{R}^n} f\left(\vec{x} + \frac{\vec{y}}{2}\right) \overline{f\left(\vec{x} - \frac{\vec{y}}{2}\right)} e^{-2i\pi\langle \vec{w}, \vec{y}\rangle} d\vec{y}.$$

By polarizing this quadratic expression, one obtains the cross-Wigner distribution of two functions $f, g \in L^2(\mathbb{R}^n)$:

$$\mathcal{W}_{f,g}(\vec{x}, \vec{w}) = \int_{\mathbb{R}^n} f\left(\vec{x} + \frac{\vec{y}}{2}\right) \overline{g\left(\vec{x} - \frac{\vec{y}}{2}\right)} e^{-2i\pi\langle \vec{w}, \vec{y}\rangle} d\vec{y}.$$

Proposition 2.51. *For $f, g \in L^2(\mathbb{R}^n)$ the cross-Wigner distribution has the following properties.*

(a) $\mathcal{W}_{f,g}$ is uniformly continuous on $\mathbb{R}^{2n}$, and

$$\|\mathcal{W}_{f,g}\|_\infty \leqslant 2^n \|f\|_2 \|g\|_2 .$$

(b) $\mathcal{W}_{f,g} = \overline{\mathcal{W}_{f,g}}$. In particular, $\mathcal{W}_f$ is real-valued.

(c) For $\vec{u}, \vec{v}, \vec{\eta}, \vec{\omega} \in \mathbb{R}^n$, we have

$$\begin{aligned}&\mathcal{W}_{T_{\vec{u}}E_{\vec{\eta}}f, T_{\vec{v}}E_{\vec{\omega}}g}(\vec{x}, \vec{w}) \\ &\qquad = e^{i\pi(\vec{u}+\vec{v})\cdot(\vec{\omega}-\vec{\eta})} e^{2i\pi\vec{x}\cdot(\vec{\eta}-\vec{\omega})} e^{-2i\pi\vec{w}\cdot(\vec{u}-\vec{v})} \mathcal{W}_{f,g}\left(\vec{x} - \frac{\vec{u}+\vec{v}}{2}, \vec{w} - \frac{\vec{\eta}+\vec{\omega}}{2}\right).\end{aligned}$$

In particular, $\mathcal{W}_f$ is covariant, that is $\mathcal{W}_{T_{\vec{u}}E_{\vec{\eta}}f}(\vec{x}, \vec{w}) = T_{\vec{u},\eta}\mathcal{W}_f(\vec{x}, \vec{w})$.

(d) $\mathcal{W}_{\hat{f},\hat{g}}(\vec{x}, \vec{w}) = \mathcal{W}_{f,g}(-\vec{w}, \vec{x})$.

(e) Moyal's formula: For $f, f', g, g' \in L^2(\mathbb{R}^n)$,

$$\langle \mathcal{W}_{f,g}, \mathcal{W}_{f',g'} \rangle_{L^2(\mathbb{R}^{2n})} = \langle f, f' \rangle \overline{\langle g, g' \rangle}.$$

(f) If $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $\mathcal{W}_{f,g} \in \mathcal{S}(\mathbb{R}^{2n})$. Here the Schwartz class is

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) \,:\, \|f\|_{(N,\alpha)} < \infty, \forall N \in \mathbb{N} \cup \{0\}, \alpha \in \Lambda \right\}$$

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where

$$\|f\|_{(N,\alpha)} = \sup_{\vec{x}\in\mathbb{R}^n} (1 + \|\vec{x}\|)^N |(D^\alpha f)(\vec{x})|$$

and Λ *is the set of multi-indices.*

One can reconstruct the modulus of f and of its Fourier transform from the Wigner distribution:

Proposition 2.52. *If* $f, \hat{f} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$*, then*

$$\int_{\mathbb{R}^n} \mathcal{W}_f(\vec{x}, \vec{w}) d\vec{w} = |f(\vec{x})|^2, \quad \int_{\mathbb{R}^n} \mathcal{W}_f(\vec{x}, \vec{w}) d\vec{x} = |f(\vec{w})|^2 .$$

In particular,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{W}_f(\vec{x}, \vec{w}) d\vec{x} d\vec{w} = \|f\|_2^2 .$$

2.8 The Symplectic Group

Throughout this section we shall be working with $2n \times 2n$ matrices, which we shall generally denote by capital calligraphic letters, and write in block form as

$$\mathcal{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c and d are $n \times n$ matrices.

2.8.1 Symplectic Linear Algebra

Recall that in Euclidean space, there is a natural symmetric, nondegenerate bilinear form: the usual inner product. The orthogonal group $O(n)$ is the group of all matrices leaving this inner product invariant,

$$\langle a\vec{x}, a\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$$

for $a \in O(n)$ and $\vec{x}, \vec{y} \in \mathbb{R}^n$.

In symplectic linear algebra, one starts with a skew-symmetric bilinear form, the matrices leaving this form invariant are called *symplectic matrices*. We begin with the matrix

$$\mathcal{J} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

which defines a skew-symmetric bilinear form, called the *symplectic form*, on $\mathbb{R}^{2n}$ by

$$[\vec{x}, \vec{y}] = \vec{x}^T \mathcal{J} \vec{y}$$

for $\vec{x}, \vec{y} \in \mathbb{R}^{2n}$. We observe that $\mathcal{J}^* = -\mathcal{J} = \mathcal{J}^{-1}$.

The *symplectic group* $Sp(n, \mathbb{R})$ is the group of all $2n \times 2n$ invertible matrices which, as operators on $\mathbb{R}^{2n}$, preserve the symplectic form:

$$Sp(n, \mathbb{R}) = \left\{ \mathcal{A} \in GL_{2n}(\mathbb{R}) \, : \, [\mathcal{A}\vec{x}, \mathcal{A}\vec{y}] = [\vec{x}, \vec{y}] \quad \forall\, \vec{x}, \vec{y} \in \mathbb{R}^{2n} \right\}.$$

The following characterizes symplectic matrices:

Proposition 2.53. *For* $\mathcal{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_{2n}(\mathbb{R})$, *the following are equivalent:*

(a) $\mathcal{A} \in Sp(n, \mathbb{R})$.

(b) $\mathcal{A}^* \mathcal{J} \mathcal{A} = \mathcal{J}$.

(c) $\mathcal{A}^{-1} = \mathcal{J} \mathcal{A}^* \mathcal{J}^{-1} = \begin{bmatrix} d^* & -b^* \\ -c^* & a^* \end{bmatrix}$.

(d) $\mathcal{A}^* \in Sp(n, \mathbb{R})$.

(e) $a^*c = c^*a,\ b^*d = d^*b,$ *and* $a^*d - c^*b = I_n$.

(f) $ab^* = ba^*,\ cd^* = dc^*,$ *and* $ad^* - bc^* = I_n$.

Proof. We have

$$[\mathcal{A}\vec{x}, \mathcal{A}\vec{y}] = (\mathcal{A}\vec{x})^T \mathcal{J} (\mathcal{A}\vec{y}) = \vec{x}^T \mathcal{A}^* \mathcal{J} \mathcal{A} \vec{y}$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^{2n}$, so (a) and (b) are equivalent. (b) and (c) are clearly equivalent. Taking the inverse transpose of (b) we get $\mathcal{A}^{-1}\mathcal{J}\mathcal{A}^{*-1} = \mathcal{J}$, and replacing $\mathcal{A}$ by $\mathcal{A}^{-1}$ we see that (b) is equivalent to (d). (e) is merely (b) written out in block form, and (f) is (b) written out in block form with $\mathcal{A}$ replaced by $\mathcal{A}^*$. $\square$

Proposition 2.54. *The subsets*

$$N = \left\{ \begin{bmatrix} I_n & a \\ 0 & I_n \end{bmatrix} : a = a^* \right\}, \qquad \overline{N} = \left\{ \begin{bmatrix} I_n & 0 \\ a & I_n \end{bmatrix} : a = a^* \right\},$$

$$L = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{*-1} \end{bmatrix} : a \in GL_n(\mathbb{R}) \right\}$$

of $GL_{2n}(\mathbb{R})$ are subgroups of $Sp(n, \mathbb{R})$. Moreover,

$$\overline{N}LN = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(n, \mathbb{R}) : \det a \neq 0 \right\}$$

Proof. The verification of the first assertion is straightforward, and therefore omitted. If

$$\begin{bmatrix} I_n & 0 \\ g & I_n \end{bmatrix} \in \overline{N}, \quad \begin{bmatrix} e & 0 \\ 0 & e^{*-1} \end{bmatrix} \in L, \quad \begin{bmatrix} I_n & f \\ 0 & I_n \end{bmatrix} \in N$$

then

$$\begin{bmatrix} I_n & 0 \\ g & I_n \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & e^{*-1} \end{bmatrix} \begin{bmatrix} I_n & f \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} e & ef \\ ge & gef + e^{*-1} \end{bmatrix},$$

so if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(n, \mathbb{R})$ and $\det a \neq 0$, we take $e = a$, $f = a^{-1}b$, and $g = ca^{-1}$, and must verify that

$$f = f^*, \quad g = g^*, \quad d = gef + e^{*-1} = ca^{-1}b + a^{*-1}.$$

But this follows easily from Proposition 2.53 (e,f). $\square$

Proposition 2.55. *$Sp(n, \mathbb{R})$ is connected and generated by $L \cup \overline{N} \cup \{\mathcal{J}\}$ and also by $L \cup N \cup \{\mathcal{J}\}$.*

2.8.2 The Metaplectic Representation

Next we introduce the Heisenberg group, defined by

$$\mathbb{H}^n = \left\{(\vec{v}, z) : \vec{v} = (\vec{x}, \vec{y}) \in \mathbb{R}^{2n},\ z \in \mathbb{R}\right\}$$

with group operation

$$(\vec{v}, z)(\vec{v}', z') = (\vec{v} + \vec{v}', z + z' - \frac{1}{2}[\vec{v}, \vec{v}']). \tag{2.11}$$

Topologically $\mathbb{H}^n$ is thus identified with $\mathbb{R}^{2n+1}$. It is often convenient to represent $\mathbb{H}^n$ as a matrix group. In fact, one easily checks that the map

$$(\vec{x}, \vec{y}, z) \in \mathbb{H}^n \mapsto \begin{bmatrix} 1 & \vec{y}^T & z + \frac{1}{2}\vec{y}^T\vec{x} \\ \vec{0} & I_n & \vec{x} \\ 0 & \vec{0} & 1 \end{bmatrix} \in GL_{n+2}(\mathbb{R})$$

is an isomorphism of topological groups, thus the Heisenberg group $\mathbb{H}^n$ is isomorphic to the closed subgroup of $GL_{n+2}(\mathbb{R})$ of the form

$$\mathbb{H}^n_{pol} = \left\{ \begin{bmatrix} 1 & \vec{y}^T & z \\ 0 & I_n & \vec{x} \\ 0 & 0 & 1 \end{bmatrix} : \vec{x}, \vec{y} \in \mathbb{R}^n, z \in \mathbb{R} \right\},$$

called the *polarized Heisenberg group.* One quickly verifies that

$$\rho(\vec{v}, z) = \rho(\vec{x}, \vec{y}, z) = e^{2i\pi z} e^{i\pi <\vec{x},\vec{y}>} T_{\vec{x}} E_{\vec{y}}$$

defines a representation of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$, called the *Schrödinger representation.* One can compose this representation with dilations by a scalar λ, to obtain a representation ρ_λ defined by

$$\rho_\lambda(\vec{x}, \vec{y}, z) = \rho(\lambda\vec{x}, \vec{y}, \lambda z).$$

Observe that the center of $\mathbb{H}^n$ is $Z = \left\{(\vec{0}, z) : z \in \mathbb{R}\right\}$ and $\rho(\vec{0}, z) = e^{2i\pi z}\mathrm{Id}$ for any $z \in \mathbb{R}$.

In representation theory, one wants to decompose representations into sums or direct integrals of basic building blocks, the irreducible representations. Representations of the Heisenberg group are well understood, and show the importance of the Schrödinger representation:

Theorem 2.56. (Stone-von Neumann) *Let π be a unitary representation of $\mathbb{H}^n$ on a Hilbert space $\mathcal{H}$, such that $\pi(0,0,z) = e^{2i\pi\lambda z} I_{\mathcal{H}}$ for some $\lambda \in \mathbb{R}\backslash\{0\}$. Then $\mathcal{H} = \bigoplus_{\alpha} \mathcal{H}_\alpha$ where the $\mathcal{H}_\alpha$'s are mutually orthogonal π-invariant subspaces of $\mathcal{H}$, and $\pi|_{\mathcal{H}_\alpha}$ is unitarily equivalent to ρ_λ. In particular, if π is irreducible then π is equivalent to ρ_λ.*

By definition of the group operation in the Heisenberg group, the symplectic group acts naturally on $\mathbb{H}^n$: each $\mathcal{A} \in Sp(n,\mathbb{R})$, defines a continuous automorphism $T_{\mathcal{A}}$ of $\mathbb{H}^n$ by $T_{\mathcal{A}}(\vec{v}, z) = (\mathcal{A}\vec{v}, z)$. In fact

$$\begin{aligned} T_{\mathcal{A}}\left((\vec{v},z)(\vec{v}',z')\right) &= T_{\mathcal{A}}\left(\vec{v}+\vec{v}', z+z'-\frac{1}{2}\left[\vec{v},\vec{v}'\right]\right) = \left(\mathcal{A}(\vec{v}+\vec{v}'), z+z'-\frac{1}{2}\left[\vec{v},\vec{v}'\right]\right) \\ &= \left(\mathcal{A}\vec{v}+\mathcal{A}\vec{v}', z+z'-\frac{1}{2}\left[\mathcal{A}\vec{v},\mathcal{A}\vec{v}'\right]\right) = \left(\mathcal{A}\vec{v},z\right)\left(\mathcal{A}\vec{v}',z'\right) \\ &= T_{\mathcal{A}}\left(\vec{v},z\right) T_{\mathcal{A}}\left(\vec{v}',z'\right) \end{aligned}$$

Composition of the Schrödinger representation with this automorphism defines a new irreducible representation $\rho_{\mathcal{A}}$ of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$,

$$\rho_{\mathcal{A}}(\vec{v},z) = \rho(\mathcal{A}\vec{v},z) \qquad ((\vec{v},z) \in \mathcal{H}^n)\,.$$

Observe that $\rho_{\mathcal{A}}(\vec{0},z) = \rho(\vec{0},z) = e^{2i\pi z}\mathrm{Id}$ for any $z \in \mathbb{R}$. Since $\rho_{\mathcal{A}}$ and ρ are irreducible unitary representation of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$ which coincide on its center Z, by Theorem 2.56, $\rho_{\mathcal{A}}$ and ρ must be equivalent. That is, there exists a (not necessarily unique) unitary operator $\mu(\mathcal{A})$ on $L^2(\mathbb{R}^n)$ such that

$$\rho(\mathcal{A}\vec{v},z) = \rho_{\mathcal{A}}(\vec{v},z) = \mu(\mathcal{A})\rho(\vec{v},z)\mu(\mathcal{A})^{-1} \tag{2.12}$$

for all $(\vec{v}, z) \in \mathbb{H}^n$. For $\mathcal{A}, \mathcal{B} \in Sp(n, \mathbb{R})$, one obtains

$$\begin{aligned}\mu(\mathcal{A}\mathcal{B})\rho(\vec{v}, z)\mu(\mathcal{A}\mathcal{B})^{-1} &= \rho_{\mathcal{A}\mathcal{B}}(\vec{v}, z) = \rho(\mathcal{A}\mathcal{B}\vec{v}, z) = \rho_{\mathcal{A}}(\mathcal{B}\vec{v}, z) \\ &= \mu(\mathcal{A})\rho(\mathcal{B}\vec{v}, z)\mu(\mathcal{A})^{-1} = \mu(\mathcal{A})\rho_{\mathcal{B}}(\vec{v}, z)\mu(\mathcal{A})^{-1} \\ &= \mu(\mathcal{A})\mu(\mathcal{B})\rho(\vec{v}, z)\mu(\mathcal{B})^{-1}\mu(\mathcal{A})^{-1}\end{aligned}$$

for all $(\vec{v}, z) \in \mathbb{H}^n$. Since ρ is irreducible and all operators are unitary, by Theorem 2.13, we obtain $\mu(\mathcal{A}\mathcal{B}) = \lambda\mu(\mathcal{A})\mu(\mathcal{B})$ for some complex scalar $|\lambda| = 1$. One can show that by proper choice of μ, $\lambda \in \{\pm 1\}$, so that $\mu(\mathcal{A}\mathcal{B}) = \pm\mu(\mathcal{A})\mu(\mathcal{B})$. Because of the factor ± 1, μ is not a proper representation. However, for the subgroups we are interested in, it will turn out to be a proper representation. μ is called the *metaplectic representation* of $Sp(n, \mathbb{R})$.

By Proposition 2.55, $Sp(n, \mathbb{R})$ is generated by matrices of three types. Applying Theorem 2.13, we can give an explicit formula of $\mu(\mathcal{A})$ for each type of generator, up to the phase factor λ. For $f \in L^2(\mathbb{R}^n)$, we have:
(i) given $\mathcal{A} = \begin{bmatrix} a & 0 \\ 0 & a^{*-1} \end{bmatrix}$, where $a \in GL_n(\mathbb{R})$, we obtain

$$\begin{aligned}\left(\mu(\mathcal{A})\rho(\vec{x}, \vec{y}, z)\mu(\mathcal{A})^{-1}f\right)(\vec{s}) &= \left(\rho(\mathcal{A}(\vec{x}, \vec{y}), z)f\right)(\vec{s}) = \left(\rho(a\vec{x}, a^{*-1}\vec{y}, z)f\right)(\vec{s}) \\ &= e^{2i\pi z}e^{i\pi\left\langle a\vec{x}, a^{*-1}\vec{y}\right\rangle}(T_{a\vec{x}}E_{a^{*-1}\vec{y}}f)(\vec{s}) \\ &= e^{2i\pi z}e^{i\pi\langle\vec{x}, \vec{y}\rangle}e^{2i\pi\langle\vec{s}-a\vec{x}, a^{*-1}\vec{y}\rangle}f(\vec{s} - a\vec{x}) \\ &= e^{2i\pi z}e^{i\pi\langle\vec{x}, \vec{y}\rangle}e^{2i\pi\langle a^{-1}\vec{s}-\vec{x}, \vec{y}\rangle}f(a(a^{-1}\vec{s} - \vec{x})) \\ &= e^{2i\pi z}e^{i\pi\langle\vec{x}, \vec{y}\rangle}\left|\det a\right|^{-1/2} e^{2i\pi\langle a^{-1}\vec{s}-\vec{x}, \vec{y}\rangle}(D_{a^{-1}}f)(a^{-1}\vec{s} - \vec{x}) \\ &= e^{2i\pi z}e^{i\pi\langle\vec{x}, \vec{y}\rangle}\left|\det a\right|^{-1/2}(E_{\vec{y}}D_{a^{-1}}f)(a^{-1}\vec{s} - \vec{x}) \\ &= e^{2i\pi z}e^{i\pi\langle\vec{x}, \vec{y}\rangle}\left|\det a\right|^{-1/2}(T_{\vec{x}}E_{\vec{y}}D_{a^{-1}}f)(a^{-1}\vec{s}) \\ &= e^{2i\pi z}e^{i\pi\langle\vec{x}, \vec{y}\rangle}(D_aT_{\vec{x}}E_{\vec{y}}D_{a^{-1}}f)(\vec{s}) \\ &= (D_a\rho(\vec{x}, \vec{y}, z)D_{a^{-1}}f)(\vec{s})\end{aligned}$$

for all $(\vec{x},\vec{y},z)\in\mathbb{H}^n$. By Theorem 2.13, it follows that

$$\mu(\mathcal{A})=\lambda D_a \tag{2.13}$$

for some $\lambda\in\mathbb{C}, |\lambda|=1$.

(ii) Given $\mathcal{A}=\begin{bmatrix} I_n & 0 \\ c & I_n \end{bmatrix}$, where $c\in M_n(\mathbb{R})$ with $c=c^*$, we obtain

$$\begin{aligned}
\left(\mu(\mathcal{A})\rho(\vec{x},\vec{y},z)\mu(\mathcal{A})^{-1}f\right)(\vec{s}) &= \left(\rho(\mathcal{A}(\vec{x},\vec{y}),z)f\right)(\vec{s}) = \rho(\vec{x},c\vec{x}+\vec{y},z)f(\vec{s}) \\
&= e^{2i\pi z}e^{i\pi\langle\vec{x},c\vec{x}+\vec{y}\rangle}(T_{\vec{x}}E_{c\vec{x}+\vec{y}}f)(\vec{s}) \\
&= e^{2i\pi z}e^{i\pi\langle\vec{x},c\vec{x}+\vec{y}\rangle}e^{2i\pi\langle\vec{s}-\vec{x},c\vec{x}+\vec{y}\rangle}f(\vec{s}-\vec{x}) \\
&= e^{2i\pi z}e^{i\pi\langle\vec{x},\vec{y}\rangle}e^{i\pi\langle c\vec{s},\vec{s}\rangle}e^{2i\pi\langle\vec{s}-\vec{x},\vec{y}\rangle}e^{-i\pi\langle c(\vec{s}-\vec{x}),\vec{s}-\vec{x}\rangle}f(\vec{s}-\vec{x}) \\
&= e^{2i\pi z}e^{i\pi\langle\vec{x},\vec{y}\rangle}e^{i\pi\langle c\vec{s},\vec{s}\rangle}e^{2i\pi\langle\vec{s}-\vec{x},\vec{y}\rangle}(N_c f)(\vec{s}-\vec{x}) \\
&= e^{2i\pi z}e^{i\pi\langle\vec{x},\vec{y}\rangle}e^{i\pi\langle c\vec{s},\vec{s}\rangle}(E_{\vec{y}}N_c f)(\vec{s}-\vec{x}) \\
&= e^{2i\pi z}e^{i\pi\langle\vec{x},\vec{y}\rangle}e^{i\pi\langle c\vec{s},\vec{s}\rangle}(T_{\vec{x}}E_{\vec{y}}N_c f)(\vec{s}) \\
&= e^{2i\pi z}e^{i\pi\langle\vec{x},\vec{y}\rangle}(N_{-c}T_{\vec{x}}E_{\vec{y}}N_c f)(\vec{s}) \\
&= (N_{-c}\rho(\vec{x},\vec{y},z)N_c f)(\vec{s})
\end{aligned}$$

for all $(\vec{x},\vec{y},z)\in\mathbb{H}^n$. By Theorem 2.13, it follows that

$$\mu(\mathcal{A})=\lambda N_{-c} \tag{2.14}$$

for some $\lambda\in\mathbb{C}, |\lambda|=1$.

(iii) Since $T_{\vec{y}}E_{-\vec{x}}=e^{-2i\pi\langle\vec{y},-\vec{x}\rangle}E_{-\vec{x}}T_{\vec{y}}$ for all $\vec{x},\vec{y}\in\mathbb{R}^n$, then we obtain

$$\begin{aligned}
\mu(\mathcal{J})\rho(\vec{x},\vec{y},z)\mu(\mathcal{J})^{-1} &= \rho(\mathcal{J}(\vec{x},\vec{y}),z) = \rho(\vec{y},-\vec{x},z) = e^{2i\pi z}e^{i\pi\langle\vec{y},-\vec{x}\rangle}T_{\vec{y}}E_{-\vec{x}} \\
&= e^{2i\pi z}e^{i\pi\langle\vec{y},-\vec{x}\rangle}e^{-2i\pi\langle\vec{y},-\vec{x}\rangle}E_{-\vec{x}}T_{\vec{y}} = e^{2i\pi z}e^{i\pi\langle\vec{y},\vec{x}\rangle}\mathcal{F}T_{\vec{x}}E_{\vec{y}}\mathcal{F}^{-1} = \mathcal{F}\rho(\vec{x},\vec{y},z)\mathcal{F}^{-1}
\end{aligned}$$

for all $(\vec{x},\vec{y},z)\in\mathbb{H}^n$. By Theorem 2.13, it follows that

$$\mu(\mathcal{J})=\lambda\mathcal{F} \tag{2.15}$$

for some $\lambda\in\mathbb{C}, |\lambda|=1$. One can show that λ can be chosen to be ±1.

2.8.3 The Extended Metaplectic Representation

Since $Sp(n,\mathbb{R})$ acts on $\mathbb{H}^n$ by automorphisms, one can form the semi-direct product of these two groups, $Sp(n,\mathbb{R}) \rtimes \mathbb{H}^n$ with the group law

$$(\mathcal{A},(\vec{v},z))\,(\mathcal{A}',(\vec{v}',z')) = (\mathcal{A}\mathcal{A}',(\vec{v},z)T_{\mathcal{A}}(\vec{v}',z'))$$

The Schrödinger representation ρ of $\mathbb{H}^n$ and the metaplectic representation μ of $Sp(n,\mathbb{R})$ fit together to form a (double-valued since μ is) unitary representation of $Sp(n,\mathbb{R}) \rtimes \mathbb{H}^n$ which is denoted by μ_e and called the *extended metaplectic representation*:

$$\mu_e(\mathcal{A},(\vec{v},z)) = \rho(\vec{v},z)\mu(\mathcal{A}).$$

Let us check that this really is a homomorphism (up to phase factor $\lambda \in \mathbb{C}$, $|\lambda| = 1$):

$$\begin{aligned}
\mu_e\,(\mathcal{A},(\vec{v},z))\mu_e\,(\mathcal{A}',(\vec{v}',z')) &= \rho(\vec{v},z)\mu(\mathcal{A})\rho(\vec{v}',z')\left(\mu(\mathcal{A})^{-1}\mu(\mathcal{A})\right)\mu(\mathcal{A}') \\
&= \rho(\vec{v},z)\rho_{\mathcal{A}}(\vec{v}',z')\mu(\mathcal{A}\mathcal{A}') = \rho(\vec{v},z)\rho(\mathcal{A}\vec{v}',z')\mu(\mathcal{A}\mathcal{A}') \\
&= \rho(\vec{v},z)\rho(T_{\mathcal{A}}(\vec{v}',z'))\mu(\mathcal{A}\mathcal{A}') = \rho((\vec{v},z)T_{\mathcal{A}}(\vec{v}',z'))\mu(\mathcal{A}\mathcal{A}') \\
&= \mu_e\,(\mathcal{A}\mathcal{A}',(\vec{v},z)T_{\mathcal{A}}(\vec{v}',z')) = \mu_e\,(\mathcal{A},((\vec{v},z))\,(\mathcal{A}',(\vec{v}',z'))) .
\end{aligned}$$

μ_e is irreducible since ρ is.

CHAPTER III

SUMS OF WAVELET REPRESENTATIONS

The important concept in the voice transform is admissibility: under what condition can a vector be reconstructed from its voice transform. Admissibility for the voice transform associated with the wavelet representation are now well understood from the paper of Laugesen et al. (2002) and the monograph by Führ (2005). Since we will need to work with transforms associated with sums of modulated wavelet representations, this chapter introduces such representation, discusses admissibility and presents methods for constructing admissible vectors as well as frames.

3.1 The Continuous Wavelet Transform

We begin by reviewing the usual continuous wavelet transform from the group theoretic point of view. Details of this part can be found in the monograph by Führ (2005).

3.1.1 The Voice Transform

Let $(\Omega, \mathcal{M}, \nu)$ be a measure space, $\mathcal{H}$ a Hilbert space and $\phi : \Omega \to \mathcal{H}$ a weakly measurable map. By this we mean that

$$\omega \in \Omega \mapsto \langle f, \phi(\omega) \rangle_{\mathcal{H}}$$

is measurable for all $f \in \mathcal{H}$. Define the *voice transform* of $f \in \mathcal{H}$ by

$$(V_\phi f)(\omega) = \langle f, \phi(\omega) \rangle_{\mathcal{H}}$$

for $\omega \in \Omega$. Thus V_ϕ is a measurable function on Ω. If the map $V_\phi : f \mapsto V_\phi f$ is a multiple of a partial isometry of $\mathcal{H}$ into $L^2(\Omega)$, that is, if there exists $c_\phi > 0$ such that

$$\|V_\phi f\|_{L^2(\Omega)} = \sqrt{c_\phi}\|f\|_{\mathcal{H}} \tag{3.1}$$

for all $f \in \mathcal{H}$, then by the polarization identity

$$\langle V_\phi f, V_\phi g\rangle_{L^2(\Omega)} = c_\phi \langle f, g\rangle_{\mathcal{H}}$$

for all $f, g \in \mathcal{H}$. It follows that

$$\begin{aligned}\langle f, g\rangle_{\mathcal{H}} &= \frac{1}{c_\phi}\langle V_\phi f, V_\phi g\rangle_{L^2(\Omega)} = \frac{1}{c_\phi}\int_\Omega (V_\phi f)(\omega)\overline{(V_\phi g)(\omega)}d\nu(\omega)\\ &= \frac{1}{c_\phi}\int_\Omega (V_\phi f)(\omega)\overline{\langle g, \phi(\omega)\rangle}d\nu(\omega) = \frac{1}{c_\phi}\int_\Omega (V_\phi f)(\omega)\langle \phi(\omega), g\rangle_{\mathcal{H}}\, d\nu(\omega)\\ &= \int_\Omega \left\langle \frac{1}{c_\phi}(V_\phi f)(\omega)\phi(\omega), g\right\rangle_{\mathcal{H}} d\nu(\omega)\end{aligned}$$

for all $f, g \in \mathcal{H}$ and one obtains the *Calderón reproducing formula*

$$f = \frac{1}{c_\phi}\int_\Omega (V_\phi f)(\omega)\phi(\omega)d\nu(\omega) \tag{3.2}$$

as a weak integral in $\mathcal{H}$. The mapping ϕ is called a *resolution of the identity* if (3.2) holds.

Resolutions of the identity arise naturally from group representations. Let G be a locally compact group with Haar measure ν, for example a matrix group, and π a representation of G on a Hilbert space $\mathcal{H}$. Fix $\psi \in \mathcal{H}$ and consider the continuous map

$$\phi : g \mapsto \pi(g)\psi$$

of G into $\mathcal{H}$. The voice transform associated with this representation is

$$(V_\psi f)(g) = \langle f, \pi(g)\psi\rangle_{\mathcal{H}}$$

for $f \in \mathcal{H}$ and condition (3.1) becomes

$$\|V_\psi f\|_{L^2(G)} = \sqrt{c_\psi}\|f\|_{\mathcal{H}}$$

for all $f \in \mathcal{H}$, i.e.,

$$\int_G |\langle f, \pi(g)\psi\rangle_{\mathcal{H}}|^2 \, d\nu(g) = c_\psi \, \|f\|^2_{\mathcal{H}}$$

for all $f \in \mathcal{H}$. If this identity holds, then ψ is called an *admissible vector*. The group G is called admissible, if at least one admissible vector exists.

3.1.2 The Classical Wavelet Transform

The wavelet transform is an example of this voice transform associated with a group representation. Here $G = H \rtimes \mathbb{R}^n$ is a subgroup of the affine group $\mathrm{Aff}_n(\mathbb{R})$ with H a closed subgroup of $GL_n(\mathbb{R})$, and π the affine representation of G on $L^2(\mathbb{R}^n)$.

Definition 3.1. *Given $\psi \in L^2(\mathbb{R}^n)$, the continuous wavelet transform W_ψ induced by ψ and the group H is defined by*

$$W_\psi f(h, \vec{x}) = \langle f, \pi(h, \vec{x})\psi\rangle_{L^2(\mathbb{R}^n)} = |\det h|^{-1/2} \int_{\mathbb{R}^n} f(\vec{y})\overline{\psi(h^{-1}(\vec{y} - \vec{x}))} d\vec{y}$$

for $f \in L^2(\mathbb{R}^n)$ and $(h, \vec{x}) \in G$.

The adjective *continuous* refers to the continuity of the translation group, consisting of all $\vec{x} \in \mathbb{R}^n$. The dilation group H, in contrast, is permitted to carry the discrete topology.

Thus G is *admissible* if and only if there exist a function $\psi \in L^2(\mathbb{R}^n)$ *(admissible vector)* and $c_\psi > 0$ such that

$$\|W_\psi f\|_{L^2(G)} = \sqrt{c_\psi}\|f\|_{L^2(\mathbb{R}^n)} \tag{3.3}$$

for all $f \in L^2(\mathbb{R}^n)$, or equivalently

$$\int_G |\langle f, \pi(h, \vec{x})\psi\rangle_{L^2(\mathbb{R}^n)}|^2 d\nu(h, \vec{x}) = \int_G |(W_\psi f)(h, \vec{x})|^2 d\nu(h, \vec{x}) = c_\psi \int_{\mathbb{R}^n} |f(\vec{y})|^2 d\vec{y} \tag{3.4}$$

for all $f \in L^2(\mathbb{R}^n)$. The Calderón reproducing formula is thus by (2.8),

$$\begin{aligned} f &= \frac{1}{c_\psi} \int_G (W_\psi f)(h,\vec{x})\ \pi(h,\vec{x})\psi\ d\nu(h,\vec{x}) \\ &= \frac{1}{c_\psi} \int_{\mathbb{R}^n} \int_H (W_\psi f)(h,\vec{x})\ \pi(h,\vec{x})\psi\ |\det h|^{-1} d\mu(h) d\vec{x} \end{aligned} \tag{3.5}$$

as a weak integral in $L^2(\mathbb{R}^n)$. Because $G = H \rtimes \mathbb{R}^n$, we also call the group H admissible if G is.

The following two theorems are by now well known. They characterize admissible functions and give criteria for an affine group $G = H \rtimes \mathbb{R}^n$ to be admissible.

Theorem 3.2. (Laugesen et al., 2002) $\psi \in L^2(\mathbb{R}^n)$ *is admissible if and only if there is* $c_\psi > 0$ *such that*

$$\int_H |\widehat{\psi}(\vec{\gamma}h)|^2 d\mu(h) = c_\psi \tag{3.6}$$

for a.e. $\vec{\gamma} \in \widehat{\mathbb{R}^n}$.

The fundamental result on admissibility of a group given by Laugesen et al. (2002) involves the notation of the ε-*stabilizer*. Given $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ and $\varepsilon \geqslant 0$, the set

$$H^\varepsilon_{\vec{\gamma}} = \{h \in H : \|\vec{\gamma}h - \vec{\gamma}\| \leq \varepsilon\}$$

is called the ε-stabilizer of $\vec{\gamma}$. Thus, the set $H_{\vec{\gamma}} \equiv H^0_{\vec{\gamma}} = \{h \in H : \vec{\gamma}h = \vec{\gamma}\}$ is the *stabilizer* of $\vec{\gamma}$. It is clear that $H^\varepsilon_{\vec{\gamma}}$ is a closed subset of H, that H_γ is a closed subgroup of H, that $H_{\vec{\gamma}} = \bigcap_{\varepsilon>0} H^\varepsilon_{\vec{\gamma}}$, and that $H^{\varepsilon_1}_{\vec{\gamma}} \subset H^{\varepsilon_2}_{\vec{\gamma}}$ when $\varepsilon_1 \leq \varepsilon_2$.

Theorem 3.3. (Laugesen et al., 2002)

(a) If H is admissible, then $\Delta \not\equiv |\det|$ and the stabilizer of $\vec{\gamma}$ is compact for a.e. $\vec{\gamma} \in \widehat{\mathbb{R}^n}$.

(b) If $\Delta \not\equiv |\det|$ and for a.e. $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ there exists an $\varepsilon > 0$ such that the ε-stabilizer of $\vec{\gamma}$ is compact, then H is admissible.

