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A truncated Painlevé expansion associated with the Tzitzéica equation: consistency and general solution

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Abstract

It is shown that the overdetermined system obtained by truncating a Painlevé expansion for solutions of the classical Tzitzéica equation may be reduced to a compatible Frobenius system. Its general solution is expressed in terms of solutions of the standard linear representation of the Tzitzéica equation. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

In a series of papers between 1907 and 1910, the Romanian geometer Tzitzéica [1–3] investigated a particular class of surfaces (affine spheres) associated with what may be regarded as a nonlinear wave equation, namely

$$(\ln h)_{xy} = h - h^{-2}. \tag{1}$$

Tzitzéica not only established invariance of (1) under a Bäcklund transformation but also constructed what is nothing but a linear representation incorporating a spectral parameter [3]. The rediscovery of the Tzitzéica equation in a solitonic context had to wait until some seventy years later [4,5]. In 1953, Jonas [6] investigated properties of another avatar of the Tzitzéica equation, namely the affine sphere equation. The latter not only has intrinsic geometric importance but affords a direct connection with an integrable anisotropic gasdynamics system [7,8].

The Painlevé test and its modifications (see [9]) for partial differential equations as introduced by Weiss et al. [10] is widely considered as a test for integrability. In 1986, Weiss [11] showed that the Tzitzéica equation passes the Painleve test. Later, in 1994, Musette and Conte [12] noted that the Tzitzéica equation admits a nontrivial truncated Painlevé expansion. Such truncations are commonly associated with Bäcklund transformations. In [13], it was subsequently shown that indeed there resides a Bäcklund transformation within the overdetermined system of equations which is obtained by truncating the Painlevé expansion. Whether this Bäcklund transformation represents the general solution of this overdetermined system remained an open problem [14,15]. Here, we examine the overdetermined system in detail in order to demonstrate that the solution presented in [13] is, in fact, the general one.

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2. The truncated Painlevé expansion

The solutions of the Tzitzéica equation (1) or, equivalently,

$$hh_{xy} - h_x h_y = h^3 - 1, \tag{2}$$

are known [12] to admit a formal Laurent expansion of the form

$$h = 2\frac{\phi_x \phi_y}{\phi^2} - 2\frac{\phi_{xy}}{\phi} + H + \sum_{k=1}^{\infty} f_k \phi^k, \tag{3}$$

where ϕ is termed the singularity manifold function [9] and the coefficients f_k are functionals of the derivatives of ϕ. Truncation of the above expansion at the 'zeroth level' imposes constraints on ϕ which are obtained by inserting the ansatz

$$h = 2\frac{\phi_x \phi_y}{\phi^2} - 2\frac{\phi_{xy}}{\phi} + H, \tag{4}$$

into the Tzitzéica equation (2) and equating the coefficients of the various powers of ϕ to zero. It is evident that there exist four relations corresponding to $\phi^0,\ \phi^{-1},\ \phi^{-2},\ \phi^{-3}$, namely

$$HH_{xy} - H_x H_y = H^3 - 1, \tag{5}$$

$$\begin{aligned} H\phi_{xxyy} &= H_y \phi_{xxy} + H_x \phi_{xyy} \\ &\quad - H_{xy}\phi_{xy} + 3H^2\phi_{xy}, \end{aligned} \tag{6}$$

$$\begin{aligned} &-2H^2\phi_{xyy}\phi_{xxy} + 2H\phi_{xyy}\big(\phi_{xy}H_x + H^2\phi_x\big) \\ &\quad + 2H\phi_{xxy}\big(\phi_{xy}H_y + H^2\phi_y\big) - 2\phi_{xy}^2 H_x H_y + 2\phi_{xy}^2 \\ &\quad - 2\phi_{xy}\phi_x H_y H^2 - 2\phi_{xy}\phi_y H_x H^2 + \phi_{xx}\phi_{yy}H^3 \\ &\quad - \phi_{xx}\phi_y H_y H^2 - \phi_x\phi_{yy}H_x H^2 + \phi_x\phi_y H_x H_y H \\ &\quad - 2\phi_x\phi_y H^4 - \phi_x\phi_y H = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} &\phi_{xyy}\phi_y H(\phi_{xx}H - \phi_x H_x) \\ &\quad + \phi_{xxy}\phi_x H(\phi_{yy}H - \phi_y H_y) \\ &\quad - \phi_{xy}\phi_{xx}\phi_{yy}H^2 + \phi_{xy}\phi_x\phi_y H_x H_y - \phi_{xy}\phi_x\phi_y \\ &\quad - \phi_{xx}\phi_y^2 H^3 - \phi_x^2\phi_{yy}H^3 \\ &\quad + \phi_x^2\phi_y H_y H^2 + \phi_x\phi_y^2 H_x H^2 = 0. \end{aligned} \tag{8}$$