This theorem is quite useful for determining the admissibility of particular groups H. For example it is clear that no compact group H can be admissible since in this case $\Delta \equiv |\det| \equiv 1$.

3.2 Sums of Wavelet Representations

3.2.1 The Modulated Wavelet Transform

For the purpose of this thesis, we need to generalize the definition of wavelet transform to include modulations. Fix a Borel function $\chi : H \to \Pi$, where $\Pi = \{z \in \mathbb{C} : |z| = 1\}$ denotes the complex unit circle. Given $h \in GL_n(\mathbb{R})$, we define a *modulated dilation* D_h^χ by

$$D_h^\chi = \chi(h) D_h.$$

Since $|\chi(h)| = 1$, then D_h^χ is still a unitary operator on $L^2(\mathbb{R}^n)$; in fact

$$\|D_h^\chi f\|_{L^2(\mathbb{R}^n)} = \|\chi(h) D_h f\|_{L^2(\mathbb{R}^n)} = |\chi(h)| \, \|D_h f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$$

for all $f \in L^2(\mathbb{R}^n)$ and $h \in H$.

We define the *modulated wavelet representation* of G on $L^2(\mathbb{R}^n)$ by

$$\pi^\chi(h, \vec{x}) = T_{\vec{x}} D_h^\chi$$

for $(h, \vec{x}) \in G$. Clearly, $\pi^\chi(h, \vec{x})$ is a unitary operator, and

$$\pi^\chi(h, \vec{x}) = T_{\vec{x}} D_h^\chi = \chi(h) T_{\vec{x}} D_h = \chi(h) \pi(h, \vec{x})$$

for $(h, \vec{x}) \in G$. Observe that π^χ is not a representation in the proper sense: it need neither be continuous, nor a homomorphism. However, if χ is a character of H, then π^χ will be a representation also.

The Fourier transform again induces a modulated wavelet representation δ^{χ} of G on $L^2(\widehat{\mathbb{R}^n})$ by

$$\begin{aligned}\delta^{\chi}(h,\vec{x}) &= \mathcal{F}\circ\pi^{\chi}(h,\vec{x})\circ\mathcal{F}^{-1} = \mathcal{F}\circ\chi(h)\pi(h,\vec{x})\circ\mathcal{F}^{-1}\\ &= \chi(h)\mathcal{F}\circ\pi(h,\vec{x})\circ\mathcal{F}^{-1} = \chi(h)\delta(h,\vec{x})\end{aligned}$$

for all $(h,\vec{x})\in G$.

Given $\psi\in L^2(\mathbb{R}^n)$, the *modulated wavelet transform* induced by ψ and the group H is defined just as the usual wavelet transform by

$$W_{\psi}^{\chi}f(h,\vec{x}) = \langle f,\pi^{\chi}(h,\vec{x})\psi\rangle_{L^2(\mathbb{R}^n)}$$

for $f\in L^2(\mathbb{R}^n)$ and $(h,\vec{x})\in G$. That is

$$W_{\psi}^{\chi}f(h,\vec{x}) = \overline{\chi(h)}\,\langle f,\pi(h,\vec{x})\psi\rangle_{L^2(\mathbb{R}^n)} = \overline{\chi(h)}W_{\psi}f(h,\vec{x})$$

and hence

$$\begin{aligned}\left\|W_{\psi}^{\chi}f\right\|_{L^2(G)}^2 = \int_G \left|W_{\psi}^{\chi}f(h,\vec{x})\right|^2 d\nu(h,\vec{x}) &= \int_G \left|\overline{\chi(h)}W_{\psi}f(h,\vec{x})\right|^2 d\nu(h,\vec{x})\\ = \int_G \left|W_{\psi}f(h,\vec{x})\right|^2 d\nu(h,\vec{x}) &= \|W_{\psi}f\|_{L^2(G)}^2\end{aligned}$$

for all $f\in L^2(\mathbb{R}^n)$ and $(h,\vec{x})\in G$. This shows that ψ is admissible for the modulated wavelet transform if and only if it is admissible for the classical wavelet transform; hence Theorem 3.2 and Theorem 3.3 still apply.

Remark 3.1. The dilation group H need not be a subgroup of $GL_n(\mathbb{k})$. It suffices that H be a locally compact group, with a continuous homomorphism $\varphi: H\to H_0$ onto a matrix subgroup of $GL_n(\mathbb{k})$. Naturally, the wavelet representation is of the form

$$\pi(\varphi(h),\vec{x})$$

and the wavelet transform

$$(W_{\phi}f)(h,\vec{x}) = \langle f,\pi(\varphi(h),\vec{x})\phi\rangle.$$

for all $f \in L^2(\mathbb{R}^n)$ and $h \in H$. Because stabilizers are required to be compact for admissibility, $\ker(\varphi)$ must be a compact subgroup of H. That is, H is a compact extension of a matrix group.

All the results in this chapter remain valid in this generalized setting, and we will make use of it in the third example of chapter IV.

3.2.2 Sums of Modulated Wavelet Representations

Consider as usual an affine group $G = H \rtimes \mathbb{R}^n$ with wavelet representation π_0 of G on $L^2(\mathbb{R}^n)$. Let J be a finite or countably infinite index set, $\{\mathcal{H}_j\}_{j\in J}$ a family of π_0-invariant closed subspaces of $L^2(\mathbb{R}^n)$, π_j the restrictions of π_0 to $\mathcal{H}_j$ and $\chi_j : H \to \Pi$ Borel functions. For each $j \in J$, consider the modulated wavelet representation of G on $\mathcal{H}_j$,

$$\pi_j^{\chi_j}(h, \vec{x}) = \chi_j(h)\pi_j(h, \vec{x})$$

for $(h, \vec{x}) \in G$. As shown in the previous section, $\pi_j^{\chi_j}(h, \vec{x})$ is a unitary operator on $L^2(\mathbb{R}^n)$ for all $(h, \vec{x}) \in G$, and as $\chi_j(h)$ is scalar, then each space $\mathcal{H}_j$ is also $\pi_j^{\chi_j}$-invariant.

Now set $\mathcal{H} = \bigoplus_{j\in J} \mathcal{H}_j$, and let

$$\pi^{\chi} = \bigoplus_{j\in J} \pi_j^{\chi_j} \tag{3.7}$$

denote the corresponding *sum of modulated wavelet representation.* For $f \in \mathcal{H}$, let us denote by f_j the component of f in $\mathcal{H}_j$, that is $f_j = P_j f$ where P_j is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_j$; thus $f = \sum_{j\in J} f_j$ with the norm

$$\|f\|_{\mathcal{H}}^2 = \sum_{j\in J} \|f_j\|_{\mathcal{H}_j}^2.$$

By (3.7), we have

$$\pi^{\chi}(h, \vec{x})\psi = \sum_{j\in J} \pi_j^{\chi_j}(h, \vec{x})\psi_j$$

for $\psi \in \mathcal{H}$ and $(h, \vec{x}) \in G$. The voice transform of G determined by π^χ and $\psi \in \mathcal{H}$ is given by

$$W_\psi^\chi f(h,\vec{x}) := \langle f, \pi^\chi(h,\vec{x})\psi\rangle_{\mathcal{H}} = \left\langle \sum_{i\in J} f_i, \sum_{j\in J} \pi_j^{\chi_j}(h,\vec{x})\psi_j \right\rangle_{\bigoplus_{j\in J}\mathcal{H}_j}$$
$$= \sum_{j\in J} \left\langle f_j, \pi_j^{\chi_j}(h,\vec{x})\psi_j \right\rangle_{\mathcal{H}_j} = \sum_{j\in J} W_{\psi_j}^{\chi_j} f_j(h,\vec{x}) \tag{3.8}$$

for $f \in \mathcal{H}$ and $(h, \vec{x}) \in G$, which, is a *sum of modulated wavelet transforms*: each $W_{\psi_j}^{\chi_j}$ is the modulated wavelet transform of G on $\mathcal{H}_j$ determined by ψ_j. We note that at most countably many terms in the series (3.8) are nonzero, and that after ordering the nonzero terms, the series converges pointwise.

3.2.3 Admissibility Conditions

Theorem 3.2 characterizes all admissible functions for the wavelet representation. In this section, we will extend this theorem to characterize all admissible vectors ψ for the sum of modulated wavelet representations. Suppose for the moment that $\psi_i \in \mathcal{H}_i$ and $\psi_j \in \mathcal{H}_j$ are admissible. That is $W_{\psi_i}^{\chi_i} f_i, W_{\psi_j}^{\chi_j} f_j \in L^2(G)$ for all $f_i \in \mathcal{H}_i$, $f_j \in \mathcal{H}_j$. One easily verifies that

$$\langle f_i, f_j\rangle_* := \left\langle W_{\psi_i}^{\chi_i} f_i, W_{\psi_j}^{\chi_j} f_j \right\rangle_{L^2(G)} = \int_G W_{\psi_i}^{\chi_i} f_i(h,\vec{x}) \overline{W_{\psi_j}^{\chi_j} f_j(h,\vec{x})} d\nu(h,\vec{x})$$

defines a continuous sesquilinear form $\mathcal{H}_i \times \mathcal{H}_j \to \mathbb{C}$, by linearity of the modulated wavelet transform.

Lemma 3.4. *Suppose, $\psi_i \in \mathcal{H}_i$ and $\psi_j \in \mathcal{H}_j$ are admissible. Then $\operatorname{Re}\langle f_i, f_j\rangle_* = 0$ for all $f_i \in \mathcal{H}_i$, $f_j \in \mathcal{H}_j$ if and only if $\langle f_i, f_j\rangle_* = 0$ for all $f_i \in \mathcal{H}_i$, $f_j \in \mathcal{H}_j$.*

Proof. Since $-if_j \in \mathcal{H}_j$ for all $f_j \in \mathcal{H}_j$ and $\langle\cdot,\cdot\rangle_*$ is sesquilinear, then

$$\operatorname{Re}\langle -if_i, f_j\rangle_* = \operatorname{Re}\left(-\langle if_i, f_j\rangle_*\right) = \operatorname{Im}\langle if_i, f_j\rangle_*$$

We thus obtain

$$\langle f_i, f_j \rangle_* = \mathrm{Re}\,\langle f_i, f_j \rangle_* + i\,\mathrm{Im}\,\langle f_i, f_j \rangle_* = \mathrm{Re}\,\langle f_i, f_j \rangle_* + i\,\mathrm{Re}\,\langle -if_i, f_j \rangle_*$$

for all $f_i \in \mathcal{H}_i,\ f_j \in \mathcal{H}_j$, from which the assertion follows easily. $\square$

The next proposition was proved in Führ (2005) for voice transform associated with general sums of representations. We present its proof for the sake of completeness.

Proposition 3.5. *Let $\psi \in \mathcal{H}$. Then ψ is admissible for π^χ if and only if*

(a) each ψ_j is admissible for $\pi_j^{\chi_j}$ with common constant $c_\psi = c_{\psi_j}$

(b) $\langle f_i, f_j \rangle_* = 0$ for all $f_i \in \mathcal{H}_i,\ f_j \in \mathcal{H}_j$ and $i \neq j$.

Proof. Recall that ψ is admissible for the sum of representations π^χ if and only if there exists a constant $c_\psi > 0$ such that

$$\left\| W_\psi^\chi f \right\|_{L^2(G)}^2 = c_\psi \left\| f \right\|_{\mathcal{H}}^2 = c_\psi \sum_{j \in J} \left\| f_j \right\|_{\mathcal{H}_j}^2$$

for all $f \in \mathcal{H}$.

Now suppose, ψ is admissible. Choose any j. Each $f \in \mathcal{H}_j$ is also an element of $\mathcal{H}$ and $W_\psi^\chi f = W_{\psi_j}^{\chi_j} f$ by (3.8). Then (a) gives

$$c_\psi \left\| f \right\|_{\mathcal{H}_j}^2 = c_\psi \left\| f \right\|_{\mathcal{H}}^2 = \left\| W_\psi^\chi f \right\|_{L^2(G)}^2 = \left\| W_{\psi_j}^{\chi_j} f \right\|_{L^2(G)}^2$$

which shows that ψ_j is admissible for $\mathcal{H}_j$ with constant c_ψ. In particular, $W_{\psi_j}^{\chi_j} f_j \in L^2(G)$ for all $f_j \in \mathcal{H}_j,\ j \in J$. Now for any $f_i \in \mathcal{H}_i$ and $f_j \in \mathcal{H}_j$ with $i \neq j$, we have by (3.8),

$$\begin{aligned}
\left\|W_\psi^\chi(f_i+f_j)\right\|_{L^2(G)}^2 &= \left\|W_{\psi_i}^{\chi_i} f_i + W_{\psi_j}^{\chi_j} f_j\right\|_{L^2(G)}^2 \\
&= \int_G \left|W_{\psi_i}^{\chi_i} f_i(h,\vec{x}) + W_{\psi_j}^{\chi_j} f_j(h,\vec{x})\right|^2 d\nu(h,\vec{x}) \\
&= \int_G \left|W_{\psi_i}^{\chi_i} f_i(h,\vec{x})\right|^2 d\nu(h,\vec{x}) + \int_G \left|W_{\psi_j}^{\chi_j} f_j(h,\vec{x})\right|^2 d\nu(h,\vec{x}) \\
&\qquad + \int_G 2\operatorname{Re}\left(W_{\psi_i}^{\chi_i} f_i(h,\vec{x})\overline{W_{\psi_j}^{\chi_j} f_j(h,\vec{x})}\right) d\nu(h,\vec{x}) \\
&= \left\|W_{\psi_i}^{\chi_i} f_i\right\|_{L^2(G)}^2 + \left\|W_{\psi_j}^{\chi_j} f_j\right\|_{L^2(G)}^2 + 2\operatorname{Re}\left\langle f_i, f_j\right\rangle_* \\
&= c_\psi \left\|f_i\right\|_{\mathcal{H}_i}^2 + c_\psi \left\|f_j\right\|_{\mathcal{H}_j}^2 + 2\operatorname{Re}\left\langle f_i, f_j\right\rangle_*
\end{aligned}$$

while also by admissibility,

$$\left\|W_\psi^\chi(f_i+f_j)\right\|_{L^2(G)}^2 = c_\psi \left\|f_i+f_j\right\|_{\mathcal{H}}^2 = c_\psi \left[\left\|f_i\right\|_{\mathcal{H}_i}^2 + \left\|f_j\right\|_{\mathcal{H}_j}^2\right]$$

which implies that $\operatorname{Re}\left\langle f_i, f_j\right\rangle_* = 0$.

Conversely, suppose that (a) and (b) hold. Since f_i, f_j were arbitrary by the Lemma 3.4, $\left\langle f_i, f_j\right\rangle_* = 0$ for all $f_i \in \mathcal{H}_i$, $f_j \in \mathcal{H}_j$. That is, $\left\{W_{\psi_j}^{\chi_j} f_j\right\}_{j\in J}$ is a collection of orthogonal vectors in $L^2(G)$ for each $f \in \mathcal{H}$, and also

$$\sum_{j\in J} \left\|W_{\psi_j}^{\chi_j} f_j\right\|_{L^2(G)}^2 = \sum_{j\in J} c_\psi \left\|f_j\right\|_{\mathcal{H}_i}^2 = c_\psi \left\|f\right\|_{\mathcal{H}}^2 < \infty.$$

It follows that $\sum\limits_{j\in J} W_{\psi_j}^{\chi_j} f_j$ converges in $L^2(G)$. By (3.8) this series converges pointwise to $W_\psi^\chi f$, so by uniqueness of limits, its $L^2(G)$-limit is $W_\psi f$ as well. Then,

$$\left\|W_\psi^\chi f\right\|_{L^2(G)}^2 = \sum_{j\in J} \left\|W_{\psi_j}^{\chi_j} f_j\right\|_{L^2(G)}^2 = \sum_{j\in J} c_\psi \left\|f_j\right\|_{\mathcal{H}_i}^2 = c_\psi \left\|f\right\|_{\mathcal{H}}^2$$

which shows that ψ is admissible. $\square$

This admissibility condition can be made more explicit by considering the action of G on the dual orbit space, as in Theorem 3.2. By the support of a Borel of a Borel function f we mean any Borel set E such that $\{\vec{x} \in \mathbb{R}^n : f(\vec{x}) \neq 0\}$

differs from E by a null set. Since each $\mathcal{H}_j$ is a $\pi_j^{\chi_j}$-invariant subspace of $L^2(\mathbb{R}^n)$, it is in particular invariant under translations. Since the Fourier transform takes translation to modulation, we now assume that each $\mathcal{H}_j$ is of the form $\mathcal{H}_j = \left\{f \in L^2(\mathbb{R}^n) : \operatorname{supp}(\widehat{f}) \subset \mathcal{O}_j\right\} \cong L^2(\mathcal{O}_j)$, for H-invariant Borel subsets $\mathcal{O}_j$ of $\widehat{\mathbb{R}^n}$. In fact, if $f \in \mathcal{H}_j$, we have

$$\pi_j^{\chi_j}(h,\vec{x})f = \left(\mathcal{F}^{-1} \circ \delta_j^{\chi_j}(h,\vec{x}) \circ \mathcal{F}\right) f = \mathcal{F}^{-1}\left(\chi_j(h)E_{-\vec{x}}D_h\widehat{f}\right) \in \mathcal{H}_j$$

as $D_h\widehat{f} \in L^2(\mathcal{O}_j)$ by H-invariance, so that $\mathcal{H}_j$ is $\pi_j^{\chi_j}$-invariant. Then the sum of modulated wavelet transforms (3.8) becomes

$$\begin{aligned} W_\psi^\chi f(h,\vec{x}) &= \sum_{j\in J} \left\langle f_j, \pi_j^{\chi_j}(h,\vec{x})\psi_j\right\rangle_{\mathcal{H}_j} = \sum_{j\in J}\left\langle \mathcal{F}f_j, \mathcal{F}\pi_j^{\chi_j}(h,\vec{x})\mathcal{F}^{-1}\mathcal{F}\psi_j\right\rangle_{L^2(\mathcal{O}_j)} \\ &= \sum_{j\in J}\left\langle \widehat{f_j}, \delta_j^{\chi_j}(h,\vec{x})\widehat{\psi_j}\right\rangle_{L^2(\mathcal{O}_j)} \end{aligned} \tag{3.9}$$

where $\delta_j^{\chi_j} = \mathcal{F}\circ\pi_j^{\chi_j}\circ\mathcal{F}^{-1}$. Thus, if we set $\widehat{\mathcal{H}} = \bigoplus_{j\in J} L^2(\mathcal{O}_j)$, $\widehat{f} = \sum_{j\in J}\widehat{f}_j \in \widehat{\mathcal{H}}$ and $\delta^\chi = \bigoplus_{j\in J}\delta_j^{\chi_j}$, then

$$W_\psi^\chi f(h,\vec{x}) = \left\langle \widehat{f}, \delta^\chi(h,\vec{x})\widehat{\psi}\right\rangle_{\widehat{\mathcal{H}}}.$$

We also let $\mathcal{O} = \bigcup_{j\in J}\mathcal{O}_j$.

The next theorem generalizes Theorem 3.2 to sums of wavelet transforms. Throughout, we may consider a function $\widehat{f} \in L^2(\mathcal{O}_j)$ as a function defined on $\widehat{\mathbb{R}^n}$, by setting it to zero outside of $\mathcal{O}_j$.

Theorem 3.6. *A vector $\psi \in \mathcal{H}$ is admissible if and only if there is a constant $c_\psi > 0$ such that*

$$\int_H \overline{\chi_i(h)\widehat{\psi_i}(\vec{\gamma}h)}\chi_j(h)\widehat{\psi_j}(\vec{\gamma}h)d\mu(h) = \delta_{i,j}c_\psi \tag{3.10}$$

for a.e. $\vec{\gamma} \in \mathcal{O}_i \cap \mathcal{O}_j$, for all $i, j \in J$.

Proof. Applying Proposition 3.5 and Theorem 3.2, we obtain that ψ is admissible if and only if

(a) $\int_H |\widehat{\psi_j}(\vec{\gamma}h)|^2 d\mu(h) = c_\psi$ for a.e. $\vec{\gamma} \in \mathcal{O}_j$ and all $j \in J$

(b) $\langle f_i, f_j \rangle_* = 0$ for all $f_i \in \mathcal{H}_i$, $f_j \in \mathcal{H}_j$ and $i \neq j$.

Since (a) is (3.10) with $i = j$ one only needs to verify that (b) is equivalent to (3.10) for $i \neq j$ provided that (a) holds.

Assume that (a) holds, that is $W_{\psi_j}^{\chi_j} f_j \in L^2(G)$ for all $f_j \in \mathcal{H}_j$. Then $\langle f_i, f_j \rangle_*$ exists and is continuous. By continuity, it suffices to show that (b) is equivalent to (3.10) on the dense subspaces $\mathcal{K}_j = \left\{ f_j \in \mathcal{H}_j \,:\, \widehat{f_j} \text{ is bounded} \right\}$ of $\mathcal{H}_j$. For all $f_i \in \mathcal{K}_i$, $f_j \in \mathcal{K}_j$, we obtain

$$
\begin{aligned}
\langle f_i, f_j \rangle_* &= \left\langle W_{\psi_i}^{\chi_i} f_i, W_{\psi_j}^{\chi_j} f_j \right\rangle_{L^2(G)} = \int_G W_{\psi_i}^{\chi_i} f_i(h,\vec{x}) \overline{W_{\psi_j}^{\chi_j} f_j(h,\vec{x})} d\nu(h,\vec{x}) \\
&= \int_G \left\langle \widehat{f_i}, \delta_i^{\chi_i}(h,\vec{x})\widehat{\psi_i} \right\rangle \overline{\left\langle \widehat{f_j}, \delta_j^{\chi_j}(h,\vec{x})\widehat{\psi_j} \right\rangle} d\nu(h,\vec{x}) \\
&= \int_G \left[\int_{\mathcal{O}_i} \widehat{f_i}(\vec{\gamma}) \overline{\delta_i^{\chi_i}(h,\vec{x})\widehat{\psi_i}(\vec{\gamma})} d\vec{\gamma} \right] \overline{\left[\int_{\mathcal{O}_j} \widehat{f_j}(\vec{\gamma}) \overline{\delta_j^{\chi_j}(h,\vec{x})\widehat{\psi_j}(\vec{\gamma})} d\vec{\gamma} \right]} d\nu(h,\vec{x}) \\
&= \int_G \left[\int_{\mathcal{O}_i} \widehat{f_i}(\vec{\gamma}) \overline{E_{-\vec{x}} D_h^{\chi_i} \widehat{\psi_i}(\vec{\gamma})} d\vec{\gamma} \right] \overline{\left[\int_{\mathcal{O}_j} \widehat{f_j}(\vec{\gamma}) \overline{E_{-\vec{x}} D_h^{\chi_j} \widehat{\psi_j}(\vec{\gamma})} d\vec{\gamma} \right]} d\nu(h,\vec{x})
\end{aligned}
$$

shift $\vec{\gamma}$ to $\vec{\gamma}h$

$$
\begin{aligned}
\langle f_i, f_j \rangle_* = \int_G & \left[\int_{\widehat{\mathbb{R}^n}} \widehat{f_i}(\vec{\gamma}) \overline{|\det h|^{1/2} e^{-2i\pi\vec{\gamma}\vec{x}} \chi_i(h) \widehat{\psi_i}(\vec{\gamma}h)} d\vec{\gamma} \right] \\
& \times \overline{\left[\int_{\widehat{\mathbb{R}^n}} \widehat{f_j}(\vec{\gamma}) \overline{|\det h|^{1/2} e^{-2i\pi\vec{\gamma}\vec{x}} \chi_j(h) \widehat{\psi_j}(\vec{\gamma}h)} d\vec{\gamma} \right]} d\nu(h,\vec{x}) \\
= \int_G & \left[\int_{\widehat{\mathbb{R}^n}} \underbrace{\widehat{f_i}(\vec{\gamma}) \overline{\chi_i(h) \widehat{\psi_i}(\vec{\gamma}h)}}_{:=\phi_h^i(\vec{\gamma})} e^{2i\pi\vec{\gamma}\vec{x}} d\vec{\gamma} \right] \\
& \times \overline{\left[\int_{\widehat{\mathbb{R}^n}} \underbrace{\widehat{f_j}(\vec{\gamma}) \overline{\chi_j(h) \widehat{\psi_j}(\vec{\gamma}h)}}_{:=\phi_h^j(\vec{\gamma})} e^{2i\pi\vec{\gamma}\vec{x}} d\vec{\gamma} \right]} |\det h| \, d\nu(h,\vec{x}).
\end{aligned}
$$

Since $f_i, f_j \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then $\phi_h^i, \phi_h^j \in L^1(\widehat{\mathbb{R}^n}) \cap L^2(\widehat{\mathbb{R}^n})$ and each of the inner integrals is an inverse Fourier transform, hence by Plancherel's theorem and

Fubini's theorem, we obtain

$$\begin{aligned}\langle f_i, f_j\rangle_* &= \int_G \overset{\vee}{\phi}{}^i_h(\vec{x})\overline{\overset{\vee}{\phi}{}^j_h(\vec{x})}\,|\det h|\,d\nu(h,\vec{x}) = \int_H\int_{\mathbb{R}^n} \overset{\vee}{\phi}{}^i_h(\vec{x})\overline{\overset{\vee}{\phi}{}^j_h(\vec{x})}d\vec{x}d\mu(h)\\ &= \int_H \langle \overset{\vee}{\phi}{}^i_h, \overset{\vee}{\phi}{}^j_h\rangle_{L^2(\mathbb{R}^n)}d\mu(h) = \int_H \left\langle \phi^i_h, \phi^j_h\right\rangle_{L^2(\widehat{\mathbb{R}^n})}d\mu(h)\\ &= \int_H\int_{\widehat{\mathbb{R}^n}} \phi^i_h(\vec{\gamma})\overline{\phi^j_h(\vec{\gamma})}d\vec{\gamma}d\mu(h)\\ &= \int_H\int_{\mathcal{O}_i\cap\mathcal{O}_j} \widehat{f}_i(\vec{\gamma})\overline{\chi_i(h)\widehat{\psi}_i(\vec{\gamma}h)}\overline{\widehat{f}_j(\vec{\gamma})\chi_j(h)\overline{\widehat{\psi_j}(\vec{\gamma}h)}}d\vec{\gamma}d\mu(h)\\ &= \int_{\mathcal{O}_i\cap\mathcal{O}_j} \widehat{f}_i(\vec{\gamma})\overline{\widehat{f}_j(\vec{\gamma})}\int_H \overline{\chi_i(h)\widehat{\psi}_i(\vec{\gamma}h)}\chi_j(h)\widehat{\psi_j}(\vec{\gamma}h)d\mu(h)d\vec{\gamma}.\end{aligned}$$

Thus

$$\begin{aligned}&\langle f_i, f_j\rangle_* = 0 \quad \forall f_i\in\mathcal{K}_i, f_j\in\mathcal{K}_j\\ &\Leftrightarrow \int_{\mathcal{O}_i\cap\mathcal{O}_j} \widehat{f}_i(\vec{\gamma})\overline{\widehat{f}_j(\vec{\gamma})}\int_H \overline{\chi_i(h)\widehat{\psi}_i(\vec{\gamma}h)}\chi_j(h)\widehat{\psi_j}(\vec{\gamma}h)d\mu(h)d\vec{\gamma} = 0 \;\; \forall \widehat{f}_i, \widehat{f}_j \in L^2(\mathcal{O}_i\cap\mathcal{O}_j).\end{aligned}$$

By choosing $\widehat{f}_i$ and $\widehat{f}_j$ characteristic functions of measurable sets, we obtain

$$\begin{aligned}&\langle f_i, f_j\rangle_* = 0 \quad \forall f_i\in\mathcal{K}_i, f_j\in\mathcal{K}_j\\ &\qquad\Leftrightarrow \int_A\int_H \overline{\chi_i(h)\widehat{\psi}_i(\vec{\gamma}h)}\chi_j(h)\widehat{\psi_j}(\vec{\gamma}h)d\mu(h)d\vec{\gamma} = 0 \quad \forall A\subset\mathcal{O}_i\cap\mathcal{O}_j \text{ measurable}\\ &\qquad\Leftrightarrow \int_H \overline{\chi_i(h)\widehat{\psi}_i(\vec{\gamma}h)}\chi_j(h)\widehat{\psi_j}(\vec{\gamma}h)d\mu(h) = 0 \quad a.e.\ \vec{\gamma}\in\mathcal{O}_i\cap\mathcal{O}_j\end{aligned}$$

which proves the theorem. □

The above Theorem yields the following well-known fact. Suppose, we are given a finite or countable partition $\{\mathcal{O}_j\}_{j\in J}$ of $\widehat{\mathbb{R}^n}$ consisting of H-invariant measurable sets. Let $\mathcal{H}_j = \mathcal{F}^{-1}(L^2(\mathcal{O}_j))$ be the corresponding π-invariant subspaces of $L^2(\mathbb{R}^n)$. For each $j\in J$, let $\psi_j\in\mathcal{H}_j$ be admissible for H, that is

$$\int_H \left|\widehat{\psi}_i(\vec{\gamma}h)\right|^2 d\mu(h) = c_\psi$$

for a.e. $\vec{\gamma}\in\mathcal{O}_i$. Then $\psi := \sum_{j\in J}\psi_j$ is admissible for $L^2(\mathbb{R}^n)$.

3.2.4 Construction of Admissible Vectors

We now generalize Theorem 3.3 to sums of modulated wavelet representations. It will turn out that the conditions of Theorem 3.3 remain valid. By Proposition 3.5, any H which is admissible for such a sum of modulated wavelet representations must satisfy (a) of Theorem 3.3 (Theorem 3.3 is formulated for $L^2(\mathbb{R}^n)$, but remains valid for spaces $\mathcal{F}^{-1}(L^2(\mathcal{O}))$ where $\mathcal{O}$ is H-invariant). We thus need only generalize part (b) of that Theorem. In addition, we provide a construction of admissible vectors whose projection onto each component space $\mathcal{H}_j$ is bandlimited, under mild assumptions on the dilation group H.

The following proofs are formulated for a countably infinite set J, but are also valid for finite J. When convenient, we will identify J with $\mathbb{N}$.

Definition 3.7. *A Borel subset S of $\widehat{\mathbb{R}^n}$ is called a transversal for the action of H on $\widehat{\mathbb{R}^n}$ if*

(a) SH is co-null in $\widehat{\mathbb{R}^n}$,

(b) $\vec{\gamma}'h = \vec{\gamma}$ for $\vec{\gamma}, \vec{\gamma}' \in S$ and $h \in H$ implies that $\vec{\gamma} = \vec{\gamma}'$.

The first property says that almost every orbit intersects the set S, while the second property states that an orbit intersects the set S at most once. Such transversals usually exists:

Proposition 3.8. (Führ 2005 and Romero 2006) *Let H be a closed subgroup of $GL_n(\mathbb{R})$ with the property that almost all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ possess compact ε-stabilizers. Then there exists a transversal S. Furthermore, the map $S \times H \to \widehat{\mathbb{R}^n}$ maps Borel set to Borel set, and $\lambda(SN) = 0$ for all null sets N in H.*

Remark 3.2. The construction in the proof of Romero (2006) gives a transversal all of whose points have compact ε-stabilizers; hence all $\vec{\gamma} \in SH$ have compact ε-stabilizers.

Theorem 3.9. *Let H be a closed subgroup of $GL_n(\mathbb{R})$, and suppose*

(a) $\Delta \not\equiv |\det|$,

(b) ε-stabilizers are compact for a.e. $\vec{\gamma} \in \mathcal{O}$.

Then H is admissible for the sum of modulated wavelet representations.

Proof. We proceed by constructing an admissible vector $\psi \in \mathcal{H}$. The idea is to construct a family of functions $\{\widehat{\psi_j}\}_{j\in J}$ of disjoint supports. This will ensure that (3.10) holds for $i \neq j$.

Since ε-stabilizers are compact for a.e. $\vec{\gamma} \in \widehat{\mathbb{R}^n}$, then by Proposition 3.8 there exists a transversal S for the action of H on $\widehat{\mathbb{R}^n}$ all of whose points have compact ε-stabilizers. Then $S \cap \mathcal{O}$ is a transversal for the action of H on $\mathcal{O}$, hence we replace S by $S \cap \mathcal{O}$. Pick a compact neighborhood V of the identity e in H and set $T = SV$. Then T is a Borel subset of $\mathcal{O}$ by Proposition 3.8, and T has positive measure, as H is 2^{nd}-countable.

Define a Borel function

$$\sigma(\vec{\gamma}) = \int_H \mathbf{1}_T(\vec{\gamma}h)d\mu(h)$$

for $\vec{\gamma} \in \mathcal{O}$. Loosely speaking, this function measures how much of an orbit passes through T, the measure being the Haar measure from H. We need to show that this function is finite valued and has a positive lower bound.

Let $\vec{\gamma} \in SH$ be arbitrary. Since S is a transversal, $\vec{\gamma}$ lies in the orbit of a unique $\vec{\gamma}_0 \in S$. That is, there exists $\tilde{h} \in H$ such that $\vec{\gamma} = \vec{\gamma}_0\tilde{h}$ and also $T \cap \mathcal{O}_H(\vec{\gamma}) = \vec{\gamma}_0 V$. Thus,

$$\sigma(\vec{\gamma}) = \int_H \mathbf{1}_{SV}(\vec{\gamma}h)d\mu(h) = \int_H \mathbf{1}_{\vec{\gamma}_0 V}(\vec{\gamma}_0\tilde{h}h)d\mu(h) = \int_H \mathbf{1}_{\vec{\gamma}_0 V}(\vec{\gamma}_0 h)d\mu(h).$$

Note that by left-invariance of the Haar measure, $\sigma(\vec{\gamma}h) = \sigma(\vec{\gamma})$ for all $h \in \mathcal{H}$, and also

$$\begin{aligned}
\vec{\gamma}_0 h \in \vec{\gamma}_0 V &\Leftrightarrow \vec{\gamma}_0 h = \vec{\gamma}_0 v \quad \text{for some } v \in V \\
&\Leftrightarrow \vec{\gamma}_0 h v^{-1} = \vec{\gamma}_0 \quad \text{for some } v \in V \\
&\Leftrightarrow h v^{-1} \in H_{\vec{\gamma}_0} \quad \text{for some } v \in V \\
&\Leftrightarrow h \in H_{\vec{\gamma}_0} V. \qquad (3.11)
\end{aligned}$$

By assumption (b), the stabilizer $H_{\vec{\gamma}_0}$ is compact, hence

$$\sigma(\vec{\gamma}) = \int_H \mathbf{1}_{\vec{\gamma}_0 V}(\vec{\gamma}_0 h) d\mu(h) = \int_H \mathbf{1}_{H_{\vec{\gamma}_0} V}(h) d\mu(h) = \mu(H_{\vec{\gamma}_0} V) < \infty.$$

Moreover, since $V \subset H_{\vec{\gamma}_0} V$, we also obtain

$$0 < \mu(V) \leqslant \mu(H_{\vec{\gamma}_0} V) = \sigma(\vec{\gamma}) < \infty.$$

We have shown that $0 < \mu(V) \leqslant \sigma(\vec{\gamma})$ for all $\vec{\gamma} \in SH$, and σ is finite valued.

Next we move T along orbits to obtain a countable, disjoint collection of sets. First, set

$$g(h) = \frac{|\det h|}{\Delta(h)}.$$

Since $g : H \to (0, \infty)$ is continuous, then $g(V)$ is a compact subset of $(0, \infty)$. Furthermore, since $1 \in g(V)$, then $g(V) \subset [a, b]$ for some $0 < a < 1 < b$. In addition, as g is not constant by assumption, there exists $h_0 \in H$ such that

$$g(h_0) < \frac{a}{b}. \qquad (3.12)$$

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We now show that $Th_0^m \cap Th_0^n = \emptyset$ for $m \neq n$. In fact,

$$\begin{aligned} Th_0^m \cap Th_0^n \neq \emptyset &\Leftrightarrow \vec{\gamma}_0 V h_0^m \cap \vec{\gamma}_0 V h_0^n \neq \emptyset \quad \text{for some } \vec{\gamma}_0 \in S \\ &\Leftrightarrow \vec{\gamma}_0 v h_0^m = \vec{\gamma}_0 \tilde{v} h_0^n \quad \text{for some } v, \tilde{v} \in V,\ \vec{\gamma}_0 \in S \\ &\Leftrightarrow \vec{\gamma}_0 v h_0^{m-n} \tilde{v}^{-1} = \vec{\gamma}_0 \quad \text{for some } v, \tilde{v} \in V,\ \vec{\gamma}_0 \in S \\ &\Leftrightarrow v h_0^{m-n} \tilde{v}^{-1} \in H_{\vec{\gamma}_0} \quad \text{for some } v, \tilde{v} \in V,\ \vec{\gamma}_0 \in S \\ &\Leftrightarrow v h_0^m \in H_{\vec{\gamma}_0} V h_0^n \quad \text{for some } v \in V,\ \vec{\gamma}_0 \in S \\ &\Leftrightarrow V h_0^m \cap H_{\vec{\gamma}_0} V h_0^n \neq \emptyset \quad \text{for some } \vec{\gamma}_0 \in S. \end{aligned}$$

Now by assumption (b), $H_{\vec{\gamma}_0}$ is a compact subgroup, hence $g(H_{\vec{\gamma}_0}) = \{1\}$. Using the homomorphism property of g, we have

$$\begin{aligned} g(Vh_0^m \cap H_{\vec{\gamma}_0} V h_0^n) &= g\left((Vh_0^{m-n} \cap H_{\vec{\gamma}_0} V) h_0^n\right) = g(Vh_0^{m-n} \cap H_{\vec{\gamma}_0} V) g(h_0^n) \\ &\subset \left(g(V) g(h_0^{m-n}) \cap g(V)\right) g(h_0^n) \subset \left([a,b] g(h_0)^{m-n} \cap [a,b]\right) g(h_0)^n = \emptyset \end{aligned}$$

for $m \neq n$ by (3.12). We conclude that $Vh_0^m \cap H_{\vec{\gamma}_0} V h_0^n = \emptyset$ for all $\vec{\gamma}_0 \in S$, $m \neq n$, and hence

$$Th_0^m \cap Th_0^n = \emptyset \qquad (m \neq n). \tag{3.13}$$

It follows that the function $\left\{\mathbf{1}_{Th_0^n}\right\}_{n=1}^{\infty}$ have a disjoint supports. We now modify these functions to make (3.10) hold for $i = j$.

We partition $\mathbb{N}$ into a union of sequences $\left\{n_k^{(j)}\right\}_{k=1}^{\infty}$, $j \in J$. Since $\lim_{m\to\infty} g(h_0)^m = \lim_{m\to\infty} \frac{\left|\det h_0^m\right|}{\Delta(h_0^m)} = 0$, then for each j we may replace the sequence $\left\{n_k^{(j)}\right\}_{k=1}^{\infty}$ by an appropriate subsequence to obtain

$$\sum_{k=1}^{\infty} \left[\frac{|\det h_0|}{\Delta(h_0)}\right]^{n_k^{(j)}} < \frac{1}{2^j}.$$

Here we have identified J with $\mathbb{N}$.

Let $\{I_k\}_{k=1}^{\infty}$ be the collection of cubes $[0,1)^n + \vec{\omega}_k$, $\vec{\omega}_k \in \mathbb{Z}^n$, and let $T_k =$

$T \cap I_k$. Then set $T_k^{(j)} = T_k h_0^{n_k^{(j)}} \cap \mathcal{O}_j$ for each j and k. By (3.13) we still have

$$T_k^i \cap T_l^j = \emptyset \tag{3.14}$$

for $i \neq j$ or $k \neq l$. Now consider the functions ψ_j whose Fourier transforms are defined on $\mathcal{O}_j$ by

$$\widehat{\psi_j} = \left[\frac{1}{\sigma}\sum_{k=1}^{\infty} \Delta(h_0)^{-n_k^{(j)}} \mathbf{1}_{T_k^{(j)}}\right]^{1/2}$$

Then by (3.14) $\widehat{\psi_i}\widehat{\psi_j} = 0$ for $i \neq j$. Furthermore,

$$\begin{aligned}
\int_{\mathcal{O}_j} \left|\widehat{\psi_j}(\vec{\gamma})\right|^2 d\vec{\gamma} &= \int_{\mathcal{O}_j} \frac{1}{\sigma(\vec{\gamma})}\left[\sum_{k=1}^{\infty} \Delta(h_0)^{-n_k^{(j)}} \mathbf{1}_{T_k^{(j)}}(\vec{\gamma})\right] d\vec{\gamma} \\
&\leqslant \frac{1}{\mu(V)}\sum_{k=1}^{\infty} \Delta(h_0)^{-n_k^{(j)}} \lambda(T_k^{(j)}) \leqslant \frac{1}{\mu(V)}\sum_{k=1}^{\infty} \frac{|\det(h_0)|^{n_k^{(j)}}}{\Delta(h_0)^{n_k^{(j)}}}\lambda(T_k) < \frac{1}{2^j\mu(V)}
\end{aligned}$$

which shows that each ψ_j is well defined as an element of $\mathcal{H}_j$, as is $\psi = \sum_{j\in J} \psi_j \in \mathcal{H}$. Furthermore, for all $\vec{\gamma} \in SH \cap \mathcal{O}_j$, by Proposition 2.7,

$$\begin{aligned}
\int_H \left|\widehat{\psi_j}(\vec{\gamma}h)\right|^2 d\mu(h) &= \int_H \frac{1}{\sigma(\vec{\gamma}h)}\left[\sum_{k=1}^{\infty} \Delta(h_0)^{-n_k^{(j)}} \mathbf{1}_{T_k^{(j)}}(\vec{\gamma}h)\right] d\mu(h) \\
&= \frac{1}{\sigma(\vec{\gamma})}\sum_{k=1}^{\infty} \int_H \Delta(h_0)^{-n_k^{(j)}} \mathbf{1}_{T_k^{(j)}}(\vec{\gamma}h) d\mu(h) \\
&= \frac{1}{\sigma(\vec{\gamma})}\sum_{k=1}^{\infty} \int_H \Delta(h_0)^{-n_k^{(j)}} \mathbf{1}_{T_k}(\vec{\gamma}h h_0^{-n_k^{(j)}}) d\mu(h) \\
&= \frac{1}{\sigma(\vec{\gamma})}\sum_{k=1}^{\infty} \int_H \mathbf{1}_{T_k}(\vec{\gamma}h) d\mu(h) = \frac{1}{\sigma(\vec{\gamma})}\int_H \sum_{k=1}^{\infty} \mathbf{1}_{T_k}(\vec{\gamma}h) d\mu(h) \\
&= \frac{1}{\sigma(\vec{\gamma})}\int_H \mathbf{1}_{\bigcup_{k=1}^{\infty} T_k}(\vec{\gamma}h) d\mu(h) = \frac{1}{\sigma(\vec{\gamma})}\int_H \mathbf{1}_T(\vec{\gamma}h) d\mu(h) = 1
\end{aligned}$$

while for all $i \neq j$ and $\vec{\gamma} \in \mathcal{O}_i \cap \mathcal{O}_j$ as $\widehat{\psi_i}\widehat{\psi_j} = 0$,

$$\int_H \chi_i(h)\widehat{\psi_i}(\vec{\gamma}h)\overline{\chi_j(h)\widehat{\psi_j}(\vec{\gamma}h)} d\mu(h) = 0.$$

Hence ψ is admissible by Theorem 3.6. $\square$

We will call a vector $f = \sum_{j\in J} f_j \in \mathcal{H}$ bandlimited, if there exists $M > 0$ such that $\|\vec{\gamma}\| \leqslant M$ for all $\vec{\gamma} \in supp(\widehat{f_j})$, and all $j \in J$. To obtain bandlimited admissible functions, bounded transversals of the following form are required:

Property A.

(a) ε-stabilizers are compact for a.e. $\vec{\gamma} \in \mathcal{O}$.

(b) Given $M > 0$, there exist $m > 0$ and a transversal S for the action of H on $\mathcal{H}$ such that

$$m < \|\vec{\gamma}\| < M$$

for all $\vec{\gamma} \in S$.

It is not difficult to show that if ε-stabilizers are compact and H possesses an expanding matrix, then property A holds. However, there exist groups H which do not contain expanding matrices, but still have property A (see Romero, 2006).

Theorem 3.10. *Let H be a closed subgroup of $GL_n(\mathbb{R})$, possessing property A. Then for each $M > 0$ and $r > 0$, there exists a bandlimited admissible ψ with $c_\psi = 1$, and $\left\|\widehat{\psi_j}\right\|_\infty \leqslant M$, and the support of each $\widehat{\psi_j}$ contained in the ball $B_r(0)$, for all $j \in J$.*

Proof. We proceed by constructing a sequence $\left\{\widehat{\psi_j}\right\}_{j \in J}$ of admissible functions with disjoint supports. By assumption (b), H is not compact, hence we can pick a precompact Borel subset V of H with $\mu(V) \geqslant 1/M^2$. By continuity of the action of H on $\widehat{\mathbb{R}^n}$, there exist $0 < \alpha < \beta$ such that

$$\alpha \leqslant \|\vec{\gamma} h\| \leqslant \beta \tag{3.15}$$

for all $\vec{\gamma}$ in the unit sphere S^{n-1} and $h \in V$.

By assumption (b), there exist a transversal S_1 for the action of H on $\mathcal{O}$ and $m_1 > 0$, such that

$$m_1 < \|\vec{\gamma}\| < r/\beta$$

for all $\vec{\gamma} \in S_1$.

Now set $T_1 = S_1V$, then by (3.15),

$$M_1 := \alpha m_1 \leqslant \|\vec{\gamma}\| \leqslant r$$

for all $\vec{\gamma} \in T_1$. Similarly, there exist a bounded transversal S_2 and $m_2 > 0$ so that

$$m_2 < \|\vec{\gamma}\| < M_1/\beta$$

for all $\vec{\gamma} \in S_2$. Setting $T_2 = S_2V$, then by (3.15),

$$M_2 := \alpha m_2 \leqslant \|\vec{\gamma}\| \leqslant M_1$$

for all $\vec{\gamma} \in T_2$. Continuing inductively, we arrive at a family of bounded transversal $\{S_j\}_{j\in J}$ and disjoint Borel subsets $T_j = S_jV$ of $\mathcal{O}$, each contained in a ball of radius no more than $r(\alpha/\beta)^{j-1}$, so that

$$\lambda(T_j) \leqslant C\left(\frac{\alpha}{\beta}\right)^{(j-1)n}$$

for all $j \in J$, where C denotes the volume of a ball of radius r.

For each j, let us define a Borel function

$$\sigma_j(\vec{\gamma}) = \int_H \mathbf{1}_{T_j}(\vec{\gamma}h)d\mu(h)$$

for $\vec{\gamma} \in \mathcal{O}_j$. Then as in the proof of the proceeding theorem,

$$0 < \mu(V) \leqslant \sigma_j(\vec{\gamma}) < \infty$$

for all $j \in J$ and $\vec{\gamma} \in S_jH \cap \mathcal{O}_j$. Consider the function ψ_j defined by

$$\widehat{\psi_j} = \frac{1}{\sqrt{\sigma_j}}\mathbf{1}_{T_j\cap\mathcal{O}_j}.$$

Then $\widehat{\psi_i}\widehat{\psi_j} = 0$ for $i \neq j$ and

$$\left\|\widehat{\psi_j}\right\|_\infty = \sup_{\vec{\gamma}\in\widehat{\mathbb{R}^n}} \frac{1}{\sqrt{\sigma_j(\vec{\gamma})}} \leqslant \frac{1}{\sqrt{\mu(V)}} \leqslant M,$$

while also,

$$\left\|\widehat{\psi_j}\right\|^2_{L^2(\mathcal{O}_j)} = \int_{\mathcal{O}_j} \left|\widehat{\psi_j}(\vec{\gamma})\right|^2 d\vec{\gamma} = \int_{\mathcal{O}_j} \frac{1}{\sigma_j(\vec{\gamma})} \mathbf{1}_{T_j}(\vec{\gamma}) d\vec{\gamma} \leqslant \frac{1}{\mu(V)} \int_{\mathcal{O}_j} \mathbf{1}_{T_j}(\vec{\gamma}) d\vec{\gamma}$$
$$\leqslant \frac{\lambda(T_j)}{\mu(V)} < M^2 C \left(\frac{\alpha}{\beta}\right)^{(j-1)n}.$$

This shows that the functions $\psi_j \in \mathcal{H}_j$ are well defined, and so is $\psi = \sum_{j\in J} \psi_j \in \mathcal{H}$. Furthermore, for all $\vec{\gamma} \in S_j H \cap \mathcal{O}_j$, we have

$$\int_H \left|\widehat{\psi_j}(\vec{\gamma}h)\right|^2 d\mu(h) = \int_H \frac{1}{\sigma_j(\vec{\gamma}h)} \mathbf{1}_{T_j}(\vec{\gamma}h) d\mu(h) = \frac{1}{\sigma_j(\vec{\gamma})} \int_H \mathbf{1}_{T_j}(\vec{\gamma}h) d\mu(h) = 1$$

while if $\vec{\gamma} \in S_i H \cap S_j H \cap \mathcal{O}_i \cap \mathcal{O}_j$ and $i \neq j$, then

$$\int_H \chi_i(h) \widehat{\psi}_i(\vec{\gamma}h) \overline{\chi_j(h) \widehat{\psi_j}(\vec{\gamma}h)} d\mu(h) = 0.$$

The assertion follows from Theorem 3.6. $\square$

3.2.5 Modulated Wavelet Frames

In Heil and Walnut (1989) it was shown how to obtain frames for wavelet representations in $L^2(\mathbb{R})$. This was later generalized by Bernier and Taylor (1996) to $L^2(\mathbb{R}^n)$ in the case of open free H-orbits, and to the general case by Romero (2006). A discrete subset P of H and a lattice Γ in $\mathbb{R}^n$ were chosen to obtain wavelet frames of the form

$$\left\{\pi\left((k, \vec{u})^{-1}\right)\psi \,:\, k \in P, \vec{u} \in \Gamma\right\}.$$

The reason inverses $\pi\left((k, \vec{u})^{-1}\right)$ are chosen is that

$$\pi\left((k, \vec{u})^{-1}\right) = \pi(k^{-1}, -k^{-1}\vec{u}) = T_{-k^{-1}\vec{u}} D_{k^{-1}} = D_{k^{-1}} T_{-\vec{u}}$$

so that the frames vectors have the form

$$(D_{k^{-1}} T_{-\vec{u}} \psi)(\vec{x}) = |\det k|^{1/2} \psi(k\vec{x} + \vec{u})$$

as with the discrete wavelet transform.

We now show how to obtain frames for sums of modulated wavelet transforms, using the ideas of the above references as a starting point.

Definition 3.11. *Let H be a closed subgroup of $GL_n(\mathbb{R})$. An N-tiling pair is a pair (P, F) of subsets of H where P is a countable subset of H and F is a pre-compact subset of H satisfying*

(a) $\bigcup_{k\in P} Fk$ *is co-null in H.*

(b) $N := \sup_{k\in P} card\,\{p \in P \,:\, Fk \cap Fp \neq \emptyset\} < \infty.$

If $N = 1$, it is called a tiling pair.

The first condition says that the translates $\{Fk\}_{k\in P}$ cover H measurably, while the second conditions say that at each set Fk may intersect at most $N-1$ other sets Fp.

Theorem 3.12. *Let (P, F) be an N-tiling pair, and suppose*

(a) H possesses property A

(b) $card(H_{\vec{\gamma}}) \leqslant M$ a.e. $\vec{\gamma} \in \mathcal{O}$.

Then given any $B \in GL_n(\mathbb{R})$ with corresponding lattice $\Gamma = B^{-1}\mathbb{Z}^n$, there exists $\psi \in \mathcal{H}$ such that

$$\left\{\pi^{\chi}\left((k, \vec{u})^{-1}\right)\psi \,:\, k \in P, \vec{u} \in \Gamma\right\}$$

is a frame for $\mathcal{H}$ with frame bounds 1 and MN.

Proof. Proceeding essentially as in the proof of Theorem 3.10 we obtain a family $\{T_j\}_{j\in J}$ of disjoint Borel subsets of $\mathcal{O}$ of positive measure, $T_j = S_jV$ where $V = F$, all contained in the parallelpiped

$$R = \left[-\frac{1}{2}, \frac{1}{2}\right]^n B$$

and satisfying

$$0 < \lambda(T_j) \leqslant C\kappa^{j-1} \tag{3.16}$$

for some $C > 0$ and $0 < \kappa < 1$.

By (3.16), we may define $\psi = \sum_{j\in J} \psi_j \in \mathcal{H}$ by

$$\widehat{\psi_j} = \frac{1}{\sqrt{\beta}} \mathbf{1}_{T_j \cap \mathcal{O}_j}$$

for $j \in J$, where $\beta = |\det B|$. Observe that the collection of function $\{e_{\vec{u}}\}_{\vec{u}\in\Gamma}$ with

$$e_{\vec{u}}(\vec{\gamma}) = \frac{1}{\sqrt{\beta}} e^{2i\pi\vec{\gamma}\vec{u}} \qquad (\vec{\gamma} \in R)$$

is an orthonormal basis of $L^2(R)$. Thus for $f \in \mathcal{H}$, we have as $\widehat{\psi_j}$ is supported in R,

$$\begin{aligned}
\left\langle f, \pi^{\chi}\left((k,\vec{u})^{-1}\right)\psi\right\rangle_{\mathcal{H}} &= \sum_{j\in J} \left\langle f_j, \pi_j^{\chi_j}\left((k,\vec{u})^{-1}\right)\psi\right\rangle_{\mathcal{H}_j} = \sum_{j\in J} \left\langle f_j, T_{-k^{-1}\vec{u}} D_{k^{-1}}^{\chi_j}\psi_j\right\rangle_{\mathcal{H}_j} \\
&= \sum_{j\in J} \left\langle \widehat{f}_j, E_{k^{-1}\vec{u}} D_{k^{-1}}^{\chi_j}\widehat{\psi_j}\right\rangle_{L^2(\mathcal{O}_j)} \\
&= \sum_{j\in J} \int_{\mathcal{O}_j} |\det k|^{-1/2}\, \widehat{f}_j(\vec{\gamma})\overline{\chi_j(k)\widehat{\psi_j}(\vec{\gamma}k^{-1})} e^{-2i\pi\vec{\gamma}k^{-1}\vec{u}} d\vec{\gamma} \\
&= \sum_{j\in J} \int_{\mathcal{O}_j} |\det k|^{1/2}\, \widehat{f}_j(\vec{\gamma}k)\overline{\chi_j(k^{-1})\widehat{\psi_j}(\vec{\gamma})} e^{-2i\pi\vec{\gamma}\vec{u}} d\vec{\gamma} \\
&= \sqrt{\beta} \int_R \underbrace{|\det k|^{1/2} \sum_{j\in J} \widehat{f}_j(\vec{\gamma}k)\overline{\chi_j(k^{-1})\widehat{\psi_j}(\vec{\gamma})}}_{:=\phi_k(\vec{\gamma})}\, \overline{e_{\vec{u}}(\vec{\gamma})} d\vec{\gamma} \\
&= \sqrt{\beta}\, \langle \phi_k, e_{\vec{u}} \rangle_{L^2(R)}
\end{aligned}$$

where $\phi_k \in L^2(R)$ and exchange of sum and integral are justified as the functions $\widehat{f}_j(\vec{\gamma}k)\widehat{\psi_j}(\vec{\gamma})$ are integrable of disjoint and bounded support. (Here, all functions are extended to R in a natural way.) The latter property also gives

$$
\begin{aligned}
\sum_{\vec{u}\in\Gamma} \left|\left\langle f, \pi^{\chi}\left((k,\vec{u})^{-1}\right)\psi\right\rangle_{\mathcal{H}}\right|^2 &= \beta \sum_{\vec{u}\in\Gamma} \left|\langle \phi_k, e_{\vec{u}}\rangle_{L^2(R)}\right|^2 = \beta \left\|\phi_k\right\|^2_{L^2(R)} \\
&= \beta \left|\det k\right| \int_R \left|\sum_{j\in J} \widehat{f}_j(\vec{\gamma}k)\overline{\chi_j(k^{-1})\widehat{\psi_j}(\vec{\gamma})}\right|^2 d\vec{\gamma} \\
&= \beta \left|\det k\right| \int_R \left[\sum_{i\in J} \widehat{f}_i(\vec{\gamma}k)\overline{\chi_i(k^{-1})\widehat{\psi_i}(\vec{\gamma})}\right] \overline{\left[\sum_{j\in J} \widehat{f}_j(\vec{\gamma}k)\overline{\chi_j(k^{-1})\widehat{\psi_j}(\vec{\gamma})}\right]} d\vec{\gamma} \\
&= \beta \sum_{j\in J} \left|\det k\right| \int_R \left|\widehat{f}_j(\vec{\gamma}k)\right|^2 \left|\widehat{\psi_j}(\vec{\gamma})\right|^2 d\vec{\gamma} \\
&= \beta \sum_{j\in J} \left|\det k\right| \int_{\mathcal{O}_j} \left|\widehat{f}_j(\vec{\gamma}k)\right|^2 \left|\widehat{\psi_j}(\vec{\gamma})\right|^2 d\vec{\gamma} \\
&= \beta \sum_{j\in J} \int_{\mathcal{O}_j} \left|\widehat{f}_j(\vec{\gamma})\right|^2 \left|\widehat{\psi_j}(\vec{\gamma}k^{-1})\right|^2 d\vec{\gamma}
\end{aligned}
$$

and hence,

$$
\begin{aligned}
\sum_{k\in P}\sum_{\vec{u}\in\Gamma} \left|\left\langle f, \pi^{\chi}\left((k,\vec{u})^{-1}\right)\psi\right\rangle_{\mathcal{H}}\right|^2 &= \beta \sum_{k\in P}\sum_{j\in J}\int_{\mathcal{O}_j} \left|\widehat{f}_j(\vec{\gamma})\right|^2 \left|\widehat{\psi_j}(\vec{\gamma}k^{-1})\right|^2 d\vec{\gamma} \\
= \sum_{j\in J}\int_{\mathcal{O}_j} \left|\widehat{f}_j(\vec{\gamma})\right|^2 \left[\sum_{k\in P} \mathbf{1}_{T_j}(\vec{\gamma}k^{-1})\right] d\vec{\gamma} &= \sum_{j\in J}\int_{\mathcal{O}_j} \left|\widehat{f}_j(\vec{\gamma})\right|^2 \left[\sum_{k\in P} \mathbf{1}_{S_jV}(\vec{\gamma}k^{-1})\right] d\vec{\gamma} \\
= \sum_{j\in J}\int_{\mathcal{O}_j} \left|\widehat{f}_j(\vec{\gamma})\right|^2 \left[\sum_{k\in P} \mathbf{1}_{S_jVk}(\vec{\gamma})\right] d\vec{\gamma}
\end{aligned}
$$

Observe that since the sets $\{Vk\}_{k\in P}$ cover H measurably,

$$
\beta \sum_{k\in P} \left|\widehat{\psi_j}(\vec{\gamma}k^{-1})\right|^2 = \sum_{k\in P} \mathbf{1}_{T_j}(\vec{\gamma}k^{-1}) = \sum_{k\in P} \mathbf{1}_{S_jVk}(\vec{\gamma}) \geqslant \mathbf{1}_{S_jA}(\vec{\gamma}).
$$

where $A = \bigcup_{k\in P} Vk$ is co-null in H. Hence by Proposition 3.8,

$$
\beta \sum_{k\in P} \left|\widehat{\psi_j}(\vec{\gamma}k^{-1})\right|^2 \geqslant 1
$$

for a.e. $\gamma \in \mathcal{O}_j$. Next suppose that $\vec{\gamma} \in S_jVk \cap S_jVl$. Then $\vec{\gamma} = \vec{\gamma}_0 vk = \vec{\gamma}_0 \tilde{v} l$ for some unique $\vec{\gamma}_0 \in S_j$, and $v, \tilde{v} \in V$. Since $card(H_{\vec{\gamma}_0}) \leqslant M$, say $H_{\vec{\gamma}_0} =$

$\{h_1, h_2, ..., h_M\}$, then

$$\begin{aligned}\vec{\gamma}_0 vk = \vec{\gamma}_0 \tilde{v} l &\Leftrightarrow \vec{\gamma}_0 vkl^{-1}\tilde{v}^{-1} = \vec{\gamma}_0 \Leftrightarrow vkl^{-1}\tilde{v}^{-1} \in H_{\vec{\gamma}_0} \\ &\Leftrightarrow vkl^{-1}\tilde{v}^{-1} = h_i \qquad \text{for some } i \in \{1, 2, ..., M\} \\ &\Leftrightarrow vk = h_i \tilde{v} l \qquad \text{for some } i \in \{1, 2, ..., M\} \\ &\Leftrightarrow vk \in \bigcup_{i=1}^{M} h_i V l.\end{aligned}$$

Now by assumption at most N of the set in $\{Vl\}_{l\in P}$ overlap, so vk can be contained in at most N of the sets $h_i V l$ for each i. It follows that vk is contained in at most MN of the sets $h_i V l$. Thus

$$\beta \sum_{k\in P} \left|\widehat{\psi_j}(\vec{\gamma} k^{-1})\right|^2 = \sum_{k\in P} \mathbf{1}_{S_j V k}(\vec{\gamma}) \leqslant MN$$

for a.e. $\vec{\gamma} \in \mathcal{O}_j$. Hence

$$\begin{aligned}\|f\|^2_{\mathcal{H}} = \sum_{j\in J} \left\|\widehat{f_j}\right\|^2_{L^2(\mathcal{O}_j)} &\leqslant \sum_{k\in P}\sum_{\vec{u}\in\Gamma} \left|\left\langle f, \pi^{\chi}\left((k,\vec{u})^{-1}\right)\psi\right\rangle_{\mathcal{H}}\right|^2 \\ &\leqslant MN \sum_{j\in J} \left\|\widehat{f_j}\right\|^2_{L^2(\mathcal{O}_j)} = MN \|f\|^2_{\mathcal{H}}.\end{aligned}$$

The proof is thus complete. $\square$

We note here that R may be chosen to be any parallelpiped covering the support of all functions $\widehat{\psi_j}$ and need not be centered at zero. For example, if $\mathcal{O}$ is the first orthant in $\widehat{\mathbb{R}^n}$, we may choose $R = [0,1]^n = \vec{\gamma}_0 + [-\frac{1}{2}, \frac{1}{2}]^n$ where $\vec{\gamma}_0 = \left(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$. One can also obtain frames from a bandlimited admissible function by using the integrated wavelets of Heinlein (2003).

Theorem 3.13. *Suppose there exists a bandlimited admissible function φ_0 for the usual wavelet transform on $L^2(\mathbb{R}^n)$, with supp$(\widehat{\varphi_0}) \cap B_\varepsilon(0) = \emptyset$ for some $\varepsilon > 0$. Then given an N-tilling pair (P, V) and $B \in GL_n(\mathbb{R})$ with corresponding lattice $\Gamma = B^{-1}\mathbb{Z}^n$, there exists $\psi \in \mathcal{H}$ such that*

$$\left\{\pi^{\chi}\left((k,\vec{u})^{-1}\right)\psi \,:\, k \in P, \vec{u} \in \Gamma\right\}$$

is a frame for $\mathcal{H}$ with frame bounds 1 *and* N.

Proof. Let (P, V) be an N-tilling pair. For each $\alpha > 0$, the function φ_α defined by

$$\widehat{\varphi_\alpha}(\vec{\gamma}) = \widehat{\varphi_0}(\alpha\vec{\gamma}) \qquad (\vec{\gamma} \in \mathbb{R}^n)$$

is also a admissible, and $\text{supp}(\widehat{\varphi_\alpha}) = \alpha^{-1}\text{supp}(\widehat{\varphi_0})$ with the same constant c_{φ_0}. Thus after replacing φ_0 by an appropriate φ_α, we may assume by compactness of $\overline{V}$ that

$$\text{supp}(\widehat{\varphi_0})\overline{V} \subset R := \left[-\frac{1}{2}, \frac{1}{2}\right]^n B$$

while also $\text{supp}(\widehat{\varphi_0})\overline{V} \cap B_m(0) = \emptyset$ for some $m > 0$. In particular, there exists $0 < m < M$ such that

$$m \leqslant \|\vec{\gamma}\| < M$$

for all $\vec{\gamma} \in \text{supp}(\widehat{\varphi_0})\overline{V}$.

Next define $\varphi_j \in \mathcal{H}_j$ by $\widehat{\varphi_j}(\vec{\gamma}) = \widehat{\varphi_0}(\alpha^j\vec{\gamma})\mathbf{1}_{\mathcal{O}_j}$, where $\alpha = M/m$. Then each φ_j is admissible for $\pi_j^{\chi_j}$ with constant c_{φ_0}. Now let ψ_j be defined by

$$\left|\widehat{\psi_j}(\vec{\gamma})\right|^2 = \frac{1}{\beta c_{\varphi_0}} \int_{V^{-1}} |\widehat{\varphi_j}(\vec{\gamma}h)|^2 \, d\mu(h)$$

for $\vec{\gamma} \in \mathcal{O}_j$, where $\beta = |\det B|$. So each $\widehat{\psi_j}$ is uniquely defined up to phase only. Each $\widehat{\psi_j}$ is supported on $\alpha^{-j}\text{supp}(\widehat{\varphi_0})\overline{V} \subset R$. Thus by choice of α, $\widehat{\psi_i}\widehat{\psi_j} = 0$ for $i \neq j$, and also

$$\begin{aligned}
\left\|\widehat{\psi_j}\right\|^2_{L^2(\mathcal{O}_j)} &= \int_{\mathcal{O}_j} \left|\widehat{\psi_j}(\vec{\gamma})\right|^2 d\vec{\gamma} = \int_{\mathcal{O}_j} \frac{1}{\beta c_{\varphi_0}} \int_{V^{-1}} |\widehat{\varphi_j}(\vec{\gamma}h)|^2 \, d\mu(h)\, d\vec{\gamma} \\
&= \frac{1}{\beta c_{\varphi_0}} \int_{V^{-1}} \int_{\mathcal{O}_j} \left|\widehat{\varphi_0}(\alpha^j\vec{\gamma}h)\right|^2 d\vec{\gamma}\, d\mu(h) \\
&= \frac{1}{\alpha^{jn}\beta c_{\varphi_0}} \int_{V^{-1}} |\det h|^{-1} \int_{\mathcal{O}_j} |\widehat{\varphi_0}(\vec{\gamma})|^2 \, d\vec{\gamma}\, d\mu(h) \\
&\leqslant \frac{K}{\alpha^{jn}} \|\widehat{\varphi_0}\|^2_{L^2(\widehat{\mathbb{R}^n})}
\end{aligned}$$

so that $\psi = \sum_{j \in J} \psi_j$ is defined in $\mathcal{H}$.

Observe that

$$\beta \sum_{k \in P} \left| \widehat{\psi_j}(\vec{\gamma} k^{-1}) \right|^2 = \sum_{k \in P} \frac{1}{c_{\varphi_0}} \int_{V^{-1}} \left| \widehat{\varphi_j}(\vec{\gamma} k^{-1} h) \right|^2 d\mu(h)$$
$$= \frac{1}{c_{\varphi_0}} \sum_{k \in P} \int_{(Vk)^{-1}} \left| \widehat{\varphi_j}(\vec{\gamma} h) \right|^2 d\mu(h). \tag{3.17}$$

Now as $\{Vk\}_{k \in P}$ covers H measurably and φ_j is admissible, for $L^2(\mathcal{O}_j)$,

$$1 = \frac{1}{c_{\varphi_0}} \int_H \left| \widehat{\varphi_j}(\vec{\gamma} h) \right|^2 \leqslant \frac{1}{c_{\varphi_0}} \sum_{k \in P} \int_{(Vk)^{-1}} \left| \widehat{\varphi_j}(\vec{\gamma} h) \right|^2 d\mu(h) = \beta \sum_{k \in P} \left| \widehat{\psi_j}(\vec{\gamma} k^{-1}) \right|^2$$

for a.e. $\vec{\gamma} \in \mathcal{O}_j$. On the other hand, as at most N of the sets $\{Vk\}_{k \in P}$ overlap, then by (3.17)

$$\beta \sum_{k \in P} \left| \widehat{\psi_j}(\vec{\gamma} k^{-1}) \right|^2 = \frac{1}{c_{\varphi_0}} \sum_{k \in P} \int_{(Vk)^{-1}} \left| \widehat{\varphi_j}(\vec{\gamma} h) \right|^2 d\mu(h) \leqslant \frac{N}{c_{\varphi_0}} \int_H \left| \widehat{\varphi_j}(\vec{\gamma} h) \right|^2 d\mu(h) = N.$$

Computing as in the proof of Theorem 3.12, we obtain that

$$\|f\|_{\mathcal{H}}^2 = \sum_{j \in J} \|\widehat{f}_j\|_{L^2(\mathcal{O}_j)}^2 \leqslant \sum_{k \in P} \sum_{\vec{u} \in \Gamma} \left| \left\langle f, \pi^{\chi} \left((k, \vec{u})^{-1} \right) \psi \right\rangle_{\mathcal{H}} \right|^2 \leqslant N \sum_{j \in J} \|\widehat{f}_j\|_{L^2(\mathcal{O}_j)}^2 = N \|f\|_{\mathcal{H}}^2.$$

This proves the theorem. $\square$

CHAPTER IV

EQUIVALENCE OF THE METAPLECTIC REPRESENTATION WITH A SUM OF WAVELET REPRESENTATIONS

We now consider a class of subgroups of the symplectic group $Sp(n, \mathbb{R})$ which are semi-direct products of a vector group by a group of automorphisms, and are isomorphic to affine groups. We show how the metaplectic representation of such a group can be equivalent to a sum of wavelet representations, thus allowing to apply the admissibility results from the previous chapter to the metaplectic representation. A large part of this chapter is devoted to three groups of examples illustrating these techniques.

The first group of examples employs simple one-parameter groups of dilations, and illuminates a number of details of interest. For example, an admissible group which only possesses admissible functions of slow decay at infinity is presented. Another example shows that the admissibility condition in Cordero et al. (2006a) is more restrictive than the usual one.

The second group of examples shows that the dilation invariant subsets U_j used to decompose the metaplectic representation into subrepresentations need not be a union of orthants, but in general take the form of hypercones.

The last example reconsiders the similitude group $SIM(2)$ discussed in Cordero et al. (2006a). We show how this group fits into the framework presented in this thesis, and are able to construct metaplectic frames.

We begin by reviewing the admissibility results for the extended metaplectic representation of Cordero et al. (2006a).