It is noted that the relation (6) has been used to simplify the relations (7), (8). Any solution of the above overdetermined system gives rise to a Bäcklund transformation since both H and

$$h = H - 2(\ln\phi)_{xy}, \tag{9}$$

obey the Tzitzéica equation (2). In fact, it has been shown [13] that a particular solution of the system (5)–(8) may be formally expressed in terms of the 'Gauß equations' for affine spheres and their associated polar-reciprocal counterparts [6]:

Theorem 1. *If H is any solution of the Tzitzéica equation* (5) *then the system*

$$\begin{aligned} \varphi_{xx} &= \frac{H_x}{H}\varphi_x + \frac{\lambda}{H}\varphi_y, & \psi_{xx} &= \frac{H_x}{H}\psi_x - \frac{\lambda}{H}\psi_y, \\ \varphi_{xy} &= H\varphi, & \psi_{xy} &= H\psi, \\ \varphi_{yy} &= \frac{H_y}{H}\varphi_y + \frac{\lambda^{-1}}{H}\varphi_x, & \psi_{yy} &= \frac{H_y}{H}\psi_y - \frac{\lambda^{-1}}{H}\psi_x, \end{aligned} \tag{10}$$

where $\lambda = \text{const}$, is compatible and the function ϕ defined by

$$\phi_x = \psi_x\varphi - \psi\varphi_x, \qquad \phi_y = \psi\varphi_y - \psi_y\varphi, \tag{11}$$

obeys the remaining equations (6)–(8) *provided that*

$$\psi_y\varphi_x + \psi_x\varphi_y = H\psi\varphi. \tag{12}$$

The latter constitutes an admissible constraint on (10). *The solution ϕ depends on* 5 *arbitrary constants of integration and the constant λ.*

As pointed out in [13], the system (10)–(12) is involutive [16,17]. Thus, for a given solution H of the Tzitzéica equation, the general solution (φ, ψ) of the subsystem (10), (12) depends on 5 constants of integration and the arbitrary constant λ. The function ϕ also depends on 6 arbitrary constants since it is defined up to an additive constant but its definition involves only products of φ and ψ. It is the purpose of this Letter to show that the relations (10)–(12) provide the general solution of the overdetermined system (5)–(8) if

$$\begin{aligned} \mu = \frac{1}{H}&\big[(H\phi_{xx} - H_x\phi_x)(H\phi_{yy} - H_y\phi_y) - \phi_x\phi_y\big] \\ &\leqslant 0. \end{aligned} \tag{13}$$

Extensive use of the computer algebra program REDUCE [18] has been made.

3. The consistency of (5)–(8)

Here, we demonstrate that the system (5)–(8) is consistent and that its general solution likewise depends on 6 arbitrary constants of integration. Thus, it is first noted that the degenerate cases $H\phi_{xx} - H_x\phi_x = 0$ or $H\phi_{yy} - H_y\phi_y = 0$ lead to the trivial condition $\phi_y\phi_x = 0$. We therefore consider the generic situation

$$\phi_y\phi_x(H\phi_{xx} - H_x\phi_x)(H\phi_{yy} - H_y\phi_y) \neq 0. \tag{14}$$