4.1 Admissibility for the Extended Metaplectic Representation

Let D be a closed subgroup of $Sp(n,\mathbb{R})$ and consider the extended metaplectic representation of $L = D \rtimes \mathbb{H}^n$ on $L^2(\mathbb{R}^n)$,

$$\mu_e(\mathcal{A},(\vec{z},t)) = \rho(\vec{z},t)\mu(\mathcal{A})$$

for $\mathcal{A} \in D$, $(\vec{z},t) = (\vec{x},\vec{y},t) \in \mathbb{H}^n$, where ρ is the Schrödinger representation of $\mathbb{H}^n$ and μ the metaplectic representation of $Sp(n,\mathbb{R})$, as presented in section 2.8.3. The voice transform associated with $\phi \in L^2(\mathbb{R}^n)$ is

$$(V_\phi f)(\mathcal{A},(\vec{z},t)) = \langle f, \mu_e(\mathcal{A},(\vec{z},t))\,\phi\rangle \qquad (f \in L^2(\mathbb{R}^2)). \tag{4.1}$$

Note that the group L is not admissible. To see this, observe that the center of $\mathbb{H}^n$ can be identified with the closed normal subgroup $Z = \{(I_{2n},(0,t)) : t \in \mathbb{R}\}$ of $D \rtimes \mathbb{H}^n$, and we have

$$\mu_e(I_{2n},(0,t)) = \rho(0,t)\mu(I_{2n}) = e^{2i\pi t}\mathrm{Id} \tag{4.2}$$

so that $\mu_e(I_{2n},(0,t+k)) = \mu_e(I_{2n},(0,t))$ for all $t \in [0,1)$ and $k \in \mathbb{Z}$. Since the Haar measure on $\mathbb{H}^n$ is the Lebesgue measure $d\vec{z}\,dt$, then by (2.3) the Haar measure on $D \rtimes \mathbb{H}^n$ is of the form $d\nu\,(\mathcal{A},(\vec{z},t)) = J(\mathcal{A}^{-1})\,d\mu(\mathcal{A})\,d\vec{z}\,dt$, where $d\mu$ denotes the Haar measure on D. Hence for all $f,\phi \in L^2(\mathbb{R}^n)$ we have

$$\begin{aligned}\|V_\phi f\|^2_{L^2(L)} &= \int_D \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}} |\langle f, \mu_e\,(\mathcal{A},(\vec{z},t))\,\phi\rangle|^2\,dt\,d\vec{z}\,J(\mathcal{A}^{-1})\,d\mathcal{A} \\ &= \int_D \int_{\mathbb{R}^{2n}} \sum_{k\in\mathbb{Z}} \int_0^1 |\langle f, \mu_e\,(\mathcal{A},(\vec{z},t))\,\phi\rangle|^2\,dt\,d\vec{z}\,J(\mathcal{A}^{-1})\,d\mathcal{A} \in \{0,\infty\}\end{aligned}$$

which shows that no ϕ is admissible. One therefore replaces L by the quotient

$$L = D \rtimes \mathbb{H}^n / Z \cong D \rtimes \mathbb{R}^{2n}.$$

Because of (4.2), μ_e factors to a representation of $D \rtimes \mathbb{R}^{2n}$ also denoted μ_e and which again is uniquely defined up to a phase factor $|\lambda| = 1$, by

$$\mu_e(\mathcal{A}, \vec{z}) = \rho(\vec{z}, 0)\mu(\mathcal{A})$$

and the voice transform associated with $\phi \in L^2(\mathbb{R}^n)$ is now

$$(V_\phi f)(\mathcal{A}, \vec{z}) = \langle f, \mu_e(\mathcal{A}, \vec{z})\phi \rangle$$

for $\mathcal{A} \in D$ and $\vec{z} \in \mathbb{R}^{2n}$. A group $K = D \rtimes \mathbb{R}^{2n}$ is thus admissible for μ_e if and only if there exists $\phi \in L^2(\mathbb{R}^n)$ such that

$$\|V_\phi f\|^2_{L^2(K)} = c_\phi \|f\|^2 \qquad \text{for all } f \in L^2(\mathbb{R}^n).$$

A characterization of admissibility in the flavour of Theorem 3.2 was derived in Cordero et al. (2006a). The Fourier transform is replaced by the Wigner distribution here.

Theorem 4.1. (Cordero et al., 2006a) *Let K be a closed subgroup of $Sp(n, \mathbb{R}) \rtimes \mathbb{R}^{2n}$, and suppose that $\phi \in L^2(\mathbb{R}^n)$ satisfies*

$$\int_K \left| \mathcal{W}_\phi(k^{-1} \cdot \vec{z}) \right| d\nu(k) \leqslant M \tag{4.3}$$

for a.e. $\vec{z} \in \mathbb{R}^{2n}$, for some $M > 0$. Then ϕ is admissible for μ_e if and only if there exists $c_\phi > 0$ such that

$$\int_K \mathcal{W}_\phi(k^{-1} \cdot \vec{z}) d\nu(k) = c_\phi$$

for a.e. $\vec{z} \in \mathbb{R}^{2n}$. Here,

$$k \cdot \vec{z} = (\mathcal{A}, \vec{q}) \cdot \vec{z} = \mathcal{A}\vec{z} + \vec{q}$$

denotes the affine action of K on $\mathbb{R}^{2n}$.

We note that there are a few shortcomings to this theorem. The Wigner distribution is difficult to compute and work with, in particular for functions not in the Schwartz class. The Wigner distribution usually takes both positive and negative values, and thus the assumption (4.3) is required. However, one of our examples will show that there exist admissible functions which do not satisfy condition (4.3). We will also show by example that there exist admissible groups not possessing admissible functions in the Schwartz class. Because of these difficulties, the nontrivial examples presented in Cordero et al. (2006a, 2006b, 2010) all work with groups K which are subgroups of $Sp(n,\mathbb{R})$, so that μ_e reduces to the metaplectic representation μ, and admissibility is proved in a different way: By an ad-hoc method, it is shown that the metaplectic representation possesses subrepresentations which are equivalent to wavelet representations, and admissibility conditions are derived from those of the wavelet representations. This chapter will work this mechanism out in a systematic way, thus producing a class of subgroups of $Sp(n,\mathbb{R})$ for which the metaplectic representation is equivalent to a sum of wavelet representations.

4.2 Semi-Direct Product Subgroups of the Symplectic Group

Every closed subgroup D of $Gl_n(\mathbb{R})$ defines an action on the vector space $Sym(n,\mathbb{R})$ of symmetric matrices by

$$a \cdot m = \alpha_a(m) = (a^{-1})^T m a^{-1} \tag{4.4}$$

for $a \in D$, $m \in Sym(n,\mathbb{R})$. If M is a D-invariant subspace of $Sym(n,\mathbb{R})$, then the corresponding semi-direct product

$$K = D \rtimes M = \{(a,m) : a \in D, m \in M\}$$

possesses the group law

$$(a,m)(a',m') = (aa', m + a\cdot m') = (aa', m + (a^{-1})^T m' a^{-1}) \tag{4.5}$$

for $(a,m),(a',m') \in K$.

This semi-direct product can be identified with a closed subgroup of $Sp(n,\mathbb{R})$. In fact, M and D are isomorphic and homeomorphic to the two closed subgroups of $Sp(n,\mathbb{R})$ of the form

$$N = \left\{ \mathcal{N}_m := \begin{bmatrix} I & 0 \\ m & I \end{bmatrix} : m \in M \right\} \cong M$$

$$L = \left\{ \mathcal{L}_a := \begin{bmatrix} a & 0 \\ 0 & \left(a^T\right)^{-1} \end{bmatrix} : a \in D \right\} \cong D$$

and $\mathcal{L}_a \mathcal{N}_m \mathcal{L}_a^{-1} = \mathcal{N}_{(a^{-1})^T m a^{-1}}$. Therefore, the action of D on M transfers to an action of L on N by

$$\mathcal{L}_a \cdot \mathcal{N}_m = \mathcal{L}_a \mathcal{N}_m \mathcal{L}_a^{-1} = \mathcal{N}_{(a^{-1})^T m a^{-1}} = \mathcal{N}_{a\cdot m}$$

for all $\mathcal{N}_m \in N$, $\mathcal{L}_a \in L$. Since

$$\mathcal{N}_m \mathcal{L}_a \mathcal{N}_{m'} \mathcal{L}_{a'} = \mathcal{N}_m \mathcal{N}_{a\cdot m'} \mathcal{L}_a \mathcal{L}_{a'} = \mathcal{N}_{m + a\cdot m'} \mathcal{L}_{aa'},$$

hence K can be represented by the closed subgroup

$$L \rtimes N = \left\{ \mathcal{N}_m \mathcal{L}_a = \begin{bmatrix} a & 0 \\ ma & \left(a^T\right)^{-1} \end{bmatrix} : a \in D, m \in M \right\} \tag{4.6}$$

of $Sp(n,\mathbb{R})$ and it is easy to see that the map $(a,m) \mapsto \mathcal{N}_m \mathcal{L}_a$ is a homeomorphism. In particular, using (2.13) and (2.14), K has a metaplectic representation given by $\mu(a,m) = N_{-m} D_a$, which is now a proper representation. Given a Borel map $\chi : D \to \Pi$, one can also define a modulated metaplectic representation μ^χ by

$$\mu^\chi(a,m) = \mu^\chi(\mathcal{N}_m \mathcal{L}_a) = N_{-m} D_a^\chi$$

that is,

$$\mu^{\chi}(a,m)f(\vec{x}) = e^{i\pi\langle m\vec{x},\vec{x}\rangle}|\det a|^{-1/2}\chi(a)f(a^{-1}\vec{x}) \tag{4.7}$$

for $f \in L^2(\mathbb{R}^n)$ and $\vec{x} \in \mathbb{R}^n$.

Equation (4.4) shows that the action of D on the vector space M is linear. That is, each $a \in D$ determines an element Ψ_a of $GL(M) = \{T : M \to M \,:\, \text{T is linear and invertible}\}$ by

$$\Psi_a(m) = a \cdot m$$

for $m \in M$. Note that, for $a, b \in D$, we obtain

(a) $\Psi_{ab}(m) = (ab)\cdot m = a\cdot(b\cdot m) = a\cdot(\Psi_b(m)) = \Psi_a(\Psi_b(m)) = (\Psi_a\Psi_b)(m)$

(b) $\Psi_a\Psi_{a^{-1}}(m) = \Psi_I(m) = I\cdot m = m$

for all $m \in M$. Hence Ψ is a homomorphism of D into $GL(M)$.

Since M has finite dimension d, after fixing a basis for M, we may identify M with $\mathbb{R}^d$ and $GL(M)$ with $GL_d(\mathbb{R})$. In the following we will often use the vector notation $\vec{m}$ when we consider an element m of M as a vector in $\mathbb{R}^n$. Hence, each $\Psi_a \in GL(M)$ may be identified with a matrix $h_a \in GL_d(\mathbb{R})$, and under this identification

$$\Psi_a(\vec{m}) = h_a\vec{m}. \tag{4.8}$$

Let us set $H = range(\Psi)$, that is

$$H := \{h_a \in GL_d(\mathbb{R}) \,:\, a \in D\}\,.$$

By (a) and (b), $h_a h_{a'} = h_{aa'}$, and also $a\cdot\vec{m} = h_a\vec{m}$. Note that Ψ is one-to-one if and only if the action of D on M is effective, that is if the global stabilizer is trivial. The map Ψ is easily seen to be continuous because the action (4.4) is. However, H need not be closed in $GL_d(\mathbb{R})$. In what follows, we will thus make two assumptions

(a) $H = range(\Psi)$ is closed,

(b) $\ker \Psi$ is compact.

The second assumption is required because stabilizers for admissible wavelet representations need to be compact.

We can thus consider the closed subgroup G of $\mathrm{Aff}_d(\mathbb{R})$,

$$G := H \rtimes \mathbb{R}^d = \left\{(h_a, \vec{m}) \,:\, h_a \in H, \vec{m} \in \mathbb{R}^d\right\}$$

with the group operation

$$(h_a, \vec{m})(h_{a'}, \vec{m}') = (h_{aa'}, \vec{m} + h_a\vec{m}'). \tag{4.9}$$

The modulated wavelet representation π^χ of G on $L^2(\mathbb{R}^n)$ is given by

$$\pi^\chi(h_a, \vec{m}) = T_{\vec{m}} D^\chi_{h_a}$$

and the Fourier transform induces a modulated wavelet representation δ^χ of G on Ł$^2(\widehat{\mathbb{R}^n})$ by

$$(\delta^\chi(h_a, \vec{m})\hat{f})(\vec{\gamma}) = (E_{-\vec{m}} D^\chi_{h_a} \hat{f})(\vec{\gamma}) = |\det h_a|^{1/2} e^{-2\pi i \vec{\gamma}\vec{m}} \chi(a) \hat{f}(\vec{\gamma} h_a) \tag{4.10}$$

for $(h_a, \vec{m}) \in G$ and $\hat{f} \in L^2(\widehat{\mathbb{R}^d})$.

By (4.8) and (4.9) the map Ψ extends to a group homomorphism $\overline{\Psi}$ of K onto G by

$$\overline{\Psi}(a, m) = (\Psi(a), m) = (h_a, \vec{m}).$$

Thus, the modulated wavelet representations π^χ and δ^χ of G can be considered as representations of K as well. Obviously, if we consider π^χ a representation of G, then $\chi : G \to \Pi$. If we consider it a representation of K, then $\chi : K \to \Pi$. Furthermore, the Haar measure on K is given by

$$d\nu(a, m) = \frac{1}{|\det h_a|} d\mu(a) d\lambda(\vec{m}) \tag{4.11}$$

where λ is the Lebesgue measure on $M \cong \mathbb{R}^d$ and μ the left Haar measure on D. In fact, let $(a', m') \in K$ be given. For all $f \in C_c(K)$, as $a \cdot m = h_a \vec{m}$,

$$\begin{aligned}\int_K f\left((a',m')(a,m)\right) d\nu(a,m) &= \int_D \int_{\mathbb{R}^d} f(a'a, \vec{m}' + h_{a'}\vec{m}) \frac{d\vec{m}\, d\mu(a)}{|\det h_a|} \\ &= \int_D \int_{\mathbb{R}^d} f(a, \vec{m}' + h_{a'}\vec{m}) \frac{d\vec{m}\, d\mu(a)}{|\det h_{a'^{-1}}|\, |\det h_a|} \\ &= \int_D \int_{\mathbb{R}^d} f(a, h_{a'}\vec{m}) \frac{|\det h_{a'}|\, d\vec{m}\, d\mu(a)}{|\det h_a|} \\ &= \int_D \int_{\mathbb{R}^d} f(a, \vec{m}) \frac{d\vec{m}\, d\mu(a)}{|\det h_a|} = \int_K f(a, \vec{m}) d\nu(a,m)\end{aligned}$$

which proves (4.11).

The following is a necessary condition for admissibility of the modulated metaplectic representation:

Proposition 4.2. *Suppose, D is admissible for the modulated metaplectic representation. Then $\Delta_D \not\equiv |\det \circ \Psi|$.*

Proof. Pick an admissible vector $\phi \in L^2(\mathbb{R}^n)$ with $c_\phi = 1$ and set $f = D_{1/2}\phi$. Then

$$\begin{aligned}\|\phi\|_2^2 = \|f\|_2^2 = \|V_\phi f\|_2^2 &= \int_K |\langle f, \mu^\chi(a,m)\phi\rangle|^2 \, d\nu(a,m) \\ &= \int_K \left| \int_{\mathbb{R}^n} 2^{n/2} \phi(2\vec{y}) e^{i\pi\langle m\vec{y},\vec{y}\rangle} |\det a|^{-1/2} \overline{\chi(a)\phi(a^{-1}\vec{y})} d\vec{y} \right|^2 d\nu(a,m) \\ &= \int_D \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^n} 2^{-n/2} \phi(a\vec{y}) e^{i\pi\left\langle \alpha_{a^{-1}}(m/4)\vec{y},\vec{y}\right\rangle} |\det a|^{1/2} \overline{\phi(\vec{y}/2)} d\vec{y} \right|^2 \frac{d\vec{m}\, d\mu(a)}{|\det h_a|} \\ &= \int_D \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^n} \phi(a^{-1}\vec{y}) e^{i\pi\langle \alpha_a(m/4)\vec{y},\vec{y}\rangle} |\det a|^{-1/2} \overline{(D_2\phi)(\vec{y})} d\vec{y} \right|^2 \frac{|\det h_a|\, d\vec{m}\, d\mu(a)}{\Delta_D(a)} \\ &= 4^d \int_D \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^n} \phi(a^{-1}\vec{y}) e^{-i\pi\langle m\vec{y},\vec{y}\rangle} |\det a|^{-1/2} \overline{(D_2\phi)(\vec{y})} d\vec{y} \right|^2 \frac{d\vec{m}\, d\mu(a)}{\Delta_D(a)} \\ &= 4^d \int_K \left| \int_{\mathbb{R}^n} (D_2\phi)(\vec{y}) e^{i\pi\langle m\vec{y},\vec{y}\rangle} |\det a|^{-1/2} \overline{\chi(a)\phi(a^{-1}\vec{y})} d\vec{y} \right|^2 \frac{|\det h_a|}{\Delta_D(a)} d\nu(a,m) \\ &= 4^d \int_K |\langle D_2\phi, \mu(a,m)\phi\rangle|^2 \frac{|\det h_a|}{\Delta_D(a)} d\nu(a,m) \\ &= 4^d \int_K |(V_\phi D_2\phi)(a,m)|^2 \frac{|\det h_a|}{\Delta_D(a)} d\nu(a,m).\end{aligned}$$

Now if $|\det h_a| = \Delta_D(a)$ for all $a \in D$, this becomes

$$\|\phi\|_2^2 = 4^d \|V_\phi D_2\phi\|_{L^2(K)}^2 = 4^d \|D_2\phi\|_2^2 = 4^d \|\phi\|_2^2$$

which is impossible as $\phi \neq 0$. $\square$

4.3 Admissibility for the Modulated Metaplectic Representation

We now look at scenarios where the modulated metaplectic representation of $K = D \rtimes M$ may be equivalent to sums of modulated wavelet representations.

For each $\vec{x} \in \mathbb{R}^n$, the map

$$m \mapsto \langle m\vec{x}, \vec{x} \rangle \qquad (m \in M)$$

defines a bounded linear functional on $M \cong \mathbb{R}^d$. By the Riesz's theorem, there is a unique vector $\Phi(\vec{x}) \in \widehat{\mathbb{R}^d}$ such that

$$\langle m\vec{x}, \vec{x} \rangle = \langle \vec{m}, -2\Phi(\vec{x}) \rangle = -2 \langle \vec{m}, \Phi(\vec{x}) \rangle \tag{4.12}$$

for all $m \in M$. (Note that on the left, the brackets denote the inner product in $\mathbb{R}^n$, while on the right, they denote the duality between $\mathbb{R}^n$, which really is an inner product in $\mathbb{R}^n$ and $\widehat{\mathbb{R}^n}$ are isomorphic.)

Note 3.1: Let us consider properties of the map $\Phi : \mathbb{R}^n \to \widehat{\mathbb{R}^d}$.

(1) $\Phi(\alpha\vec{x}) = \alpha^2\Phi(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$, α scalar.

(2) In particular $\Phi(-\vec{x}) = \Phi(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$. Thus Φ is not one-to-one.

(3) Each component of the row vector $\Phi(\vec{x})$ is a polynomial of degree 2 in the components of $\vec{x}$, and hence Φ is a smooth function on $\mathbb{R}^n$.

Note 3.2: For $\vec{x} \in \mathbb{R}^n$, $m \in M$, and $a \in D$, we have

$$\begin{aligned}\langle \vec{m}, \Phi(\vec{x})h_a \rangle &= \Phi(\vec{x})h_a\vec{m} = \langle h_a\vec{m}, \Phi(\vec{x}) \rangle = \langle a \cdot \vec{m}, \Phi(\vec{x}) \rangle = -\frac{1}{2} \langle (a \cdot \vec{m})\vec{x}, \vec{x} \rangle \\ &= -\frac{1}{2} \left\langle (a^{-1})^T m a^{-1}\vec{x}, \vec{x} \right\rangle = -\frac{1}{2} \left\langle m a^{-1}\vec{x}, a^{-1}\vec{x} \right\rangle = \left\langle \vec{m}, \Phi(a^{-1}\vec{x}) \right\rangle.\end{aligned}$$

Thus

$$\Phi(a^{-1}\vec{x}) = \Phi(\vec{x})h_a \tag{4.13}$$

for all $a \in D$ and $\vec{x} \in \mathbb{R}^n$.

As we want the map Φ to be locally invertible, we will from now on assume that $d = n$. Suppose U is an open D-invariant subset of $\mathbb{R}^n$ on which Φ is injective and has non-vanishing Jacobian determinant $J_\Phi(\vec{x})$. By D-invariant we mean of course that U is invariant under dilation

$$a \cdot \vec{x} = a^{-1}\vec{x} \in U$$

for $a \in D, \vec{x} \in U$. D-invariance implies that $L^2(U)$ will be a μ^χ-invariant closed subspace of $L^2(\mathbb{R}^n)$ as can be seen from (4.7). The remaining assumptions guarantee that the restriction $\Phi|_U$ maps U homeomorphically onto an open subset $\mathcal{O}$ of $\widehat{\mathbb{R}^n}$. For simplicity, write Φ for $\Phi|_U$. Furthermore, by (4.13) $\mathcal{O}$ is H-invariant. It follows from (4.10) that $L^2(\mathcal{O})$ is a δ^χ-invariant closed subspace of $L^2(\widehat{\mathbb{R}^n})$. Since U is open, (4.13) implies that the map $\Psi : a \mapsto h_a$ is one-to-one, so that K and G are isomorphism groups. To see this, pick $\vec{x}_0 \in U$ and an open ball $B_\varepsilon(0)$ such that $\vec{x}_0 + B_\varepsilon(0) \subset U$. Now suppose that $h_a = I_n$. Then for all $\vec{x} \in B_\varepsilon(0)$, we have by (4.13),

$$\begin{aligned} a^{-1}\vec{x} &= a^{-1}(\vec{x}_0 + \vec{x}) - a^{-1}\vec{x}_0 = \Phi^{-1}\left(\Phi(\vec{x}_0 + \vec{x})h_a\right) - \Phi^{-1}\left(\Phi(\vec{x}_0)h_a\right) \\ &= \Phi^{-1}\left(\Phi(\vec{x}_0 + \vec{x})\right) - \Phi^{-1}\left(\Phi(\vec{x}_0)\right) = (\vec{x}_0 + \vec{x}) - \vec{x}_0 = \vec{x}. \end{aligned}$$

As $a^{-1}\vec{x} = \vec{x}$ for all $\vec{x} \in B_\varepsilon(0)$, then $a^{-1}\vec{x} = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$, so that $a = I_n$.

Proposition 4.3. *The map $Q : L^2(U) \to L^2(\mathcal{O})$ defined by*

$$(Qf)(\vec{\gamma}) = |J_{\Phi^{-1}}(\vec{\gamma})|^{1/2} f(\Phi^{-1}(\vec{\gamma})) \tag{4.14}$$

for $f \in L^2(U)$ and $\vec{\gamma} \in \mathcal{O}$, is a linear and surjective isometry.

Proof. Clearly the map Q is linear. By (4.14) we have

$$\begin{aligned}\|Qf\|^2_{L^2(\mathcal{O})} &= \int_{\mathcal{O}} |Qf(\vec{\gamma})|^2\, d\vec{\gamma} = \int_{\mathcal{O}} \left|f(\Phi^{-1}(\vec{\gamma}))\right|^2 |J_{\Phi^{-1}}(\vec{\gamma})|\, d\vec{\gamma} \\ &= \int_U |f(\vec{x})|^2\, d\vec{x} = \|f\|^2_{L^2(U)}\end{aligned}$$

which shows that $Qf \in L^2(\mathcal{O})$ and that Q is an isometry. Finally, let $g \in L^2(\mathcal{O})$ and set

$$f(\vec{x}) = |J_\Phi(\vec{x})|^{1/2} g(\Phi(\vec{x}))$$

for $\vec{x} \in U$. Then

$$\|f\|^2_{L^2(U)} = \int_U |f(\vec{x})|^2\, d\vec{x} = \int_U |g(\Phi(\vec{x}))|^2 \, |J_\Phi(\vec{x})|^{1/2}\, d\vec{x} = \int_{\mathcal{O}} |g(\vec{\gamma})|^2\, d\vec{\gamma} = \|g\|^2_{L^2(\mathcal{O})}$$

which shows that $f \in L^2(U)$. Furthermore,

$$(Qf)(\vec{\gamma}) = |J_{\Phi^{-1}}(\vec{\gamma})|^{1/2} f(\vec{\Phi}^{-1}(\vec{\gamma})) = |J_{\Phi^{-1}}(\vec{\gamma})|^{1/2} |J_\Phi(\Phi^{-1}(\vec{\gamma}))|^{1/2} g(\vec{\gamma}) = g(\vec{\gamma})$$

for all $\vec{\gamma} \in \mathcal{O}$. Thus Q is surjective. $\square$

Proposition 4.4. *The restrictions $\mu^\chi|_{L^2(U)}$ and $\delta^\chi|_{L^2(\mathcal{O})}$ are equivalent. In particular,*

$$E_{-\vec{m}} = QN_{-m}Q^{-1} \qquad \textit{and} \qquad D^\chi_{h_a} = QD^\chi_a Q^{-1}$$

for all $m \in M$ and $a \in D$, when these operators are restricted to the respective invariant subspaces.

Proof. For simplicity, we use Φ to denote the restriction $\Phi|_U$.

Let $m \in M$ be given. By using (4.12), we obtain

$$\begin{aligned}(QN_{-m}f)(\vec{\gamma}) &= |J_{\Phi^{-1}}(\vec{\gamma})|^{1/2}\, (N_{-m}f)(\Phi^{-1}(\vec{\gamma})) \\ &= |J_{\Phi^{-1}}(\vec{\gamma})|^{1/2}\, e^{i\pi\left\langle m\Phi^{-1}(\vec{\gamma}), \Phi^{-1}(\vec{\gamma})\right\rangle} f(\Phi^{-1}(\vec{\gamma})) \\ &= |J_{\Phi^{-1}}(\vec{\gamma})|^{1/2}\, e^{-2i\pi\left\langle \vec{m}, \Phi(\Phi^{-1}(\vec{\gamma}))\right\rangle} f(\Phi^{-1}(\vec{\gamma})) \\ &= |J_{\Phi^{-1}}(\vec{\gamma})|^{1/2}\, e^{-2i\pi\langle \vec{m}, \vec{\gamma}\rangle} f(\Phi^{-1}(\vec{\gamma})) \\ &= e^{-2i\pi\langle \vec{m}, \vec{\gamma}\rangle} (Q_i f)(\vec{\gamma}) = (E_{-\vec{m}}Qf)(\vec{\gamma})\end{aligned}$$

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for $f \in L^2(U)$ and $\vec{\gamma} \in \mathcal{O}$.

Next let $a \in D$. Now (4.13) yields that

$$a^{-1}\Phi^{-1}(\vec{\gamma}) = \Phi^{-1}(\vec{\gamma}h_a)$$

for all $\vec{\gamma} \in \mathcal{O}$ and then by the Chain rule

$$|\det a^{-1}|J_{\Phi^{-1}}(\vec{\gamma}) = J_{\Phi^{-1}}(\vec{\gamma}h_a)|\det h_a|. \tag{4.15}$$

Thus

$$\begin{aligned}
(QD_a^\chi f)(\vec{\gamma}) &= |J_{\Phi^{-1}}(\vec{\gamma})|^{1/2} (D_a^\chi f)(\Phi^{-1}(\vec{\gamma})) \\
&= \chi(a)\, |J_{\Phi^{-1}}(\vec{\gamma})|^{1/2} \, |\det a|^{-1/2} f(a^{-1}\Phi^{-1}(\vec{\gamma})) \\
&= \chi(a)\, |J_{\Phi^{-1}}(\vec{\gamma})|^{1/2} \, |\det a|^{-1/2} f(\Phi^{-1}(\vec{\gamma}h_a)) \\
&= \chi(a)\, |\det h_a|^{1/2} \, |J_{\Phi^{-1}}(\vec{\gamma}h_a)|^{1/2} f(\Phi^{-1}(\vec{\gamma}h_a)) \\
&= \chi(a)\, |\det h_a|^{1/2} \, (Qf)(\vec{\gamma}h_a) = (D_{h_a}^\chi Qf)(\vec{\gamma})
\end{aligned}$$

$f \in L^2(U)$ and $\vec{\gamma} \in \mathcal{O}$. It follows that

$$Q\mu^\chi(a,m)Q^{-1} = QN_{-m}D_a^\chi Q^{-1} = E_{-\vec{m}}D_{h_a}^\chi = \delta^\chi(h_a,\vec{m})$$

for $(a,m) \in K$, which proves the proposition. □

Now suppose that $\mathbb{R}^n$ decomposes measurably into a collection $\{U_j\}_{j\in J}$ of D-invariant open subsets on each of which Φ is injective and has non-vanishing Jacobian determinant $J_\Phi(\vec{x})$, $\vec{x} \in U_j$. Correspondingly, we have a decomposition

$$L^2(\mathbb{R}^n) = \bigoplus_{j\in J} L^2(U_j)$$

into μ^χ-invariant subspaces. We set $\mathcal{O}_j = \Phi(U_j)$, $\Phi_j = \Phi|_{U_j}$ and let $Q_j : L^2(U_j) \to L^2(\mathcal{O}_j)$ denote the unitary operators of Proposition 4.4.

Define a unitary operator $Q : L^2(\mathbb{R}^n) \to \mathcal{H} = \underset{j\in J}{\oplus} L^2(\mathcal{O}_j)$ by

$$Q = \underset{j\in J}{\oplus} Q_j,$$

that is for $\phi \in L^2(\mathbb{R}^n)$,

$$Q\phi = \sum_{j\in J} Q_j\phi_j$$

where ϕ_j denotes the restriction of ϕ to U_j.

Correspondingly, let δ_j^χ denote the modulated wavelet representations of K on $L^2(\mathcal{O}_j)$. Then $\mu^\chi|_{L^2(U_j)}$ and δ_j^χ are equivalent for all $j \in J$ by Proposition 4.4, $\delta_j^\chi = Q_j\, \mu^\chi|_{L^2(U_j)}\, Q_j^{-1}$. Set

$$\delta^\chi = \underset{j\in J}{\oplus} \delta_j^\chi.$$

Then $\delta^\chi = Q\mu^\chi Q^{-1}$.

Proposition 4.5. *The modulated metaplectic representation μ^χ and the sum of modulated wavelet representations δ^χ are equivalent representations of K.*

Proof. Since each $L^2(U_j)$ is μ_j^χ-invariant, we have

$$Q\mu^\chi(a,m)Q^{-1} = Q\left(\underset{j\in J}{\oplus} \mu^\chi|_{L^2(U_j)}(a,m)\right)Q^{-1} = \underset{j\in J}{\oplus} Q_j\, \mu^\chi|_{L^2(U_j)}(a,m)Q_j^{-1}$$
$$= \underset{j\in J}{\oplus} \delta_j^\chi(a,m) = \delta^\chi(a,m).$$

$\square$

By Proposition 4.5, we have

$$\|V_\phi f\|^2_{L^2(K)} = \int_K |\langle f, \mu^\chi(a,m)\phi\rangle|^2\, d\nu(a,m) = \int_K |\langle Qf, \delta^\chi(a,m)Q\phi\rangle|^2\, d\nu(a,m)$$
$$= \|W_{Q\phi}Qf\|^2_{L^2(K)} = \|W_{Q\phi}g\|^2_{L^2(K)}$$

where $g = Qf$. Since Q is a unitary operator, then

$$\begin{aligned} & \|V_\phi f\|^2_{L^2(K)} = c_\phi \|f\|^2_{L^2(\mathbb{R}^n)} \qquad \forall f \in L^2(\mathbb{R}^n) \\ \Leftrightarrow & \|W_{Q\phi} Qf\|^2_{L^2(K)} = c_\phi \|Qf\|^2_{L^2(\mathbb{R}^n)} \qquad \forall f \in L^2(\mathbb{R}^n) \\ \Leftrightarrow & \|W_{Q\phi} g\|^2_{L^2(K)} = c_\phi \|g\|^2_{L^2(\mathbb{R}^n)} \qquad \forall g \in \bigoplus_{j\in J} L^2(\mathcal{O}_j) \end{aligned}$$

That is, $\phi \in L^2(\mathbb{R}^n)$ is admissible for the modulated metaplectic representation μ^χ of K if and only if $Q\phi$ is admissible for the sum of modulated wavelet representation δ^χ. By applying Theorem 3.6, we can now specify an admissibility condition for the metaplectic representation of $K = D \rtimes M$:

Proposition 4.6. $\phi \in L^2(\mathbb{R}^n)$ *is admissible for* μ^χ *if and only if there exists* $c_\phi > 0$ *such that*

$$\int_D \overline{\chi_i(a) Q_i \phi_i(\vec{\gamma} h_a)} \chi_j(a) Q_j \phi_j(\vec{\gamma} h_a) d\mu(a) = \delta_{i,j} c_\phi \tag{4.16}$$

for a.e. $\vec{\gamma} \in \Phi(U_i) \cap \Phi(U_j)$.

In many of the applications, the sets $\Phi(U_j)$ will all coincide, hence the sets U_j will be homeomorphic. We fix one of the sets U_{j_0} and let $F_j : U_{j_0} \to U_j$ be given by

$$F_j = \Phi_j^{-1} \circ \Phi_{j_0}, \tag{4.17}$$

where Φ_j denotes the restriction of Φ to U_j for each j. By (4.15), we have for all $\vec{\gamma} = \Phi_{j_0}(\vec{x})$ and $\vec{x} \in U_{j_0}$,

$$\begin{aligned} (Q_j \phi_j)(\Phi_{j_0}(\vec{x}) h_a) &= \left| J_{\Phi_j^{-1}}(\Phi_{j_0}(\vec{x}) h_a) \right|^{1/2} \phi_j\left(\Phi_j^{-1}(\Phi_{j_0}(\vec{x}) h_a)\right) \\ &= \left| J_{\Phi_j^{-1}}(\Phi_{j_0}(\vec{x})) \right|^{1/2} |\det a|^{-1/2} |\det h_a|^{-1/2} \phi_j\left(a^{-1} \Phi_j^{-1}(\Phi_{j_0}(\vec{x}))\right) \\ &= \frac{1}{|J_\Phi(F_j(\vec{x}))|^{1/2} |\det a|^{1/2} |\det h_a|^{1/2}} \phi_j\left(a^{-1} F_j(\vec{x})\right). \end{aligned}$$

Then (4.16) can be written as

$$\int_D \overline{\chi_i(a)\phi_i\left(a^{-1}F_i(\vec{x})\right)}\chi_j(a)\phi_j\left(a^{-1}F_j(\vec{x})\right) \times \frac{1}{\left|J_\Phi(F_i(\vec{x}))J_\Phi(F_j(\vec{x}))\right|^{1/2}|\det a|\,|\det h_a|}d\mu(a) = \delta_{i,j}c_\phi \tag{4.18}$$

for a.e. $\vec{x} \in U_{j_0}$.

We observe that Proposition 4.5 allows for constructing frames for the metaplectic representation: If $\{\pi\left((k,\vec{u})^{-1}\right)\psi\}_{k\in P,\vec{u}\in\Gamma}$ is a frame for the representation δ^χ of K as in Theorem 3.12 and 3.13, then the collection $\{\mu\left((k,\vec{u})^{-1}\right)\phi\}_{k\in P,\vec{u}\in\Gamma}$ will be a frame for the metaplectic representation μ^χ of K, where $\phi = Q^{-1}\psi$.

4.4 Example 1: A Simple Dilation Group

Let us consider the n-dimensional subspace

$$M = \left\{m(\vec{u}) := \mathrm{diag}(-u_1, -u_2, ..., -u_n) \,:\, \vec{u} = (u_1, u_2, ..., u_n)^T \in \mathbb{R}^n\right\}$$

of diagonal matrices of $Sym(n,\mathbb{R})$, and let D be the one-parameter group,

$$D = \left\{a(t) := A^{-t} \,:\, t \in \mathbb{R}\right\}$$

generated by a diagonal fixed matrix $A = \mathrm{diag}(a_1, a_2, ..., a_n)$, $a_k > 0$ for all k and $a_k \neq 1$ for at least one k. The action (4.4) of D on M is given by

$$\left[a(t)^{-1}\right]^T m(\vec{u})a(t)^{-1} = A^t m(\vec{u})A^t = m(A^{2t}\vec{u}) \tag{4.19}$$

and is effective. By (4.6), the corresponding semi-direct product $K = D \rtimes M$ can be represented as the subgroup

$$K \cong \left\{ k(t,\vec{u}) = \begin{bmatrix} A^{-t} & 0 \\ m(\vec{u})A^{-t} & A^t \end{bmatrix} \,:\, t \in \mathbb{R}, \vec{u} \in \mathbb{R}^n \right\}$$

of $Sp(n,\mathbb{R})$, and the group law of K is

$$k(t,\vec{u})k(t',\vec{u}') = k(t+t',\vec{u}+A^{2t}\vec{u}').$$

By (4.11) the left Haar measure ν on K is given by

$$d\nu(k(t,\vec{u})) = (a_1a_2\cdots a_n)^{-2t}dt\,d\vec{u}.$$

Let us compute the map Φ. Since $m(\vec{u})$ is a diagonal matrix, we have for $\vec{x}=(x_1,x_2,...,x_n)^T\in\mathbb{R}^n$,

$$\langle m(\vec{u})\vec{x},\vec{x}\rangle = -\sum_{k=1}^{n}u_kx_k^2 = -2\left\langle \vec{u},\frac{1}{2}(x_1^2,x_2^2,...,x_n^2)^T\right\rangle \tag{4.20}$$

so we obtain

$$\Phi(\vec{x}) = \frac{1}{2}(x_1^2,x_2^2,...,x_n^2).$$

The Jacobian of Φ is

$$J_\Phi(\vec{x}) = x_1x_2\cdots x_n.$$

Since $J_\Phi(\vec{x})=0$ if and only if $x_k=0$ for some k, this leads to a splitting of $\mathbb{R}^n$ into 2^n orthants

$$U_\alpha = \left\{\vec{x}=(x_1,x_2,...,x_n)^T\in\mathbb{R}^n : \operatorname{sgn}(x_k)=\alpha_k\right\}$$

where $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$ is a multi-index with $\alpha_k\in\{-1,1\}$ for all k. For each $\vec{x}\in U_\alpha$, we have as $a_k>0$ for all k,

$$a(t)^{-1}\vec{x} = A^t\vec{x} = \operatorname{diag}(a_1,a_2,...,a_n)^t\vec{x} = (a_1^tx_1,a_2^tx_2,...,a_n^tx_n)^T\in U_\alpha$$

that is, each U_α is D-invariant.

We can thus make use of the technique discussed in the previous section to show that the metaplectic representation μ of K is equivalent to a sum of wavelet representations. In fact, by (4.20) μ is of the form

$$\begin{aligned}\mu(k(t,\vec{u}))f(\vec{x}) &= (N_{-m(\vec{u})}D_{a(t)}f)(\vec{x})\\ &= e^{-i\pi\sum\limits_{k=1}^{n}u_kx_k^2}(a_1a_2\cdots a_n)^{t/2}f(a_1^tx_1,a_2^tx_2,...,a_n^tx_n)\end{aligned}$$

for $f \in L^2(\mathbb{R}^n)$ and $\vec{x} \in \mathbb{R}^n$.

Observe that the restrictions Φ_α of Φ to U_α map the sets U_α homeomorphically onto the first orthant

$$\mathcal{O} := \left\{ \vec{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_n) \in \widehat{\mathbb{R}^n} \, : \, \gamma_k > 0 \ \forall k \right\}$$

of $\widehat{\mathbb{R}^n}$. In fact, the inverse maps are given by

$$\Phi_\alpha^{-1}(\vec{\gamma}) = (\alpha_1 \sqrt{2\gamma_1}, \alpha_2 \sqrt{2\gamma_2}, ..., \alpha_n \sqrt{2\gamma_n}), \qquad (\alpha_k = \operatorname{sgn}(x_k)),$$

and the Jacobians of Φ_α^{-1} at $\vec{\gamma} \in \mathcal{O}$ are

$$J_{\Phi_\alpha^{-1}}(\vec{\gamma}) = \frac{\alpha_1}{\sqrt{2\gamma_1}} \frac{\alpha_2}{\sqrt{2\gamma_2}} \cdots \frac{\alpha_n}{\sqrt{2\gamma_n}}.$$

By (4.19), K is isomorphic to the closed subgroup $G = H \rtimes \mathbb{R}^n$ of $\mathrm{Aff}_n(\mathbb{R}^n)$, where

$$H = \left\{ h(t) := A^{2t} \, : \, t \in \mathbb{R} \right\}.$$

As ε-stabilizers are trivial and H is abelian, by Theorem 3.3, H is admissible if and only if $\det A = a_1 a_2 \cdots a_n \neq 1$, which we will assume from here on.

Let $U = U_{(1,1,...,1)}$ denote the first orthant. By (4.17) the maps $F_\alpha = \Phi_\alpha^{-1} \circ \Phi\big|_U$ are defined by

$$F_\alpha(x_1, x_2, ..., x_n) = (\alpha_1 x_1, \alpha_2 x_2, ..., \alpha_n x_n).$$

Thus

$$|J_\Phi\left(F_\alpha(\vec{x})\right)| = |J_\Phi(\alpha_1 x_1, \alpha_2 x_2, ..., \alpha_n x_n)| = x_1 x_2 \cdots x_n$$

for all $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\vec{x} = (x_1, x_2, ..., x_n) \in U$. Then by (4.18), $\phi \in L^2(\mathbb{R}^n)$ is admissible for μ if and only if

$$\int_{\mathbb{R}} \overline{\phi\left(a_1^t \beta_1 x_1, a_2^t \beta_2 x_2, ..., a_n^t \beta_n x_n\right)} \phi\left(a_1^t \alpha_1 x_1, a_2^t \alpha_2 x_2, ..., a_n^t \alpha_n x_n\right) \times \frac{dt}{(x_1 x_2 \cdots x_n)(\det A)^t} = \delta_{\alpha,\beta} c_\psi \tag{4.21}$$

for almost every $\vec{x}$ in the first orthant. Without loss of generality we may assume that $a_1 \neq 1$, and by shift-invariance of the Lebesgue measure, the above simplifies to

$$\int_{\mathbb{R}} \overline{\phi\left(a_1^t\beta_1, a_2^t\beta_2 x_2, ..., a_n^t\beta_n x_n\right)} \phi\left(a_1^t\alpha_1, a_2^t\alpha_2 x_2, ..., a_n^t\alpha_n x_n\right) \times \frac{dt}{(x_2\cdots x_n)(\det A)^t} = \delta_{\alpha,\beta} c_\psi \tag{4.22}$$

for almost every $(x_2, ..., x_n)^T$ in the first orthant of $\mathbb{R}^{n-1}$.

An admissible function to which Theorem 4.1 does not apply. We let $n = 1$ and $a_1 = e$. Hence,

$$K = \left\{ k(t,u) = \begin{bmatrix} e^{-t} & 0 \\ -ue^{-t} & e^t \end{bmatrix} : t, u \in \mathbb{R} \right\}.$$

By (4.22) and a change of variables, a function $\phi \in L^2(\mathbb{R})$ is admissible for μ if and only if

$$\int_0^\infty \overline{\phi\left((-1)^l z\right)} \phi\left((-1)^j z\right) \frac{dz}{z^2} = \delta_{j,l} c_\phi \qquad (j, l \in \{0, 1\})$$

In particular,

$$\phi = 2 \cdot \mathbf{1}_{[-4,-2]} + \sqrt{2} \cdot \mathbf{1}_{[1,2]}$$

is admissible, with $c_\phi = 1$.

We now show that Theorem 4.1 can not be applied here. The Wigner distribution of ϕ is

$$\mathcal{W}_\phi(x,\omega) = \int_{\mathbb{R}} \phi\left(x + \frac{y}{2}\right) \overline{\phi\left(x - \frac{y}{2}\right)} e^{-2\pi i \omega y} dy$$

that is

$$\begin{aligned} \mathcal{W}_\phi(x,w) = 4 &\int_{\mathbb{R}} \mathbf{1}_{[-4,-2]}\left(x + \frac{y}{2}\right) \mathbf{1}_{[-4,-2]}\left(x - \frac{y}{2}\right) e^{-2\pi i w y} dy \\ &+ 2\sqrt{2} \int_{\mathbb{R}} \mathbf{1}_{[-4,-2]}\left(x + \frac{y}{2}\right) \mathbf{1}_{[1,2]}\left(x - \frac{y}{2}\right) e^{-2\pi i w y} dy \\ &+ 2\sqrt{2} \int_{\mathbb{R}} \mathbf{1}_{[1,2]}\left(x + \frac{y}{2}\right) \mathbf{1}_{[-4,-2]}\left(x - \frac{y}{2}\right) e^{-2\pi i w y} dy \\ &+ 2 \int_{\mathbb{R}} \mathbf{1}_{[1,2]}\left(x + \frac{y}{2}\right) \mathbf{1}_{[1,2]}\left(x - \frac{y}{2}\right) e^{-2\pi i w y} dy. \end{aligned}$$

Note that

$$\mathbf{1}_{[-4,-2]}\left(x+\frac{y}{2}\right)=1 \Leftrightarrow y\in[-8-2x,-4-2x]$$
$$\mathbf{1}_{[-4,-2]}\left(x-\frac{y}{2}\right)=1 \Leftrightarrow y\in[4+2x,8+2x]$$
$$\mathbf{1}_{[1,2]}\left(x+\frac{y}{2}\right)=1 \Leftrightarrow y\in[2-2x,4-2x]$$
$$\mathbf{1}_{[1,2]}\left(x-\frac{y}{2}\right)=1 \Leftrightarrow y\in[-4+2x,-2+2x].$$

Thus, for $1\leqslant x\leqslant 3/2$, we have

$$\mathbf{1}_{[-4,-2]}\left(x+\frac{y}{2}\right)\mathbf{1}_{[-4,-2]}\left(x-\frac{y}{2}\right)=1 \Leftrightarrow y\in[2-2x,-2+2x]$$

and, hence

$$\begin{aligned}\mathcal{W}_\phi(x,w)&=2\int_{2-2x}^{-2+2x}e^{-2\pi iwy}dy=-2\frac{e^{-2\pi iwy}}{2\pi iw}\bigg|_{y=2-2x}^{-2+2x}=2\frac{e^{4\pi iw(x-1)}-e^{-4\pi i(x-1)}}{2\pi iw}\\&=\frac{2\sin(4\pi w(x-1))}{\pi w}=8(x-1)\mathrm{sinc}(4\pi w(x-1))\end{aligned}$$

where $\mathrm{sinc}\,x=\frac{\sin x}{x}$.

Now the inverse matrix of $k(t,u)$ is

$$k(t,u)^{-1}=\begin{bmatrix}e^t & 0\\ e^{-t}u & e^{-t}\end{bmatrix}$$

and hence for $\vec{z}=(x,w)^T\in\mathbb{R}^2$ with $1\leqslant x\leqslant 3/2$, we have

$$\begin{aligned}\int_K\left|\mathcal{W}_\phi(k(t,u)^{-1}\cdot\vec{z})\right|d\nu(k(t,u))&=\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{W}_\phi(e^tx,e^{-t}ux+e^{-t}w)\right|e^{-2t}du\,dt\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{W}_\phi(e^tx,u)\right|\frac{du\,dt}{e^tx}=\int_0^\infty\int_{\mathbb{R}}\left|\mathcal{W}_\phi(y,u)\right|\frac{du\,dy}{y^2}\\&\geqslant\int_1^{3/2}\frac{8(y-1)}{y^2}\int_{\mathbb{R}}\left|\mathrm{sinc}(4\pi u(y-1))\right|du\,dy=\infty\end{aligned}$$

as the sinc-function is not integrable. It follows that Theorem 4.1 cannot be used to prove admissibility of ϕ.

A group which possesses no admissible function in the Schwartz class. We let $n=2$, choose $r>0$ and $a_1>0$ and set $a_2=a_1^{-1/(2r+1)}$, so that

$a_2 < 1$ and $\det A = a_1 a_2 > 1$. If $\phi \in L^2(\mathbb{R}^2)$ is admissible, then by (4.22) in particular,

$$\int_{\mathbb{R}} \left|\phi\left(a_1^t, a_2^t x_2\right)\right|^2 \frac{dt}{x_2 a_1^t a_2^t} = c_\phi > 0$$

for a.e. $x_2 > 0$. Now suppose that ϕ is in Schwartz class, or more generally, $\phi \in L^2(\mathbb{R}^2)$ satisfies

$$|\phi(\vec{x})| \leqslant \frac{M}{\|\vec{x}\|^{r+\varepsilon}} \qquad (\vec{x} \neq 0)$$

for some $\varepsilon > 0$. Then for all $x_2 > 1$, the condition (4.22) becomes

$$\begin{aligned}
\int_{\mathbb{R}} \left|\phi\left(a_1^t, a_2^t x_2\right)\right|^2 \frac{dt}{x_2 a_1^t a_2^t} &= \int_0^\infty \left|\phi\left(a_1^t, a_2^t x_2\right)\right|^2 \frac{dt}{x_2 a_1^t a_2^t} + \int_0^\infty \left|\phi\left(a_1^{-t}, a_2^{-t} x_2\right)\right|^2 \frac{a_1^t a_2^t dt}{x_2} \\
&\leqslant \int_0^\infty \frac{M^2}{a_1^{2t(r+\varepsilon)} x_2} dt + \int_0^\infty \frac{M^2}{a_2^{-2t(r+\varepsilon)} x_2^{2(r+\varepsilon)}} \frac{a_2^{-t(2r+1)} a_2^t}{x_2} dt \\
&\leqslant \frac{M^2}{x_2} \left[\int_0^\infty \frac{1}{a_1^{2t(r+\varepsilon)}} dt + \int_0^\infty a_2^{2\varepsilon t} dt\right] \to 0 \qquad \text{as } x_2 \to \infty.
\end{aligned}$$

Hence ϕ cannot be admissible.

The dilation group need not be connected. For ease of exposition we choose $n = 2$; the case of general n proceeds similarly. The group D is modified to include reflections along the coordinate axes,

$$D = \left\{ \, a(t,p,q) := A^{-t} R^p S^q \;\middle|\; t \in \mathbb{R},\ p, q \in \{0,1\} \, \right\}$$

where $A = \left[\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right]$ with $a_1, a_2 > 0$, $a_1, a_2 \neq 1$, $R = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. This group is not connected; the reflections divide D into four connectedness components. Since

$$\left[a(t,p,q)^{-1}\right]^T m(\vec{u})\, a(t,p,q)^{-1} = m(A^{2t}\vec{u}) \tag{4.23}$$

then (4.23) still determines a linear action of D on M, and the group operation on $K = D \rtimes M$ is

$$k(t,p,q,\vec{u})k(t',p',q',\vec{u}') = k(t+t', (p+p') \mod 2, (q+q') \mod 2, \vec{u} + A^{2t}\vec{u}')$$

so that

$$H = \{\ h(t) = A^{2t}\ \big|\ t \in \mathbb{R}\ \}.$$

Thus, the map $\Psi : D \to H$ has kernel $\{a(0,p,q) : p,q \in \{0,1\}\}$ and $G = H \rtimes \mathbb{R}^2$ is a quotient of K.

By the presence of the reflections, none of the four quadrants is D-invariant, so the considerations leading to (4.21) can not be applied. However, we can split $L^2(\mathbb{R}^2)$ into a direct sum of four closed subspaces

$$L^2_{i,j}(\mathbb{R}^2) = \left\{ f \in L^2(\mathbb{R}^2) \mid D_R f = (-1)^i f \quad \text{and} \quad D_S f = (-1)^j f\ \right\}$$

for $i,j \in \{0,1\}$. The projections of $L^2(\mathbb{R}^2)$ onto these subspaces are given by

$$P_{i,j} = \frac{1}{4}\left[\,\mathrm{Id} + (-1)^i D_R + (-1)^j D_S + (-1)^{i+j} D_R D_S\,\right].$$

Recall that the metaplectic representation of K is

$$\mu(k(t,p,q,\vec{u})) = N_{-m(\vec{u})} D_{a(t,p,q)} = N_{-m(\vec{u})} D_{A^{-t}} D_{R^p} D_{S^q}.$$

We observe that the subspaces $L^2_{i,j}(\mathbb{R}^2)$ are μ-invariant. In fact, as D is an abelian group, it follows that

$$P_{i,j} D_{a(t,p,q)} = D_{a(t,p,q)} P_{i,j}$$

for all $a(t,p,q) \in D$. On the other hand by (4.23), $m(\vec{u})$ is invariant under the action of $R = a(0,1,0)$, hence for all $f \in L^2(\mathbb{R}^2)$ and $\vec{x} \in \mathbb{R}^2$,

$$\begin{aligned}\left(D_R N_{-m(\vec{u})} f\right)(\vec{x}) &= e^{i\pi\langle m(\vec{u}) R^{-1}\vec{x}, R^{-1}\vec{x}\rangle} f\left(R^{-1}\vec{x}\right) \\ &= e^{i\pi\langle m(\vec{u})\vec{x},\vec{x}\rangle} \left(D_R f\right)(\vec{x}) = \left(N_{-m(\vec{u})} D_R f\right)(\vec{x}),\end{aligned}$$

and similarly,

$$D_S N_{-m(\vec{u})} = N_{-m(\vec{u})} D_S.$$

We conclude that

$$P_{i,j} N_{-m(\vec{u})} = N_{-m(\vec{u})} P_{i,j}$$

as well. It follows that $P_{i,j}\mu(k(t,p,q,\vec{u})) = \mu(k(t,p,q,\vec{u}))P_{i,j}$, hence $L^2_{i,j}(\mathbb{R}^2) = \text{range}(P_{i,j})$ is μ-invariant.

Now each of these four subspaces can be identified with $L^2(U)$, U denoting the first quadrant, via the unitary maps $V_{i,j} : L^2_{i,j}(\mathbb{R}^2) \to L^2(U)$ given by

$$(V_{i,j}f)(\vec{x}) = 2f(\vec{x}) \qquad (\vec{x} \in U)$$

for $f \in L^2_{i,j}(\mathbb{R}^2)$. Let us compute the corresponding representations $\mu_{i,j} = V_{i,j}\mu V_{i,j}^{-1}$ of K on $L^2(U)$. For each $f \in L^2_{i,j}(\mathbb{R}^2)$ and $\vec{x} \in U$ we have

$$\begin{aligned}\left(V_{i,j}D_{a(t,p,q)}f\right)(\vec{x}) &= 2\left(D_{A^{-t}}D_{R^p}D_{S^q}f\right)(\vec{x}) = 2|\det A|^{t/2}\left(D_{R^p}D_{S^q}f\right)(A^t\vec{x}) \\ &= 2|\det A|^{t/2}(-1)^{ip}(-1)^{jq}f(A^t\vec{x}) = (-1)^{ip+jq}\,|\det A|^{t/2}\,(2f)(A^t\vec{x}) \\ &= \chi_{i,j}(t,p,q)\,\left(D_{A^{-t}}V_{i,j}f\right)(\vec{x})\end{aligned}$$

where $\chi_{i,j}$ is the character on D given by $\chi_{i,j}(t,p,q) = (-1)^{ip+jq}$. That is

$$V_{i,j}D_{a(t,p,q)}V_{i,j}^{-1} = \chi_{i,j}(t,p,q)D_{A^{-t}}. \tag{4.24}$$

Also,

$$\left(V_{i,j}N_{-m(\vec{u})}f\right)(\vec{x}) = 2\left(N_{-m(\vec{u})}f\right)(\vec{x}) = \left(N_{-m(\vec{u})}(2f)\right)(\vec{x}) = \left(N_{-m(\vec{u})}V_{i,j}\right)(\vec{x})$$

which shows that

$$V_{i,j}N_{-m(\vec{u})}V_{i,j}^{-1} = N_{-m(\vec{u})}.$$

It follows that each $\mu_{i,j}^{\chi_{i,j}}$ is a modulated metaplectic representation,

$$\mu_{i,j}^{\chi_{i,j}}(t,p,q,\vec{u}) = V_{i,j}N_{-m(\vec{u})}D_{a(t,p,q)}V_{i,j}^{-1} = \chi_{i,j}(t,p,q)N_{-m(\vec{u})}D_{A^{-t}},$$

and that the metaplectic representation of K on $L^2(\mathbb{R}^2)$ is equivalent to the sum of representations $\bigoplus_{i,j}\mu_{i,j}^{\chi_{i,j}}$ of K on $\bigoplus_{i,j}L^2(U)$. We now apply the results of section 4.2 to show that K is admissible for this sum of representations, and hence for the metaplectic representation μ, and obtain an admissibility condition.

Let $\phi \in L^2(\mathbb{R}^2)$. The corresponding element in $\bigoplus_{i,j} L^2(U)$ is $\sum_{i,j} V_{i,j}P_{i,j}$. Next let $Q : L^2(U) \to L^2(\mathcal{O})$ be as in Proposition 4.4, which shows that each $\mu_{i,j}^{\chi_{i,j}}$ is equivalent to a modulated wavelet representation $\delta_{i,j}^{\chi_{i,j}}$ on $L^2(\mathcal{O})$, by $\delta_{i,j}^{\chi_{i,j}} = Q\mu_{i,j}^{\chi_{i,j}}Q^{-1}$. Then Q amplifies to a unitary map $\tilde{Q} = \bigoplus_{i,j} Q : \bigoplus_{i,j} L^2(U) \to \bigoplus_{i,j} L^2(\mathcal{O})$ between the sums of the four function spaces, and $\tilde{Q}$ implements an equivalence between the representation $\tilde{\mu} = \bigoplus_{i,j} \mu_{i,j}^{\chi_{i,j}}$ and the sum of modulated wavelet representations $\tilde{\delta} = \bigoplus_{i,j} \delta_{i,j}^{\chi_{i,j}}$. Hence ϕ is admissible for $\tilde{\mu}$ if and only if $\tilde{Q}\phi = \bigoplus_{i,j} QV_{i,j}P_{i,j}\phi$ is admissible for $\tilde{\delta}$ if and only if

$$\sum_{p=0}^{1}\sum_{q=0}^{1} \int_{\mathbb{R}} \chi_{i,j}(t,p,q)\overline{\chi_{k,l}(t,p,q)}\ [QV_{i,j}P_{i,j}\phi](\vec{\gamma}A^{2t})\ \overline{[QV_{k,l}P_{k,l}\phi]\,(\vec{\gamma}A^{2t})}dt$$
$$= c_\phi \delta_{i,k}\delta_{j,l} \tag{4.25}$$

for a.e. $\gamma \in \mathcal{O}$. (Clearly the Haar measure on D is given by $\sum_{p\in\{0,1\}}\sum_{q\in\{0,1\}} dt$.) The above can be summarized in the diagram

$$\begin{array}{ccccccc}
\mu & \xrightarrow{\text{split}} & \bigoplus_{i,j} \mu|_{L^2_{i,j}(\mathbb{R}^2)} & \xrightarrow{\bigoplus_{i,j} \mathrm{ad}V_{i,j}} & \bigoplus_{i,j} \mu_{i,j}^{\chi_{i,j}} & \xrightarrow{\mathrm{ad}\tilde{Q}} & \delta = \bigoplus_{i,j} \delta_{i,j}^{\chi_{i,j}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L^2(\mathbb{R}^2) & \xrightarrow{\bigoplus_{i,j} P_{i,j}} & \bigoplus_{i,j} L^2_{i,j}(\mathbb{R}^2) & \xrightarrow{\bigoplus_{i,j} V_{i,j}} & \bigoplus_{i,j} L^2(U) & \xrightarrow{\tilde{Q}=\bigoplus_{i,j} Q} & \bigoplus_{i,j} L^2(\mathcal{O})
\end{array}.$$

Now recall that

$$(Qf)(\vec{\gamma}A^{2t}) = \left|J_{\Phi^{-1}}(\vec{\gamma}A^{2t})\right|^{1/2} f\left(\Phi^{-1}(\vec{\gamma}A^{2t})\right)$$

for $f \in L^2(U)$ and where $\Phi = \Phi\Big|_U$, and that $\Phi^{-1}(\vec{\gamma}A^{2t}) = A^t\Phi^{-1}(\vec{\gamma})$. Setting $\vec{x} = \Phi^{-1}(\vec{\gamma})$ it follows from (4.25) that ϕ is admissible if and only if

$$\sum_{p=0}^{1}\sum_{q=0}^{1} \int_{\mathbb{R}} \chi_{i,j}(t,p,q)\overline{\chi_{k,l}(t,p,q)}\ [V_{i,j}P_{i,j}\phi](A^t\vec{x})\ \overline{[V_{k,l}P_{k,l}\phi]\,(A^t\vec{x})}$$
$$\times \frac{dt}{x_1x_2\,|\det A|^t} = c_\phi\delta_{i,k}\delta_{j,l}$$

a.e. $\vec{x} = (x_1, x_2)^T \in U$, for $i, j, k, l \in \{0, 1\}$. Because by (4.24) $D_{A^{-t}} V_{i,j} = V_{i,j} D_{A^{-t}}$ and by definition of $V_{i,j}$, this is equivalent to

$$4 \sum_{p=0}^{1} \sum_{q=0}^{1} (-1)^{(i+k)p} (-1)^{(j+l)q} \int_{\mathbb{R}} [P_{i,j}\phi](A^t \vec{x}) \, \overline{[P_{k,l}\phi](A^t \vec{x})} \, \frac{dt}{x_1 x_2 \,|\det A|^t} = c_\phi \delta_{i,k} \delta_{j,l}.$$

Observe that the left-hand side vanishes when $i \neq k$ or $j \neq l$, as the integral is independent of p and q, and thus the admissibility condition reduces to

$$16 \int_{\mathbb{R}} \Big| [P_{i,j}\phi](A^t \vec{x}) \Big|^2 \frac{dt}{x_1 x_2 \,|\det A|^t} = c_\phi$$

a.e. $\vec{x} = (x_1, x_2)^T \in U$, for $i, j \in \{0, 1\}$. As before, we may choose $x_1 = 1$; hence ϕ is admissible if and only if

$$16 \int_{\mathbb{R}} \Big| [P_{i,j}\phi](a_1^t, a_2^t x_2) \Big|^2 \frac{dt}{x_2 \, a_1^t a_2^t} = c_\phi$$

a.e. $x_2 > 0$, for $i, j \in \{0, 1\}$. By the change of variables $y = a_1^t$ this becomes

$$16 \int_0^\infty \Big| [P_{i,j}\phi](y, xy^\alpha) \Big|^2 \frac{dy}{x \, \ln a_1 \, y^{\alpha+2}} = c_\phi$$

a.e. $x > 0$, for $i, j \in \{0, 1\}$, where $\alpha = \ln a_2 / \ln a_1$. Using the definition of the projections $P_{i,j}$, this condition can be rewritten as

$$\int_0^\infty \Big| \phi(y, xy^\alpha) + (-1)^i \phi(-y, xy^\alpha) + (-1)^j \phi(y, -xy^\alpha) + (-1)^{i+j} \phi(-y, -xy^\alpha) \Big|^2 d\mu_x(y)$$
$$= c_\phi$$

a.e. $x > 0$, for $i, j \in \{0, 1\}$, where we have set $d\mu_x(y) = \frac{dy}{x \, \ln a_1 \, y^{\alpha+2}}$. Expanding, we obtain

$$\begin{aligned}
&\int_0^\infty \Big[\big|\phi(y, xy^\alpha)\big|^2 + \big|\phi(y, xy^\alpha)\big|^2 + \big|\phi(y, xy^\alpha)\big|^2 + \big|\phi(y, xy^\alpha)\big|^2 \Big] d\mu_x(y) \\
&\quad + (-1)^i \int_0^\infty 2Re\Big[\phi(y, xy^\alpha) \, \overline{\phi(-y, xy^\alpha)} \, + \, \phi(y, -xy^\alpha) \, \overline{\phi(-y, -xy^\alpha)} \Big] \, d\mu_x(y) \\
&\quad + (-1)^j \int_0^\infty 2Re\Big[\phi(y, xy^\alpha) \, \overline{\phi(y, -xy^\alpha)} \, + \, \phi(-y, xy^\alpha) \, \overline{\phi(-y, -xy^\alpha)} \Big] \, d\mu_x(y) \\
&\quad + (-1)^{i+j} \int_0^\infty 2Re\Big[\phi(y, xy^\alpha) \, \overline{\phi(-y, -xy^\alpha)} \, + \, \phi(-y, xy^\alpha) \, \overline{\phi(y, -xy^\alpha)} \Big] \, d\mu_x(y) \\
&\quad = c_\phi
\end{aligned}$$

a.e. $x > 0$, for $i, j \in \{0, 1\}$. In short,

$$a_x + (-1)^i b_x + (-1)^j c_x + (-1)^{i+j} d_x = c_\phi \qquad (i, j \in \{0, 1\})$$

The solution of this system of four equations in four unknowns is $a_x = c_\phi$, $b_x = c_x = d_x = 0$. Hence, ϕ is admissible if and only if

$$\int_0^\infty \left[\left|\phi(y, xy^\alpha)\right|^2 + \left|\phi(-y, xy^\alpha)\right|^2 + \left|\phi(y, -xy^\alpha)\right|^2 + \left|\phi(-y, -xy^\alpha)\right|^2 \right] \frac{dy}{x \ln a_1 \, y^{\alpha+2}} = c_\phi$$

for a.e. $x > 0$.

4.5 Example 2: The D-Invariant Subsets are Cones

In most of the examples, we have a decomposition of $L^2(\mathbb{R}^n)$ into μ-invariant subspaces $L^2(U_j)$, where each U_j is an orthant or a union of orthants. In general, since $\Phi(\alpha\vec{x}) = \alpha^2\Phi(\vec{x})$, Φ maps cones to cones, and hence one expects the sets U_j to have at least the form of cones. In this example, they will indeed turn out to be not orthants, but cones.

Let $n \geqslant 3$ and consider the n-dimensional subspace of $Sym(n, \mathbb{R})$,

$$M = \left\{ m(u, \vec{v}) := \begin{bmatrix} -uI_{n-1} & -\vec{v} \\ -\vec{v}^T & -u \end{bmatrix} \,:\, u \in \mathbb{R},\, \vec{v} \in \mathbb{R}^{n-1} \right\}.$$

Then fix a closed subgroup F of the orthogonal group $O(n-1)$, and consider the closed subgroup of $GL_n(\mathbb{R})$,

$$D = \left\{ a(t, b) := e^{-t} \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \,:\, t \in \mathbb{R},\, b \in F \right\} \cong \mathbb{R} \times F.$$

For each $a(t, b) \in D$ and $m(u, \vec{v}) \in M$, we have

$$(a(t, b)^{-1})^T m(u, \vec{v}) a(t, b)^{-1} = m(e^{2t}u, e^{2t}b\vec{v}). \qquad (4.26)$$

The corresponding semi-direct product $K = D \rtimes M$ can be represented as the subgroup of $Sp(n,\mathbb{R})$ of $Sp(n,\mathbb{R})$ of the form

$$K = \left\{ k(t,b,u,\vec{v}) = \begin{bmatrix} a(t,b) & 0 \\ m(u,\vec{v})a(t,b) & \left(a(t,b)^T\right)^{-1} \end{bmatrix} : t,u \in \mathbb{R}, \vec{v} \in \mathbb{R}^{n-1}, b \in F \right\},$$

and the group law of K is

$$k(t,b,u,\vec{v})k(t',b',u',\vec{v}') = k(t+t', bb', u + e^{2t}u', \vec{v} + e^{2t}b\vec{v}')$$

Let us compute the map Φ. For $\vec{x} = (\vec{x}_0, x_n)^T \in \mathbb{R}^n$,

$$\langle m(u,\vec{v})\vec{x}, \vec{x} \rangle = -u \left\|\vec{x}\right\|^2 - 2x_n \vec{v}^T \vec{x}_0 = -2 \left\langle \begin{bmatrix} u \\ \vec{v} \end{bmatrix}, \frac{1}{2} \begin{bmatrix} \left\|\vec{x}\right\|^2 \\ 2x_n\vec{x}_0 \end{bmatrix} \right\rangle$$

so we obtain

$$\Phi(\vec{x}) = \frac{1}{2}\left(\left\|\vec{x}\right\|^2, 2x_n\vec{x}_0^T\right).$$

The Jacobian of Φ is

$$J_\Phi(\vec{x}) = (-x_n)^{n-2}\left(\left\|\vec{x}_0\right\|^2 - x_n^2\right).$$

It follows that $J_\Phi(\vec{x}) = 0$ if and only if $x_n = 0$ or $|x_n| = \|\vec{x}_0\|$. These two hypersurfaces lead to a splitting of $\mathbb{R}^n$ into four open hypercones

$$U_1 = \left\{\vec{x} = (\vec{x}_0, x_n)^T \in \mathbb{R}^n \,:\, 0 < \|\vec{x}_0\| < x_n\right\}$$

$$U_2 = \left\{\vec{x} = (\vec{x}_0, x_n)^T \in \mathbb{R}^n \,:\, x_n < -\|\vec{x}_0\| < 0\right\}$$

$$U_3 = \left\{\vec{x} = (\vec{x}_0, x_n)^T \in \mathbb{R}^n \,:\, 0 < x_n < \|\vec{x}_0\|\right\}$$

$$U_4 = \left\{\vec{x} = (\vec{x}_0, x_n)^T \in \mathbb{R}^n \,:\, -\|\vec{x}_0\| < x_n < 0\right\}$$

and for each $\vec{x} \in U_j$, $j = 1, ..., 4$, we obtain

$$a(t,b)^{-1}\vec{x} = e^t \begin{bmatrix} b^T & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x}_0 \\ x_n \end{bmatrix} = \begin{bmatrix} e^t b^T \vec{x}_0 \\ e^t x_n \end{bmatrix} \in U_j$$

which shows that the U_i are D-invariant subsets. Consequently, the metaplectic representation of K is given by,

$$\mu(k(t,b,u,\vec{v}))f(\vec{x}) = (N_{-m(u,\vec{v})}D_{a(t,b)}f)(\vec{x}) = e^{-i\pi(u\|\vec{x}\|^2+2x_n\vec{v}^T\vec{x}_0)}e^{nt/2}f\left(e^tb^T\vec{x}_0, e^tx_n\right)$$

for $f \in L^2(\mathbb{R}^n)$ and $\vec{x} \in \mathbb{R}^n$, and we have a decomposition $L^2(\mathbb{R}^n) = \bigoplus_{j=1}^{4} L^2(U_j)$ into μ-invariant subspaces.