In this case, relation (8) may be solved for the derivative ϕ_{xxy}. Substitution of the latter into (7) is then readily shown to lead to a quadratic algebraic equation in ϕ_{xyy} which may be solved for ϕ_{xyy} provided that the associated discriminant is non-negative, that is

$$\Delta = \mu\big(2b - \mu\phi_{xy}^2\big) \geqslant 0, \tag{15}$$

where

$$b = \mu\big(\phi_{xy}^2 - H\phi_x\phi_y\big) - \phi_x^2\phi_y^2. \tag{16}$$

Hence, the derivatives ϕ_{xxy} and ϕ_{xyy} are found. These may now be inserted into (6) and the compatibility condition

$$\frac{\partial\phi_{xxy}}{\partial y} = \frac{\partial\phi_{xyy}}{\partial x}, \tag{17}$$

resulting in two linear algebraic equations for ϕ_{xxx} and ϕ_{yyy}. The corresponding determinant of this linear system turns out to be non-vanishing by virtue of the assumption (14). Consequently, all third-order derivatives of the function ϕ are determined:

$$\begin{aligned}
\phi_{xxx} &= \frac{3(\mu\phi_{xy}H + S)}{2\phi_y H(\phi_{yy}H - \phi_y H_y)} + \phi_x\frac{H_{xx}}{H},\\
\phi_{xxy} &= \frac{\mu\phi_{xy}H - S + 2\phi_x\phi_y\phi_{xy}}{2\phi_x H(\phi_{yy}H - \phi_y H_y)} + \phi_{xy}\frac{H_x}{H} + \phi_x H,\\
\phi_{xyy} &= \frac{\mu\phi_{xy}H + S + 2\phi_x\phi_y\phi_{xy}}{2\phi_y H(\phi_{xx}H - \phi_x H_x)} + \phi_{xy}\frac{H_y}{H} + \phi_y H,\\
\phi_{yyy} &= \frac{3(\mu\phi_{xy}H - S)}{2\phi_x H(\phi_{xx}H - \phi_x H_x)} + \frac{\phi_y H_{yy}}{H},
\end{aligned} \tag{18}$$

where the quantity S is defined by

$$S^2 = H^2\Delta. \tag{19}$$

It is directly verified that the remaining compatibility conditions

$$\frac{\partial\phi_{xxx}}{\partial y} = \frac{\partial\phi_{xxy}}{\partial x}, \qquad \frac{\partial\phi_{yyy}}{\partial x} = \frac{\partial\phi_{xyy}}{\partial y}, \tag{20}$$

are indeed satisfied. For computational purposes, it here proves convenient to be aware of the relations

$$\mu_x = 2\frac{\mu\phi_{xy}H + S}{\phi_y H}, \qquad \mu_y = 2\frac{\mu\phi_{xy}H - S}{\phi_x H}. \tag{21}$$

Thus, it has been shown that the system (5)–(8) is consistent and may be reduced to the system (18) if and only if the condition (15) holds. The general solution of the Frobenius system (18) is seen to contain 6 arbitrary constants of integration.

4. The general solution of (5)–(8)

In the preceding, it has been established that for a given solution H of the Tzitzéica equation (5) both the general solution of the Frobenius system (18) and ϕ constructed in terms of the solution of the linear system (10) depend on 6 arbitrary constants (of integration). This result is an indication that the latter indeed gives rise to the general solution of the overdetermined system (5)–(8). In order to show this rigorously, we now regard ϕ as a known solution of (18) and construct a solution (φ, ψ) of the system (10) subject to the constraint (12). Let us suppose for the moment that there exists such a pair (φ, ψ). Then, the relations (11) may be solved for the derivatives φ_x and φ_y since we assume that $\psi \neq 0$. The compatibility condition

$$\frac{\partial\varphi_x}{\partial y} = \frac{\partial\varphi_y}{\partial x} \tag{22}$$

produces $(10)_3$ which may be written as

$$\psi\phi_{xy} = \psi_y\phi_x + \psi_x\phi_y. \tag{23}$$

Moreover, $(10)_1$ assumes the form

$$2\lambda\varphi\psi_y = \phi_x H_x - \lambda\phi_y - H\phi_{xy}. \tag{24}$$

The latter two equations may now be solved for the derivatives ψ_x and ψ_y and hence the following expressions for the first-order derivatives of φ and ψ have been found:

$$\begin{aligned}
\varphi_x &= -\frac{\phi_x}{2\psi} + \frac{\lambda(2\psi^2\varphi\phi_{xy} - \phi_x\phi_y)}{2\psi(H_x\phi_x - H\phi_{xy})},\\
\varphi_y &= \frac{\phi_y}{2\psi} + \frac{H_x\phi_x - H\phi_{xy}}{2\lambda\psi},
\end{aligned}$$

$$\psi_x = \frac{\phi_x}{2\varphi} + \frac{\lambda(2\psi^2\varphi\phi_{xy} - \phi_x\phi_y)}{2\varphi(H_x\phi_x - H\phi_{xy})},$$
$$\psi_y = -\frac{\phi_y}{2\varphi} + \frac{H_x\phi_x - H\phi_{xy}}{2\lambda\varphi}. \tag{25}$$

It turns out that the condition resulting from the compatibility requirement

$$\frac{\partial\psi_x}{\partial y} = \frac{\partial\psi_y}{\partial x}, \tag{26}$$

and the remaining relations $(10)_{2,4,5,6}$, (12) are not independent. In fact, they reduce to

$$\lambda^2 = \frac{\mu H(-\phi_{xy}^2 + \phi_x\phi_y H) - S\phi_{xy} + \phi_x^2\phi_y^2 H}{H\phi_y^2(H_y\phi_y - H\phi_{yy})^2},$$
$$\lambda\varphi\psi = \frac{\mu\phi_{xy}H + S}{2H\phi_y(H_y\phi_y - H\phi_{yy})}, \tag{27}$$

and differential consequences thereof provided that

$$\phi_{xyy}\phi_y - \phi_{xy}\phi_{yy} = \phi_y^2\left(\frac{\phi_{xy}}{\phi_y}\right)_y \neq 0. \tag{28}$$

Differentiation reveals that $(27)_1$ is nothing but a first integral of the Frobenius system (18) and $(27)_2$ is compatible with the differential relations (25). The case $(\phi_{xy}/\phi_y)_y = 0$ leads to constraints on H so that the solution H of the Tzitzéica equation is not generic. Thus, we are in a position to formulate the following theorem:

Theorem 2. *If ϕ constitutes a solution of the overdetermined system* (5)–(8) *and therefore the Frobenius system* (18) *and the constant λ is given by* $(27)_1$ *then the functions φ and ψ defined by the compatible system* (25) *subject to the admissible constraint* $(27)_2$ *satisfy the system* (10)–(12).

For completeness, it is observed that for the above theorem to hold, the right-hand side of $(27)_1$ is required to be positive. This condition may be analyzed by studying $(27)_1$ written in the form

$$(H\phi_{yy} - H_y\phi_y)^2\phi_y^2\lambda^2 + b = -\frac{S\phi_{xy}}{H}. \tag{29}$$

Thus, if we solve the latter for S and insert the result into (19) then we obtain

$$\left[\lambda^2\phi_y(H\phi_{yy} - H_y\phi_y) - \phi_x(H\phi_{xx} - H_x\phi_x)\right]^2$$
$$+ 2\lambda^2\phi_{xy}^2\mu = 0. \tag{30}$$

Accordingly, μ as given by (13) must be non-positive. Now, since the discriminant $\Delta = \mu(2b - \mu\phi_{xy}^2) \geqslant 0$ and $\mu \leqslant 0$, it follows that $b \leqslant \mu\phi_{xy}^2/2$ and hence $b \leqslant 0$. If we choose the sign of S such that

$$\frac{S\phi_{xy}}{H} \leqslant 0, \tag{31}$$

then it is guaranteed that $\lambda^2 > 0$. Indeed, in terms of φ and ψ, we find that

$$\mu = -2\varphi^2\psi^2 \leqslant 0,$$
$$\mu\phi_{xy}^2 - 2b = 2\left(\frac{\varphi_y^2\psi_x\psi^2 - H\varphi_y\varphi\psi^3 + \psi_x\psi_y^2\varphi^2}{\psi_y}\right)^2$$
$$\geqslant 0. \tag{32}$$

We have therefore established that truncation of the Painlevé expansion for the Tzitzéica equation leads to the Bäcklund transformation set down in [13] if $\mu \leqslant 0$.

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