The restrictions Φ_j of Φ to U_j, $j = 1, ..., 4$ map the each set U_j diffeomorphically onto the open hypercone

$$\mathcal{O} := \left\{\vec{\gamma} = (\gamma_1, \vec{\gamma}_0) = (\gamma_1, \gamma_2, ..., \gamma_n) \in \widehat{\mathbb{R}^n} \,:\, 0 < \|\vec{\gamma}_0\| < \gamma_1\right\}$$

in $\widehat{\mathbb{R}^n}$ for all $j = 1, ..., 4$. In fact, the inverse maps are given by

$$\Phi_j^{-1}(\vec{\gamma}) = \left((-1)^q\frac{\vec{\gamma}_0}{s_p(\vec{\gamma})}, (-1)^q s_p(\vec{\gamma})\right)^T, \qquad (j = 2p+q+1; \quad p, q \in \{0,1\})$$

where $s_p(\vec{\gamma}) = \sqrt{\gamma_1 + (-1)^p l(\vec{\gamma})}$ and $l(\vec{\gamma}) = \sqrt{\gamma_1^2 - \|\vec{\gamma}_0\|^2}$ for all $\vec{\gamma} \in \mathcal{O}$. Furthermore, the Jacobians of Φ_j^{-1} at $\vec{\gamma} \in \mathcal{O}$ are

$$J_{\Phi_j^{-1}}(\vec{\gamma}) = \frac{(-1)^{p+q+1}}{2l(\vec{\gamma})s_p^{n-2}(\vec{\gamma})}.$$

It is easier to express $\vec{x}_0 \in \mathbb{R}^{n-1}$ in spherical coordinates, $\vec{x}_0 = r\vec{w}$ where $\vec{w} \in S^{n-2}$ and $r > 0$. Then

$$\Phi(\vec{x}) = \Phi(\vec{x}_0, x_n) = \Phi(r\vec{w}, x_n) = \left(\frac{1}{2}(x_n^2 + r^2), x_n r\vec{w}^T\right).$$

Similarly, if $\vec{\gamma}_0 = \rho\vec{\kappa}$ with $\vec{\kappa} \in S^{n-2}$ and $\rho > 0$, then

$$l(\vec{\gamma}) = l(\gamma_1, \rho\vec{\kappa}) = \sqrt{\gamma_1^2 - \rho^2}.$$

Note that $l(\vec{\gamma})$ and hence $s_p(\vec{\gamma})$ are independent of $\vec{\kappa}$. Thus,

$$\Phi_j^{-1}(\vec{\gamma}) = \Phi_j^{-1}(\gamma_1, \rho\vec{\kappa}) = \left((-1)^q\frac{\rho\vec{\kappa}}{s_p(\gamma_1,\rho)}, (-1)^q s_p(\gamma_1,\rho)\right)^T.$$

So if we choose $U = U_1$, then

$$l(\vec{\gamma}) = l\left(\Phi(r\vec{w}, x_n)\right) = \sqrt{\frac{1}{4}(x_n^2 + r^2)^2 - (rx_n)^2} = \frac{x_n^2 - r^2}{2}$$

for all $\vec{x} \in U$ and $\vec{\gamma} = \Phi(\vec{x})$, giving by (4.17),

$$\begin{aligned} F_j(\vec{x}) &= \Phi_j^{-1}\left(\Phi(r\vec{w}, x_n)\right) = \Phi_j^{-1}\left(\frac{1}{2}(x_n^2 + r^2), rx_n\vec{w}^T\right) \\ &= \left((-1)^q \frac{rx_n\vec{w}^T}{s_p(r, x_n)}, (-1)^q s_p(r, x_n)\right)^T \end{aligned}$$

and

$$\begin{aligned} \left|J_\Phi\left(F_j(\vec{x})\right)\right| &= \left|J_\Phi\left((-1)^q \frac{rx_n\vec{w}^T}{s_p(r, x_n)}, (-1)^q s_p(r, x_n)\right)\right| \\ &= s_p^{n-2}(r, x_n)\left(\frac{(rx_n)^2}{s_p^2(r, x_n)} - s_p^2(r, x_n)\right) \end{aligned}$$

where

$$s_p(r, x_n) = \sqrt{\frac{1}{2}(x_n^2 + r^2) + (-1)^p \frac{1}{2}(x_n^2 - r^2)} = \begin{cases} x_n & \text{if } p = 0 \\ r & \text{if } p = 1 \end{cases}.$$

By (4.26),

$$H = \left\{ h(t, b) = e^{2t} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} : t \in \mathbb{R},\, b \in F \right\}.$$

so that $\det\left(h(t, b)\right) = e^{2nt}$, while $\det\left(a(t, b)\right) = e^{-nt}$. Furthermore, the left Haar measure ν on D is given by

$$d\mu(a(t, b)) = dt\, db$$

where db is a left Haar measure of F.

Thus, if S is a transversal for the action of F on the sphere S^{n-2} in $\mathbb{R}^{n-1}$, then the set $\tilde{S} = \{(r\vec{w}, 1) \,:\, \vec{w} \in S^{n-2},\, 0 < r < 1\}$ will be a transversal for the action of D on U, as U is a cone and

$$a(t, b)^{-1} \begin{bmatrix} r\vec{w} \\ 1 \end{bmatrix} = \begin{bmatrix} e^t r b^T \vec{w} \\ e^t \end{bmatrix}.$$

Now we can determine the admissibility condition for the metaplectic representation μ of K. By (4.18), $\phi \in L^2(\mathbb{R}^n)$ is admissible if and only if

$$\int_{\mathbb{R}} \int_F \overline{\phi\left(\frac{(-1)^d e^t b^T \vec{w}}{s_c(r)}, (-1)^d e^t s_c(r)\right)} \phi\left(\frac{(-1)^q e^t b^T \vec{w}}{s_p(r)}, (-1)^q e^t s_p(r)\right) \times \frac{db\, dt}{[s_p(r)s_c(r)]^{n/2-1}(1-r^2)e^{nt}} = \delta_{c,p}\delta_{d,q}c_\phi \tag{4.27}$$

a.e. $\vec{w} \in S^{n-2}$ and $0 < r < 1$, where $p, q, c, d \in \{0, 1\}$ and $s_p(r) = r^p$.

4.5.1 Admissibility in Case $n = 3$

In the particular case where $n = 3$ and F is the rotation group $SO(2)$, we can specify this admissibility condition more precisely. We parameterize $SO(2)$ as

$$F = \left\{ R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : \theta \in [0, 2\pi) \right\}$$

and give it Haar measure $d\theta$. Hence $D \cong \mathbb{R} \times SO(2)$. Express a vector $\vec{x}_0$ in the polar coordinates, $\vec{x} = (\vec{x}_0, x_3) = (re^{i\eta}, x_3)$ with $r > 0$ and $0 \leqslant \eta < 2\pi$ and pick the transversal

$$\tilde{S} = \{(r, 1) : 0 < r < 1\}$$

for the action of D on $U = \{(re^{i\theta}, x_3) : r < x_3\}$ in these coordinates. Thus $\phi \in L^2(\mathbb{R}^3)$ is admissible for the metaplectic representation μ of $K = D \rtimes M$ if and only if

$$\int_{\mathbb{R}} \int_0^{2\pi} \overline{\phi\left(\frac{(-1)^d r e^t e^{i\theta}}{s_c(r)}, (-1)^d e^t s_c(r)\right)} \times \phi\left(\frac{(-1)^q r e^t e^{i\theta}}{s_p(r)}, (-1)^q e^t s_p(r)\right) \frac{d\theta\, dt}{\sqrt{s_p(r)s_c(r)}(1-r^2)e^{3t}} = \delta_{c,p}\delta_{d,q}c_\phi$$

a.e. $r \in (0, 1)$, where $p, q, c, d \in \{0, 1\}$. Switching to Cartesian coordinates, $\vec{y} = e^t e^{i\theta}$, this condition can be written

$$\int_{\mathbb{R}^2} \overline{\phi\left((-1)^d \left(r^{1-c}\vec{y}, r^c \|\vec{y}\|\right)\right)} \phi\left((-1)^q \left(r^{1-p}\vec{y}, r^p \|\vec{y}\|\right)\right) \frac{d\vec{y}}{\|\vec{y}\|^5\, r^{(c+p)/2}(1-r^2)} = \delta_{c,p}\delta_{d,q}c_\phi$$

a.e. $r \in (0, 1)$.

4.5.2 Classification in Case $n = 3$

Let $n = 3$ and fix (α, β), $\alpha \neq 0$. Consider the one-parameter group of the form

$$D_{\alpha,\beta} = \left\{ a_{\alpha,\beta}(t) := e^{-\alpha t} \begin{bmatrix} \cos\beta t & \sin\beta t & 0 \\ -\sin\beta t & \cos\beta t & 0 \\ 0 & 0 & 1 \end{bmatrix} \,:\, t \in \mathbb{R} \right\}.$$

Each $D_{\alpha,\beta}$ is a subgroup of the group $\mathbb{R} \times SO(2)$ considered in subsection 4.5.1. The corresponding subgroups $K_{\alpha,\beta} = D_{\alpha,\beta} \rtimes M$ can be represented as subgroups of $Sp(3, \mathbb{R})$

$$K_{\alpha,\beta} \cong \left\{ k_{\alpha,\beta}(t, u, \vec{v}) = \begin{bmatrix} a_{\alpha,\beta}(t) & 0 \\ m(u, \vec{v}) a_{\alpha,\beta}(t) & \left(a_{\alpha,\beta}(t)^T\right)^{-1} \end{bmatrix} \,:\, t, u \in \mathbb{R}, \vec{v} \in \mathbb{R}^2 \right\}.$$

We classify these groups up to isomorphism, and derive the admissibility condition for the metaplectic representation. It turns out that the classification of the isomorphic affine groups $G_{\alpha,\beta} = H_{\alpha,\beta} \rtimes \mathbb{R}^3$ is easier to accomplish. The subgroup H of $GL_n(\mathbb{R})$ corresponding to $D_{\alpha,\beta}$ is the one-parameter group

$$H_{\alpha,\beta} = \left\{ h_{\alpha,\beta}(t) = e^{2\alpha t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta t & \sin\beta t \\ 0 & -\sin\beta t & \cos\beta t \end{bmatrix} \,:\, t \in \mathbb{R} \right\}$$

and hence $G_{\alpha,\beta} = H_{\alpha,\beta} \rtimes \mathbb{R}^3$ is of the form

$$G_{\alpha,\beta} = \left\{ \begin{bmatrix} h_{\alpha,\beta}(t) & \vec{x} \\ 0 & 1 \end{bmatrix} \,:\, t \in \mathbb{R},\, \vec{x} \in \mathbb{R}^n \right\}.$$

In general, we classify subgroups of the affine group $\mathrm{Aff}_n(\mathbb{R})$ of the form

$$G_B = \left\{ \begin{bmatrix} e^{tB} & \vec{x} \\ 0 & 1 \end{bmatrix} \,:\, t \in \mathbb{R},\ \vec{x} \in \mathbb{R}^n \right\}$$

where B is a fixed non-skew symmetric $n \times n$ matrix. This classification may be known, but we are not aware of any reference. Since the groups G_B are simply connected Lie groups (using the argument presented in section 5.2), it suffices to classify the corresponding Lie algebras $\mathfrak{g}_B$.

Theorem 4.7. *Two Lie algebras $\mathfrak{g}_{\tilde{B}}$ and $\mathfrak{g}_B$ are isomorphic if and only if $\tilde{B}$ is similar to a nonzero scalar multiple of B.*

Proof. Observe that the Lie algebra $\mathfrak{g}_B$ of G_B (and similarly $\mathfrak{g}_{\tilde{B}}$) is easily seen to be of the form

$$\mathfrak{g}_B = \left\{ \begin{bmatrix} sB & \vec{x} \\ 0 & 0 \end{bmatrix} : \ s \in \mathbb{R}, \ \vec{x} \in \mathbb{R}^n \right\},$$

and thus decomposes into the direct sum of vector spaces $\mathfrak{g}_B = V_M \oplus V_X$ where

$$V_M = \left\{ sM \ : \ s \in \mathbb{R}, \ M = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \right\} \simeq \mathbb{R}$$

$$V_X = \left\{ X_{\vec{x}} = \begin{bmatrix} 0 & \vec{x} \\ 0 & 0 \end{bmatrix} \ : \ \vec{x} \in \mathbb{R}^n \right\} \simeq \mathbb{R}^n,$$

and the only nontrivial Lie brackets are determined by $[M, X_{\vec{x}}] = X_{B\vec{x}}$.

Now if $\tilde{B} = \alpha SBS^{-1}$, where $\alpha \neq 0$ and S is an invertible matrix, then the Lie algebras $\mathfrak{g}_{\tilde{B}}$ and $\mathfrak{g}_B$ are isomorphic. In fact define a vector space isomorphism $T : \mathfrak{g}_B \to \mathfrak{g}_{\tilde{B}}$ by

$$T(M) = \frac{1}{\alpha}\tilde{M} \qquad \text{and} \qquad T(X_{\vec{x}}) = X_{S\vec{x}}$$

then

$$\begin{aligned} [T(M), T(X_{\vec{x}})] &= \left[\frac{1}{\alpha}\tilde{M}, X_{S\vec{x}}\right] = \frac{1}{\alpha}X_{\tilde{B}S\vec{x}} = \frac{1}{\alpha}X_{\alpha SBS^{-1}S\vec{x}} = X_{SB\vec{x}} \\ &= T(X_{B\vec{x}}) = T\left([M, X_{\vec{x}}]\right). \end{aligned}$$

Conversely, let $T : \mathfrak{g}_B \to \mathfrak{g}_{\tilde{B}}$ be a Lie algebra isomorphism. In light of the vector space decomposition $V_M \oplus V_X$, and the fact that T maps the nilradical V_X onto the nilradical $V_{\tilde{X}}$, we may represent T by a matrix

$$\begin{bmatrix} a & 0 \\ \vec{v} & S \end{bmatrix}$$

where $a \neq 0$, $\vec{v} \in \mathbb{R}^n$ and $S \in GL_n(\mathbb{R})$. That is

$$T(M) = a\tilde{M} + X_{\vec{v}} \qquad \text{and} \qquad T(X_{\vec{x}}) = X_{S\vec{x}}.$$

Thus

$$[T(M), T(X_{\vec{x}})] = \left[a\tilde{M} + X_{\vec{v}}, X_{S\vec{x}}\right] = \left[a\tilde{M}, X_{S\vec{x}}\right] = X_{a\tilde{B}S\vec{x}}.$$

On the other hand

$$T\left([M, X_{\vec{x}}]\right) = T(X_{B\vec{x}}) = X_{SB\vec{x}}.$$

Since T is a Lie algebra isomorphism, then $SB\vec{x} = a\tilde{B}S\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$, which shows that

$$\tilde{B} = \frac{1}{a} SBS^{-1}$$

and proves the theorem. □

Note that each $h_{\alpha,\beta}(t)$ is of the form $e^{B_{\alpha,\beta}t}$ where

$$B_{\alpha,\beta} = \begin{bmatrix} 2\alpha & 0 & 0 \\ 0 & 2\alpha & \beta \\ 0 & -\beta & 2\alpha \end{bmatrix}.$$

Since $B_{\alpha,\beta}$ is similar to a multiple of $B_{\tilde{\alpha},\tilde{\beta}}$ if and only if $\left|\frac{\beta}{\alpha}\right| = \left|\frac{\tilde{\beta}}{\tilde{\alpha}}\right|$ (for $\alpha \neq 0$, $\tilde{\alpha} \neq 0$), we thus have shown:

Proposition 4.8. *$G_{\alpha,\beta}$ and $G_{\tilde{\alpha},\tilde{\beta}}$ are isomorphic if and only if $\left|\frac{\beta}{\alpha}\right| = \left|\frac{\tilde{\beta}}{\tilde{\alpha}}\right|$.*

Next we discuss admissibility condition of the groups $K_{\alpha,\beta} \cong G_{\alpha,\beta}$. By Proposition 4.8, we may assume that $\alpha = 1$ and $\beta > 0$. Recall that $U = \{(re^{i\theta}, x_3) : 0 < r < x_3,\, \theta \in [0, 2\pi)\}$. One quickly verifies that

$$\tilde{S} = \left\{(re^{i\theta}, 1) : r \in (0,1),\, \theta \in [0, 2\pi)\right\}$$

is a transversal for the action of $D_{1,\beta}$ on U. As it is enough to verify the admissibility condition for elements $\vec{x}$ in the transversal, then by (4.16), $\phi \in L^2(\mathbb{R}^3)$ is admissible for μ if and only if

$$\int_{\mathbb{R}} \overline{\phi\left((-1^d)\frac{re^t}{s_c(r)}e^{i(\theta+\beta t)}, (-1)^d e^t s_c(r)\right)}\phi\left((-1^q)\frac{re^t}{s_p(r)}e^{i(\theta+\beta t)}, (-1)^q e^t s_p(r)\right)$$
$$\times \frac{dt}{[s_p(r)s_c(r)]^{n/2-1}(1-r^2)e^{3t}} = \delta_{c,p}\delta_{d,q}c_\psi$$

a.e. $r \in (0,1)$ and $\theta \in [0, 2\pi)$ where $s_p(r) = r^p$ and $p, q, c, d \in \{0, 1\}$.

4.6 Example 3: The Two-Fold Covering of the SIM(2) Group

Let $n = 2$ and $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Consider the subspace

$$M = \left\{ m(\vec{u}) := \begin{bmatrix} -u_1 & -u_2 \\ -u_2 & u_1 \end{bmatrix} : \vec{u} = (u_1, u_2)^T \in \mathbb{R}^2 \right\}$$

of $Sym(2, \mathbb{R})$, and the group

$$D = \left\{a(t,\theta) := t^{-1/2}R_{-\theta/2} : t > 0,\, \theta \in [0, 4\pi)\right\}$$

of dilations and rotations in the plane. Since every invertible symmetric 2×2 matrix whose column vectors are orthogonal is of form $m(\vec{u})$, then D acts on M by

$$\left[a(t,\theta)^{-1}\right]^T m(\vec{u})a(t,\theta)^{-1} = m(tR_{-\theta}\vec{u}) \tag{4.28}$$

with global stabilizer $D_o = \{a(1,0), a(1,2\pi)\}$. Thus, the corresponding semi-direct product $K = D \rtimes M$ can be represented as the subgroup

$$K \cong \left\{ k(t,\theta,\vec{u}) := \begin{bmatrix} t^{-1/2}R_{-\theta/2} & 0 \\ t^{-1/2}m(\vec{u})R_{-\theta/2} & t^{1/2}R_{-\theta/2} \end{bmatrix} : t > 0, \vec{u} \in \mathbb{R}^2,\, \theta \in [0,4\pi) \right\}$$

of $Sp(2,\mathbb{R})$ with the group law

$$k(t,\theta,\vec{u})k(t',\theta,'\vec{u}') = k(tt', (\theta+\theta') \bmod 4\pi, \vec{u} + tR_{-\theta}\vec{u}').$$

Cordero et al. (2006a) showed that K/D_o is isomorphic to the group of similitudes $SIM(2)$, and that the restriction of the symplectic representation of K to the subspace of even functions in $L^2(\mathbb{R}^2)$ factors to a representation of K/D_o which is equivalent to the wavelet representation of $SIM(2)$. In addition, admissibility conditions for the metaplectic representation of a class of subgroups of the form $SO(2) \rtimes G_\beta$ of $Sp(2,\mathbb{R})$ were derived. A simple reparametrization shows that these groups are actually all identical to K, and we show now how the admissibility condition in Cordero et al. (2006a) arises from a sum of wavelet representations. This will allow us to introduce metaplectic frames as well.

By (4.28), the group H is of the form

$$H = \{\ h(t,\theta) := tR_{-\theta} \ :\ t > 0,\ \theta \in [0,2\pi)\ \}$$

and hence,

$$G = H \rtimes \mathbb{R}^2 = \left\{\ g(t,\theta,\vec{u}) := \begin{bmatrix} tR_{-\theta} & \vec{u} \\ 0 & 1 \end{bmatrix} : t > 0,\ \theta \in [0,2\pi),\ \vec{u} \in \mathbb{R}^2\ \right\}$$

with the group law

$$g(t,\theta,\vec{u},)g(t',\theta',\vec{u}') = g(tt', (\theta+\theta') \bmod 2\pi, \vec{u} + tR_{-\theta}\vec{u}').$$

which is the similitude group $SIM(2)$.

We note that the map $\Psi : a \in D \mapsto h_a \in H$ is not an isomorphism, but its kernel is D_o. It follows that $H \cong D/D_o$, and similarly, as D_o is naturally embedded in K as the normal subgroup $K_o = \{k(1,0,0), k(1,2\pi,0)\}$, that $G \cong K/K_o$.

The metaplectic representation of K is given by

$$\mu\left[k(t,\theta,\vec{u})\right] = N_{-m(\vec{u})} D_{t^{-1/2}} D_{R_{-\theta/2}} \tag{4.29}$$

that is,

$$\left(\mu\left[\, k(t,\theta,\vec{u})\,\right] f\right)(\vec{x}) = e^{i\pi\langle m(\vec{u})\vec{x},\vec{x}\rangle} t^{1/2} f(t^{1/2} R_{\theta/2}\vec{x}) \tag{4.30}$$

for $f \in L^2(\mathbb{R}^2)$, $\vec{x} \in \mathbb{R}^2$ and $k(t,\theta,\vec{u}) \in K$.

Observe that

$$\langle m(\vec{u})\vec{x},\vec{x}\rangle = -u_1(x_1^2 - x_2^2) - 2u_2x_1x_2 = -2\left\langle \vec{u}, \frac{1}{2}\left(x_1^2 - x_2^2, 2x_1x_2\right)\right\rangle$$

and hence,

$$\Phi(\vec{x}) = \left(\frac{x_1^2 - x_2^2}{2}, x_1x_2\right).$$

Now Φ maps the two half planes $\{\vec{x} \in \mathbb{R}^2 \,:\, x_1 > 0\}$ and $\{\vec{x} \in \mathbb{R}^2 \,:\, x_1 < 0\}$ homeomorphically onto a dense open subset of $\mathbb{R}^2$. However, none of these half planes is rotation invariant, thus the process discussed in the previous examples can not be employed. One way to bypass this obstacle would be to omit rotations and consider the closed subgroup

$$K_1 = \mathbb{R}^+ \rtimes M = \left\{k(t,0,\vec{u}) \,:\, t > 0,\ \vec{u} \in \mathbb{R}^2\right\}.$$

As each of the two half planes is invariant under scalar dilation, an admissible function g for K_1 may be obtained as shown in the previous examples. Since K is an extension of K_1 by the compact group $SO(2)$, g will also be admissible for K. However, this construction does not yield all of the admissible functions. We therefore choose to proceed differently, by decomposing μ into a sum of subrepresentations.

For convenience, express elements of $\mathbb{R}^2$ in polar form, $\vec{x} = re^{i\eta}$, $0 \leqslant \eta < 2\pi$, and elements of $\widehat{\mathbb{R}^2}$ as $\vec{\gamma} = qe^{i\varphi}$, $0 \leq \varphi < 2\pi$. Then split $L^2(\mathbb{R}^2)$ as

$$L^2(\mathbb{R}^2) = L^2_{even}(\mathbb{R}^2) \oplus L^2_{odd}(\mathbb{R}^2),$$

the closed subspaces of a.e. even, respectively odd functions. One notes from (4.30) that both of these subspaces are μ-invariant, hence μ splits into a sum of subrepresentations $\mu_1 \oplus \mu_o$ along these subspaces. Observe that the map $V : f \mapsto \tilde{f}$ given by $\tilde{f}\left(re^{i\eta}\right) = e^{i\eta} f\left(re^{i\eta}\right)$ maps $L^2_{odd}(\mathbb{R}^2)$ isometrically onto $L^2_{even}(\mathbb{R}^2)$, and carries μ_o to a representation μ_2 of $L^2_{even}(\mathbb{R}^2)$ by

$$\begin{aligned}
\left(\mu_2\left(k(t,\theta,\vec{u})\right) f\right)\left(re^{i\eta}\right) &= \left(V\mu_o(k(,t,\theta,\vec{u}))V^{-1}f\right)\left(re^{i\eta}\right) \\
&= e^{i\eta}\left(N_{-m(\vec{u})} D_{t^{-1/2}} D_{R_{-\theta/2}} V^{-1} f\right)\left(re^{i\eta}\right) \\
&= e^{i\eta} e^{i\pi\langle m(\vec{u}) re^{i\eta}, re^{i\eta}\rangle}\left(D_{t^{-1/2}} D_{R_{-\theta/2}} V^{-1} f\right)\left(re^{i\eta}\right) \\
&= e^{i\eta} e^{i\pi\langle m(\vec{u}) re^{i\eta}, re^{i\eta}\rangle} t^{1/2}\left(V^{-1} f\right)\left(t^{1/2} re^{i\eta - \frac{\theta}{2}}\right) \\
&= e^{i\frac{\theta}{2}} e^{i\pi\langle m(\vec{u}) re^{i\eta}, re^{i\eta}\rangle} t^{1/2} f\left(t^{1/2} re^{i\eta - \frac{\theta}{2}}\right) \\
&= \left(e^{i\frac{\theta}{2}} N_{-m(\vec{u})} D_{t^{-1/2}} D_{R_{-\theta/2}} f\right)\left(re^{i\eta}\right) \qquad (4.31)
\end{aligned}$$

for $f \in L^2_{even}(\mathbb{R}^2)$. Thus, the metaplectic representation μ of K on $L^2(\mathbb{R}^2)$ is equivalent to the sum $\mu_1 \oplus \mu_2$ of representations on $L^2_{even}(\mathbb{R}^2) \oplus L^2_{even}(\mathbb{R}^2)$.

Note that in polar coordinates,

$$\Phi(\vec{x}) = \Phi\left(re^{i\eta}\right) = \left(\frac{r^2}{2} e^{2i\eta}\right). \qquad (4.32)$$

Furthermore, Φ maps the half plane $U = \{re^{i\eta} : r > 0,\ 0 \leqslant \eta < \pi\}$ bijectively onto $\mathcal{O} = \widehat{\mathbb{R}^2} \backslash \{0\}$. Observe that

$$J_\Phi(\vec{x}) = J_\Phi(re^{i\eta}) = r^2 = \|\vec{x}\|^2$$

and when Φ is restricted to U, then

$$\Phi^{-1}(\vec{\gamma}) = \Phi^{-1}(qe^{i\varphi}) = \sqrt{2q} e^{i\varphi/2}$$

and also

$$J_{\Phi^{-1}}(\vec{\gamma}) = J_{\Phi^{-1}}(qe^{i\varphi}) = \frac{1}{J_{\Phi}(\Phi^{-1}(qe^{i\varphi}))} = \frac{1}{2q} = \frac{1}{2\|\vec{\gamma}\|}.$$

Now $L^2_{even}(\mathbb{R}^2)$ can be identified with $L^2(U)$ in a natural way via the map $f(\vec{x}) \mapsto \sqrt{2}f(\vec{x})$ for $\vec{x} \in U$. Composing this map with the unitary operator Q of Proposition 4.3, we obtain a new unitary operator $Q : L^2_{even}(\mathbb{R}^2) \to L^2(\mathcal{O})$ by

$$(Qf)(\vec{\gamma}) = \sqrt{2}\ |J_{\Phi^{-1}}(\vec{\gamma})|^{1/2} f\left(\Phi^{-1}(\vec{\gamma})\right)$$

or equivalently,

$$(Qf)(qe^{i\varphi}) = \frac{1}{\sqrt{q}} f\left(\sqrt{2q}e^{i\varphi/2}\right) \qquad \left(f \in L^2_{even}(\mathbb{R}^2)\right) \tag{4.33}$$

whose inverse is

$$\left(Q^{-1}\widehat{\psi}\right)(re^{i\eta}) = \frac{r}{\sqrt{2}}\widehat{\psi}\left(\frac{r^2}{2}e^{2i\eta}\right) \qquad \left(\widehat{\psi} \in L^2(\mathcal{O})\right). \tag{4.34}$$

It is easy to see that the assertion of Proposition 4.4 still holds for this operator Q, and in particular,

$$QD_{R_{-\theta/2}}Q^{-1} = D_{R_{-\theta}}.$$

Thus $\mu_1 \oplus \mu_2$ is equivalent to a sum $\delta = \delta_1 \oplus \delta_2$ of modulated wavelet representations on $L^2(\mathcal{O}) \oplus L^2(\mathcal{O})$, where by (4.29), (4.31) and Proposition 4.4,

$$\delta_1(t, \theta, \vec{u}) = E_{-\vec{u}}D_tD_{R_{-\theta}} \qquad \text{and} \qquad \delta_2(t, \theta, \vec{u}) = e^{i\theta/2}E_{-\vec{u}}D_tD_{R_{-\theta}}.$$

This sequence of equivalences can be represented as a diagram

$$\begin{array}{ccccccc}
\mu & \xrightarrow{\text{split}} & \mu_1 \oplus \mu_o & \xrightarrow{\text{Id}\oplus\text{ad}V} & \mu_1 \oplus \mu_2 & \xrightarrow{\text{ad}Q\oplus\text{ad}Q} & \delta_1 \oplus \delta_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L^2(\mathbb{R}^2) & \xrightarrow{\text{split}} & \begin{array}{c} L^2_{even}(\mathbb{R}^2) \\ \oplus L^2_{odd}(\mathbb{R}^2) \end{array} & \xrightarrow{\text{Id}\oplus V} & \begin{array}{c} L^2_{even}(\mathbb{R}^2) \\ \oplus L^2_{even}(\mathbb{R}^2) \end{array} & \xrightarrow{Q\oplus Q} & L^2(\mathcal{O}) \oplus L^2(\mathcal{O})
\end{array}$$

Since D acts transitively on $\mathcal{O}$ we may pick the singleton consisting of $\vec{\gamma}_o = qe^{i\varphi}$ with $q = 1$ and $\varphi = 0$ as transversal. By Theorem 3.6, $\phi = \phi_1 \oplus \phi_2 \in L^2_{even}(\mathbb{R}^2) \oplus L^2_{even}(\mathbb{R}^2)$ is admissible for $\mu_1 \oplus \mu_2$ if and only if

$$\int_D |(Q\phi_j)(\vec{\gamma}_o R_{-\theta} t)|^2 \, d\mu(t,\theta) = c_\phi \qquad (j = 1,2)$$

and

$$\int_D (Q\phi_1)(\vec{\gamma}_o R_{-\theta} t) \, \overline{e^{i\theta/2}(Q\phi_2)(\vec{\gamma}_o R_{-\theta} t)} \, d\mu(t,\theta) = 0.$$

Normalising the Haar measure on D to $d\mu(t,\theta) = \frac{dt\,d\theta}{4\pi t}$,

$$\int_0^\infty \int_0^{4\pi} |(Q\phi_j)(\vec{\gamma}_o R_{-\theta} t)|^2 \, \frac{dt\,d\theta}{4\pi t} = c_\phi \qquad (j = 1,2) \tag{4.35}$$

and

$$\int_0^\infty \int_0^{4\pi} (Q\phi_1)(\vec{\gamma}_o R_{-\theta} t) \, \overline{e^{i\theta/2}(Q\phi_2)(\vec{\gamma}_0 R_{-\theta} t)} \, \frac{dt\,d\theta}{4\pi t} = 0. \tag{4.36}$$

We observe that the inner integral in (4.36) can be written as

$$\int_0^{2\pi} (Q\phi_1)(\vec{\gamma}_o R_{-\theta} t) \, \overline{e^{i\theta/2}(Q\phi_2)(\vec{\gamma}_o R_{-\theta} t)} \, d\theta - \int_0^{2\pi} (Q\phi_1)(\vec{\gamma}_o R_{-\theta} t) \, \overline{e^{i\theta/2}(Q\phi_2)(\vec{\gamma}_o R_{-\theta} t)} \, d\theta$$

and thus vanishes, hence the admissibility condition reduces to (4.35). Employing (4.33), it becomes

$$\int_0^\infty \int_0^{4\pi} \left|\phi_j(\sqrt{2t} e^{i\theta/2})\right|^2 \, \frac{dt\,d\theta}{4\pi t^2} = c_\phi \qquad (j = 1,2). \tag{4.37}$$

Replacing ϕ_2 by $V^{-1}\phi_2$ it follows that $\phi = \phi_1 \oplus \phi_2 \in L^2_{even}(\mathbb{R}^2) \oplus L^2_{odd}(\mathbb{R}^2)$ is admissible for μ if and only if (4.37) holds. Switching over to Cartesian coordinates, this admissibility condition is equivalent to

$$\frac{2}{\pi} \int_{\mathbb{R}^2} \frac{|\phi_j(\vec{x})|^2}{\|\vec{x}\|^4} \, d\vec{x} = c_\phi \quad (j = 1,2).$$

That is $\phi \in L^2(\mathbb{R}^2)$ is admissible if and only if its even and odd parts are orthogonal vectors of equal length in the space $L^2(\mathbb{R}^2, \frac{2dx}{\pi\|x\|^4})$. Using symmetry and the fact that $\phi_1(\vec{x}) = [\phi(\vec{x}) + \phi(-\vec{x})]/2$ and $\phi_2(\vec{x}) = [\phi(\vec{x}) - \phi(-\vec{x})]/2$, one easily shows that this condition is equivalent to the admissibility conditions derived in Cordero et al. (2006a),

$$\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{|\phi(\vec{x})|^2}{\|\vec{x}\|^4}\, d\vec{x} = c_\phi \qquad \text{and} \qquad \int_{\mathbb{R}^2} \frac{\phi(\vec{x})\ \overline{\phi(-\vec{x})}}{\|\vec{x}\|^4}\, d\vec{x} = 0.$$

Note that here we have normalized the Haar measure of $SO(2)$ to one; this is why the constant in front of the first integral differs from that in Cordero et al. (2006a).

4.6.1 Metaplectic Frames

Having established an equivalence of the metaplectic representation of K with a sum of wavelet representations, we can now make use of the rich theory wavelet frames to introduce frames for the metaplectic representation. As an illustration, we construct a Parseval frame along the lines of the proof of Theorem 3.12.

Since the global stabilizer D_o of K is not trivial, but we want to obtain a tight frame, we begin with the group $G = H \rtimes \mathbb{R}^n$. Fix an even positive integer $p \geq 8$, set

$$F = \left\{ h(t,\theta) \in H \,:\, \frac{1}{2} < t \leqslant 2,\, 0 \leqslant \theta < \frac{2\pi}{p} \right\}$$

and

$$P = \left\{ h\Big(4^d, \frac{2\pi l}{p}\Big) \in H \,:\, d \in \mathbb{Z},\ l \in \mathbb{Z}_p \right\}.$$

Then (P, F) is a tiling for H. We again choose the singleton $S = \{\vec{\gamma}_o\}$ as transversal, and let

$$T_1 = SF = \{\vec{\gamma}_o h(t,\theta) \,:\, h(t,\theta) \in V\} = \left\{ te^{-i\theta} \,:\, \frac{1}{2} < t \leqslant 2,\, 0 \leqslant \theta < \frac{2\pi}{p} \right\}$$

and

$$T_2 = SR_{-\frac{2\pi}{p}}F = \left\{\vec{\gamma}_o R_{-\frac{2\pi}{p}} h(t,\theta) \ : \ \ h(t,\theta) \in F\right\}$$
$$= \left\{te^{-i\theta} \ : \ \frac{1}{2} < t \leqslant 2, \ \frac{2\pi}{p} \leqslant \theta < \frac{4\pi}{p}\right\}.$$

Observe that $T_1 \cup T_2$ is contained in the unit square $R = [0,1] \times [-1,0]$ by choice of p. Now set

$$\widehat{\psi_1}(qe^{i\varphi}) = \mathbf{1}_{T_1}(qe^{i\varphi}), \qquad \widehat{\psi_2}(qe^{i\varphi}) = e^{i\varphi/2}\mathbf{1}_{T_2}(qe^{i\varphi}).$$

These two functions are defined as in the proof of Theorem 3.12, except that $\widehat{\psi_2}$ has a phase factor $e^{i\varphi/2}$ which does not affect the proof of Theorem 3.12. By the theorem

$$\left\{ \delta\left(g\left(4^d, \frac{2\pi l}{p}, \vec{u}\right)^{-1}\right)\left(\widehat{\psi_1} + \widehat{\psi_2}\right) \ : \ d \in \mathbb{Z}, \ l \in \mathbb{Z}_p, \ \vec{u} \in \mathbb{Z}^2 \right\}$$

is a Parseval frame for the sum of modulated wavelet representations of $G = H \rtimes \mathbb{R}^2$ on $L^2(\mathcal{O}) \oplus L^2(\mathcal{O})$.

Since the map $\Psi : K \to G$ has kernel D_o, for each $g\left(4^d, \frac{2\pi l}{p}, \vec{u}\right) \in G$, there exist two elements in K with $\Psi\left(k\left(4^d, \frac{2\pi l}{p}, \vec{u}\right)\right) = g\left(4^d, \frac{2\pi l}{p}, \vec{u}\right)$, namely $k\left(4^d, \frac{2\pi l}{p}, \vec{u}\right)$ and $k\left(4^d, 2\pi + \frac{2\pi l}{p}, \vec{u}\right)$; we choose first one. Hence

$$\left\{ \delta\left(k\left(4^d, \frac{2\pi l}{p}, \vec{u}\right)^{-1}\right)\left(\widehat{\psi_1} + \widehat{\psi_2}\right) \ : \ d \in \mathbb{Z}, \ l \in \mathbb{Z}_p, \ \vec{u} \in \mathbb{Z}^2 \right\}$$

is a frame for the sum of modulated wavelet representations δ of $K = D \rtimes M$ on $L^2(\mathcal{O}) \oplus L^2(\mathcal{O})$. Next we transfer this frame back to a frame of $L^2(\mathbb{R}^2)$ for the modulated metaplectic representation. The elements in the two copies of $L^2_{even}(\mathbb{R}^2)$ corresponding to $\widehat{\psi_j}$ $(j = 1, 2)$ are by (4.33),

$$\phi_1(re^{i\eta}) = \left(Q^{-1}\widehat{\psi_1}\right)(re^{i\eta}) = \frac{r}{\sqrt{2}}\left[\mathbf{1}_{S_1} + \mathbf{1}_{(-S_1)}\right](re^{i\eta})$$

and

$$\phi_2(re^{i\eta}) = \left(Q^{-1}\widehat{\psi_2}\right)\left(re^{i\eta}\right) = \frac{re^{i\eta}}{\sqrt{2}}\ \left[\mathbf{1}_{S_2} - \mathbf{1}_{(-S_2)}\right]\left(re^{i\eta}\right)$$

where

$$S_1 = \left\{te^{-i\theta}\ :\ 1 < t \leqslant 2,\ 0 \leqslant \theta < \frac{\pi}{p}\right\},\ \ S_2 = \left\{te^{-i\theta}\ :\ 1 < t \leqslant 2,\ \frac{\pi}{p} \leqslant \theta < \frac{2\pi}{p}\right\}.$$

Replacing ϕ_2 by $V^{-1}\phi_2$, we obtain

$$\phi_2(re^{i\eta}) = \frac{r}{\sqrt{2}}\Big[\mathbf{1}_{S_2} - \mathbf{1}_{(-S_2)}\Big]\left(re^{i\eta}\right) \in L^2_{odd}(\mathbb{R}^2).$$

Thus, if we let $\phi = \phi_1 + \phi_2$, then the collection

$$\left\{g_{d,l,\vec{u}} := \mu\left(k\left(4^d, \frac{2\pi l}{p}, \vec{u}\right)^{-1}\right)\phi\ :\ d \in \mathbb{Z},\ l \in \mathbb{Z}_p,\ \vec{u} \in \mathbb{Z}^2\right\}$$

will be a Parseval frame for $L^2(\mathbb{R}^2)$. Now as

$$k\left(4^d, \frac{2\pi l}{p}, \vec{u}\right)^{-1} = k\left(4^{-d}, 4\pi - \frac{2\pi l}{p}, -4^{-d}R_{\frac{2\pi l}{p}}\vec{u}\right)$$

it follows from (4.30) that

$$g_{d,l,\vec{u}}\left(\vec{x}\right) = 2^{-d}e^{i\pi\langle m(-4^{-d}R_{\frac{2\pi l}{p}}\vec{u})\vec{x},\vec{x}\rangle}\phi\left(2^{-d}R_{-\frac{\pi l}{p}}\vec{x}\right),$$

and a detailed computation shows that these functions are of the form

$$g_{d,l,\vec{u}}\left(\vec{x}\right) = 2^{-(d+1/2)}\,e^{-i\pi 4^{-d}\langle m(R_{\frac{2\pi l}{p}}\vec{u})\vec{x},\vec{x}\rangle}\|\vec{x}\|\left[\mathbf{1}_{S^{(0,0)}_{l,d}} + \mathbf{1}_{S^{(0,1)}_{l,d}} + \mathbf{1}_{S^{(1,0)}_{l,d}} - \mathbf{1}_{S^{(1,1)}_{l,d}}\right]\left(\vec{x}\right)$$

where

$$S^{(\alpha,\beta)}_{l,d} = \left\{te^{-i\theta} \in \mathbb{R}^2\ :\ 2^d < t \leqslant 2^{d+1},\ \tfrac{\pi(\beta p+\alpha+l)}{p} \leqslant \theta < \tfrac{\pi(\beta p+\alpha+l+1)}{p}\right\}$$

for $\alpha, \beta \in \{0, 1\}$.

CHAPTER V

EXTENSIONS OF THE HEISENBERG GROUP BY DILATIONS

In this chapter, we extend the Heisenberg group by a one-parameter group of matrix dilations. We then classify these extensions up to isomorphism and show that they can be represented as subgroups of the symplectic group of the form discussed in this thesis.

5.1 The Groups $G_{a,B,c}$

Recall from chapter II that the Heisenberg group $\mathbb{H}^n$ can be represented in matrix form as the polarized Heisenberg group

$$\mathbb{H}^n = \left\{ h(\vec{x}, \vec{y}, z) = \begin{bmatrix} 1 & \vec{y}^T & z \\ 0 & I_n & \vec{x} \\ 0 & 0 & 1 \end{bmatrix} \ : \ \vec{x}, \vec{y} \in \mathbb{R}^n, \ z \in \mathbb{R} \right\}.$$

In this parametrization, the group law is

$$h(\vec{x}, \vec{y}, z)\, h(\vec{x}', \vec{y}', z') = h(\vec{x} + \vec{x}', \vec{y} + \vec{y}', z + z' + \vec{y}^T \vec{x}').$$

Now fix nonnegative real numbers r and s, and an exponential $n \times n$ matrix $A = e^B$, with at least one of r, s, A not the identity. Next let D denote the closed subgroup of $GL_{n+2}(\mathbb{R})$ consisting of block-diagonal matrices of the form

$$D = \left\{ d(t) := \operatorname{diag}(r^t, A^t, s^t) \ : \ t \in \mathbb{R} \right\}.$$

We assume that A is not an orthogonal matrix in case $r = s = 1$; this is equivalent to B being not skew-symmetric and guarantees that the map $t \to d(t)$ is one-to-one. The group D acts on the Heisenberg group $\mathbb{H}^n$ by conjugation,

$$d(t)\, h(\vec{x},\vec{y},z)\, d(t)^{-1} = h(s^{-t}A^t\vec{x}, r^t(A^T)^{-t}\vec{y}, r^ts^{-t}z); \tag{5.1}$$

and this action gives rise to a semi-direct product

$$D \rtimes \mathbb{H}^n = \{(d(t), h(\vec{x},\vec{y},z)) \ : \ d(t) \in D,\ h(\vec{x},\vec{y},z) \in \mathbb{H}^n\}$$

with the group operation

$$\begin{aligned}
&(d(t), h(\vec{x},\vec{y},z))\ (d(t'), h(\vec{x}',\vec{y}',z')) \\
&\quad = \left(d(t+t'), h(\vec{x} + s^{-t}A^t\vec{x}', \vec{y} + r^t(A^T)^{-t}\vec{y}', z + r^ts^{-t}z' + s^{-t}\vec{y}^TA^t\vec{x}'\right).
\end{aligned}$$

As the action (5.1) is by conjugation, this semi-direct product can be presented in matrix form as

$$D \rtimes \mathbb{H}^n = H_{r,A,s} := \{h(\vec{x},\vec{y},z)\, d(t) \ : \ h(\vec{x},\vec{y},z) \in \mathbb{H}^n,\ d(t) \in D\},$$

that is,

$$H_{r,A,s} = \left\{ h(t,\vec{x},\vec{y},z) := \begin{bmatrix} r^t & \vec{y}^TA^t & s^tz \\ 0 & A^t & s^t\vec{x} \\ 0 & 0 & s^t \end{bmatrix} \ : \ \vec{x},\vec{y} \in \mathbb{R}^n,\ t,z \in \mathbb{R} \right\}$$

and we may rewrite the group operation as

$$\begin{aligned}
&h(t,\vec{x},\vec{y},z)\, h(t',\vec{x}',\vec{y}',z') \\
&\quad = h(t+t', \vec{x} + s^{-t}A^t\vec{x}', \vec{y} + r^t(A^T)^{-t}\vec{y}', z + r^ts^{-t}z' + s^{-t}\vec{y}^TA^t\vec{x}').
\end{aligned}$$

5.2 Classification of the Group $G_{a,B,c}$

In order to classify the groups $H_{r,A,s}$, we choose a simpler parametrization, replacing $\vec{y}^TA^t$ with $\vec{y}^T$, $s^t\vec{x}$ with $\vec{x}$ and s^tz with z. Furthermore, to distinguish between these parametrizations, we rename these groups to $G_{r,A,s}$,

$$G_{r,A,s} = \left\{ g(t,\vec{x},\vec{y},z) := \begin{bmatrix} r^t & \vec{y}^T & z \\ 0 & A^t & \vec{x} \\ 0 & 0 & s^t \end{bmatrix} \;:\; \vec{x},\vec{y} \in \mathbb{R}^n,\; t,z \in \mathbb{R} \right\}$$

and the group operation is given by

$$g(t,\vec{x},\vec{y},z)\, g(t',\vec{x}',\vec{y}',z') = g(t+t', s^{t'}\vec{x} + A^t\vec{x}', (A^T)^{t'}\vec{y} + r^t\vec{y}', s^{t'}z + r^t z' + \vec{y}^T\vec{x}').$$

It will be simpler to write the dilations in exponential form and relabel the groups yet again,

$$G_{a,B,c} = \left\{ \begin{bmatrix} e^{at} & \vec{y}^T & z \\ 0 & e^{Bt} & \vec{x} \\ 0 & 0 & e^{ct} \end{bmatrix} \;:\; \vec{x},\vec{y} \in \mathbb{R}^n,\; t,z \in \mathbb{R} \right\}$$

where $a, c \in \mathbb{R}$, $B \in M_n(\mathbb{R})$ and B is not skew-symmetric in case $a = c = 0$, with group law

$$\begin{aligned} &g(t,\vec{x},\vec{y},z)\, g(t',\vec{x}',\vec{y}',z') \\ &\quad = g(t+t', e^{ct'}\vec{x} + e^{Bt}\vec{x}', e^{B^T t'}\vec{y} + e^{at}\vec{y}', e^{ct'}z + e^{at}z' + \vec{y}^T\vec{x}'). \end{aligned} \tag{5.2}$$

The groups $G_{a,B,c}$ are simply connected. In fact being semi-direct products, they are topological products $D \times \mathbb{H}^n$. Now $\mathbb{H}^n$ carries the topolgy of $\mathbb{R}^{2n+1}$, and is thus simply connected. Since products of simply connected spaces are again simply connected, it suffices to show that D is homeomorphic to $\mathbb{R}$.

Clearly, $t \mapsto d(t)$ is continuous. We need to show that its inverse is also continuous. If $a \neq 0$, then $t \mapsto e^{at}$ is a homeomorphism of $\mathbb{R}$ into $(0,\infty)$. Thus, if $d(t_n) \to d(t_0)$, then $e^{at_n} \to e^{at_0}$ and hence $t_n \to t_0$. A similarly argument applies if $c \neq 0$. It is left to consider the case $a = c = 0$. Then by assumption, B is not skew-symmetric and hence the map $t \mapsto e^{Bt}$ is one-to-one, as can be seen, for example, from the Jordan normal form of e^{Bt}. Suppose first that, $e^{Bt_n} \to I$. Then for sufficiently large n, $\|e^{Bt_n} - I\| < 1$ and hence $\log(e^{Bt_n})$ exists, and $e^{\log(e^{Bt_n})} = e^{Bt_n}$. But as the map $t \mapsto e^{Bt}$ is one-to-one, we conclude that $\log(e^{Bt_n}) = Bt_n$. Since the log function is continuous in a neighborhood of the identity, it follows that $Bt_n = \log(e^{Bt_n}) \to \log(I) = 0$, and hence, as $B \neq 0$, $t_n \to 0$. In general, suppose $d(t_n) \to d(t_0)$. Then $e^{Bt_n} \to e^{Bt_0}$ so that $e^{B(t_n - t_0)} \to I$. By the above, it follows that $t_n - t_0 \to 0$ or equivalently, $t_n \to t_0$. This shows that the inverse of $t \mapsto d(t)$ is also continuous and hence D is homeomorphic to $\mathbb{R}$, so that $G_{a,B,c}$ is simply connected.

Because the groups $G_{a,B,c}$ are simply connected, it is enough to classify their Lie algebras $\mathfrak{g}_{a,B,c}$.

We now show that $\mathfrak{g}_{a,B,c}$ is the set of matrices of the form

$$\left\{ \begin{bmatrix} sa & \vec{y}^T & z \\ 0 & sB & \vec{x} \\ 0 & 0 & sc \end{bmatrix} : s, z \in \mathbb{R}, \vec{x}, \vec{y} \in \mathbb{R}^n \right\}. \tag{5.3}$$

In fact, consider the differentiable curve $\gamma : \mathbb{R} \to G_{a,B,c}$ given by

$$\gamma(t) = \begin{bmatrix} e^{ast} & t\vec{y}^T & tz \\ 0 & e^{Bst} & t\vec{x} \\ 0 & 0 & e^{cst} \end{bmatrix}$$

which satisfies $\gamma(0) = I_{n+2}$. One quickly verifies that

$$\gamma'(0) = \begin{bmatrix} sa & \vec{y}^T & z \\ 0 & sB & \vec{x} \\ 0 & 0 & sc \end{bmatrix}$$

so that the set (5.3) is a subset of $\mathfrak{g}_{a,B,c}$. Conversely, let $\gamma : (-\varepsilon, \varepsilon) \to G_{a,B,c}$ be any differentiable curve with $\gamma(0) = I_{n+2}$. As the map $t \mapsto d(t)$ is one-to-one, we can write this curve as

$$\gamma(t) = \begin{bmatrix} e^{a\gamma_1(t)} & \gamma_3(t)^T & \gamma_4(t) \\ 0 & e^{B\gamma_1(t)} & \gamma_2(t) \\ 0 & 0 & e^{c\gamma_1(t)} \end{bmatrix}$$

with $\gamma_1, \gamma_4 : (-\varepsilon, \varepsilon) \to \mathbb{R}$ and $\gamma_2, \gamma_3 : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ and $\gamma_i(0) = 0$ for all i. Computing the derivative at 0 we obtain by chain rule,

$$\gamma'(0) = \begin{bmatrix} \gamma_1'(0)a & \gamma_3'(0)^T & \gamma_4'(0) \\ 0 & \gamma_1'(0)B & \gamma_2'(0) \\ 0 & 0 & \gamma_1'(0)c \end{bmatrix}$$

which is an element of the set (5.3).

Now $\mathfrak{g}_{a,B,c}$ has a decomposition as a direct sum of vector spaces

$$\mathfrak{g}_{a,B,c} = V_M \oplus V_X \oplus V_Y \oplus V_Z$$

where

$$V_M = \left\{ sM \,:\, s \in \mathbb{R},\ M = \begin{bmatrix} a & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & c \end{bmatrix} \right\} \simeq \mathbb{R}$$

$$V_X = \left\{ X_{\vec{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \vec{x} \\ 0 & 0 & 0 \end{bmatrix} \,:\, \vec{x} \in \mathbb{R}^n \right\} \simeq \mathbb{R}^n$$

$$V_Y = \left\{ Y_{\vec{y}} = \begin{bmatrix} 0 & \vec{y}^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \,:\, \vec{y} \in \mathbb{R}^n \right\} \simeq \mathbb{R}^n$$

$$V_Z = \left\{ Z_z = \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \,:\, z \in \mathbb{R} \right\} \simeq \mathbb{R}.$$

and the nontrivial Lie algebra brackets are

$$\begin{aligned} [M, X_{\vec{x}}] &= X_{(B-cI)\vec{x}}, & [M, Y_{\vec{y}}] &= Y_{(aI-B^T)\vec{y}}, \\ [M, Z_z] &= Z_{(a-c)z}, & [Y_{\vec{y}}, X_{\vec{x}}] &= Z_{\vec{y}^T\vec{x}}. \end{aligned} \tag{5.4}$$

The Lie algebras $\mathfrak{g} = \mathfrak{g}_{a,B,c}$ are all solvable: In fact, we have

$$\begin{aligned} \mathfrak{g}^1 &= [\mathfrak{g}, \mathfrak{g}] \subseteq V_X \oplus V_y \oplus V_Z \\ \mathfrak{g}^2 &= \left[\mathfrak{g}^1, \mathfrak{g}^1\right] \subseteq [V_X \oplus V_y \oplus V_Z, V_X \oplus V_y \oplus V_Z] = V_Z \\ \mathfrak{g}^3 &= \left[\mathfrak{g}^2, \mathfrak{g}^2\right] \subseteq [V_Z, V_Z] = \{0\}. \end{aligned}$$

The classification now involves several steps. Throughout, I will denote the $n \times n$ identity matrix I_n.

Proposition 5.1. *If any of the following conditions hold, then $\mathfrak{g}_{\tilde{a},\tilde{B},\tilde{c}}$ and $\mathfrak{g}_{a,B,c}$ are isomorphic Lie algebras.*

1. *$\tilde{a} = a$, $\tilde{c} = c$ and $\tilde{B}$ is similar to B, say $\tilde{B} = SBS^{-1}$ for some $S \in GL_n(\mathbb{R})$.*
2. *$(\tilde{a}, \tilde{B}, \tilde{c}) = \alpha(a + k, B + kI, c + k)$ for some scalars $\alpha \neq 0$, k.*
3. *$(\tilde{a}, \tilde{B}, \tilde{c}) = (c, B^T, a)$.*

Proof. Throughout, we let $\tilde{M} = diag(\tilde{a}, \tilde{B}, \tilde{c})$. Furthermore, elements of the subspaces $\tilde{V}_X, \tilde{V}_Y$ and $\tilde{V}_Z$ of $\mathfrak{g}_{\tilde{a},\tilde{B},\tilde{c}}$ will still be denoted by $X_{\vec{x}}$, $Y_{\vec{y}}$ and Z_z, respectively.

1. Suppose that $\tilde{a} = a$, $\tilde{c} = c$ and $\tilde{B} = SBS^{-1}$. Define a vector space isomorphism $T : \mathfrak{g}_{a,B,c} \to \mathfrak{g}_{\tilde{a},\tilde{B},\tilde{c}}$ by

$$T(M) = \tilde{M}, \qquad T(X_{\vec{x}}) = X_{S\vec{x}}, \qquad T(Y_{\vec{y}}) = Y_{(S^{-1})^T\vec{y}}, \qquad T(Z_z) = Z_z.$$

A straightforward computation shows that T preserves Lie brackets,

$$\begin{aligned}
\big[T(M),T(X_{\vec{x}})\big] &= \big[\tilde{M},X_{S\vec{x}}\big] = X_{(\tilde{B}-\tilde{c}I)S\vec{x}} = X_{(SBS^{-1}-cI)S\vec{x}} = X_{S(B-cI)\vec{x}} \\
&= T(X_{(B-cI)\vec{x}}) = T\big([M,X_{\vec{x}}]\big) \\
\big[T(M),T(Y_{\vec{y}})\big] &= \big[\tilde{M},Y_{(S^{-1})^T\vec{y}}\big] = Y_{(\tilde{a}I-\tilde{B}^T)(S^{-1})^T\vec{y}} = Y_{(aI-(SBS^{-1})^T)(S^{-1})^T\vec{y}} \\
&= Y_{(S^{-1})^T(aI-B^T)\vec{y}} = T(Y_{(aI-B^T)\vec{y}}) = T\big([M,Y_{\vec{y}}]\big) \\
\big[T(M),T(Z_z)\big] &= \big[\tilde{M},Z_z\big] = Z_{(\tilde{a}-\tilde{c})z} = Z_{(a-c)z} = T(Z_{(a-c)z}) = T\big([M,Z_z]\big) \\
\big[T(Y_{\vec{y}}),T(X_{\vec{x}})\big] &= \big[Y_{(S^{-1})^T\vec{y},X_{S\vec{x}}}\big] = Z_{\vec{y}^TS^{-1}S\vec{x}} = Z_{\vec{y}^T\vec{x}} = T(Z_{\vec{y}^T\vec{x}}) = T\big([Y_{\vec{y}},X_{\vec{x}}]\big).
\end{aligned}$$

2. Next suppose that $(\tilde{a},\tilde{B},\tilde{c}) = \alpha(a+k,B+kI,c+k)$ with $\alpha \neq 0$. Define a vector space isomorphism $T : \mathfrak{g}_{a,B,c} \to \mathfrak{g}_{\tilde{a},\tilde{B},\tilde{c}}$ by

$$T(M) = \frac{1}{\alpha}\tilde{M}, \qquad T(X_{\vec{x}}) = X_{\vec{x}}, \qquad T(Y_{\vec{y}}) = Y_{\vec{y}}, \qquad T(Z_z) = Z_z.$$

We only need to verify that Lie brackets involving M are preserved,

$$\begin{aligned}
\big[T(M),T(X_{\vec{x}})\big] &= \left[\frac{1}{\alpha}\tilde{M},X_{\vec{x}}\right] = X_{(\frac{1}{\alpha}\tilde{B}-\frac{1}{\alpha}\tilde{c}I)\vec{x}} = X_{([B+kI]-[c+k]I)\vec{x}} \\
&= X_{(B-cI)\vec{x}} = T(X_{(B-cI)\vec{x}}) = T\big([M,X_{\vec{x}}]\big)
\end{aligned}$$

$$\begin{aligned}
\big[T(M),T(Y_{\vec{y}})\big] &= \left[\frac{1}{\alpha}\tilde{M},Y_{\vec{y}}\right] = Y_{(\frac{1}{\alpha}\tilde{a}I-\frac{1}{\alpha}\tilde{B}^T)\vec{y}} = Y_{([a+k]I-(B^T+kI])\vec{y}} \\
&= Y_{(aI-B^T)\vec{y}} = T(Y_{(aI-B^T)\vec{y}}) = T\big([M,Y_{\vec{y}}]\big) \\
\big[T(M),\,T(Z_z)\big] &= \left[\frac{1}{\alpha}\tilde{M},Z_z\right] = Z_{(\frac{1}{\alpha}\tilde{a}-\frac{1}{\alpha}\tilde{c})z} = Z_{([a+k]-[c+k])z} \\
&= Z_{(a-c)z} = T(Z_{(a-c)z}) = \big([M,Z_z]\big).
\end{aligned}$$

3. Finally, suppose that $(\tilde{a},\tilde{B},\tilde{c}) = (c,B^T,a)$. Define a vector space isomorphism $T : \mathfrak{g}_{a,B,c} \to \mathfrak{g}_{\tilde{a},\tilde{B},\tilde{c}}$ by

$$T(M) = -\tilde{M}, \qquad T(X_{\vec{x}}) = Y_{\vec{x}}, \qquad T(Y_{\vec{y}}) = X_{\vec{y}}, \qquad T(Z_z) = Z_z.$$

Then T preserves Lie brackets, as

$$\begin{aligned}
\big[T(M), T(X_{\vec{x}})\big] &= \big[-\tilde{M}, Y_{\vec{x}}\big] = Y_{(\tilde{B}^T-\tilde{a}I)\vec{x}} = Y_{(B-cI)\vec{x}} \\
&= T(X_{(B-cI)\vec{x}}) = T\big([M, X_{\vec{x}}]\big) \\
\big[T(M), T(Y_{\vec{y}})\big] &= \big[-\tilde{M}, X_{\vec{y}}\big] = X_{(\tilde{c}I-\tilde{B})\vec{y}} = X_{(aI-B^T)\vec{y}} \\
&= T(Y_{(aI-B^T)\vec{y}}) = T\big([M, Y_{\vec{y}}]\big) \\
\big[T(M),\, T(Z_z)\big] &= \big[-\tilde{M}, Z_z\big] = Z_{(\tilde{c}-\tilde{a})z} = Z_{(a-c)z} = T(Z_{(a-c)z}) = T\big([M, Z_z]\big) \\
\big[T(Y_{\vec{y}}),\, T(X_{\vec{x}})\big] &= \big[X_{\vec{y}}, Y_{\vec{x}}\big] = Z_{\vec{x}^T\vec{y}} = Z_{\vec{y}^T\vec{x}} = T(Z_{\vec{y}^T\vec{x}}) = T\big([Y_{\vec{y}}, X_{\vec{x}}]\big).
\end{aligned}$$

This proves the proposition. □

Proposition 5.2. *Suppose B is a block-diagonal matrix, $B = diag(B_1, ..., B_l)$. Let $\tilde{B}$ denote the matrix obtained from B by replacing the i-th block B_i with $pI - B_i^T$, where $p = a + c$. Then $\mathfrak{g}_{a,\tilde{B},c}$ and $\mathfrak{g}_{a,B,c}$ are isomorphic.*

Proof. By the first part of the previous proposition, we may assume that the block to be modified is the first block,

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \qquad \text{and} \qquad \tilde{B} = \begin{bmatrix} pI_m - B_1^T & 0 \\ 0 & B_2 \end{bmatrix}.$$

Let $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$ be the corresponding decomposition of $\mathbb{R}^n$. Define a vector space isomorphism $T : \mathfrak{g}_{a,B,c} \to \mathfrak{g}_{a,\tilde{B},c}$ by

$$\begin{aligned}
&T(M) = \tilde{M}, && T(X_{(\vec{x}_1,\vec{x}_2)}) = -Y_{(\vec{x}_1,0)} + X_{(0,\vec{x}_2)}, \\
&T(Y_{(\vec{y}_1,\vec{y}_2)}) = X_{(\vec{y}_1,0)} + Y_{(0,\vec{y}_2)}, && T(Z_z) = Z_z
\end{aligned}$$

where $\tilde{M} = diag(a, \tilde{B}, c)$, and $\vec{x}_1, \vec{y}_1 \in \mathbb{R}^m$, $\vec{x}_2, \vec{y}_2 \in \mathbb{R}^{n-m}$. Then

$$\begin{aligned}
\big[T(M), T(X_{(\vec{x}_1,\vec{x}_2)})\big] &= \big[\tilde{M}, -Y_{(\vec{x}_1,0)} + X_{(0,\vec{x}_2)}\big] = \big[\tilde{M}, -Y_{(\vec{x}_1,0)}\big] + \big[\tilde{M}, X_{(0,\vec{x}_2)}\big] \\
&= -Y_{(aI-\tilde{B}^T)(\vec{x}_1,0)} + X_{(\tilde{B}-cI)(0,\vec{x}_2)} = -Y_{(B_1\vec{x}_1 - c\vec{x}_1,0)} + X_{(0,B_2\vec{x}_2 - c\vec{x}_2)} \\
&= T(X_{(B_1\vec{x}_1-c\vec{x}_1,B_2\vec{x}_2-c\vec{x}_2)}) = T(X_{B(\vec{x}_1,\vec{x}_2)-c(\vec{x}_1,\vec{x}_2)}) = T\big(\,[M, X_{(\vec{x}_1,\vec{x}_2)}]\,\big) \\
\big[T(M), T(Y_{(\vec{y}_1,\vec{y}_2)})\big] &= \big[\tilde{M}, X_{(\vec{y}_1,0)} + Y_{(0,\vec{y}_2)}\big] = \big[\tilde{M}, X_{(\vec{y}_1,0)}\big] + \big[\tilde{M}, Y_{(0,\vec{y}_2)}\big] \\
&= X_{(\tilde{B}-cI)(\vec{y}_1,0)} + Y_{(aI-\tilde{B}^T)(0,\vec{y}_2)} = X_{(a\vec{y}_1 - B_1^T\vec{y}_1,0)} + Y_{(0,a\vec{y}_2 - B_2^T\vec{y}_2)} \\
&= T(Y_{(a\vec{y}_1-B_1^T\vec{y}_1,a\vec{y}_2-B_2^T\vec{y}_2)}) = T(Y_{a(\vec{y}_1,\vec{y}_2)-B^T(\vec{y}_1,\vec{y}_2)}) = T\big(\,[M, Y_{(\vec{y}_1,\vec{y}_2)}]\,\big) \\
\big[T(M), T(Z_z)\big] &= \big[\tilde{M}, Z_z\big] = Z_{(a-c)z} = T(Z_{(a-c)z}) = T\big([M, Z_z]\big) \\
\big[T(Y_{(\vec{y}_1,\vec{y}_2)}),\, T(X_{(\vec{x}_1,\vec{x}_2)})\big] &= \big[X_{(\vec{y}_1,0)} + Y_{(0,\vec{y}_2)}, -Y_{(\vec{x}_1,0)} + X_{(0,\vec{x}_2)}\big] \\
&= \big[X_{(\vec{y}_1,0)}, -Y_{(\vec{x}_1,0)}\big] + \big[Y_{(0,\vec{y}_2)}, X_{(0,\vec{x}_2)}\big] = Z_{\vec{x}_1^T\vec{y}_1} + Z_{\vec{y}_2^T\vec{x}_1} \\
&= T(Z_{(\vec{y}_1,\vec{y}_2)^T(\vec{x}_1,\vec{x}_2)}) = T\big(\,[Y_{(\vec{y}_1,\vec{y}_2)}, X_{(\vec{x}_1,\vec{x}_2)}]\,\big).
\end{aligned}$$

This shows that T preserves the Lie brackets and proves the proposition. $\square$

By Proposition 5.1, we may normalize the algebras $\mathfrak{g}_{a,B,c}$ so that $c = 0$, $a \in \{0,1\}$ and B is in real Jordan normal form. Then $p = a$, and Proposition 5.2 allows us to assume that $\mathrm{Re}(\lambda) \geqslant \frac{p}{2}$ for each eigenvalue λ of B. Denote the normalized Lie algebras by $\mathfrak{g}_{p,B}$. The Lie brackets (5.4) are thus of the form

$$[M, X_{\vec{x}}] = X_{B\vec{x}}, \quad [M, Y_{\vec{y}}] = Y_{(pI-B^T)\vec{y}}, \quad [M, Z_z] = Z_{pz}, \quad [Y_{\vec{y}}, X_{\vec{x}}] = Z_{\vec{y}^T\vec{x}}. \quad (5.5)$$

Let us investigate the structure of $\mathfrak{g}_{p,B}$ for various values of p and types of B.

Begin with the case $p = 1$. It is easy to see from (5.5) that each Lie algebra $\mathfrak{g} = \mathfrak{g}_{1,B}$ has trivial center. In addition, the nilradical is $\mathfrak{h} = V_X \oplus V_Y \oplus V_Z$. To see this, note that $\mathfrak{g}$ itself is not nilpotent:

$$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] \supseteq \operatorname{span}\big\{X_{B\vec{x}}, Y_{(I-B^T)\vec{y}} \ :\ \vec{x}, \vec{y} \in \mathbb{R}^n\big\}$$

and continuing inductively,

$$\mathfrak{g}_j = [\mathfrak{g}, \mathfrak{g}_{j-1}] \supseteq \operatorname{span}\left\{X_{B^j\vec{x}}, Y_{(I-B^T)^j\vec{y}} \;:\; \vec{x}, \vec{y} \in \mathbb{R}^n\right\}$$

for all j. Now if B is not nilpotent then obviously, $\{X_{B^j\vec{x}} \;:\; \vec{x} \in \mathbb{R}^n\} \neq \{0\}$ for all j. On the other hand, if B is nilpotent, then $I - B^T$ is not nilpotent, so that $\left\{Y_{(I-B^T)^j\vec{y}} \;:\; \vec{y} \in \mathbb{R}^n\right\} \neq \{0\}$ for all j. Thus, $\mathfrak{g}_j \neq \{0\}$ for all j. In addition, $\mathfrak{h}$ is a maximal ideal in $\mathfrak{g}$, and this ideal is nilpotent, as

$$\mathfrak{h}_1 = [\mathfrak{h}, \mathfrak{h}] = V_Z$$

$$\mathfrak{h}_2 = [\mathfrak{h}, \mathfrak{h}_1] = [\mathfrak{h}, V_Z] = \{0\}.$$

Thus, $\mathfrak{h}$ is the nilradical of $\mathfrak{g}$.

Next consider the case $p = 0$. Clearly, $\mathfrak{g} = \mathfrak{g}_{0,B}$ has center V_Z. We split B into its nilpotent and invertible parts B_0 and B_1 respectively, and by a change of basis we may assume that B has form $B = \operatorname{diag}(B_0, B_1)$. Now whenever B is not nilpotent, we may further normalize to $\det(B_1) = 1$; after this normalization, the eigenvalues of B still have non-negative real parts. Furthermore arguing as in case $p = 1$, one sees that $\{X_{B^j\vec{x}} \;:\; \vec{x} \in \mathbb{R}^n\} \neq \{0\}$ for all j, so that $\mathfrak{g}$ has nilradical $V_X \oplus V_Y \oplus V_Z$. On the other hand, if B itself is a nilpotent matrix of degree $k \geqslant 1$, then $\mathfrak{g}_{0,B}$ is a nilpotent Lie algebra of nilpotency $k+1 \geqslant 2$. For suppose, $B^k = 0$ while $B^{k-1} \neq 0$. Then

$$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] = \operatorname{span}\left\{X_{B\vec{x}}, Y_{B^T\vec{y}}, Z_z \;:\; \vec{x}, \vec{y} \in \mathbb{R}^n,\; z \in \mathbb{R}\right\}$$

and by induction,

$$\mathfrak{g}_j = [\mathfrak{g}, \mathfrak{g}_{j-1}] = \operatorname{span}\left\{X_{B^j\vec{x}}, Y_{(B^T)^j\vec{y}}, Z_z \;:\; \vec{x}, \vec{y} \in \mathbb{R}^n,\; z \in \mathbb{R}\right\} \neq \{0\}$$

for all $j \leqslant k$. Observe that as $B^k = 0$, then $\mathfrak{g}_k = \{Z_z \;:\; z \in \mathbb{R}\} = V_Z$. It follows that

$$\mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_k] = [\mathfrak{g}, V_Z] = \{0\}.$$

There is one additional exceptional case, namely $(a, B, c) = (k, kI, k)$. Here, we normalize to $k = 1$ and denote this normalized Lie algebra by $\mathfrak{g}_{2,I}$. It has center $V_M \oplus V_Z$.

It follows that two Lie algebras $\mathfrak{g}_{\tilde{p},\tilde{B}}$ and $\mathfrak{g}_{p,B}$ can only be isomorphic if $\tilde{p} = p$. Furthermore, $\mathfrak{g}_{0,\tilde{B}}$ and $\mathfrak{g}_{0,B}$ can only be isomorphic if B and $\tilde{B}$ have identical degrees of nilpotency. In fact, we have:

Theorem 5.3. *Two normalized Lie algebras $\mathfrak{g}_{p,\tilde{B}}$ and $\mathfrak{g}_{p,B}$ are isomorphic if and only if $\tilde{B}$ and B are similar.*

Proof. The sufficiency implication is an immediate consequence of Proposition 5.1. To prove necessity, let T be a Lie algebra isomorphism mapping $\mathfrak{g}_{p,B}$ onto $\mathfrak{g}_{p,\tilde{B}}$. Using the vector space decomposition of these Lie algebras $\mathfrak{g}_{p,B} = V_M \oplus V_X \oplus V_Y \oplus V_Z$ and $\mathfrak{g}_{p,\tilde{B}} = \tilde{V}_M \oplus \tilde{V}_X \oplus \tilde{V}_Y \oplus \tilde{V}_Z$, we may represent T by the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

with each a_{ij} corresponding to a linear mapping between Euclidean spaces. (Here we write elements of the Lie algebras as column vectors.)

First suppose that $p = 1$. As T maps nilradical to nilradical, and the center V_Z of the nilradical to the center $\tilde{V}_Z$ of the nilradical, it follows that T has matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{5.6}$$

with $a_{11} \neq 0$, $a_{44} \neq 0$ by invertibility of T. Next suppose that $p = 0$ and B, hence $\tilde{B}$, is not nilpotent. (The case $p = 0$ and B nilpotent will be treated later.) Then

T maps nilradical to nilradical, and the center V_Z of $\mathfrak{g}_{0,B}$ to the center $\tilde{V}_Z$ of $\mathfrak{g}_{0,\tilde{B}}$; hence again has matrix representation (5.6).

In both cases, as T maps the ideal V_Z onto the ideal $\tilde{V}_Z$, it determines a Lie algebra isomorphism T_0 between the quotient algebras,

$$T_0 \quad : \quad \mathfrak{k} := \mathfrak{g}_{p,B}/V_Z \quad \to \quad \tilde{\mathfrak{k}} := \mathfrak{g}_{p,\tilde{B}}/\tilde{V}_Z. \tag{5.7}$$

Observe that $\mathfrak{k}$ (and similarly $\tilde{\mathfrak{k}}$) is isomorphic as a vector space to $V_M \oplus V_X \oplus V_Y$ and has nontrivial Lie brackets

$$[M, X_{\vec{x}}] = X_{B\vec{x}}, \quad [M, Y_{\vec{y}}] = Y_{(pI-B^T)\vec{y}},$$

hence T_0 has matrix representation

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

with $a_{11} \neq 0$. That is,

$$\begin{aligned} T_0(M) &= a_{11}\tilde{M} + \tilde{X}_{a_{21}} + \tilde{Y}_{a_{31}} \\ T_0(X_{\vec{x}}) &= \tilde{X}_{a_{22}\vec{x}} + \tilde{Y}_{a_{32}\vec{x}} \\ T_0(Y_{\vec{y}}) &= \tilde{X}_{a_{23}\vec{y}} + \tilde{Y}_{a_{33}\vec{y}}. \end{aligned}$$

Note that

$$\big[T_0(M), T_0(X_{\vec{x}})\big] = T_0\big([M, X_{\vec{x}}]\big) = T_0(X_{B\vec{x}}) = \tilde{X}_{a_{22}B\vec{x}} + \tilde{Y}_{a_{32}B\vec{x}}$$
$$\big[T_0(M), T_0(Y_{\vec{y}})\big] = T_0\big([M, Y_{\vec{y}}]\big) = T_0(Y_{(pI-B^T)\vec{y}}) = \tilde{X}_{a_{23}(pI-B^T)\vec{y}} + \tilde{Y}_{a_{33}(pI-B^T)\vec{y}}$$

while also

$$\big[T_0(M), T_0(X_{\vec{x}})\big] = \tilde{X}_{a_{11}\tilde{B}a_{22}\vec{x}} + \tilde{Y}_{a_{11}(pI-\tilde{B}^T)a_{32}\vec{x}}$$
$$\big[T_0(M), T_0(Y_{\vec{y}})\big] = \tilde{X}_{a_{11}\tilde{B}a_{23}\vec{y}} + \tilde{Y}_{a_{11}(pI-\tilde{B}^T)a_{33}\vec{y}}.$$

Comparing coefficients, we see that T_0 preserves Lie brackets if and only if

$$a_{11}\begin{bmatrix} \tilde{B} & 0 \\ 0 & pI-\tilde{B}^T \end{bmatrix}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & pI-B^T \end{bmatrix}. \tag{5.8}$$

Now the matrix

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \tag{5.9}$$

still represents an isomorphism of vector spaces and satisfies system (5.8), hence we may assume the T_0 is of this simplified form.

Next we determine the value of a_{11}. If $p = 1$, then by (5.6) and as T preserves Lie brackets,

$$a_{44}Z_z = T(Z_z) = T\left([M, Z_z]\right) = [T(M), T(Z_z)] = \left[a_{11}\tilde{M}, \tilde{Z}_{a_{44}z}\right] = a_{11}a_{44}Z_z,$$

and it follows that $a_{11} = 1$. On the other hand if $p = 0$, using the fact that the matrix $\left[\begin{smallmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{smallmatrix}\right]$ is invertible by (5.9), equation (5.8) shows that $a_{11}\mathrm{diag}(\tilde{B}, -\tilde{B}^T)$ and $\mathrm{diag}(B, -B^T)$ are similar, hence so are their invertible parts $a_{11}\mathrm{diag}(\tilde{B}_1, -\tilde{B}_1^T)$ and $\mathrm{diag}(B_1, -B_1^T)$. However, the determinants of the invertible parts have been normalized to one, it thus follows that $a_{11} = 1$ as well.

Recall that we assume throughout that B and $\tilde{B}$ are in real Jordan normal form, and we are now ready to show that they have identical Jordan blocks. Consider the adjoint action of M, $\mathrm{ad}_M : \mathfrak{k} \to \mathfrak{k}$ given by $\mathrm{ad}_M(V) = [M, V]$ for $V \in \mathfrak{k}$. Since

$$\mathrm{ad}_M(X_{\vec{x}}) = X_{B\vec{x}}, \qquad \text{and} \qquad \mathrm{ad}_M(Y_{\vec{y}}) = Y_{(pI-B^T)\vec{y}},$$

then V_X and V_Y are both ad_M-invariant subspaces, and the restriction of ad_M to $V_X \oplus V_Y \simeq \mathbb{R}^n \oplus \mathbb{R}^n$ can be represented by the matrix

$$\begin{bmatrix} B & 0 \\ 0 & pI-B^T \end{bmatrix}.$$

Since each eigenvalue λ_k of B has real part equal or greater than $p/2$, it follows that the collection of eigenvalues and Jordan blocks of ad_M is obtained from that of B as follows. A Jordan block J_k of B corresponding to some eigenvalue λ_k (for ease of notation, we consider a complex conjugate pair of eigenvalues as a single eigenvalue here) gives rise first to an identical Jordan block on V_X belonging to the very same eigenvalue $\lambda_k^+ = \lambda_k$ of ad_M so that $\mathrm{Re}(\lambda_k^+) \geqslant p/2$, and secondly to a Jordan block on V_Y derived from $pI - J_k^T$ and belonging to the eigenvalue $\lambda_k^- = p - \lambda_k$, so that $\mathrm{Re}(\lambda_k^-) \leqslant p/2$. As we have identified V_X and V_Y each with $\mathbb{R}^n$ in the obvious way, these two Jordan blocks operate on the same subspace of $\mathbb{R}^n$. A similar statement holds for $\mathrm{ad}_{\tilde{M}}$.

Now since T_0 maps $V = V_X \oplus V_Y$ onto $\tilde{V} = \tilde{V}_X \oplus \tilde{V}_Y$, maps M to $\tilde{M}$ and preserves Lie brackets, we have

$$\mathrm{ad}_{\tilde{M}} = T_0|_V \circ \mathrm{ad}_M \circ (T_0|_V)^{-1}.$$

Thus, $\mathrm{ad}_{\tilde{M}}$ and ad_M have identical eigenvalues and identical Jordan blocks.

The above description shows that there is a one-to-one correspondence between the Jordan blocks of B belonging to eigenvalues $\mathrm{Re}(\lambda) > p/2$ and those of ad_M belonging to eigenvalues $\mathrm{Re}(\lambda) > p/2$, the latter blocks operating on V_X only. As $\tilde{B}$ and $\mathrm{ad}_{\tilde{M}}$ have the same property, B and $\tilde{B}$ must have identical Jordan blocks for this range of eigenvalues. On the other hand, as each Jordan block of B belonging to an eigenvalue $\mathrm{Re}(\lambda) = p/2$ determines a pair of identical Jordan blocks of ad_M, one acting on V_X and the other on V_Y, and a similar statement is true for $\tilde{B}$ and $\mathrm{ad}_{\tilde{M}}$, there is also a one-to-one correspondence between the Jordan blocks of B and $\tilde{B}$ belonging to an eigenvalue $\mathrm{Re}(\lambda) = p/2$. This shows that B and $\tilde{B}$ are similar matrices.

It is left to discuss the case where $p = 0$ and $B, \tilde{B}$ are both nilpotent of degree k, so that $\mathfrak{g}_{0,B}$ and $\mathfrak{g}_{0,\tilde{B}}$ are both of nilpotency $k+1$. As B and $\tilde{B}$ are nilpotent, all their eigenvalues are zero, so it suffices to verify that they both have Jordan blocks of identical sizes. Again, T maps center to center and thus factors as in (5.7). Note that $\mathfrak{k}$ and $\tilde{\mathfrak{k}}$ are both of nilpotency k only. In fact, considering the lower central series

$$\mathfrak{k}_j = [\mathfrak{k}, \mathfrak{k}_{j-1}] = [M, \mathfrak{k}_{j-1}] = \operatorname{span}\left\{X_{B^j\vec{x}}, Y_{(B^T)^j\vec{y}} \;:\; \vec{x}, \vec{y} \in \mathbb{R}^n\right\} \tag{5.10}$$

where $\mathfrak{k}_0 = \mathfrak{k}$, we see that $\mathfrak{k}_j \neq \{0\}$ for $j < k$, and by nilpotency of B, $\mathfrak{k}_k = \{0\}$. For each $1 \leqslant r \leqslant k$, let n_r (respectively $\tilde{n}_r$) denote the number of Jordan blocks of B, hence of $-B^T$, (respectively $\tilde{B}$ and $-\tilde{B}^T$) which are nilpotent of degree r. Since a Jordan block of nilpotency r has size r as well, (5.10) shows that a Jordan block of B of nilpotency r will result in two component subspaces of $\mathfrak{k}_j$ of dimension $r - j$ each, provided that $r > j$. Counting the dimensions of the component subspaces of each $\mathfrak{k}_j$ we obtain

$$\dim(\mathfrak{k}_0) = 1 + 2\sum_{r=1}^{k} r n_r \qquad \dim(\tilde{\mathfrak{k}}_0) = 1 + 2\sum_{r=1}^{k} r\tilde{n}_r$$
$$\dim(\mathfrak{k}_j) = 2\sum_{r=j+1}^{k}(r-j)n_r \qquad \dim(\tilde{\mathfrak{k}}_j) = 2\sum_{r=j+1}^{k}(r-j)\tilde{n}_r$$

where $1 \leqslant j \leqslant k-1$. Now as $\mathfrak{k}_j$ and $\tilde{\mathfrak{k}}_j$ are isomorphic Lie algebras, it follows that $n_r = \tilde{n}_r$ for all $1 \leqslant r \leqslant k$, and hence B and $\tilde{B}$ have Jordan blocks of identical sizes. Thus, the proof of the theorem is complete. □

5.3 The Metaplectic Representations of the Groups $G_{a,B,c}$

We now show that the groups $G_{a,B,c}$ and hence $H_{r,A,s}$, are isomorphic to subgroups of the symplectic group $Sp(n+1, \mathbb{R})$. With applications in mind, we normalize only mildly by assuming that $c = 0$, and setting $p = a$. We thus consider groups

$$G_{p,B} = \left\{ g(t,\vec{x},\vec{y},z) := \begin{bmatrix} e^{pt} & \vec{y}^T & z \\ 0 & e^{Bt} & \vec{x} \\ 0 & 0 & 1 \end{bmatrix} \ : \ \vec{x},\vec{y} \in \mathbb{R}^n, t,z \in \mathbb{R} \right\}$$

with $p \in \mathbb{R}$ and $B \in M_n(\mathbb{R})$ fixed, so that the group law is

$$g(t,\vec{x},\vec{y},z)g(t',\vec{x}',\vec{y}',z') = g(t+t',\vec{x}+e^{Bt}\vec{x}', e^{B^Tt'}\vec{y}+e^{pt}\vec{y}', z+e^{pt}z'+\vec{y}^T\vec{x}').$$

Consider the $n+1$ dimensional subspace of $Sym(n+1,\mathbb{R})$,

$$M = \left\{ m(z,\vec{x}) := \begin{bmatrix} -z & -\vec{x}^T \\ -\vec{x} & 0 \end{bmatrix} \ : \ \vec{x} \in \mathbb{R}^n, \ z \in \mathbb{R} \right\}$$

and the closed subgroup of $GL_{n+1}(\mathbb{R})$,

$$D_{p,B} = \left\{ a(t,\vec{y}) := \begin{bmatrix} e^{-pt/2} & 0 \\ -\frac{1}{2}e^{-pt/2}e^{-B^Tt}\vec{y} & e^{pt/2}e^{-B^Tt} \end{bmatrix} \ : \ t \in \mathbb{R}, \ \vec{y} \in \mathbb{R}^n \right\}.$$

The group law in $D_{p,B}$ is

$$a(t,\vec{y})a(t',\vec{y}') = a(t+t', e^{B^Tt'}\vec{y}+e^{pt}\vec{y}'). \tag{5.11}$$

Now M is invariant under the $D_{p,B}$-action, in fact

$$\left(a(t,\vec{y})^{-1}\right)^T m(z,\vec{x})\, a(t,\vec{y})^{-1} = m(e^{pt}z+\vec{y}^T\vec{x}, e^{Bt}\vec{x}) \tag{5.12}$$

for $m(z,\vec{x}) \in M$ and $a(t,\vec{y}) \in D_{p,B}$. We thus obtain a semi-direct product

$$K = K_{p,B} = \{k(t,\vec{x},\vec{y},z) \ : \ t,z \in \mathbb{R}, \ \vec{x},\vec{y} \in \mathbb{R}^n\}$$

which can be represented as a closed subgroup of $Sp(n+1,\mathbb{R})$ by

$$K_{p,B} = \left\{ k(t,\vec{x},\vec{y},z) = \begin{bmatrix} a(t,\vec{y}) & 0 \\ m(z,\vec{x})a(t,\vec{y}) & \left(a(t,\vec{y})^T\right)^{-1} \end{bmatrix} \ : \ t,z \in \mathbb{R}, \ \vec{x},\vec{y} \in \mathbb{R}^n \right\}$$

and by (5.11) and (5.12) possesses the group law

$$k(t,\vec{x},\vec{y},z)\,k(t',\vec{x}',\vec{y}',z') = k(t+t',\vec{x}+e^{Bt}\vec{x}', e^{B^Tt'}\vec{y}+e^{pt}\vec{y}', z+e^{pt}z'+\vec{y}^T\vec{x}').$$

Note that this is the same group law as in $G_{p,B}$. In fact, since $D_{p,B}$ acts effectively on M, then by the discussion in chapter IV, $K_{p,B}$ is isomorphic to an affine group $H_{p,B} \rtimes \mathbb{R}^{n+1}$, where by (5.12),

$$H_{p,B} = \left\{ h(t,\vec{y}) := \begin{bmatrix} e^{pt} & \vec{y}^T \\ 0 & e^{Bt} \end{bmatrix} \;:\; \vec{y}\in\mathbb{R}^n,\, t\in\mathbb{R} \right\}.$$

and hence

$$H_{p,B} \rtimes \mathbb{R}^{n+1} = \left\{ \begin{bmatrix} h(t,\vec{y}) & \left[\begin{smallmatrix} z \\ \vec{x} \end{smallmatrix}\right] \\ 0 & 1 \end{bmatrix} \;:\; h(t,\vec{y}) \in H_{p,B},\; (z,\vec{x})^T \in \mathbb{R}^{n+1} \right\}$$

which is precisely the group $G_{p,B}$.

We now discuss admissibility of $K_{p,B}$ for the metaplectic representation, which is given by

$$\mu\left(k(t,\vec{x},\vec{y},z)\right) = N_{-m(z,\vec{x})} D_{a(t,\vec{y})}.$$

Now as

$$a(t,\vec{y})^{-1}(s,\vec{w}) = \begin{bmatrix} e^{pt/2} & 0 \\ \frac{1}{2}e^{-pt/2}\vec{y} & e^{-pt/2}e^{B^Tt} \end{bmatrix} \begin{bmatrix} s \\ \vec{w} \end{bmatrix} = \begin{bmatrix} e^{pt/2}s \\ e^{-pt/2}\left(\frac{s}{2}\vec{y}+e^{B^Tt}\vec{w}\right) \end{bmatrix} \tag{5.13}$$

and

$$\langle m(z,\vec{x})(s,\vec{w}),(s,\vec{w})\rangle = -s^2z - 2s\vec{w}^T\vec{x} = -2\left\langle [z,\vec{x}], \frac{1}{2}\left[s^2, 2s\vec{w}\right] \right\rangle \tag{5.14}$$

it follows that

$$\mu\left(k(t,\vec{x},\vec{y},z)\right)f(s,\vec{w})$$
$$= e^{-i\pi(s^2z+2s\vec{w}^T\vec{x})}e^{(1-n)pt/4}e^{tr(B)t/2}\, f\left(e^{pt/2}s, e^{-pt/2}\left(\frac{s}{2}\vec{y}+e^{B^Tt}\vec{w}\right)\right)$$

for $f \in L^2(\mathbb{R}^{n+1})$ and $(s, \vec{w}) \in \mathbb{R}^{n+1}$. Next we compute the map $\Phi : \mathbb{R}^{n+1} \to \widehat{\mathbb{R}^{n+1}}$. Equation (5.14) shows that

$$\Phi(s, \vec{w}) = \frac{1}{2}(s^2, 2s\vec{w}^T)$$

and its Jacobian is computed as

$$J_\Phi(s, \vec{w}) = s^{n+1}. \tag{5.15}$$

We note that $J_\Phi(s, \vec{w}) = 0$ if and only if $s = 0$. This leads to a splitting of $\mathbb{R}^{n+1}$ into two open half spaces

$$U_1 = \mathbb{R}^{n+1}_- := \left\{(s, \vec{w}) \in \mathbb{R}^{n+1} \,:\, s < 0\right\}$$
$$U_2 = \mathbb{R}^{n+1}_+ := \left\{(s, \vec{w}) \in \mathbb{R}^{n+1} \,:\, s > 0\right\}$$

which by (5.13) are both $D_{p,B}$-invariant.

The restrictions Φ_j of Φ to U_j, $j = 1, 2$ map the sets U_j homeomorphically onto the open half space

$$\mathcal{O}_+ := \left\{(\omega, \vec{\gamma}) \in \widehat{\mathbb{R}^{n+1}} \,:\, \omega > 0\right\}.$$

of $\widehat{\mathbb{R}^{n+1}}$, and the inverse maps are given by

$$\Phi_j^{-1}(\omega, \vec{\gamma}) = \left((-1)^j\sqrt{2\omega},\, \frac{(-1)^j\vec{\gamma}}{\sqrt{2\omega}}\right)^T$$

for all $(\omega, \vec{\gamma}) \in \mathcal{O}_+$. Furthermore, by (5.15), the Jacobians of Φ_j^{-1} at $(\omega, \vec{\gamma}) \in \mathcal{O}_+$ are

$$J_{\Phi_j^{-1}}(\omega, \vec{\gamma}) = \frac{1}{J_\Phi\left(\Phi_j^{-1}(\omega, \vec{\gamma})\right)} = \frac{(-1)^{(n+1)j}}{(2\omega)^{(n+1)/2}}.$$

Now as the sets U_j are $D_{p,B}$-invariant, it follows that the two complementary closed subspaces $L^2(U_j)$ of $L^2(\mathbb{R}^n)$ are both μ-invariant. Thus, by the discussion in section 4.2, the metaplectic representation of $K_{p,B}$ is equivalent to the sum $\delta = \delta_1 \oplus \delta_2$ of wavelet representations of $G_{p,B} \cong K_{p,B}$ on $L^2(\mathcal{O}_+) \oplus L^2(\mathcal{O}_+)$, where δ_i denotes the wavelet representation of $K_{1,B}$ on the corresponding copy of $L^2(\mathcal{O}_+)$. We therefore can make use of the admissibility results for the wavelet representation.

The $H_{p,B}$-orbit of a point $(\omega, \vec{\gamma}) \in \mathcal{O}_+$ is

$$(\omega, \vec{\gamma})h(t, \vec{y}) = (\omega, \vec{\gamma}) \begin{bmatrix} e^{pt} & \vec{y}^T \\ 0 & e^{Bt} \end{bmatrix} = \left(e^{pt}\omega,\, \omega\vec{y}^T + \vec{\gamma}e^{Bt}\right).$$

Now when $p = 0$, then the $H_{0,B}$-stabilizers of a point $(\omega, \vec{\gamma}) \in \mathcal{O}_+$ is of the form

$$H_{0,B}(\omega, \vec{\gamma}) = \left\{ a(t, \vec{y}) \in H_{0,B} \ : \ \vec{y}^T = \frac{\vec{\gamma}}{\omega}\left[I - e^{Bt}\right] \right\}$$

and is not compact so that the group is not admissible. We thus will assume that $p \neq 0$ in what follows. For $p \neq 0$ the stabilizers are easily see to be trivial.

Now we determine what function ϕ are admissible for the metaplectic representation. One way would be use (4.18) directly. In order make the connection with the wavelet representation clear once more, we will use (4.16) instead. We may choose the singleton consisting of $(\omega, \vec{\gamma}) = (1, 0)$ as transversal, whose orbit is

$$\mathcal{O}_{(1,0)} = \left\{ (e^{pt}, \vec{y}^T) \ : \ \vec{y} \in \mathbb{R}^n, t \in \mathbb{R} \right\} = \mathcal{O}_+.$$

Since $\mathcal{O}_+$ is a free, open orbit, $H_{p,B}$ is admissible by the results of Bernier and Taylor (1996). (Alternatively, one can verify that $\Delta \neq |\det|$ and ε-stabilizers are compact.) One easily verifies that the left Haar measure on $H_{p,B}$ is

$$d\mu(h(t, \vec{y})) = e^{-pnt}\, dt\, d\vec{y}.$$

By Proposition 3.6, a vector $\psi = \psi_1 + \psi_2 \in L^2(\mathcal{O}_+) \oplus L^2(\mathcal{O}_+)$ is admissible for δ, if and only if there is a constant $c_\psi > 0$ such that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} \overline{\psi_i(e^{pt}, \vec{y}^T)}\psi_j(e^{pt}, \vec{y}^T) e^{-pnt}\, dt\, d\vec{y} = \delta_{i,j} c_\psi. \tag{5.16}$$

Next, we determine the corresponding admissibility condition for the metaplectic representation μ. By Proposition 4.3, the unitary maps $Q_j : L^2(U_j) \to L^2(\mathcal{O}_+)$, $j = 1, 2$ are defined by

$$(Q_j\phi_j)(\omega, \vec{\gamma}) = \frac{1}{(2\omega)^{(n+1)/4}}\phi_j\left((-1)^j\sqrt{2\omega}, \frac{(-1)^j\vec{\gamma}^T}{\sqrt{2\omega}}\right).$$

for $\phi_j = \phi_{|U_j} \in L^2(U_j)$ and $(\omega, \vec{\gamma}) \in \mathcal{O}_+$. Thus for $\phi \in L^2(\mathbb{R}^n)$,

$$(Q_j\phi)(e^{pt}, \vec{y}^T) = \frac{1}{(2e^{pt})^{(n+1)/4}}\phi\left((-1)^j\sqrt{2e^{pt}}, \frac{(-1)^j\vec{y}}{\sqrt{2e^{pt}}}\right).$$

By (5.16) and Proposition 4.6, $\phi \in L^2(\mathbb{R}^n)$ is admissible for μ if and only if

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}} \overline{\phi\left((-1)^i\sqrt{2e^{pt}}, \frac{(-1)^i\vec{y}}{\sqrt{2e^{pt}}}\right)}\phi\left((-1)^j\sqrt{2e^{pt}}, \frac{(-1)^j\vec{y}}{\sqrt{2e^{pt}}}\right)\frac{e^{-pnt}dt\, d\vec{y}}{(2e^{pt})^{(n+1)/2}} = \delta_{i,j}c_\phi,$$

Setting $\vec{x} = (x_1, \vec{x}_0)^T$ where $x_1 = \sqrt{2e^{pt}}$, $\vec{x}_0 = \vec{y}/x_1$, this can be simplified to

$$\int_{\mathbb{R}^{n+1}_+} \overline{\phi\left((-1)^i\vec{x}\right)}\phi\left((-1)^j\vec{x}\right)\frac{2^{n+1}}{|p|x_1^{2n+2}}\,d\vec{x} = \delta_{i,j}c_\phi. \tag{5.17}$$

Note that this admissibility condition does not depend on the choice of dilation matrix B !

5.4 Connection with Known Groups

In order to relate the groups $G_{1,B}$ with some of the groups already discussed in the literature, we reparametrize all groups involved by changing the variable $\vec{y}$ to $e^{B^Tt}\vec{y}$. (This means we revert back from the groups $G_{r,A,s}$ to the groups $H_{r,A,s}$.) Thus,

$$D = D_{p,B} = \left\{a(t,\vec{y}) = \begin{bmatrix} e^{-pt/2} & 0 \\ -\frac{1}{2}e^{-pt/2}\vec{y} & e^{pt/2}e^{-B^Tt}\end{bmatrix} : \vec{y} \in \mathbb{R}^n, t \in \mathbb{R}\right\}.$$

with group law

$$a(t,\vec{y})\,a(t',\vec{y}') = a(t+t', \vec{y} + e^{(pI-B^T)t}\vec{y}').$$

Since the D-action now is

$$\left(a(t,\vec{y})^{-1}\right)^T m(z,\vec{x})\, a(t,\vec{y})^{-1} = m(e^{pt}z + \vec{y}^T e^{Bt}\vec{x}, e^{Bt}\vec{x}), \tag{5.18}$$

it follows that the group law on $K = K_{p,B} = D_{p,B} \rtimes M$ is now parametrized as

$$k(t,\vec{x},\vec{y},z)\, k(t',\vec{x}',\vec{y}',z') = k(t+t', \vec{x}+e^{Bt}x', \vec{y}+e^{(pI-B^T)t}\vec{y}', z+e^{pt}z'+\vec{y}^T e^{Bt}\vec{x}') \tag{5.19}$$

and the metaplectic representation of K thus turns out to be parametrized by

$$\begin{aligned}\mu(k(t,\vec{x},\vec{y},z))f(s,\vec{w}) &= [N_{-m(z,\vec{x})}D_{a(t,\vec{y})}f](s,\vec{w}) \\ &= e^{-i\pi(s^2z+2s\vec{w}^T\vec{x})}e^{(1-n)pt/4}e^{tr(B)t/2}f\left(e^{pt/2}s, e^{-pt/2}e^{B^Tt}\left(\frac{s}{2}\vec{y}+\vec{w}\right)\right).\end{aligned}$$

for $f \in L^2(\mathbb{R}^{n+1})$ and $(s,\vec{w}) \in \mathbb{R}^{n+1}$.

Furthermore, we have

$$H_{p,B} = \left\{ h(t,\vec{y}) = \begin{bmatrix} e^{pt} & \vec{y}^T e^{Bt} \\ 0 & e^{Bt} \end{bmatrix} \;:\; \vec{y} \in \mathbb{R}^n,\; t \in \mathbb{R} \right\}$$

and the group operation on the affine group

$$G_{p,B} = H_{p,B} \rtimes \mathbb{R}^{n+1} = \left\{ g(t,\vec{x},\vec{y},z) := \begin{bmatrix} e^{pt} & \vec{y}^T e^{Bt} & z \\ 0 & e^{Bt} & \vec{x} \\ 0 & 0 & 1 \end{bmatrix} \;:\; \vec{x},\vec{y} \in \mathbb{R}^n, t,z \in \mathbb{R} \right\}$$

follows the law (5.19).

The $H_{p,B}$-orbit of the transversal $(\omega,\vec{\gamma}) = (1,0)$ becomes

$$\mathcal{O}_{(1,0)} = \left\{ (e^{pt}, \vec{y}^T e^{Bt}) \;:\; \vec{y} \in \mathbb{R}^n, t \in \mathbb{R} \right\}$$

and the left Haar measure of $H_{p,B}$ is now parametrized to

$$d\mu(h(t,\vec{y})) = e^{(tr(B)-pn)t}\, dt\, d\vec{y}. \tag{5.20}$$

By Proposition 3.6, a vector $\psi = \psi_1 + \psi_2 \in L^2(\mathcal{O}_+) \oplus L^2(\mathcal{O}_+)$ is admissible for δ if and only if there is a constant $c_\psi > 0$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}} \overline{\psi_i(e^{pt}, \vec{y}^T e^{Bt})}\psi_j(e^{pt}, \vec{y}^T e^{Bt}) e^{(tr(B)-pn)t}\, dt\, d\vec{y} = \delta_{i,j} c_\psi. \tag{5.21}$$

A change of variables leads back to admissibility condition (5.16) and hence to (5.17).

We now show that some of the groups discussed in the recent literature fall into the class of the groups $G_{p,B}$. First choose $p = 2$ and $B = I_n$. Thus the dilation group is

$$H_{2,I_n} = \left\{ h(t,\vec{y}) = \begin{bmatrix} e^{2t} & e^t \vec{y}^T \\ 0 & e^t I_n \end{bmatrix} : \vec{y} \in \mathbb{R}^n, t \in \mathbb{R} \right\}$$

which by (5.20) has Haar measure $d\mu(h(t,\vec{y})) = e^{-nt}\, dt\, d\vec{y}$. Furthermore

$$G_{2,I_n} = \left\{ g(t,\vec{x},\vec{y},z) := \begin{bmatrix} e^{2t} & \vec{y}^T e^t & z \\ 0 & e^t I_n & \vec{x} \\ 0 & 0 & 1 \end{bmatrix} : \vec{x},\vec{y} \in \mathbb{R}^n, t, z \in \mathbb{R} \right\}$$

with group operation

$$g(t,\vec{x},\vec{y},z)g(t',\vec{x}',\vec{y}',z') = g(t+t', \vec{x} + e^t x', \vec{y} + e^t \vec{y}', z + e^{2t} z' + e^t \vec{y}^T \vec{x}').$$

Now we switch from the polarized Heisenberg group to the Heisenberg group by changing z to $u = 2z - \vec{y}^T \vec{x}$. The group law is now

$$g(t,\vec{x},\vec{y},u)g(t',\vec{x}',\vec{y}',u') = g(t+t', \vec{x} + e^t x', \vec{y} + e^t \vec{y}', u + e^{2t} u' + e^t\, [(\vec{x},\vec{y}),(\vec{x}',\vec{y}')]),$$

and we have obtained the group $\mathbb{H}_e^n$ of Cordero et al. (2010).

Next we choose $p = 1$ and $B = I_n$, to obtain the group

$$H_{1,I_n} = \left\{ h(t,\vec{y}) = \begin{bmatrix} e^t & e^t \vec{y}^T \\ 0 & e^t I_n \end{bmatrix} : \vec{y} \in \mathbb{R}^n, t \in \mathbb{R} \right\}$$

whose Haar measure by (5.20) is $d\mu(h(t,\vec{y})) = dt\, d\vec{y}$. The group law on

$$G_{1,I_n} = \left\{ g(t,\vec{x},\vec{y},z) := \begin{bmatrix} e^t & \vec{y}^T e^t & z \\ 0 & e^t I_n & \vec{x} \\ 0 & 0 & 1 \end{bmatrix} \ : \ \vec{x},\vec{y} \in \mathbb{R}^n, t,z \in \mathbb{R} \right\}$$

is

$$g(t,\vec{x},\vec{y},z)g(t',\vec{x}',\vec{y}',z') = g(t+t',\vec{x}+e^t x',\vec{y}+e^t\vec{y}',z+e^{2t}z'+e^t\vec{y}^T\vec{x}').$$

This is the group $(TDS)_n$ of King (2009).

Finally, we choose $p = 1$ and $B = \mathrm{diag}(\frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1)$. The group $G_{1,B} = H_{1,B} \rtimes \mathbb{R}^{n+1}$ is the group $(CDS)_n$ of Czaja and King (preprint).

We observe that the admissibility conditions given by the authors of these last three examples coincide with (5.17).

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CHAPTER VI

CONCLUSION

The results of this thesis fall into three parts.

In the first part, finite and countably infinite sums of modulated wavelet representations were introduced. Theorem 3.6 gives a characterization for a vector to be admissible for the sum of wavelet representations. Theorem 3.9 shows that a subgroup of the affine group is admissible for a sum of wavelet representations if and only if it is admissible for the usual wavelet representation, and its proof presents a concrete way on how to obtain an admissible vector. Theorem 3.10 then shows how to construct a bandlimited admissible vector, provided that the dilation group possesses an expanding matrix. This condition also allows the construction of frames as outlined in Theorems 3.12 and 3.13.

The second part considered subgroups of $Sp(n, \mathbb{R})$ which arise as semidirect products $K = D \rtimes M$ and are isomorphic to or compact extension of subgroups $G = H \rtimes \mathbb{R}^n$ of the affine group. By examples, it was shown that in many cases, the metaplectic representation of $D \rtimes M$ decomposes into a finite sum of subrepresentations, each of which is equivalent to a modulated wavelet representation of G, and hence of K. The admissibility results on sums of wavelet representations can thus be used to obtain concrete conditions for a function to be admissible for the metaplectic representation, and even to construct metaplectic frames. It was also shown that the concept of admissibility for the metaplectic representation given in Cordero et al. (2006a) is more narrow than the usual one. Finally, these techniques clarify the relationship between the metaplectic and wavelet representations of the group $SIM(2)$ derived ad-hoc in Cordero et al. (2006a), and show how frames can be introduced to the metaplectic representations of $SIM(2)$ and its two-fold covering.

In the third part, a one-parameter matrix group of dilations on the Heisenberg group $\mathbb{H}^n$ was introduced, generalizing previous results for $\mathbb{H}^1$ in Schulz and Taylor (1999). In Theorem 5.3, the extensions of $\mathbb{H}^n$ by such one-parameter groups were classified up to isomorphism using Lie-algebra techniques. It was shown that these extensions are subgroups of $Sp(n, \mathbb{R})$ as discussed in part 2, and hence admissibility conditions for the metaplectic representation could be derived. It the was shown that the groups TDS in Cordero et al. (2006a), $(TDS)_n$ in King (2009) and $(CDS)_n$ in Czaja and King (Preprint) are special cases of this construction.

While Proposition 3.8 gives a necessary condition for subgroups $D \rtimes M$ of $Sp(n, \mathbb{R})$ to be admissible, there is no sufficiency result yet which mirrors part 2 of Theorem 3.9 and provides conditions sufficient for admissibility of the metaplectic representations. The difficulty here is that in general, no analogue of the Plancherel Theorem is available. Work towards this goal, together with algorithms for constructing admissible functions, could be the direction of future research.

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